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$\mathbb{C}P^N$ SIGMA MODELS VIA THE SU(2) COHERENT STATES APPROACH

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Dedicated to Luc Vinet on the occasion of his sixtieth anniversary

Abstract. In this paper we present results obtained from the unification of $\mathbf{SU}(2)$ coherent states with $\mathbb{C}P^N$ sigma models defined on the Riemann sphere having finite actions. The set of coherent states generated by a vector belonging to a carrier space of an irreducible representation of the group gives rise to a map from the sphere into the set of rank-1 Hermitian projectors in that space. The map can be identified with a particular solution of the $\mathbb{C}P^N$ sigma model, where $N+1$ is equal to the dimension of the representation space. In particular a choice of the generating vector as the highest weight vector of the representation gives rise to the map known as a Veronese immersion. Using a description of the matrix elements of these representations in terms of Jacobi polynomials, we obtain an explicit parametrization of the solutions of the

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$\mathbb{C}P^N$ models, which has not been previously found. We relate the analytical properties of the solutions, which are known to belong to separate classes—holomorphic, anti-holomorphic and various types of mixed ones—to the weight corresponding to the chosen weight vector. Some examples of the described constructions are elaborated in detail in this paper.

1. Introduction. In this paper we study properties of solutions of $\mathbb{C}P^N$ sigma models in the group theoretical perspective, in particular using the language and methods of the theory of generalized coherent states. Such an analysis is facilitated by the introduction of a Hermitian projector formalism into the formulation of $\mathbb{C}P^N$ sigma models, [26], and the recognition of the group theoretical character of some of the solutions of $\mathbb{C}P^N$ sigma models in [15]. However, the interpretation within the framework of the theory of coherent states seems to have been noticed only recently, [20], adding a new application to this rapidly developing area of modern physics (see e.g. [8] and references therein).

We show that the systems of coherent states generated by the weight vectors of an irreducible representation of the $\mathbf{SU}(2)$ group can be described as a map into Hermitian rank-one projectors giving rise to harmonic transforms of the Veronese surface in the projective space associated to with the carrier space of the representation. Further, we formulate the invariant recurrence relations for $\mathbb{C}P^N$ models ([9]) in terms of shift operators for the representation. We also point to the possibility of parametrizing solutions of the $\mathbb{C}P^N$ model in terms of Jacobi polynomials. The connection between these two analytical descriptions constitute the main goal of this paper. It allows us to provide a unification of the $\mathbf{SU}(2)$ coherent states with the surfaces associated with $\mathbb{C}P^N$ sigma models, immersed in the $\mathfrak{su}(N+1)$ algebra.

This paper is organized as follows. In Section 2 we introduce basic notions on $\mathbb{C}P^N$ sigma models where we focus on the use of the projector formalism and the generalized Weierstrass formula for the immersion of surfaces into $\mathfrak{su}(N+1)$ Lie algebras. In Section 3 we recall the main elements of the $\mathbb{C}P^N$ theory from the group-theoretical point of view, which allows us to perform further computations, including the introduction of coherent states and covariant maps. Then, an explicit parametrization of coherent states for spin representations of $\mathbf{SU}(2)$ are expressed in terms of the Jacobi polynomials. Section 4 contains final remarks and possible future developments.

2. Solutions of $\mathbb{C}P^N$ models expressed in terms of projectors. The dynamical fields in the $\mathbb{C}P^N$ sigma models are maps from the unit sphere S^2 to the complex projective space $\mathbb{C}P^N$. Such maps can be described in terms of functions (fields) $z = (z_0, z_1, \dots, z_n) : S^2 \to S^{2N+1}$ taking values in the unit sphere $S^{2N+1} = \{z \in \mathbb{C}^{N+1} \mid |z|^2 = 1\}$, where the norm $|z| = \langle z^\dagger, z\rangle^{1/2}$ is derived from the standard Hermitian inner product $\langle z, w\rangle = z^\dagger \cdot w = \sum_{j=0}^{N} \overline{z}_j w_j$.

The independent variables of the fields of the $\mathbb{C}P^N$ model are pairs $(\xi^1, \xi^2) \in \mathbb{R}^2$ often written in complex form by $\xi = \xi^1 + i\xi^2$, and its complex conjugate $\overline{\xi} = \xi^1 - i\xi^2$. The covariant derivatives D_μ of the field $z \in S^{2N+1}$ are given as

$$D_\mu z = \partial_\mu z - (z^\dagger \cdot \partial_\mu z)z, \qquad \partial_\mu = \partial_{\xi^\mu},\ \mu = 1, 2. \tag{1}$$

The dynamics of the $\mathbb{C}P^N$ sigma model defined on the Riemann sphere $S^2 = \mathbb{C}\cup\{\infty\}$ are determined by the stationary points of the action functional $\mathcal{A} = \int_{S^2} \mathcal{L}(z)\, d\xi\, d\bar{\xi}$, where the Lagrangian density $\mathcal{L}$ is (see e.g. [26])

$$\mathcal{L}(z) = \frac{1}{4}(D_\mu z)^\dagger \cdot (D_\mu z). \tag{2}$$

The Euler–Lagrange (E–L) equations take the form

$$D_\mu D_\mu z + (D_\mu z)^\dagger \cdot (D_\mu z) z = 0, \tag{3}$$

subject to the algebraic constraint

$$z^\dagger z = 1. \tag{4}$$

Since $\mathcal{L}(z)$ is constant on fibers of the Hopf fibration $S^1 \to S^{2N+1} \to \mathbb{C}P^N$, obtained form the circle S^1 action on S^{2N+1} by coordinatewise multiplication, we observe that $\mathcal{L}(z)$ depends only on the map $[z] : \Omega \subset \mathbb{C} \to \mathbb{C}P^N$, where for $z \in S^{2N+1}$ we denote by $[z] = \{e^{i\psi} z \,|\, \psi \in \mathbb{R}\}$ the element of the projective space $\mathbb{C}P^N$ corresponding to z.

However, it is convenient to describe the models (3) in terms of the homogeneous, "unnormalized" field $\xi \mapsto f = (f_0, \ldots, f_N) \in \mathbb{C}^{N+1} \setminus \{0\}$ related to the "z's", for which

$$z = \frac{f}{(f^\dagger \cdot f)^{1/2}}\,. \tag{5}$$

Using the standard notion of complex derivatives ∂, $\bar{\partial}$ with respect to ξ and $\bar{\xi}$ given by

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial \xi^1} - i\,\frac{\partial}{\partial \xi^2}\right), \qquad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial \xi^1} + i\,\frac{\partial}{\partial \xi^2}\right),$$

we see that the homogeneous variable f satisfies an unconstrained form of the E–L equations

$$\left(\mathbb{I}_{N+1} - \frac{f \otimes f^\dagger}{f^\dagger \cdot f}\right) \cdot \left[\partial\bar{\partial} f - \frac{1}{f^\dagger \cdot f}\big((f^\dagger \cdot \bar{\partial} f)\partial f + (f^\dagger \cdot \partial f)\bar{\partial} f\big)\right] = 0, \tag{6}$$

where $\mathbb{I}_{N+1}$ is the $(N+1) \times (N+1)$ identity matrix.

An even more compact form of the E–L equations (6) is obtained by expressing them as a conservation law for the rank-one Hermitian projector

$$P = \frac{f \otimes f^\dagger}{f^\dagger \cdot f}\,, \tag{7}$$

satisfying $P^2 = P$, $P^\dagger = P$ associated with the field f. In this formulation the action functional takes the form

$$\mathcal{A}(P) = \int_{S^2} \mathrm{tr}(\partial P \bar{\partial} P)\, d\xi\, d\bar{\xi}. \tag{8}$$

In terms of the projector P the E–L equations (3) take the simple form

$$[\partial\bar{\partial} P, P] = 0, \tag{9}$$

or equivalently can be written as the conservation law

$$\partial[\bar{\partial} P, P] + \bar{\partial}[\partial P, P] = 0. \tag{10}$$

This gives the following expression for the matrix-valued (or more precisely $\mathfrak{su}(N+1)$-valued) 1-form

$$dX = i\big(-[\bar{\partial} P, P]\, d\xi + [\partial P, P]\, d\bar{\xi}\big) \tag{11}$$

which is closed and therefore can be integrated in any simply connected domain, e.g. on the surface of the sphere S^2, leading to an immersion of the domain into the Lie algebra $\mathfrak{su}(N+1)$. We will expand on that point later on.

Based on the Gram–Schmidt orthogonalization procedure, a method for constructing an entire class of solutions admitting the finite action $\mathcal{A}$ of the $\mathbb{C}P^N$ model was proposed by Din and Zakrzewski [6], later studied by Sasaki [19], and improved by Eells and Wood [7].

Under the assumption that the model is defined on the Riemann sphere $S^2 = \mathbb{C}\cup\{\infty\}$ and that its action (8) is finite, all the solutions can be obtained from a basic solution given in terms of the holomorphic nonconstant function by successive applications of the "raising" operator:

$$P_+ : f \in \mathbb{C}^{N+1} \setminus \{0\} \to P_+ f = \Big(\mathbb{I}_{N+1} - \frac{f \otimes f^\dagger}{f^\dagger \cdot f}\Big)\partial f \quad \text{for } \overline{\partial} f = 0 \tag{12}$$

or analogously in terms of a basic antiholomorphic solution given by an antiholomorphic nonconstant function under the application of the "lowering" operator:

$$P_- : f \in \mathbb{C}^{N+1} \setminus \{0\} \to P_- f = \Big(\mathbb{I}_{N+1} - \frac{f \otimes f^\dagger}{f^\dagger \cdot f}\Big)\overline{\partial} f \quad \text{for } \partial f = 0. \tag{13}$$

This method allows us to construct three classes of solutions: holomorphic, antiholomorphic and mixed solutions, which are determined by

$$f_k := P_+^k f, \quad k = 0, 1, \dots, N, \quad \text{where } P_+^0 = \mathrm{id},\ P_+^{N+1} f = 0. \tag{14}$$

Here the operator P_+^k is obtained by applying the operator P_+ k times successively. As a result we not only have information about all harmonic maps $S^2 \mapsto \mathbb{C}P^N$ but also an orthogonal basis in $\mathbb{C}^{N+1}$ of solutions of the $\mathbb{C}P^N$ model (10) [6].

