50th SEMINAR “SOPHUS LIE”
BANACH CENTER PUBLICATIONS, VOLUME 113
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2017

ELLIPTIC OPERATORS IN THE BUNDLE OF SYMMETRIC TENSORS

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Abstract. Differential operators: the gradient grad and the divergence div are defined and examined in the bundles of symmetric tensors on a Riemannian manifold. For the second order operator div grad which appears to be elliptic and a manifold with boundary a system of natural boundary conditions is constructed and investigated. There are 2^{k+1} conditions in the bundle S^k of symmetric tensors of degree k. This is in contrast to the bundle of skew-symmetric forms where (for analogous differential operators) there are always four such conditions independently of the degree of forms (i.e. independently of k). All the 2^{k+1} conditions are investigated in detail. In particular, it is proved that each of them is self-adjoint and elliptic. Such ellipticity of a given boundary condition has an essential significance for the existing of a discrete spectrum and an orthonormal basis in L^2 consisting of smooth sections that are the eigenvectors of the operator and satisfy the boundary condition. Some special cases, e.g. $k = 1$ or the cases that the boundary is umbilical or totally geodesic, are also discussed.

1. Introduction. For many years symmetric tensors have been intensively studied in the mathematics and physics literature (cf. e.g. [WP], [Rn], [DS] or [Fx]). Let us mention only so called symmetric Killing tensors or conformal symmetric Killing tensors on a Riemannian manifold M being the counterparts of Killing and conformal Killing forms [HMS]. The motivation and importance came from the fact that symmetric Killing tensors define first integrals of the equations of motion, i.e. functions which are constant on geodesics. Conformal Killing tensors still define first integrals for null geodesics. Special role is played by symmetric Killing 2-tensors. These tensors arise in connection with special topics in differential geometry and analytical mechanics: geodesic equivalence and

2010 *Mathematics Subject Classification*: Primary: 58J32, 53C15, 53C21; Secondary: 58G05, 53C21, 58C40.
Key words and phrases: symmetric tensors, the gradient, the divergence, elliptic operators, elliptic boundary conditions, Weitzenböck type formula.
The paper is in final form and no version of it will be published elsewhere.

DOI: 10.4064/bc113-0-11

separation of variables. They also play an important role in the theory of correspondent dynamical systems of Levi-Civita [Bi]. Killing and conformal Killing tensors appear in several other areas of mathematics, e.g. in connection with geometric inverse problems, integrable systems and Einstein–Weyl geometry (cf. e.g. [SM], [Sl] or [Jk]).

In many cases the considered classes of tensors constitute kernels of some natural differential operators. The operators have usually some invariance properties with respect to the action of Lie groups related to the geometric structure of M. The operators investigated here are mostly related to the orthogonal group, i.e. to the Riemannian structure.

A special role is played by the first order differential operator d^s being the symmetric part of the Levi-Civita covariant derivative considered on the bundle of symmetric tensors and, in contrast to the skew-symmetric case, depends on the geometric structure of M. Its kernel consist of Killing tensors. The trace free part of d^s defines in tern the conformal Killing tensors. By the way it is worth to notice that also in contrast to the well known exterior derivative d (acting in the bundle of skew-symmetric forms) the two operators: d^s and its trace free part (both acting in the bundle of symmetric forms) are elliptic in the sense of injectivity of their symbols (cf. Proposition 6.4). The shape of operators formally adjoint to d^s both in the bundle of symmetric forms and in the bundle of vector valued symmetric forms is derived (Theorems 3.3 and 3.4) in Section 3.

In Section 4 the operators of gradient grad and divergence div in the bundle of symmetric tensors are introduced and investigated in detail. The two operators have nice algebraic properties. They are differentiations in the algebra of symmetric forms in the sense of Propositions 4.1 and 4.3. They are formally adjoint each to the other with respect to global (integral) scalar product (Theorem 4.1). Their composition div grad is a second order elliptic differential operator.

It is interesting that, similarly to the skew-symmetric case, the negative of div grad differs from the classical Laplace operator Δ^s introduced by Sampson in [Sn] by a zero order operator (tensor) depending on the curvature, i.e. Weitzenböck type formula holds also for symmetric tensors (Theorem 5.1), Section 5.

The main results of the paper are contained in Section 6 where the boundary properties of the operators are examined. The operator div grad is strongly elliptic, so, it seems to be interesting to investigate its behavior at the boundary. It is formally selfadjoint since $-\operatorname{div}$ is formally adjoint to grad. As usual in such cases, a crucial role is played by the shape of the boundary integral determining the obstruction for the full selfadjointness. The boundary integral is derived explicitly in Theorem 6.1. The integrand involves there the gradient operator grad. And grad has its values in the bundle $S^k \otimes TM$ of symmetric forms with values in the tangent bundle TM. The bundle TM splits at the boundary onto its tangent and normal parts (formula (30)). This forces a similar splitting for grad (formula (31)). This splitting is a starting point in setting a system of boundary conditions for div grad. In particular, we construct such a system by the method described by Branson–Pierzchalski in their manuscript in 2004 [BP]. The rule of constructing of such systems was next repeated—for the Stein–Weiss gradients—in [KP]. Using this rule we define a system of geometrically natural conditions. There are 2^{k+1} conditions in the bundle S^k of symmetric tensors of degree k, so quiet a big variety of different conditions, especially for higher degrees. Let us notice the contrast to the bundle of skew-symmetric forms where independently of the degree of forms (i.e. independently of k) there are always four natural boundary conditions (cf. e.g. [ØP] and [KP]).

All the 2^{k+1} conditions are investigated in detail. In particular, it is proved that each of them is self-adjoint (Proposition 6.2) and elliptic (Theorem 6.2). In the proof of *ellipticity at the boundary* the method described in [Gy] is used. A so called *auxiliary bundle* is constructed. Such bundle is constructed there for each boundary condition separately. Here, our original construction of the auxiliary bundle, enables proving the ellipticity for the all 2^{k+1} particular boundary conditions simultaneously. In addition, elliptic boundary conditions are, in the special case $k = 1$, presented explicitly in Example 6.1. It is also shown that the proof of the theorem simplifies essentially under some additional assumptions on the geometry of the boundary. The cases of totally umbilical or totally geodesic boundary are mentioned.

Notice finally that the *ellipticity at the boundary*—investigated in Section 6—has its strong consequence contained in Proposition 6.6 and saying that for each of the considered boundary condition there is a complete in L^2 system of smooth sections of S^k satisfying that condition. The existence of such a system enables then investigating the boundary value problems by application of standard methods of harmonic analysis in each particular case.

2. Symmetric tensors. Let V be a vector space over $\mathbb{R}$, $\dim V = n$.

For $k = 1, 2, \dots$ the space of all k-linear mappings

$$\varphi : \underbrace{V \times \ldots \times V}_{k} \to \mathbb{R}$$

is called the space of *k-tensors* over V and denoted by T^{*k}. We also accept the convention $T^{*0} = \mathbb{R}$.

A k-tensor φ is called *symmetric* if $\varphi(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \varphi(v_1, \dots, v_k)$ for all vectors $v_1, \dots, v_k \in V$ and any permutation $\sigma \in S_k$ of the set $\{1, \dots, k\}$. The space of all k-symmetric tensors will be denoted by S^k. Its elements will be called here shortly *k-forms*. We accept the convention $S^0 = \mathbb{R}$.

The *symmetrization* is the linear operation $\mathrm{Sym} : T^{*k} \to T^{*k}$ defined by

$$(\mathrm{Sym}\, \psi)(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \psi(v_{\sigma(1)}, \dots, v_{\sigma(k)}),$$

where $v_1, \dots, v_k \in V$.

It is easy to check that the symmetrization is a projection, i.e. that

$$\mathrm{Sym}^2 = \mathrm{Sym} \qquad \text{and} \qquad \mathrm{Im}\,\mathrm{Sym} = S^k,$$

so, in particular, Sym is the identity on S^k.