The raising and lowering operators $P_\pm$ of solution (10) can also be expressed in terms of projector operators through the formulas given in [9].

$$P_{k+1} = \Pi_+(P_k) = \frac{\overline{\partial} P_k P_k \partial P_k}{\mathrm{tr}(\overline{\partial} P_k P_k \partial P_k)}\,, \qquad k = 0, 1, \dots, N \tag{15}$$

$$P_{k-1} = \Pi_-(P_k) = \frac{\partial P_k P_k \overline{\partial} P_k}{\mathrm{tr}(\partial P_k P_k \overline{\partial} P_k)}\,, \qquad \text{where } P_k = \frac{f_k \otimes f_k^\dagger}{f_k^\dagger \cdot f_k}\,, \tag{16}$$

$$\sum_{j=0}^{N} P_j = \mathbb{I}_N, \qquad P_k P_j = \delta_{kj} P_j. \tag{17}$$

As a result equation (7) gives an isomorphism between the equivalence classes of the $\mathbb{C}P^N$ model and the set of rank-one Hermitian projectors P_k. To each of these solutions (14) we can associate a surface in the $\mathfrak{su}(N+1)$ algebra [14] and, using equation (11) as in [13], we obtain a sequence of surfaces

$$X_k = -i\Big(P_k + 2\sum_{j=0}^{k-1} P_j\Big) + i c_k \mathbb{I}_{N+1} \in \mathfrak{su}(N+1), \quad c_k = \frac{1+2k}{N+1}\,. \tag{18}$$

Here, the c_k's are integration constants, ensuring that the integrals are skew-Hermitian and traceless, and therefore belong to the Lie algebra $\mathfrak{su}(N+1)$. The $\mathfrak{su}(N+1)$-valued matrix functions $X_k(\xi,\bar{\xi})$ constitute the generalized Weierstrass formula for the immersion of 2D surfaces into $\mathbb{R}^{N(N+2)}$, isomorphic to the Lie algebra $\mathfrak{su}(N+1)$, [16]. The matrix-valued functions X_k satisfy the following cubic matrix equations (the minimal polynomial identity) [10]:

$$(X_k - ic_k\mathbb{I}_{N+1})(X_k - i(c_k-1)\mathbb{I}_{N+1})(X_k - i(c_k-2)\mathbb{I}_{N+1}) = 0, \qquad 0<k<N \tag{19}$$

for any mixed solution of equation (10) in the $\mathbb{C}P^N$ model. For any holomorphic ($k=0$) or anti-holomorphic ($k=N$) solutions of equation (10) in the $\mathbb{C}P^N$ model, the minimal polynomial for the functions X_0 and X_N is quadratic,

$$\begin{aligned}(X_0 - ic_0\mathbb{I}_{N+1})(X_0 - i(c_0-1)\mathbb{I}_{N+1}) &= 0,\\ (X_N + ic_0\mathbb{I}_{N+1})(X_N + i(c_0-1)\mathbb{I}_{N+1}) &= 0, \qquad \text{where } c_0 + c_N = 2.\end{aligned}$$

The projectors P_k fulfill the completeness relation $\sum_{k=0}^{N} P_k = \mathbb{I}_{N+1}$, which implies in turn that the immersion functions X_k satisfy the linear relation $\sum_{k=0}^{N}(-1)^k X_k = 0$. The raising and lowering operators $\chi_\pm$ for the sequence of surfaces were devised in [9]. These operators map between the surfaces as follows

$$X_{k+1} = \chi_+(X_k) = X_k - i(\Pi_+(P_k) + P_k) + \frac{2i}{N+1}\,\mathbb{I}_{N+1}, \tag{20}$$

$$X_{k-1} = \chi_-(X_k) = X_k + i(\Pi_-(P_k) + P_k) - \frac{2i}{N+1}\,\mathbb{I}_{N+1}. \tag{21}$$

As a result a certain geometric characterization of the surfaces immersed in the $\mathfrak{su}(N+1)$ algebra can be performed (see e.g. [12, 9, 18]). The geometric properties are such that the surfaces X_k are conformally parametrized, and we can derive the first and second fundamental forms, principal curvatures, topological charges, Willmore functionals and Euler–Poincaré characteristics of the surfaces.

We would like to illustrate the above procedure by describing a particular solution of the $\mathbb{C}P^N$ sigma model, which will be discussed in detail in Section 3, coming from the classical Veronese imbeddings, cf. [3]. For the case of the $\mathbb{C}P^2$ model from the holomorphic Veronese map $f_0 = (1, \sqrt{2}\,\xi, \xi^2) \in \mathbb{C}^3 \setminus \{0\}$ we obtain a sequence of projectors

$$\begin{aligned}
P_0 &= \frac{f_0\otimes f_0^\dagger}{f_0^\dagger\cdot f_0} = \frac{1}{(1+|\xi|^2)^2}\begin{bmatrix} 1 & 2^{1/2}\bar{\xi} & \bar{\xi}^2\\ 2^{1/2}\xi & 2|\xi|^2 & 2^{1/2}|\xi|^2\bar{\xi}\\ \xi^2 & 2^{1/2}|\xi|^2\xi & |\xi|^4\end{bmatrix},\\
P_1 &= \frac{f_1\otimes f_1^\dagger}{f_1^\dagger\cdot f_1} = \frac{1}{(1+|\xi|^2)^2}\begin{bmatrix} 2|\xi|^2 & 2^{1/2}(|\xi|^2-1)\bar{\xi} & -2\bar{\xi}^2\\ 2^{1/2}(|\xi|^2-1)\xi & (|\xi|^2-1)^2 & -2^{1/2}(|\xi|^2-1)\bar{\xi}\\ -2\xi^2 & -2^{1/2}(|\xi|^2-1)\xi & 2|\xi|^2\end{bmatrix},\\
P_2 &= \frac{f_2\otimes f_2^\dagger}{f_2^\dagger\cdot f_2} = \frac{1}{(1+|\xi|^2)^2}\begin{bmatrix} |\xi|^4 & -2^{1/2}|\xi|^2\bar{\xi} & \bar{\xi}^2\\ -2^{1/2}|\xi|^2\xi & 2|\xi|^2 & -2^{1/2}\bar{\xi}\\ \xi^2 & -2^{1/2}\xi & 1\end{bmatrix}.
\end{aligned} \tag{22}$$

The corresponding $\mathfrak{su}(3)$-valued forms give the following immersions for the surfaces

$$X_0 = \frac{i}{(1+|\xi|^2)^2}\begin{bmatrix} \frac{1}{3}(|\xi|^4+2|\xi|^2-2) & -\sqrt{2}\,\bar{\xi} & -\bar{\xi}^2 \\ -\sqrt{2}\,\xi & \frac{1}{3}(|\xi|^4-4|\xi|^2+1) & -\sqrt{2}\,|\xi|^2\bar{\xi} \\ -\xi^2 & -\sqrt{2}\,|\xi|^2\xi & -\frac{1}{3}(2|\xi|^4-2|\xi|^2-1) \end{bmatrix},$$

$$X_1 = \frac{i}{(1+|\xi|^2)^2}\begin{bmatrix} |\xi|^2-1 & -\sqrt{2}\,\bar{\xi} & 0 \\ -\sqrt{2}\,\xi & 0 & -\sqrt{2}\,\bar{\xi} \\ 0 & -\sqrt{2}\,\xi & -|\xi|^2+1 \end{bmatrix}, \tag{23}$$

$$X_2 = \frac{i}{(1+|\xi|^2)^2}\begin{bmatrix} -\frac{1}{3}(1-2|\xi|^4+2|\xi|^2) & -\sqrt{2}\,|\xi|^2\bar{\xi} & \bar{\xi}^2 \\ -\sqrt{2}\,|\xi|^2\xi & -\frac{1}{3}(1+|\xi|^4-4|\xi|^2) & -\sqrt{2}\,\bar{\xi} \\ \xi^2 & -\sqrt{2}\,\xi & -\frac{1}{3}(|\xi|^4+2|\xi|^2-2) \end{bmatrix}.$$

Thus we see that the only solutions of (10) with a finite action (8) are given by holomorphic, mixed and antiholomorphic projectors P_k, $k = 0, 1, 2$, obtained from the successive action of the contracting operator P_+ applied to P_0.

An advantage of the presented approach is that, without referring to any additional considerations, the recurrence relations give a very useful tool for constructing the sequence of successive surfaces X_k associated with the $\mathbb{C}P^N$ sigma model obtained from the knowledge of the previous one. The geometrical setting allows us to study certain global properties of successive surfaces X_k as illustrated by the example of surfaces associated with the $\mathbb{C}P^2$ model.

We now demonstrate according to [9] that the surfaces X_k and X_l associated with the $\mathbb{C}P^{N-1}$ model do not intersect if $k \neq l$, with the exception of X_0 and X_1 in the $\mathbb{C}P^1$ model since X_0 and X_1 are equal in that case.

Proof. If l and k are two different indices of the induced surfaces, where $l > k$, then we obtain by subtracting (18) from the analogous expression for X_l

$$P_l - P_k + 2\sum_{j=k}^{l-1} P_j - \frac{2(l-k)}{N}\,\mathbb{I}_N = 0. \tag{24}$$

Multiplying equation (24) by P_k and using the orthogonality condition (17) we get

$$P_k\left(\mathbb{I}_N - \frac{2(l-k)}{N}\right) = 0. \tag{25}$$

On the other hand, multiplying both sides of the expression (24) by P_{l-1} we obtain

$$P_{l-1}\left(\mathbb{I}_N - \frac{l-k}{N}\right) = 0 \quad \text{for } k < l-1 \tag{26}$$

and

$$P_k\left(\mathbb{I}_N - \frac{2}{N}\right) = 0 \quad \text{for } k = l-1. \tag{27}$$

The equations (25), (26) and (27) can only be satisfied when $N = 2$, $l = 1$ and $k = 0$. This implies that $X_1 = X_0$. ∎

We now demonstrate that the immersion functions X_k and X_m make a constant angle relative to the Euclidean inner product (A, B) of the $\mathfrak{su}(N+1)$ matrices

$$(A, B) = -\tfrac{1}{2}\,\mathrm{tr}(A \cdot B). \tag{28}$$

That is, the angle Φ_{km} between the immersion functions X_k and X_m is independent of the choice of projector P_0 and of the coordinates ξ and $\bar{\xi}$. Hence, the angle Φ_{km} between two different immersion functions, X_k and X_m for $k < m$, is given ([9]) by

$$\cos\Phi_{km} = \frac{c_k(2-c_m)}{\{[c_k(2-c_k)-1/N][c_m(2-c_m)-1/N]\}^{1/2}}\,. \tag{29}$$

The formula (29) can be obtained either by using the scalar product from the generalized Weierstrass formula for immersion for X_k and X_m (18) (taking into account the fact that the projectors $P_0, P_1, \ldots, P_k$ are mutually orthogonal) or by operating directly on the eigenvalues of the immersion functions. In both cases we obtain for $m > k$

$$(X_k, X_m) = -\tfrac{1}{2}\,\mathrm{tr}(X_k \cdot X_m) = \tfrac{N}{2} c_k(2-c_m), \tag{30}$$

or in the case when $k = m$

$$(X_k, X_k) = -\tfrac{1}{2}\,\mathrm{tr}(X_k \cdot X_k) = \tfrac{1}{2}[N c_k(2-c_k) - 1]. \tag{31}$$

It is easy to show that $\cos\Phi_{km} \in (0, 1)$ except in the case where $N = 2$ (and clearly for $k = 0$, $m = 1$), for which the surfaces coincide and $\cos\Phi_{km} = 1$. Equation (29) is symmetric with respect to a transformation $k \leftrightarrow N-1-m$, as can be seen e.g. in tables of $\cos\Phi_{km}$ for the $\mathbb{C}P^2$ and $\mathbb{C}P^3$ models in Sections 3.4 and 4.