Assume that V is equipped with a scalar product $\langle \cdot, \cdot \rangle$. The scalar product can be transmitted to T^{*k}—the space of k-tensors by the formula:

For $\xi, \eta \in T^{*k}$

$$\langle \xi, \eta \rangle = \sum_{i_1,\dots,i_k=1}^{n} \xi(e_{i_1},\dots,e_{i_k})\eta(e_{i_1},\dots,e_{i_k}),$$

where $e_1,\dots,e_n$ is an orthonormal basis in V. One can easily check that the definition is independent of the choice of such basis.

S^k as a subspace of T^{*k} inherits the scalar product $\langle \cdot,\cdot \rangle$ from the ambient space. In addition, the scalar product $\langle \cdot,\cdot \rangle$ in $S^k \otimes V$ is given by

$$\langle \varphi \otimes v, \psi \otimes w \rangle = \langle \varphi, \psi \rangle \langle v, w \rangle,$$

where $\varphi, \psi \in S^k$, $v, w \in V$.

In S^k and $S^k \otimes V$ there will be considered also another scalar product

$$\langle \cdot \,|\, \cdot \rangle = \frac{1}{k!} \langle \cdot,\cdot \rangle. \tag{1}$$

Before giving its alternative description (cf. (3)) let us define the symmetric product of symmetric tensors: $\odot : S^k \times S^l \to S^{k+l}$, $k, l \in \mathbb{N}$:

For $\varphi \in S^k$, $\psi \in S^l$ set

$$\begin{aligned}(\varphi \odot \psi)&(v_1,\dots,v_k,v_{k+1},\dots,v_{k+l})\\ &= \sum_{\sigma \in \mathrm{sh}(k,l)} \varphi(v_{\sigma(1)},\dots,v_{\sigma(k)})\psi(v_{\sigma(k+1)},\dots,v_{\sigma(k+l)}),\end{aligned}$$

where $v_1,\dots,v_k,v_{k+1},\dots,v_{k+l} \in V$ and where $\mathrm{sh}(k,l)$ denotes the set of all shuffles of type (k,l) (cf. [Fr]). Recall that a shuffle of type (k,l) is any permutation of the set $\{1,\dots,k+l\}$ which is increasing on each of the two sets $\{1,\dots,k\}$ and $\{k+1,\dots,k+l\}$.

In addition, for products that contain vector valued tensors we set:

For $\varphi \in S^k$, $\Psi = \psi \otimes Y \in S^l \otimes V$

$$\varphi \odot \Psi = (\varphi \odot \psi) \otimes Y.$$

For $\Phi = \varphi \otimes X \in S^k \otimes V$, $\psi \in S^l$

$$\Phi \odot \psi = (\varphi \odot \psi) \otimes X.$$

In particular, if $k = 1$ we have for $\varphi \in S^1$ and $\psi \in S^l$

$$(\varphi \odot \psi)(v_1,\dots,v_{l+1}) = \sum_{j=1}^{l+1} \varphi(v_j)\psi(v_1,\dots,\hat{v}_j,\dots,v_{l+1}), \tag{2}$$

where $v_1,\dots,v_{l+1} \in V$.

One can easily check the following facts:

PROPOSITION 2.1. *For any* $\varphi_1,\dots,\varphi_k,\psi_1,\dots,\psi_k \in T^{*1}$

$$\langle \varphi_1 \odot \ldots \odot \varphi_k \,|\, \psi_1 \odot \ldots \odot \psi_k \rangle = \mathrm{perm}(\langle \varphi_i, \psi_j \rangle), \tag{3}$$

where

$$\mathrm{perm}\,(a_{ij}) = \sum_{\sigma \in S_k} a_{1\sigma(1)} \cdots a_{k\sigma(k)}.$$

Proposition 2.2. *Let* $e_1, \dots, e_n$ *be a basis of* V *and* $e_1^*, \dots, e_n^*$ *be the dual basis (it is a basis in* T^{*1}*). Then the set of* k*-tensors*

$$e_{i_1}^* \odot \ldots \odot e_{i_k}^* \tag{4}$$

where $i_1, \dots, i_k \in \{1, \dots, n\}$ *forms a basis of* S^k*. So,* $\dim S^k = \binom{n+k-1}{k}$*.*

The tensors (4) *can be alternatively ordered to*

$$e_1^{*\,\alpha_1} \odot \ldots \odot e_n^{*\,\alpha_n},$$

where $\alpha_1, \dots, \alpha_n \in \mathbb{N} \cup \{0\}$*,* $\alpha_1 + \ldots + \alpha_n = k$*. We accept the convention that* $e_j^{*\,\alpha_j} = 1$ *for* $\alpha_j = 0$*.*

The operation ι_v of *substitution* of a vector $v \in V$ is defined as the mapping $\iota_v : S^k \to S^{k-1}$ of the form

$$(\iota_v \varphi)(v_1, \dots, v_{k-1}) = \varphi(v, v_1, \dots, v_{k-1}) \quad \text{for } k > 0,$$
$$\iota_v \varphi = 0 \quad \text{for } k = 0,$$

where $v_1, \dots, v_{k-1} \in V$, $\varphi \in S^k$.

Directly by the definition of substitution and (2) we see that the substitution is a derivation of the algebra of symmetric tensors in the following sense.

Proposition 2.3. *Let* $v \in V$*,* $\varphi \in S^k$*,* $\psi \in S^l$*. Then*

$$\iota_v(\varphi \odot \psi) = \iota_v \varphi \odot \psi + \varphi \odot \iota_v \psi.$$

Definition 2.1. For $k = 1, 2, \dots$ define the operator $\mathfrak{a} : S^k \to S^{k-1} \otimes V$ by

$$\mathfrak{a}\varphi = \sum_{i=1}^{n} \iota_{e_i} \varphi \otimes e_i \tag{5}$$

and, for $k = 0$, by

$$\mathfrak{a}\varphi = 0, \tag{6}$$

where $e_1, \dots, e_n$ is an orthonormal basis in V and $\varphi \in S^k$.

One can check that the definition of $\mathfrak{a}$ is independent of the choice of orthonormal frame.

Define the isomorphism $\flat : V \to T^{*1}$ by

$$v^\flat(w) = \langle v, w \rangle, \qquad v, w \in V. \tag{7}$$

By (7) and Definition 2.1 we get directly

Theorem 2.1. *For* $k = 0, 1, \dots$ *the operator* $\mathfrak{a}^* : S^k \otimes V \to S^{k+1}$ *adjoint to* $\mathfrak{a}$ *with respect to the scalar product* (1) *is given by*

$$\mathfrak{a}^*(\psi \otimes v) = v^\flat \odot \psi,$$

where $\psi \in S^k$*.*

One can also easily prove (cf. [Ka]) the following

Proposition 2.4. *For* $k = 0, 1, \dots$

$$\mathfrak{a}^* \mathfrak{a} = k \, \mathrm{id}_{|S^k}.$$

In particular, for $k = 1, 2, \dots$ $\mathfrak{a}^*$ *is surjective and* $\mathfrak{a}$ *is injective.*

Define now two trace operators. First, the trace operator acting on vector forms.

DEFINITION 2.2. The *trace operator* $\mathrm{tr} : T^{*k} \otimes V \to T^{*k-1}$ is defined by the formula

$$\mathrm{tr}(\varphi \otimes X) = \iota_X \varphi \qquad \text{for } k > 0$$
$$\mathrm{tr}(\varphi \otimes X) = 0 \qquad \text{for } k = 0.$$

Next, the trace operator acting on scalar forms. In contrast to the previous one it will be marked with $\tilde{}$.

DEFINITION 2.3. The *trace operator* $\tilde{\mathrm{tr}} : T^{*k} \to T^{*k-2}$ is defined by the formula

$$\tilde{\mathrm{tr}}\,\varphi = \sum_{i=1}^{n} \iota_{e_i}\iota_{e_i}\varphi \qquad \text{for } k = 2, 3, \ldots \tag{8}$$

$$\tilde{\mathrm{tr}}\,\varphi = 0 \qquad \text{for } k = 0, 1. \tag{9}$$

Here $e_1, \ldots, e_n$ is an orthonormal frame of V.

One can easily see that the right hand side of (8) is independent of the choice of frame.

We will use the same symbols for the restrictions of operators tr and $\tilde{\mathrm{tr}}$ to subspaces of $T^{*k} \otimes V$ and T^{*k}, respectively. In particular, we will use the same symbols for the restrictions of these operators to the subspaces $S^k \otimes V$ and S^k, respectively.