Finally we wish to point out the relation of the above construction to the linear spectral problem (LSP) associated with the $\mathbb{C}P^N$ sigma model as formulated in the papers [25, 5]

$$\partial\Phi_k = \frac{2}{1+\lambda}\,[\partial P_k, P_k]\Phi_k, \quad \bar{\partial}\Phi_k = \frac{2}{1-\lambda}[\bar{\partial} P_k, P_k]\Phi_k, \qquad 0 \le k \le N,\ \lambda \in \mathbb{C}. \tag{32}$$

The explicit solutions Φ_k of the LSP (32) which tend to the identity matrix $\mathbb{I}$ as $\lambda \to \infty$, were found in [5] to be

$$\Phi_k = \mathbb{I}_{N+1} + \frac{4\lambda}{(1-\lambda)^2}\sum_{j=0}^{k-1} P_j - \frac{2}{1-\lambda}\,P_k \in SU(N+1), \qquad \lambda = it, \tag{33}$$

$$\Phi_k^\dagger = \Phi_k^{-1} = \mathbb{I}_{N+1} - \frac{4\lambda}{(1+\lambda)^2}\sum_{j=0}^{k-1} P_j - \frac{2}{1+\lambda}\,P_k, \qquad t \in \mathbb{R}. \tag{34}$$

The immersion functions X_k may be expressed in terms of the wavefunctions Φ_k in two ways, either by the Sym–Tafel immersion formula [21] for completely integrable models

$$X_k = \alpha(\lambda)\Phi_k^{-1}\partial_\lambda\Phi_k + ic_k\mathbb{I}_{N+1} \in \mathfrak{su}(N+1), \tag{35}$$

(where $\alpha(\lambda)$ is an arbitrary function of the spectral parameter λ), or using its asymptotics for large values of λ [9],

$$X_k = \frac{i}{2}\lim_{\lambda\to\infty}[\lambda(\mathbb{I}_{N+1} - \Phi_k)] + ic_k\mathbb{I}_{N+1} \in \mathfrak{su}(N+1)\,. \tag{36}$$

So in conformal coordinates we obtain, as a result, a sequence of surfaces X_k whose structural equations are identical to the equations of motion (10) for the $\mathbb{C}P^N$ model, see [11],

$$[\partial\overline{\partial}X_k, X_k] = 0, \qquad k = 0, \dots, N. \tag{37}$$

The main goal of this paper is to provide a unification of the $SU(2)$ coherent states and the surfaces associated with $\mathbb{C}P^N$ sigma models immersed in the $\mathfrak{su}(N+1)$ algebra. Through this link, we derive the generating vectors as weight vectors of the representations in terms of the Jacobi polynomials parametrizing the solutions of the $\mathbb{C}P^N$ model.

3. Covariant construction of maps from S^2 into $\mathbb{C}P^N$. In order to establish the notation used in the sequel we shall recall in some detail the classical construction of the $\mathbf{SU}(2)$-covariant coverings of $S^2 \simeq \mathbb{C}P^1$ by the unit sphere $S^3 \simeq \mathbf{SU}(2)$. Let

$$\mathbf{SL}(2,\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,b,c,d \in \mathbb{C},\ ad - bc = 1 \right\}, \tag{38}$$

and let $\mathbf{SU}(2)$ be its unitary subgroup

$$\mathbf{SU}(2) = \left\{ \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix} \,\middle|\, u,v \in \mathbb{C},\ |u|^2 + |v|^2 = 1 \right\}. \tag{39}$$

Every matrix $g \in \mathbf{SU}(2)$ is determined by its first column, which is a vector of unit length in $\mathbb{C}^2$. In fact, setting

$$\xi = \begin{bmatrix} u \\ v \end{bmatrix} \in S^3 \subset \mathbb{C}^2, \quad \text{and} \quad J\xi = \begin{bmatrix} -\overline{v} \\ \overline{u} \end{bmatrix}, \tag{40}$$

we may write

$$g(\xi) = \begin{bmatrix} \xi & J\xi \end{bmatrix} = \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix}. \tag{41}$$

Thus $\mathbf{SU}(2)$ can be identified with the unit sphere S^3 contained in $\mathbb{R}^4 \simeq \mathbb{C}^2$. The polar coordinates in $\mathbb{R}^4$, $\mathbb{R}_+ \times S^3 \ni (t,\xi) \mapsto t\xi \in \mathbb{R}^4 \setminus \{0\}$ become a bijection of $\mathbb{R}^4 \setminus \{0\}$ with $\mathbb{R}_+ \times \mathbf{SU}(2)$, the multiplicative group of nonzero quaternions.

We take the customary Euler angles (θ, φ, ψ) as parameters for $\mathbf{SU}(2)$ and set $u = \cos(\theta/2)e^{i(\varphi+\psi)}$, $v = i\sin(\theta/2)e^{i(\psi-\varphi)}$, where $0 < \theta < \pi$, $0 \le \varphi < 2\pi$, $-2\pi \le \psi < 2\pi$,

$$\begin{aligned} g = g(\theta,\varphi,\psi) &= \begin{bmatrix} \cos(\theta/2)e^{i(\varphi+\psi)/2} & i\sin(\theta/2)e^{i(\varphi-\psi)/2} \\ i\sin(\theta/2)e^{i(\psi-\varphi)/2} & \cos(\theta/2)e^{-i(\varphi+\psi)/2} \end{bmatrix} \\ &= \begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix} \begin{bmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix} \begin{bmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{bmatrix}. \end{aligned} \tag{42}$$

In particular, the subgroup $K \subset \mathbf{SU}(2)$ of diagonal matrices of the form

$$d(\varphi) = g(0, 2\varphi, 0) = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{bmatrix}, \qquad 0 \le \varphi < 2\pi, \tag{43}$$

is isomorphic to $\mathbf{U}(1)$ and can be identified with the unit circle S^1 in the plane $\mathbb{C} \simeq \mathbb{R}^2$.

Let us recall the well-known identification of $\mathbb{R}^3$ with the space of traceless Hermitian 2 by 2 matrices obtained in terms of the Pauli matrices σ_α,

$$(x_1, x_2, x_3) \longleftrightarrow x \cdot \sigma = \sum_{\alpha=1}^{3} x_\alpha \sigma_\alpha = \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix}. \tag{44}$$

Thus there is a well-defined action of the group $\mathbf{SU}(2)$ on $\mathbb{R}^3$ by means of

$$(g \cdot x) \cdot \sigma = g^*(x \cdot \sigma)g. \tag{45}$$

This is an orthogonal action, since $|x| = -\det x \cdot \sigma = -\det g^*(x \cdot \sigma)g = |g \cdot x|$, and gives the familiar representation of $\mathbf{SU}(2)$ by rotations of $\mathbb{R}^3$.[1]

In particular, the orbit of $e_3 = (0, 0, 1) \in \mathbb{R}^3$ provides a convenient identification of the unit sphere $S^2 \subset \mathbb{R}^3$ with the homogeneous (coset) space $\mathbf{SU}(2)/\mathbf{U}(1)$. Thus for any $\xi \in S^3$ we may define an element $H(\xi) \in \mathbb{R}^3$ by the formula

$$g(\xi)^*(e_3 \cdot \sigma)g(\xi) = H(\xi) \cdot \sigma, \tag{46}$$

and since $|H(\xi)| = 1$, we actually have a map

$$H : \xi \in S^3 \mapsto H(\xi) \in S^2. \tag{47}$$

Moreover, since $H(\lambda\xi) = H(\xi)$ if $|\lambda| = 1$, each orbit of S^1 in S^3 is mapped to a single point, hence H is a fibration of S^3 by means of S^1, called *the Hopf fibration.*

Let us now look at the complex projective space $\mathbb{C}P^1$ of (complex) dimension 1. Given $(\xi_0, \xi_1) \neq 0$ we denote by $l(\xi_0, \xi_1)$ the complex line in $\mathbb{C}^2$ passing through (ξ_0, ξ_1) and the origin 0, and by Π the map (canonical projection) assigning the line $l(\xi_0, \xi_1)$ to that point (ξ_0, ξ_1). Any pair of complex numbers (ξ_0, ξ_1), where ξ_0 and ξ_1 are not both equal to zero, determining the line $l = l(\xi_0, \xi_1)$, is called a set of homogeneous coordinates of l and denoted by $[\xi_0 : \xi_1]$. Thus the quotient $\mathbb{C}^2_*/\mathbb{C}_*$, where the asterisk $*$ at the subscript position signifies the removal of 0, may be identified with the set of one-dimensional complex subspaces in $\mathbb{C}^2$—the complex projective space $\mathbb{C}P^1$. Furthermore, by restricting the projection Π to the unit sphere $S^3 \subset \mathbb{C}^2_*$, we see that $\Pi : S^3 \to \mathbb{C}P^1$ is surjective, so we may identify $\mathbb{C}P^1$ with S^3/S^1.

On the other hand, it is customary to identify $\mathbb{C}P^1$ with the compactified complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ by using the so-called inhomogeneous coordinates in $\mathbb{C}P^1$

$$[\xi_0 : \xi_1] \mapsto \begin{cases} w = \frac{\xi_1}{\xi_0} & \text{if } \xi_0 \neq 0 \\ w' = \frac{\xi_0}{\xi_1} & \text{if } \xi_1 \neq 0. \end{cases}$$

We shall denote by χ the map $\chi : \mathbb{C}P^1 \to \overline{\mathbb{C}}$ identifying the two spaces given by the top line above, so that

$$\chi([\xi_0 : \xi_1]) = \begin{cases} \frac{\xi_1}{\xi_0} & \text{if } \xi_0 \neq 0 \\ \infty & \text{if } \xi_0 = 0. \end{cases}$$

From this formula we get the familiar expression for the stereographic projection of S^2 from the South Pole $s \in S^2$ onto the complex plane

$$w = \frac{\xi_1}{\xi_0} = \frac{x_1 + ix_2}{1 + x_3}\,.$$

[1] We use g^* to denote the Hermitian conjugate of g.

The standard matrix action of the group $\mathbf{SL}(2,\mathbb{C})$ on the space $\mathbb{C}^2$ gives rise to the action on $\mathbb{C}P^1$

$$[\xi_0 \,:\, \xi_1] \mapsto g\cdot[\xi_0 \,:\, \xi_1] = [a\xi_0 + b\xi_1 \,:\, c\xi_0 + d\xi_1] = \left[1 \,:\, \frac{c\xi_0 + d\xi_1}{a\xi_0 + b\xi_1}\right],$$

inducing in turn the action on the compactified plane $\overline{\mathbb{C}}$ by means of the homographic (linear fractional) maps

$$\zeta \mapsto g\cdot\zeta = \frac{c+d\zeta}{a+b\zeta} \quad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } a + b\zeta \neq 0. \tag{48}$$

This situation is summarized in the well-known commutative diagram

PROPOSITION 1. *Let $H : S^3 \to S^2$ be the Hopf fibration, $\Pi : S^3 \to \mathbb{C}P^1 \simeq S^3/S^1$ be the projection of the sphere S^3 on the projective space corresponding to the orbit map and $\Phi : \overline{\mathbb{C}} \to S^2$ be the inverse mapping to the stereographic projection from the South Pole $s = -e_3 = (0,0,-1) \in S^2$. The diagram*

$$\begin{CD} S^3 @>H>> S^2 \\ @V{\Pi}VV @AA{\Phi}A \\ \mathbb{C}P^1 @>\chi>> \overline{\mathbb{C}} \end{CD} \tag{49}$$

is a commutative diagram of maps intertwining the respective actions of the group $\mathbf{SU}(2)$.

A similar construction of the projective space $\mathbb{C}P^N$ can be carried out at the general level (where N is an arbitrary natural number). We define

$$\mathbb{C}P^N = \mathbb{C}_*^{N+1}/\mathbb{C}_* \simeq \mathbf{SU}(N+1)/\mathbf{S}(\mathbf{U}(1)\times\mathbf{U}(N)). \tag{50}$$

The unit sphere $S^{2N+1} \subset \mathbb{C}^{N+1}$ also admits the Hopf fibration $S^1 \to S^{2N+1} \to \mathbb{C}P^N$, where the action of S^1 on S^{2N+1} is the componentwise multiplication by $\lambda \in S^1$. The space $\mathbb{C}P^N$ is equipped with a natural Riemannian metric, the so-called Fubini–Study metric, but we shall not need an explicit form of it. The situation is to a great extent analogous to the previous one, and at the higher-dimensional level we also have the following diagram.

THEOREM 1. *Let ξ be a homomorphism of $K \simeq \mathbf{U}(1)$ into $\mathbb{C}_*$, the multiplicative group of nonzero complex numbers, i.e. a character of $\mathbf{U}(1)$. Then each smooth (i.e. infinitely differentiable) map $F : \mathbf{SU}(2) \to \mathbb{C}^{N+1}$ such that*

$$F(gh) = \xi(h)F(g), \qquad g \in \mathbf{SU}(2),\ h \in K, \tag{51}$$

induces a unique smooth mapping $\Phi : S^2 \to \mathbb{C}P^N$ by the relation $\Phi(g\cdot e_1) = \Pi(F(g))$. This can be expressed by means of the commutativity of the diagram

$$\begin{CD} \mathbf{SU}(2) \simeq S^3 @>F>> \mathbb{C}^{N+1} \\ @V{H}VV @VV{\Pi}V \\ S^2 @>\Phi>> \mathbb{C}P^N. \end{CD} \tag{52}$$

3.1. Coherent states and covariant maps. Of fundamental importance for a quantum mechanical description of physical systems is that a state of a system is determined by a ray (one-dimensional subspace) in a Hilbert space rather than a single vector. Hence the space of states is properly described as the projective space PH of a certain Hilbert space H rather than H itself. In the case when the Hilbert space is finite-dimensional, by fixing a basis we may identify the projective space PH with the standard projective space $\mathbb{C}P^N$.

The well-known Perelomov definition of generalized coherent states, cf. [17], is a natural source of maps satisfying equation (51) (we shall refer to them as "covariant maps").

Definition 1 (Generalized coherent states, Perelomov). Given a representation T of a group G in a Hilbert space $\mathcal{H}$ and $\psi_0 \in \mathcal{H}$, such that $T(h)\psi_0 = \alpha(h)\psi_0$, with $\alpha(h) \in S^1$ for h belonging to a subgroup $H \subset G$, the image of the orbit $\{T(g)\psi_0 \,|\, g \in G\}$ in the set of states $\mathbf{P}(H)$ is said to be a system of (*generalized*) *coherent states of type* (T, ψ_0).

In this paper we are going to describe an application of that construction in the context of $\mathbb{C}P^N$ sigma models.

3.2. A brief review of irreducible representations of $\mathbf{SU}(2)$. One realizes representations of the group $\mathbf{SL}(2,\mathbb{C})$ on $\mathcal{P}(\mathbb{C}^2)$, the space of complex-valued polynomials in two complex variables z_1, z_2. The action is transferred from the standard (right) matrix action of $\mathbf{SL}(2,\mathbb{C})$ on row vectors in $\mathbb{C}^2$. For any polynomial $p(z) = p(z_1, z_2)$ in $\mathcal{P}(\mathbb{C}^2)$ we set

$$g \cdot p(z) = p(zg) = p(az_1 + cz_2, bz_1 + dz_2) \quad \text{for } g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbf{SL}(2,\mathbb{C}). \tag{53}$$

Let d be a non-negative integer and $\mathcal{P}^d = \mathcal{P}^d(\mathbb{C}^2) \subset \mathcal{P}(\mathbb{C}^2)$ be the subspace of homogeneous polynomials of degree d in z_1, z_2. Clearly $\dim \mathcal{P}^d = d + 1$. It is easy to observe that the action of $\mathbf{SL}(2,\mathbb{C})$ leaves invariant each of the subspaces $\mathcal{P}^d$ and hence the restriction of the action (53) to $\mathcal{P}^d$ defines a representation $\mathbf{SL}(2,\mathbb{C})$ in $\mathcal{P}^d$. This action is known to be irreducible for each integer d, and so is its restriction to the subgroup $\mathbf{SU}(2) \subset \mathbf{SL}(2,\mathbb{C})$ of unitary matrices. Dealing with these representations we shall often employ the notation used in the quantum theory of spin, and parametrize representations by the value $j = d/2$ with the meaning of the total spin. Accordingly, the representation space $\mathcal{P}^{2j}$ will be denoted by $\mathcal{H}^{(j)}$, where $\dim \mathcal{H}^{(j)} = 2j + 1$, and for $g \in \mathbf{SU}(2)$ the operators of the representation will be denoted by $T^{(j)}(g)$. Explicitly, with $p \in \mathcal{H}^{(j)}$ and $z = (z_1, z_2) \in \mathbb{C}^2$ we have

$$T^{(j)}(g)p(z) = p(uz_1 + vz_2, -\overline{v}z_1 + \overline{u}z_2) \quad \text{for } g = \begin{bmatrix} u & -\overline{v} \\ v & \overline{u} \end{bmatrix} \in \mathbf{SU}(2). \tag{54}$$

3.3. Matrix elements of irreducible representations of $\mathbf{SU}(2)$. It is easy to observe that polynomials of the form $z_1^{j+m} z_2^{j-m}$ are joint eigenvectors for the action of elements belonging to the diagonal subgroup K in $\mathbf{SU}(2)$, cf. (43), with eigenvalue $\chi_m(d(\varphi)) = e^{i2m\varphi}$. They are the weight vectors for the representation $T^{(j)}$. In the context of the quantum mechanical description of spin, the weight vectors correspond to the states with fixed spin j and projection m of spin on the third axis, which are usually denoted by $|j,m\rangle$.

We fix a basis of $\mathcal{H}^{(j)}$ consisting of weight vectors normalized as follows:

$$w_m^{(j)}(z) = \frac{z_1^{j+m} z_2^{j-m}}{[(j+m)!(j-m)!]^{1/2}}, \qquad m = -j, -j+1, \ldots, j-1, j. \tag{55}$$

This gives an orthonormal basis in $\mathcal{H}^{(j)}$ with respect to the inner product (Fischer inner product) which is defined in the following way. To every $p \in \mathcal{H}^{(j)}$ we assign a differential polynomial (homogeneous partial differential operator) in the variables z_1, z_2, denoted by $p(\partial)$, by the following formula:

$$p(z) = \sum_{\alpha+\beta=2j} p_{\alpha\,\beta} z_1^\alpha z_2^\beta \quad \longrightarrow \quad p(\partial) = \sum_{\alpha+\beta=2j} p_{\alpha\,\beta} \partial_1^\alpha \partial_2^\beta.$$

Then, with the bar denoting complex conjugation, we set for $p, q \in \mathcal{H}^{(j)}$

$$(p\,|\,q) = \overline{q}(\partial)p. \tag{56}$$

It can be verified that this is indeed an inner product and that the set $\{w_m^{(j)}\}$ is an orthonormal basis with respect to it. Moreover, the representation $T^{(j)}$ of the group $\mathbf{SU}(2)$ is unitary, cf. e.g. [24, Ch. 8].

The matrix elements $t_{km}^{(j)}(g)$ of the representation $T^{(j)}$ with respect to the basis of the weight vectors $\{w_m^{(j)}\}$ are defined by the expansion

$$T^{(j)}(g) w_m^{(j)}(z) = \sum_{k=-j}^{j} t_{km}^{(j)}(g) w_k^{(j)}(z). \tag{57}$$

Using the parametrization (41) of $\mathbf{SU}(2)$ we have an explicit formula

$$t_{km}^{(j)}(g(\xi)) = [(j+m)!(j-m)!(j+k)!(j-k)!]^{-1/2} \times \partial_1^{j+k} \partial_2^{j-k} (uz_1 + vz_2)^{j+m} (-\overline{v}z_1 + \overline{u}z_2)^{j-m}. \tag{58}$$

This defines matrix elements $t_{km}^{(j)}$ as functions on the unit sphere S^3 in $\mathbb{R}^4$, however one can extend them to homogeneous polynomials on $\mathbb{R}^4$ by the formula

$$\widetilde{t}_{km}^{(j)}(x) = |x|^{2j} t_{km}^{(j)}(g(|x|^{-1}x)), \quad \text{for } x \in \mathbb{R}^4 \setminus \{0\}.$$

Remarkably enough, these extensions turn out to be harmonic polynomials.

Theorem 2. *The matrix elements* $\xi \mapsto t_{km}^{(j)}(g(\xi))$ *are spherical harmonics of degree* $2j$*, i.e. their homogeneous extensions to* $\mathbb{R}^4$ *are harmonic polynomials.*

Before studying the matrix elements in their full generality we consider the special cases where $j = 1$ or $j = 3/2$.