3. Symmetric derivatives and their adjoints. All the objects and morphisms are assumed to be smooth, i.e. of class C^∞.

In order to deliberate differential operators, vector bundles over a manifold M instead of a simple vector space will be considered. Since a vector bundle can just be treated as a vector space parametrized by points of M, some symbols used in the previous section for vector spaces will be used (from now on) to denote suitable bundles.

Let (M, g) be an oriented Riemannian manifold of dimension n, i.e. an oriented manifold M equipped with a Riemannian metric g. For any $p \in M$,

$$\langle \cdot, \cdot \rangle = g_p(\cdot, \cdot) : T_p \times T_p \to \mathbb{R}$$

is then a scalar product in T_p where $T_p = T_pM$ denotes the tangent space to M at p, i.e. the fiber at p of the tangent bundle T. Denote by $C^\infty(M)$ the ring of smooth functions on M. Let $T^* = T^*M$ be the cotangent bundle. Denote by $T^{*k} = T^{*k}M$ the bundle of covariant k-tensors on M and by $S^k = S^kM$ its subbundle of k-symmetric tensors (k-forms). For any bundle E over M denote by $C^\infty(E)$ the $C^\infty(M)$—module of sections of E.

Let $\nabla : C^\infty(T) \times C^\infty(T) \to C^\infty(T)$ be the *Levi-Civita covariant derivative* on M (written as $\nabla(X, Y) = \nabla_X Y$). It is—as any covariant derivative—linear (=tensorial) with respect to the first argument and has a property of differentiation with respect to the other. Moreover, it is the only one that satisfies the following two conditions:

$$\nabla_X Y - \nabla_Y X = [X, Y] \quad (\textit{torsionless}),$$
$$X\langle Y_1, Y_2\rangle = \langle \nabla_X Y_1, Y_2\rangle + \langle Y_1, \nabla_X Y_2\rangle \quad (\textit{compatibility with metric})$$

for any X, X_1, X_2 and Y, Y_1, Y_2 from $C^\infty(T)$, where $[X, Y]$ is the Lie bracket of X and Y (cf. [GKM]).

Transmit ∇ in the canonical way to the dual bundle T^* ($=S^1$) and next—by the Leibniz rule—to any tensor bundle over M. In particular, to the bundle S^k.

The extended connection is denoted by the same symbol ∇. By the definition,

$$(\nabla_X \varphi)(X_1, \ldots, X_k) = X(\varphi(X_1, \ldots, X_k)) - \sum_{j=1}^{k} \varphi(X_1, \ldots, \nabla_X X_j, \ldots, X_k)$$

for $\varphi \in C^\infty(S^k)$ and $X, X_1, \ldots, X_k \in C^\infty(T)$.

If we use the convention:

$$(\nabla \varphi)(X, X_1, \ldots, X_k) = (\nabla_X \varphi)(X_1, \ldots, X_k),$$

the covariant derivative obtained may be treated as the map:

$$\nabla : C^\infty(S^k) \to C^\infty(T^* \otimes S^k).$$

By the above definitions we have:

PROPOSITION 3.1. *Let $\varphi \in C^\infty(S^k)$ and $X_1, \ldots, X_{k+1} \in C^\infty(T)$. Then*

$$(\operatorname{Sym} \nabla \varphi)(X_1, \ldots, X_{k+1}) = \frac{1}{k+1} \sum_{j=1}^{k+1} (\nabla_{X_j} \varphi)(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1}).$$

Proof. Just calculations. ■

DEFINITION 3.1. The operator $\mathrm{d}^s : C^\infty(S^k) \to C^\infty(S^{k+1})$ given by

$$\mathrm{d}^s = (k+1) \operatorname{Sym} \nabla \tag{10}$$

is called the *symmetric derivative.*

By the definition and Proposition 3.1 we get directly

PROPOSITION 3.2.

$$\mathrm{d}^s \varphi(X_1, \ldots, X_{k+1}) = \sum_{j=1}^{k+1} (\nabla_{X_j} \varphi)(X_1, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

for $\varphi \in C^\infty(S^k)$, $X_1, \ldots, X_{k+1} \in C^\infty(T)$.