3.4. Spin $l = 1$ case and classical Veronese surfaces. We shall examine in this and the following section the coherent states systems associated with the weight vectors for the spin 1 case and shall show that they are in fact the classical Veronese surface and its harmonic transforms. The physical content of this discussion will perhaps be clearer if we employ the notation of physical literature connected with the quantum mechanical description of spin. The weight vectors $w_m^{(j)}$ are usually denoted by $|j, m\rangle$, where j is the total spin of the system, and m is the spin projection. For $j = 1$ we have

$$\begin{aligned} |1,\,1\rangle &\simeq w_1^{(1)}(z) = 2^{-1/2} z_1^2;\\ |1,0\rangle &\simeq w_0^{(1)}(z) = z_1 z_2;\\ |1,-1\rangle &\simeq w_{-1}^{(1)}(z) = 2^{-1/2} z_2^2. \end{aligned} \tag{59}$$

Setting in (57) $j = 1$ we obtain by a direct calculation the following form of the orbit maps $\mathbf{SU}(2) \ni g \mapsto T^{(1)}(g)|1, m\rangle \in \mathcal{H}^{(1)}$ corresponding to the weight vectors $|1, m\rangle$

$$F_1(g(\xi)) = T^{(1)}(g(\xi))|1,\,1\rangle = u^2|1,\,1\rangle + 2^{1/2}uv|1,\,0\rangle + v^2|1,-1\rangle; \tag{60}$$

$$F_0(g(\xi)) = T^{(1)}(g(\xi))|1,\,0\rangle = -2^{1/2}\overline{v}u|1,\,1\rangle + (|u|^2 - |v|^2)|1,\,0\rangle + 2^{1/2}v\overline{u}|1,-1\rangle; \tag{61}$$

$$F_{-1}(g(\xi)) = T^{(1)}(g(\xi))|1,-1\rangle = \overline{v}^2|1,\,1\rangle - 2^{1/2}\overline{vu}|1,\,0\rangle + \overline{u}^2|1,-1\rangle. \tag{62}$$

Since for $d(\varphi)$ given by (43) and $m = 1, 0, -1$

$$F_m(g(\xi)d(\varphi)) = e^{2im\varphi}F_m(g(\xi)),$$

by identifying the space $\mathcal{H}^{(1)}$ with $\mathbb{C}^3$ by means of the basis $\{|1, m\rangle\}$ and referring to diagram (52) of Theorem 1 we see that each of these maps induces a map from the sphere S^2 to the projective space $\mathbb{C}P^2$. Now setting $\zeta = g{\cdot}0 = v/u \in \mathbb{C}_*$ and parametrizing the sphere by the complex plane we obtain the following result.

Proposition 2. *The maps* (60–62) *in terms of coordinates with respect to the basis* $\{|1, m\rangle\}$ *coincide with what in* [3] *is called the Veronese sequence—the* Veronese surface *and its harmonic transforms*:

$$\zeta \mapsto \phi_1(\zeta) = [(1, 2^{1/2}\zeta, \zeta^2)] \in \mathbb{C}P^2; \tag{63}$$

$$\zeta \mapsto \phi_0(\zeta) = [(-2^{1/2}\overline{\zeta}, 1 - |\zeta|^2, 2^{1/2}\zeta)] \in \mathbb{C}P^2; \tag{64}$$

$$\zeta \mapsto \phi_{-1}(\zeta) = [(\overline{\zeta}^2, -2^{1/2}\overline{\zeta}, 1)] \in \mathbb{C}P^2. \tag{65}$$

It may be worthwhile to express this map in terms of the Euler angles (42). If we note that $\zeta = \frac{v}{u} = i\tan(\theta/2)e^{-i\psi}$, it follows from (63–65) that

$$\begin{aligned} \phi_1(\theta,\psi) &= \big[(1, 2^{1/2}i\tan(\theta/2)e^{-i\psi}, -2\tan^2(\theta/2)e^{-i2\psi})\big] \in \mathbb{C}P^2;\\ \phi_0(\theta,\psi) &= \big[(2^{1/2}i\tan(\theta/2)e^{i\psi}, 1 - \tan^2(\theta/2), 2^{1/2}i\tan(\theta/2)e^{-i\psi})\big] \in \mathbb{C}P^2;\\ \phi_{-1}(\theta,\psi) &= \big[(-2\tan^2(\theta/2)e^{i2\psi}, 2^{1/2}i\tan(\theta/2)e^{i\psi}, 1)\big] \in \mathbb{C}P^2. \end{aligned}$$

Remark 1. The Veronese surface (63) is well known and much studied in the complex differential geometry. It is a conformal minimal immersion of S^2 into the projective space $\mathbb{C}P^2$ with constant curvature 2.

The harmonic transforms (64–65) of the Veronese surface also represent conformal minimal immersions with constant curvature. However, unlike the first one, they are not holomorphic.

It is worth noting that the highest, lowest respective, weight vectors give rise to holomorphic, antiholomorphic respective, maps into $\mathbb{C}P^2$.

The construction of "harmonic transforms" was used in disguise and in a different context (sigma-models) already in the classic paper of Din and Zakrzewski [6].

In order to relate this with the formulation given in [9] we explicitly state the expressions for projector-valued functions corresponding to the coherent states.

Corollary 1. *For* $m = 1, 0, -1$ *we define*

$$P_m(\zeta) = T(g)|1, m\rangle \otimes \langle m, 1|T^*(g), \qquad \text{where } \zeta = g \cdot 0.$$

The projection-valued fields corresponding to the weight vectors $|1, m\rangle$ *are as in* (22)

$$P_1(\zeta) = \frac{1}{(1+|\zeta|^2)^2}\begin{bmatrix} 1 & 2^{1/2}\overline{\zeta} & \overline{\zeta}^2 \\ 2^{1/2}\zeta & 2|\zeta|^2 & 2^{1/2}\overline{\zeta}|\zeta|^2 \\ \zeta^2 & 2^{1/2}\zeta|\zeta|^2 & |\zeta|^4 \end{bmatrix};$$

$$P_0(\zeta) = \frac{1}{(1+|\zeta|^2)^2}\begin{bmatrix} 2|\zeta|^2 & -2^{1/2}\overline{\zeta}(1-|\zeta|^2) & -2\overline{\zeta}^2 \\ -2^{1/2}\zeta(1-|\zeta|^2) & (1-|\zeta|^2)^2 & 2^{1/2}\overline{\zeta}(1-|\zeta|^2) \\ -2\zeta^2 & 2^{1/2}\zeta(1-|\zeta|^2) & 2|\zeta|^2 \end{bmatrix};$$

$$P_{-1}(\zeta) = \frac{1}{(1+|\zeta|^2)^2}\begin{bmatrix} |\zeta|^4 & -2^{1/2}\overline{\zeta}|\zeta|^2 & \overline{\zeta}^2 \\ -2^{1/2}\zeta|\zeta|^2 & 2|\zeta|^2 & -2^{1/2}\overline{\zeta} \\ \zeta^2 & -2^{1/2}\zeta & 1 \end{bmatrix}.$$

We note that our indexing here deviates from what was used in formulas following (22) in Section 2, since we want to conform with the customary ordering of weights. The present indices $m = 1, 0, -1$ stand for $k = 0, 1, 2$ we used them in what follows. The projectors $P_m(\zeta)$ coincide with the ones constructed from the recurrence relations (15) and (16). In fact, the operators P_+ and P_- termed in [9] raising and lowering operators respectively, which are at the origin of those recurrence relations can be modeled algebraically by means of the raising and lowering operators (shift operators) of the corresponding representation of $\mathbf{SU}(2)$.

Recall that in the case of the spin 1 representation, the shift operators act on the weight vectors (59) as differential operators

$$\pi_- = 2^{-1/2} z_2 \frac{\partial}{\partial z_1}; \qquad \pi_+ = 2^{-1/2} z_1 \frac{\partial}{\partial z_2},$$

so that the weight vectors form a chain (a ladder) obtained by successive applications of π_+ or π_-

$$\pi_- w_1 = w_0, \qquad \pi_- w_0 = w_{-1}, \qquad \pi_- w_{-1} = 0;$$
$$\pi_+ w_{-1} = w_0, \qquad \pi_+ w_0 = w_1, \qquad \pi_+ w_1 = 0.$$

Their matrices with respect to the basis of the weight vectors (w_1, w_0, w_1) are

$$\Pi_- = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \Pi_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The covariant map with respect to the group action from one system to another may be obtained by sending $T(g)|1,-1\rangle$ to $T(g)|1,0\rangle$ and similarly $T(g)|1,0\rangle$ to $T(g)|1,1\rangle$. Formally we consider the maps $\Pi_+(g) = T(g)\Pi_+T(g)^*$ and $\Pi_-(g) = T(g)\Pi_-T(g)^*$ and compute their matrices with respect to the basis of weight vectors. After somewhat tedious but straightforward computations we obtain

$$\Pi_+(g) = \begin{bmatrix} -2^{1/2}uv & u^2 & 0 \\ -v^2 & 0 & u^2 \\ 0 & -v^2 & 2^{1/2}uv \end{bmatrix}; \quad \Pi_-(g) = \begin{bmatrix} -2^{1/2}\overline{uv} & -\overline{v}^2 & 0 \\ \overline{u}^2 & 0 & -\overline{v}^2 \\ 0 & \overline{u}^2 & 2^{1/2}\overline{uv} \end{bmatrix}.$$

One can check that $\Pi_+(g) : T(g)|1,0\rangle \mapsto T(g)|1,1\rangle$, or $\Pi_-(g)T(g)|1,1\rangle \mapsto T(g)|1,0\rangle$, explicitly

$$\Pi_+(g)T(g)|1,0\rangle = \Pi_+(g)\begin{bmatrix} -2^{1/2}u\overline{v} \\ |u|^2 - |v|^2 \\ 2^{1/2}\overline{u}v \end{bmatrix} = \begin{bmatrix} u^2 \\ 2^{1/2}uv \\ v^2 \end{bmatrix};$$

$$\Pi_-(g)T(g)|1,1\rangle = \Pi_+(g)\begin{bmatrix} |u|^2 \\ 2^{1/2}uv \\ v^2 \end{bmatrix} = \begin{bmatrix} -2^{1/2}u\overline{v} \\ |u|^2 - |v|^2 \\ 2^{1/2}\overline{u}v \end{bmatrix}.$$

To end this overview we restate, with the new indexing, the formulas (23) for the $\mathfrak{su}(3)$-valued forms X_m, which give the immersions into the Lie algebra $\mathfrak{su}(3)$ of the surfaces belonging to the Veronese sequence studied here and state their main geometrical properties.

$$X_1 = \frac{i}{(1+|\zeta|^2)^2}\begin{bmatrix} \frac{1}{3}(|\zeta|^4 + 2|\zeta|^2 - 2) & -2^{1/2}\overline{\zeta} & -\overline{\zeta}^2 \\ -2^{1/2}\zeta & \frac{1}{3}(|\zeta|^4 - 4|\zeta|^2 + 1) & -2^{1/2}|\zeta|^2\overline{\zeta} \\ -\zeta^2 & -2^{1/2}|\zeta|^2\zeta & -\frac{1}{3}(2|\zeta|^4 - 2|\zeta|^2 - 1) \end{bmatrix},$$

$$X_0 = \frac{i}{(1+|\zeta|^2)^2}\begin{bmatrix} |\zeta|^2 - 1 & -2^{1/2}\overline{\zeta} & 0 \\ -2^{1/2}\zeta & 0 & -2^{1/2}\overline{\zeta} \\ 0 & -2^{1/2}\zeta & -|\zeta|^2 + 1 \end{bmatrix}, \tag{66}$$

$$X_{-1} = \frac{i}{(1+|\zeta|^2)^2}\begin{bmatrix} -\frac{1}{3}(1 - 2|\zeta|^4 + 2|\zeta|^2) & -2^{1/2}|\zeta|^2\overline{\zeta} & \overline{\zeta}^2 \\ -2^{1/2}|\zeta|^2\zeta & -\frac{1}{3}(1 + |\zeta|^4 - 4|\zeta|^2) & -2^{1/2}\overline{\zeta} \\ \zeta^2 & -2^{1/2}\zeta & -\frac{1}{3}(|\zeta|^4 + 2|\zeta|^2 - 2) \end{bmatrix}.$$

The angles between the immersion functions X_k and X_l associated with the $\mathbb{C}P^2$ model are given by

$k\backslash l$	1	0	−1
1	$5/\sqrt{33}$	$\sqrt{3/11}$	$1/3$
0	$\sqrt{3/11}$	$9/11$	$\sqrt{3/11}$
−1	$1/3$	$\sqrt{3/11}$	$5/\sqrt{33}$

We now explore certain geometrical characteristics of surfaces (66) immersed in the $\mathfrak{su}(3)$ algebra and express them in terms of the projectors P_m. Using the known expression for the Gaussian curvatures

$$\mathcal{K}_m = -2\,\frac{\partial\overline{\partial}\ln|\mathrm{tr}(\partial P_m \cdot \overline{\partial} P_m)|}{\mathrm{tr}(\partial P_m \cdot \overline{\partial} P_m)}, \tag{67}$$

one checks easily that each surface X_m has constant and positive curvature, that is

$$\mathcal{K}_1 = \mathcal{K}_{-1} = 2, \qquad \mathcal{K}_0 = 1. \tag{68}$$

Also the norms $\|\cdot\| = (\cdot,\cdot)^{1/2}$ of the mean curvature vectors

$$\mathcal{H}_m = -4i\, \frac{[\partial P_m, \bar{\partial} P_m]}{\mathrm{tr}(\partial P_m \cdot \bar{\partial} P_m)} \tag{69}$$

are constant and positive

$$\|\mathcal{H}_1\| = \|\mathcal{H}_{-1}\| = 4, \qquad \|\mathcal{H}_0\| = 2. \tag{70}$$

The Willmore functionals are defined by

$$W_m = \int_{S^2} \mathrm{tr}([\partial P_m, \bar{\partial} P_m])^2 \, d\zeta^1 \, d\zeta^2, \tag{71}$$

and by computing the integrals we get

$$W_1 = W_{-1} = 4\pi, \qquad W_0 = 2\pi. \tag{72}$$

The topological charges associated with the surfaces X_m are defined by

$$Q_m = \frac{-2}{\pi} \int_{S^2} \mathrm{tr}(P_m \cdot [\partial P_m, \bar{\partial} P_m]) \, d\zeta^1 \, d\zeta^2,$$

and are

$$Q_1 = 2, \qquad Q_0 = 1, \qquad Q_{-1} = -2.$$

The Euler–Poincaré characters are determined by

$$\Delta_m = \frac{-2}{\pi} \int_{S^2} \partial\bar{\partial} \ln|\mathrm{tr}(\partial P_m \cdot \bar{\partial} P_m)| \, d\zeta^1 \, d\zeta^2, \tag{73}$$

and we obtain the same value for the surfaces X_m, i.e.

$$\Delta_1 = \Delta_0 = \Delta_{-1} = 2. \tag{74}$$

This means that the surfaces X_m are homeomorphic to ovaloids, since $\mathcal{K}_m > 0$.

4. The case of spin $l = 3/2$. The case of spin $l = 3/2$ can be discussed in an analogous way as the previous one. The weight vectors $|3/2, m\rangle$ may be identified with the polynomials

$$w_m^{(3/2)}(z) = \frac{z_1^{3/2+m} z_2^{3/2-m}}{[(3/2+m)!(3/2-m)!]^{1/2}},$$

hence by virtue of the formula (57) the orbit of the highest weight vector $|3/2, 3/2\rangle$ can be parametrized as

$$S^3 \ni \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \sqrt{3!}\, u^3 \big(\big|\tfrac{3}{2}, \tfrac{3}{2}\big\rangle + \sqrt{3}\,\zeta \big|\tfrac{3}{2}, \tfrac{1}{2}\big\rangle + \sqrt{3}\,\zeta^2 \big|\tfrac{3}{2}, -\tfrac{1}{2}\big\rangle + \zeta^3 \big|\tfrac{3}{2}, -\tfrac{3}{2}\big\rangle\big),$$

where we have set as before $\zeta = v/u$, the stereographic parameter of the S^2 sphere. Thus the basic holomorphic solution of the $\mathbb{C}P^3$ sigma model is obtained by projecting this orbit to the $\mathbb{C}P^3$ space. This gives the Veronese immersion

$$\mathbb{C} \ni \zeta \mapsto (1, \sqrt{3}\,\zeta, \sqrt{3}\,\zeta^2, \zeta^3) \in \mathbb{C}^4$$

and its corresponding projector-valued field

$$P_{3/2} = \frac{1}{(1+|\zeta|^2)^3}\begin{bmatrix} 1 & \sqrt{3}\,\overline{\zeta} & \sqrt{3}\,\overline{\zeta}^2 & \overline{\zeta}^3 \\ \sqrt{3}\,\zeta & 3\,|\zeta|^2 & 3\,|\zeta|^2\,\overline{\zeta} & \sqrt{3}\,\,|\zeta|^2\,\overline{\zeta}^2 \\ \sqrt{3}\,\zeta^2 & 3\zeta\,|\zeta|^2 & 3\,|\zeta|^4 & \sqrt{3}\,\,|\zeta|^4\,\overline{\zeta} \\ \zeta^3 & \sqrt{3}\,\zeta^2\,|\zeta|^2 & \sqrt{3}\,\zeta\,|\zeta|^4 & |\zeta|^6 \end{bmatrix}.$$

The corresponding $\mathfrak{su}(4)$-valued immersion function $X_{3/2}$ takes the form

$$X_{3/2} = \frac{i}{(1+|\zeta|^2)^3}\left[\begin{matrix} \frac{1}{4}(|\zeta|^6+3|\zeta|^4+3|\zeta|^2-3) & -\sqrt{3}\,\overline{\zeta} \\ -\sqrt{3}\,\zeta & \frac{1}{4}(|\zeta|^6+3|\zeta|^4-9|\zeta|^2+1) \\ -\sqrt{3}\,\zeta^2 & -3\zeta|\zeta|^2 \\ -\zeta^3 & -\sqrt{3}\,\zeta^2|\zeta|^2 \end{matrix}\right.$$

$$\left.\begin{matrix} -\sqrt{3}\,\overline{\zeta}^2 & -\overline{\zeta}^3 \\ -3|\zeta|^2\overline{\zeta} & -\sqrt{3}\,|\zeta|^2\overline{\zeta}^2 \\ \frac{1}{4}(|\zeta|^6-9|\zeta|^4+3|\zeta|^2+1) & -\sqrt{3}\,|\zeta|^4\overline{\zeta} \\ -\sqrt{3}\,\zeta|\zeta|^4 & \frac{1}{4}(-3|\zeta|^6+3|\zeta|^4+3|\zeta|^2+1) \end{matrix}\right].$$

In an entirely analogous way one can compute the coherent state generated by the weight vector $|3/2,1/2\rangle$. It is parametrized by the formula

$$S^3 \ni \begin{bmatrix} u \\ v \end{bmatrix} \mapsto T(g)\big|\tfrac{3}{2},\tfrac{1}{2}\big\rangle = -\sqrt{3!}\,u^2\overline{v}\big|\tfrac{3}{2},\tfrac{3}{2}\big\rangle + u(|u|^2-2|v|^2)\big|\tfrac{3}{2},\tfrac{1}{2}\big\rangle$$
$$+ v(2|u|^2-|v|^2)\big|\tfrac{3}{2},-\tfrac{1}{2}\big\rangle + \sqrt{3!}\,v^2\overline{u}\big|\tfrac{3}{2},-\tfrac{3}{2}\big\rangle\big),$$

After proper normalization we get

$$\mathbb{C} \ni \zeta \mapsto \big(3|\zeta|^2, \sqrt{3}\,\zeta(2|\zeta|^2-1), \sqrt{3}\,\zeta^2(|\zeta|^2-2),\ -3\zeta^3\big) \in \mathbb{C}^4$$

and the projector-valued field corresponding to this imbedding is given by

$$(1+|\zeta|^2)^3 P_{1/2} =$$
$$\begin{bmatrix} 3|\zeta|^2 & \sqrt{3}\,(2|\zeta|^2-1)\overline{\zeta} & \sqrt{3}\,(|\zeta|^2-2)\overline{\zeta}^2 & -3\overline{\zeta}^3 \\ \sqrt{3}\,\zeta(2|\zeta|^2-1) & (1-2|\zeta|^2)^2 & (|\zeta|^2-2)(2|\zeta|^2-1)\overline{\zeta} & -\sqrt{3}\,(2|\zeta|^2-1)\overline{\zeta}^2 \\ \sqrt{3}\,\zeta^2(|\zeta|^2-2) & \zeta(|\zeta|^2-2)(2|\zeta|^2-1) & |\zeta|^2(|\zeta|^2-2)^2 & -\sqrt{3}\,|\zeta|^2(|\zeta|^2-2)\overline{\zeta} \\ -3\zeta^3 & -\sqrt{3}\,\zeta^2(2|\zeta|^2-1) & -\sqrt{3}\,\zeta|\zeta|^2(|\zeta|^2-2) & 3|\zeta|^4 \end{bmatrix}.$$