It appears that the symmetric derivative has a local expression similar to that one for the exterior derivative acting in skew-symmetric forms. The proof for the latter case may be found e.g. in [Yu]. An analogous proof (cf. [Ka]) leads in our case of symmetric forms to the following

PROPOSITION 3.3. *Let $e_1, \ldots, e_n$ be a local frame of sections of T and let $e_1^*, \ldots, e_n^*$ be the dual frame, then*

$$\mathrm{d}^s \varphi = \sum_{j=1}^{n} e_j^* \odot \nabla_{e_j} \varphi \tag{11}$$

for $\varphi \in C^\infty(S^k)$.

From the last proposition one can derive in particular that d^s is a derivation of the algebra of the symmetric forms:

Proposition 3.4. *For $\varphi \in C^\infty(S^k)$ and $\psi \in C^\infty(S^l)$ we have*

$$\mathrm{d}^s(\varphi \odot \psi) = (\mathrm{d}^s \varphi) \odot \psi + \varphi \odot (\mathrm{d}^s \psi). \tag{12}$$

Proof. The formula (12) is a consequence of (11) and the fact that ∇_{e_j} is a derivation. ∎

Notice the following fact that can be derived directly from the properties of the operators ι and ∇.

Proposition 3.5. *For $X, Y \in C^\infty(T)$,*

$$\iota_X \nabla_Y = \nabla_Y \iota_X - \iota_{\nabla_Y X}.$$

Extend the symmetric derivative d^s (and denote the extension by the same letter) to the bundle $S^k \otimes T$ as follows:

Definition 3.2. Define the operator $\mathrm{d}^s : C^\infty(S^k \otimes T) \to C^\infty(S^{k+1} \otimes T)$ by

$$\mathrm{d}^s(\varphi \otimes X) = \mathrm{d}^s \varphi \otimes X + \varphi \odot \nabla X, \tag{13}$$

for $\varphi \otimes X \in C^\infty(S^k \otimes T)$ where ∇X is treated as 1-form with values in T. Locally this form can be given by $\nabla X = \sum_{j=1}^n e_j^* \otimes \nabla_{e_j} X$.

Also, in analogy to Proposition 3.3, we get

Proposition 3.6. *Let $e_1, \ldots, e_n$ be a local frame of sections of T and let $e_1^*, \ldots, e_n^*$ be the dual frame, then, for $\Phi \in C^\infty(S^k \otimes T)$,*

$$\mathrm{d}^s \Phi = \sum_{j=1}^n e_j^* \odot \nabla_{e_j} \Phi. \tag{14}$$

Let E be any vector bundle over M and $\langle \cdot, \cdot \rangle$ be a scalar product in E. Define the *global scalar product* $(\cdot, \cdot)$ in the space of sections of E by

$$(\cdot, \cdot) = \int_M \langle \cdot, \cdot \rangle \, \Omega_M,$$

where Ω_M is the volume form on M defined by the orientation and the metric g.

The global scalar product is then defined only for such pairs of sections that the integral exists and is finite. This is always a case when, e.g., at least one section is of compact support.

That way, for the bundle S^k, we have two global scalar products, $(\cdot, \cdot)$ and $(\cdot \,|\, \cdot)$. They (cf. (1)) are related by

$$(\cdot \,|\, \cdot) = \frac{1}{k!} (\cdot, \cdot). \tag{15}$$

To get an explicit shape of the operator formally adjoint to ∇ with respect to the global scalar product (15) we will consider for a while the bundle of skew-symmetric forms on M with the exterior multiplication $\wedge$ and the operator of exterior derivation d.

Recall that the *Hodge star operator* $\star$ in the bundle of skew-symmetric forms is defined by

$$\alpha \wedge \star\beta = g(\alpha, \beta) \, \Omega_M,$$

where Ω_M is the volume form on M and α, β are skew-symmetric k-forms and, finally, g is the Riemannian scalar product extended to the bundle of skew-symmetric forms by

$$g(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}, \psi_{j_1} \wedge \cdots \wedge \psi_{j_k}) = \det(g(\varphi_{i_r}, \psi_{j_s}))_{1 \le r,s \le k},$$

for $\varphi_{i_r}, \psi_{j_s} \in C^\infty(T^*)$.