The corresponding immersion function $X_{1/2}$ is obtained from (18) and reads

$$X_{1/2} = \frac{i}{(1+|\zeta|^2)^3}\left[\begin{matrix} \frac{1}{4}(3|\zeta|^6+9|\zeta|^4-3|\zeta|^2-5) & -\sqrt{3}\,(2|\zeta|^2+1)\overline{\zeta} \\ -\sqrt{3}\,\zeta(2|\zeta|^2+1) & \frac{1}{4}((3|\zeta|^2-7)|\zeta|^4+|\zeta|^2-1) \\ -\sqrt{3}\,\zeta^2|\zeta|^2 & -\zeta(2|\zeta|^4+|\zeta|^2+2) \\ \zeta^3 & -\sqrt{3}\,\zeta^2 \end{matrix}\right.$$

$$\left.\begin{matrix} -\sqrt{3}\,|\zeta|^2\overline{\zeta}^2 & \overline{\zeta}^3 \\ (-2|\zeta|^4-|\zeta|^2-2)\overline{\zeta} & -\sqrt{3}\,\overline{\zeta}^2 \\ \frac{1}{4}((1-|\zeta|^2)|\zeta|^4-7|\zeta|^2+3) & -\sqrt{3}\,|\zeta|^2(|\zeta|^2+2)\overline{\zeta} \\ -\sqrt{3}\,\zeta|\zeta|^2(|\zeta|^2+2) & \frac{1}{4}(-5|\zeta|^6-3|\zeta|^4+9|\zeta|^2+3) \end{matrix}\right].$$

The remaining projection-valued fields and immersion functions are computed in an analogous way—we just give the results.

$$P_{-1/2} = \frac{1}{(1+|\zeta|^2)^3}\begin{bmatrix} 3|\zeta|^4 & \sqrt{3}\,|\zeta|^2(|\zeta|^2-2)\overline{\zeta} & -\sqrt{3}\,(2|\zeta|^2-1)\overline{\zeta}^2 & 3\overline{\zeta}^3 \\ \sqrt{3}\,\zeta|\zeta|^2(|\zeta|^2-2) & |\zeta|^2(|\zeta|^2-2)^2 & (2-|\zeta|^2)(2|\zeta|^2-1)\overline{\zeta} & \sqrt{3}\,(|\zeta|^2-2)\overline{\zeta}^2 \\ -\sqrt{3}\,\zeta^2(2|\zeta|^2-1) & -\zeta(|\zeta|^2-2)(2|\zeta|^2-1) & (1-2|\zeta|^2)^2 & -\sqrt{3}\,(2|\zeta|^2-1)\overline{\zeta} \\ 3\zeta^3 & \sqrt{3}\,\zeta^2(|\zeta|^2-2) & -\sqrt{3}\,\zeta(2|\zeta|^2-1) & 3|\zeta|^2 \end{bmatrix};$$

$$X_{-1/2} = \frac{i}{(1+|\zeta|^2)^3}\begin{bmatrix} \frac{1}{4}(5|\zeta|^6+3|\zeta|^4-9|\zeta|^2-3) & -\sqrt{3}\,|\zeta|^2(|\zeta|^2+2)\overline{\zeta} & \sqrt{3}\,\overline{\zeta}^2 & \overline{\zeta}^3 \\ -\sqrt{3}\,\zeta|\zeta|^2(|\zeta|^2+2) & \frac{1}{4}(|\zeta|^6-|\zeta|^4+7|\zeta|^2-3) & -(2|\zeta|^4+|\zeta|^2+2)\overline{\zeta} & \sqrt{3}\,|\zeta|^2\overline{\zeta}^2 \\ \sqrt{3}\,\zeta^2 & -\zeta(2|\zeta|^4+|\zeta|^2+2) & \frac{1}{4}(-3|\zeta|^6+7|\zeta|^4-|\zeta|^2+1) & -\sqrt{3}\,(2|\zeta|^2+1)\overline{\zeta} \\ \zeta^3 & \sqrt{3}\,\zeta^2|\zeta|^2 & -\sqrt{3}\,\zeta(2|\zeta|^2+1) & \frac{1}{4}(-3|\zeta|^6-9|\zeta|^4+3|\zeta|^2+5) \end{bmatrix}$$

and finally

$$P_{-3/2} = \frac{1}{(1+|\zeta|^2)^3}\begin{bmatrix} |\zeta|^6 & -\sqrt{3}\,|\zeta|^4\overline{\zeta} & \sqrt{3}\,|\zeta|^2\overline{\zeta}^2 & -\overline{\zeta}^3 \\ -\sqrt{3}\,\zeta|\zeta|^4 & 3|\zeta|^4 & -3|\zeta|^2\overline{\zeta} & \sqrt{3}\,\overline{\zeta}^2 \\ \sqrt{3}\,\zeta^2|\zeta|^2 & -3\zeta|\zeta|^2 & 3|\zeta|^2 & -\sqrt{3}\,\overline{\zeta} \\ -\zeta^3 & \sqrt{3}\,\zeta^2 & -\sqrt{3}\,\zeta & 1 \end{bmatrix}$$

$$X_{-3/2} = \frac{i}{(1+|\zeta|^2)^3}\begin{bmatrix} \frac{1}{4}(3|\zeta|^6-3|\zeta|^4-3|\zeta|^2-1) & -\sqrt{3}\,|\zeta|^4\overline{\zeta} & \sqrt{3}\,|\zeta|^2\overline{\zeta}^2 & -\overline{\zeta}^3 \\ -\sqrt{3}\,\zeta|\zeta|^4 & \frac{1}{4}(-|\zeta|^6+9|\zeta|^4-3|\zeta|^2-1) & -3|\zeta|^2\overline{\zeta} & \sqrt{3}\,\overline{\zeta}^2 \\ \sqrt{3}\,\zeta^2|\zeta|^2 & -3\zeta|\zeta|^2 & \frac{1}{4}(-|\zeta|^6-3|\zeta|^4+9|\zeta|^2-1) & -\sqrt{3}\,\overline{\zeta} \\ -\zeta^3 & \sqrt{3}\,\zeta^2 & -\sqrt{3}\,\zeta & \frac{1}{4}(-|\zeta|^6-3|\zeta|^4-3|\zeta|^2+3) \end{bmatrix}$$

The angles between the immersion functions X_k and X_l associated with the $\mathbb{C}P^3$ model have the form

$k\backslash l$	$3/2$	$1/2$	$-1/2$	$-3/2$
$3/2$	$3/8$	$5/8$	$3/8$	$1/8$
$1/2$	$5/8$	$11/8$	$9/8$	$3/8$
$-1/2$	$3/8$	$9/8$	$11/8$	$5/8$
$-3/2$	$1/8$	$3/8$	$5/8$	$3/8$

The immersion functions X_k are considered as position vectors whose endpoints trace out the two-dimensional surfaces in the $\mathfrak{su}(N+1)$ algebra. This implies that the position vectors make a constant angle with each other, independent of the variables ζ and $\overline{\zeta}$. Furthermore, within a particular $\mathbb{C}P^N$ model and corresponding coherent state, the angle is the same for all choices of holomorphic solutions P_k of the Euler–Lagrange equations (9).

The Gaussian curvatures are positive and constant, that is

$$K_{3/2} = K_{-3/2} = \frac{4}{3}\,, \qquad K_{1/2} = K_{-1/2} = 4\,\frac{\sqrt{13}}{7} \tag{75}$$

and the norm of the mean curvature vector are also positive and constant

$$\mathcal{H}_{3/2} = \mathcal{H}_{-3/2} = 4, \qquad \mathcal{H}_{1/2} = \mathcal{H}_{-1/2} = 4\,\frac{\sqrt{13}}{7}\,. \tag{76}$$

The Willmore functionals are

$$W_{3/2} = W_{-3/2} = \frac{9}{2}\,\pi, \qquad W_{1/2} = W_{-1/2} = \frac{13}{2}\,\pi. \tag{77}$$

The topological charges take the form

$$Q_{3/2} = 6, \qquad Q_{1/2} = 2, \qquad Q_{-1/2} = -2, \qquad Q_{-3/2} = -6. \tag{78}$$

The Euler–Poincaré characters are

$$\Delta_{3/2} = \Delta_{1/2} = \Delta_{-1/2} = \Delta_{-3/2} = 4. \tag{79}$$

This means that the surfaces X_m associated with the $\mathbb{C}P^3$ model are homeomorphic to ovaloids in view that $K_m > 0$.

4.1. Explicit parametrization in terms of Jacobi polynomials. As is well known, cf. [22] or [24] for example, using the parametrization (42) of $\mathbf{SU}(2)$ by Euler angles one can express the matrix elements $t^{(l)}_{jk}$ in (58) in terms of Jacobi polynomials $P^{(k-j,-j-k)}_{l+j}$. To be more precise, let us call the restrictions $t^{(l)}_{jk}(g(\theta,0,0))$ of matrix elements to the subgroup of $\mathbf{SU}(2)$, consisting of matrices

$$\begin{bmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix},$$

the reduced matrix elements. Then recalling that the vectors $\{w^{(l)}_m\}$ defined by (55) are the weight vectors with respect to the diagonal unitary group $K = \mathbf{U}(1) \subset \mathbf{SU}(2)$ and putting

$$\tau_{jk}(\varphi,\psi) = e^{2i(j\varphi+k\psi)}, \qquad \text{for } k,j = -l,\dots,l \tag{80}$$

we have

$$t^{(l)}_{jk}(g(\theta,\varphi,\psi)) = \tau_{jk}(\varphi,\psi)\,t^{(l)}_{jk}(g(\theta,0,0))$$

and the reduced matrix elements $t^{(l)}_{jk}(g(\theta,0,0))$ can be written as $P^{(k-j,-j-k)}_{l+j}(\cos\theta)$. For simplicity, we adopt here the definition of the Jacobi polynomials $P^{(\alpha,\beta)}_k(x)$ with real parameters (α,β) by means of the Rodrigues-type formula

$$P^{(\alpha,\beta)}_k(x) = \frac{(-1)^k}{2^k\,k!}\,(1-x)^{-\alpha}(1+x)^{-\beta}\,\frac{d^k}{dx^k}\big[(1-x)^{k+\alpha}(1+x)^{k+\beta}\big], \tag{81}$$

allowing α, β to be arbitrary real parameters, since the usual assumption $\alpha, \beta > -1$ is needed to ensure integrability of the weight $(1-x)^\alpha(1+x)^\beta$ over the interval $(-1,1)$, which will not be the question here. Before discussing the general case, let us examine the cases of spin 1 and spin 3/2 discussed above.

4.1.1. *The case of spin* 1. The Jacobi polynomials $P_{1+j}^{(k-j,-k-j)}(x)$ for $j,k=-1,0,1$ are given by the following table:

	$k=1$	$k=0$	$k=-1$
$j=1$	$P_2^{(0,-2)}(x)=\frac{1}{4}(x+1)^2$	$P_2^{(-1,-1)}(x)=\frac{1}{4}(x^2-1)$	$P_2^{(-2,0)}(x)=\frac{1}{4}(x-1)^2$
$j=0$	$P_1^{(1,-1)}(x)=x+1$	$P_1^{(0,0)}(x)=x$	$P_1^{(-1,1)}(x)=x-1$
$j=-1$	$P_0^{(2,0)}(x)=1$	$P_0^{(1,1)}(x)=1$	$P_0^{(0,2)}(x)=1$

It is now straightforward to write down expressions for the matrix elements involving the Jacobi polynomials.

$$\begin{aligned}
t_{11}^{(1)}(\theta,\varphi,\psi) &= \cos^2(\theta/2)e^{i(\psi+\varphi)} &&= \cos^{-2}(\theta/2)e^{i(\psi+\varphi)}P_2^{(0,-2)}(\cos\theta),\\
t_{01}^{(1)}(\theta,\varphi,\psi) &= 2^{-1/2}i\sin\theta e^{i\psi} &&= 2^{-1/2}i\,\frac{\sin(\theta/2)}{\cos(\theta/2)}\,e^{i\psi}P_1^{(1,-1)}(\cos\theta),\\
t_{-11}^{(1)}(\theta,\varphi,\psi) &= -\sin^2(\theta/2)e^{i(\psi-\varphi)} &&= -\sin^2(\theta/2)e^{i(\psi-\varphi)}P_0^{(2,0)}(\cos\theta),\\
t_{10}^{(1)}(\theta,\varphi,\psi) &= 2^{-1/2}i\sin\theta e^{i\varphi} &&= -2^{3/2}i\sin^{-1}\theta e^{i\varphi}P_2^{(-1,-1)}(\cos\theta),\\
t_{00}^{(1)}(\theta,\varphi,\psi) &= \cos\theta &&= P_1^{(0,0)}(\cos\theta),\\
t_{-10}^{(1)}(\theta,\varphi,\psi) &= 2^{-1/2}i\sin\theta e^{-i\varphi} &&= 2^{-1/2}i\sin\theta e^{-i\varphi}P_0^{(1,1)}(\cos\theta),\\
t_{1-1}^{(1)}(\theta,\varphi,\psi) &= -\sin^2(\theta/2)e^{i(\varphi-\psi)} &&= \sin^{-2}(\theta/2)e^{i(\varphi-\psi)}P_2^{(-2,0)}(\cos\theta),\\
t_{0-1}^{(1)}(\theta,\varphi,\psi) &= 2^{-1/2}i\sin\theta e^{-i\psi} &&= -2^{-1/2}i\,\frac{\cos\theta/2}{\sin\theta/2}\,e^{-i\psi}P_1^{(-1,1)}(\cos\theta),\\
t_{-1-1}^{(1)}(\theta,\varphi,\psi) &= \cos^2(\theta/2)e^{-i(\psi+\varphi)} &&= \cos^2(\theta/2)e^{-i(\psi+\varphi)}P_0^{(0,2)}(\cos\theta).
\end{aligned}$$

It may be noted that the middle (0-th) column, i.e. $\{t_{j0}^{(1)}(\theta,\varphi,\psi)\}$ consists of standard spherical harmonics of degree 1 (with respect to the variables (ψ,φ)).

4.1.2. *The case of spin* 3/2. The Jacobi polynomials $P_{3/2+j}^{(k-j,-k-j)}(x)$ relevant for this case are given in the following table.

	$k=\frac{3}{2}$	$k=\frac{1}{2}$	$k=-\frac{1}{2}$	$k=-\frac{3}{2}$
$j=\frac{3}{2}$	$P_3^{(0,-3)}(x)$ $=\frac{1}{8}(x+1)^3$	$P_3^{(-1,-2)}(x)$ $=\frac{1}{8}(x-1)(x+1)^2$	$P_3^{(-2,-1)}(x)$ $=\frac{1}{8}(1-x)^2(1+x)$	$P_3^{(-3,0)}(x)$ $=\frac{1}{8}(x-1)^3$
$j=\frac{1}{2}$	$P_2^{(1,-2)}(x)$ $=\frac{3}{4}(1+x)^2$	$P_2^{(0,-1)}(x)$ $=\frac{1}{4}(x+1)(3x-1)$	$P_2^{(-1,0)}(x)$ $=\frac{1}{4}(1+3x)(x-1)$	$P_2^{(-2,1)}(x)$ $=\frac{3}{4}(1-x)^2$
$j=-\frac{1}{2}$	$P_1^{(2,-1)}(x)$ $=\frac{3}{2}(1+x)$	$P_1^{(1,0)}(x)$ $=\frac{1}{2}(3x+1)$	$P_1^{(0,1)}(x)$ $=\frac{1}{2}(3x-1)$	$P_1^{(-1,2)}(x)$ $=\frac{3}{2}(x-1)$
$j=-\frac{3}{2}$	$P_0^{(3,0)}(x)=1$	$P_0^{(2,1)}(x)=1$	$P_0^{(1,2)}(x)=1$	$P_0^{(0,3)}(x)=1$

In this case, for reasons of space, we give below only the reduced matrix elements, of which the full form can be obtained by combining the formulas for the reduced matrix elements $t^{(3/2)}_{jk}(\theta,0,0)$ with the factor $\tau_{jk}(\varphi,\psi)$ from (80).

$$t^{(3/2)}_{(3/2)(3/2)}(\theta) = \cos^3\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(1/2)(3/2)}(\theta) = \sqrt{3}i\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(-1/2)(3/2)}(\theta) = -\sqrt{3}\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(-3/2)(3/2)}(\theta) = -i\sin^3\tfrac{\theta}{2}$$

$$t^{(3/2)}_{(3/2)(1/2)}(\theta) = \sqrt{3}i\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(1/2)(1/2)}(\theta) = \cos^3\tfrac{\theta}{2} - 2\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(-1/2)(1/2)}(\theta) = i(2\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2} - \sin^3\tfrac{\theta}{2})$$
$$t^{(3/2)}_{(-3/2)(1/2)}(\theta) = -\sqrt{3}\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$t^{(3/2)}_{(3/2)(-1/2)}(\theta) = \sqrt{3}i\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(1/2)(-1/2)}(\theta) = \cos^3\tfrac{\theta}{2} - 2\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(-1/2)(-1/2)}(\theta) = i(2\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2} - \sin^3\tfrac{\theta}{2})$$
$$t^{(3/2)}_{(-3/2)(-1/2)}(\theta) = -\sqrt{3}\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$

$$t^{(3/2)}_{(3/2)(-3/2)}(\theta) = -i\sin^3\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(1/2)(-3/2)}(\theta) = -\sqrt{3}\sin^2\tfrac{\theta}{2}\cos\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(-1/2)(-3/2)}(\theta) = \sqrt{3}i\sin\tfrac{\theta}{2}\cos^2\tfrac{\theta}{2}$$
$$t^{(3/2)}_{(-3/2)(-3/2)}(\theta) = \cos^3\tfrac{\theta}{2}$$

4.2. Matrix elements of SU(2) irreducible representations and general Veronese immersions. We return to the general case of the spin j representation of $\mathbf{SU}(2)$, acting in the space $\mathcal{H}^{(j)} = \mathcal{P}^{2j}(\mathbb{C}^2)$ with $\dim\mathcal{H}^{(j)} = 2j+1$ which is given by the formula

$$(T^{(j)}(g)p)(z_1,z_2) = p(uz_1+vz_2, -\overline{v}z_1+\overline{u}z_2) \qquad \text{for } p\in\mathcal{H}^{(j)}.$$

The matrix elements

$$t^{(j)}_{km}(g) = (w^{(j)}_k \,|\, T^{(j)}(g)w^{(j)}_m)$$

can be given in the following explicit form.

THEOREM 3 (cf. [22, 24]). *Given any half-integer j and $-j\le k,\ m\le j$ set $\alpha = k-m$, $\beta = k+m$ and $n = j-k$. The matrix elements of the representation $T^{(j)}$ are given by*

$$t^{(j)}_{km}(g) = t^{(j)}_{km}(g(\theta,\varphi,\psi)) = \tau_{km}(\varphi,\psi)t^{(j)}_{km}(g(\theta,0,0)) \tag{82}$$

where

$$t^{(j)}_{km}(g(\theta,0,0)) = c(j,k,m)(\cos(\theta/2))^{-\beta}(\sin(\theta/2))^{\alpha}P^{(\alpha,-\beta)}_n(\cos\theta) \tag{83}$$

and

$$c(j,k,m) = i^{m-k}\frac{[(j+k)!(j-k)!]^{1/2}}{[(j+m)!(j-m)!]^{1/2}}\,.$$

Since for any m the coherent state map

$$\mathbf{SU}(2)\ni g\mapsto (t^{(j)}_{-j\,m}(g), t^{(j)}_{-j+1\,m}(g),\ldots,t^{(j)}_{j\,m}(g))\in\mathbb{C}^{2j+1}$$

satisfies the assumptions of Theorem 1, due to diagram (52), it induces an imbedding of the sphere S^2 into the projective space $\mathbb{C}P^{2j}$. Comparing with the results of the paper [3], we see that it gives a conformal minimal imbedding belonging to the Veronese family.

It might be interesting to investigate the implications of the direct parametrization of this map on the study of the geometry in question. Some work in this direction is in progress.

5. Final remarks and future developments. The links between different analytic descriptions of $\mathbf{SU}(2)$ coherent states and the $\mathbb{C}P^N$ sigma models (defined on the Riemann sphere with finite actions) can be generalized to more general sigma models than the one proposed in this paper. An analysis of the complex Grassmannian sigma models taking values on the homogeneous spaces

$$G(m,n) = \frac{\mathbf{SU}(N)}{\mathbf{S}(\mathbf{U}(m)\times\mathbf{U}(n))}, \qquad N = m+n, \tag{84}$$

similar to the one carried out in Section 2 can provide us with a more general explicit form for coherent states. These models share many common properties with the $\mathbb{C}P^N$ models presented here. Namely, they possess an infinite number of local and/or nonlocal conserved quantities, as well as infinite-dimensional dynamical symmetries generating the Kac–Moody algebra. Both the Grassmannian sigma model equations and the $\mathbb{C}P^N$ sigma model have a Hamiltonian structure, complete integrability, and the existence of multisoliton solutions, where the linear spectral problem is well established [26]. Several classes of solutions of both equations are known. These solutions can be expressed in terms of holomorphic functions and functions obtained from them by a procedure similar to the one presented in this paper, which allows us to generate a complete set of solutions (more general than the ones constructed from the $\mathbb{C}P^N$ model). It is evident that our approach can be applied to the complex Grassmannian sigma model which can describe much more diverse types of coherent states. This task will be undertaken in our future work.

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