Now we are going to use the simple fact that one-tensors may be treated both as symmetric and skew-symmetric ones. Other-word, the spaces of symmetric and skew-symmetric 1-forms coincide. In particular, every contraction of symmetric forms that is a 1-tensor may then also be treated as skew-symmetric 1-form.

Let $e_1, \ldots, e_n$ be a local orthonormal frame on M. For $\varphi \in C^\infty(S^k)$ and $\psi \in C^\infty(T^* \otimes S^k)$ the *contraction* $\varphi \lrcorner \psi$ is the 1-form is defined by

$$\varphi \lrcorner \psi(e_r) = \sum_{i_1,\ldots,i_k=1}^{n} \varphi(e_{i_1}, \ldots, e_{i_k})\psi(e_r, e_{i_1}, \ldots, e_{i_k}). \tag{16}$$

Proposition 3.7. *For $\varphi \in C^\infty(S^k)$ and $\psi \in C^\infty(S^{k+1})$ we have*

$$\mathrm{d}(\star(\varphi \lrcorner \psi)) = \big(\langle \nabla\varphi, \psi\rangle + \langle \varphi, \tilde{\mathrm{tr}}\, \nabla\psi\rangle\big)\, \Omega_M. \tag{17}$$

Here d *denotes the usual exterior derivative in the bundle of skew-symmetric forms.*

Proof. Let $p \in M$. Take a local positively oriented orthonormal basis $e_1, \ldots, e_n$ in a neighborhood of p such that $\nabla_{e_i} e_j = 0$ at p. Take $\varphi \in C^\infty(S^k)$ and $\psi \in C^\infty(T^* \otimes S^k)$ then $\star(\varphi \lrcorner \psi)$ is a $(n-1)$-skew-symmetric form and $\mathrm{d}(\star(\varphi \lrcorner \psi))$ is a n-skew-symmetric form on M. So we have:

$$\begin{aligned}
&\mathrm{d}(\star(\varphi \lrcorner \psi))(e_1, \ldots, e_n) \\
&\quad = \sum_{r=1}^{n} (-1)^{r-1} e_r(\star(\varphi \lrcorner \psi))(e_1, \ldots, \hat{e}_r, \ldots, e_n) = \sum_{r=1}^{n} e_r(\varphi \lrcorner \psi)(e_r).
\end{aligned}$$

This, by (16), can be continued with

$$\begin{aligned}
&\sum_{r=1}^{n} e_r\Big(\sum_{i_1,\ldots,i_k=1}^{n} \varphi(e_{i_1}, \ldots, e_{i_k})\psi(e_r, e_{i_1}, \ldots, e_{i_k})\Big) \\
&\quad = \sum_{r,i_1,\ldots,i_k=1}^{n} e_r\big(\varphi(e_{i_1}, \ldots, e_{i_k})\big)\psi(e_r, e_{i_1}, \ldots, e_{i_k}) \\
&\qquad + \sum_{r,i_1,\ldots,i_k=1}^{n} \varphi(e_{i_1}, \ldots, e_{i_k}) e_r\big(\psi(e_r, e_{i_1}, \ldots, e_{i_k})\big) \\
&\quad = \big(\langle \nabla\varphi, \psi\rangle + \langle \varphi, \tilde{\mathrm{tr}}\, \nabla\psi\rangle\big)\, \Omega_M(e_1, \ldots, e_n),
\end{aligned}$$

at p. ■

Integrating (17) over M and applying the Stokes theorem we obtain

Theorem 3.1. *The operator*

$$\tilde{\mathrm{tr}}\, \nabla : C^\infty(T^* \otimes S^k) \to C^\infty(S^k)$$

is formally adjoint to $\nabla : C^\infty(S^k) \to C^\infty(T^* \otimes S^k)$ with respect to the global scalar product $(\cdot,\cdot)$, i.e., for $\psi \in C^\infty(T^* \otimes S^k)$ and $\varphi \in C^\infty(S^k)$ we have

$$(\nabla\varphi, \psi) = (\varphi, -\, \tilde{\mathrm{tr}}\, \nabla\psi)$$

if only φ or ψ is of compact support.

A similar theorem holds also for vector valued forms:

THEOREM 3.2. *The operator*

$$-\tilde{\mathrm{tr}}\,\nabla : C^\infty(T^* \otimes S^k \otimes T) \to C^\infty(S^k \otimes T)$$

is formally adjoint to $\nabla : C^\infty(S^k \otimes T) \to C^\infty(T^* \otimes S^k \otimes T)$ *with respect to the global scalar product* $(\cdot,\cdot)$*, i.e., for* $\psi \in C^\infty(T^* \otimes S^k \otimes T)$ *and* $\varphi \in C^\infty(S^k \otimes T)$ *we have*

$$(\nabla\varphi, \psi) = (\varphi, -\tilde{\mathrm{tr}}\,\nabla\psi)$$

if only φ *or* ψ *is of compact support.*

Proof. The proof is similar to that of Theorem 3.1. ∎

The last theorems may be used in deriving the operator formally adjoint to d^s.

Notice first the following

PROPOSITION 3.8. *For* $\varphi \in C^\infty(S^k)$ *and* $\psi \in C^\infty(S^{k+1})$ *we have*

$$(\mathrm{d}^s\,\varphi, \psi) = (k+1)(\varphi, -\tilde{\mathrm{tr}}\,\nabla\psi) \tag{18}$$

if only φ *or* ψ *is of compact support.*

Proof. By the definition of symmetric derivative and Theorem 3.1 we get

$$\begin{aligned}\int_M \langle \mathrm{d}^s\,\varphi, \psi\rangle &= \int_M \langle (k+1)\,\mathrm{Sym}\,\nabla\varphi, \psi\rangle = (k+1)\int_M \langle \mathrm{Sym}\,\nabla\varphi, \psi\rangle \\ &= (k+1)\int_M \langle \nabla\varphi, \psi\rangle = (k+1)\int_M \langle \varphi, -\tilde{\mathrm{tr}}\,\nabla\psi\rangle = (k+1)(\varphi, -\tilde{\mathrm{tr}}\,\nabla\psi).\end{aligned}$$ ∎

THEOREM 3.3. *With respect to the global scalar product* $(\cdot\,|\,\cdot)$ *the operator*

$$\mathrm{d}^{s\,*} : C^\infty(S^{k+1}) \to C^\infty(S^k)$$

formally adjoint to $\mathrm{d}^s : C^\infty(S^k) \to C^\infty(S^{k+1})$ *is of the form*

$$\mathrm{d}^{s\,*} = -\tilde{\mathrm{tr}}\,\nabla_{|C^\infty(S^{k+1})},$$

i.e., for $\varphi \in C^\infty(S^k)$ *and* $\psi \in C^\infty(S^{k+1})$*, we have*

$$(\mathrm{d}^s\,\varphi\,|\,\psi) = (\varphi\,|\,-\tilde{\mathrm{tr}}\,\nabla\psi),$$

if only φ *or* ψ *is of compact support.*

Proof. Multiplying both sides of (18) by $\frac{1}{(k+1)!}$ we get

$$\frac{1}{(k+1)!}(\mathrm{d}^s\,\varphi, \psi) = \frac{1}{k!}(\varphi, -\tilde{\mathrm{tr}}\,\nabla\psi).$$

By (15) we get the assertion. ∎

Similar facts hold also for vector valued forms:

PROPOSITION 3.9. *For* $\Phi \in C^\infty(S^k \otimes T)$ *and* $\Psi \in C^\infty(S^{k+1} \otimes T)$ *we have*

$$(\mathrm{d}^s\,\Phi, \Psi) = (k+1)(\Phi, -\tilde{\mathrm{tr}}\,\nabla\Psi),$$

if only Φ *or* Ψ *is of compact support.*

Proof. The proof is similar to that of Proposition 3.8. ∎

THEOREM 3.4. *With respect to the global scalar product* $(\cdot\,|\,\cdot)$ *the operator*

$$\mathrm{d}^{s\,*} : C^\infty(S^{k+1}\otimes T) \to C^\infty(S^k\otimes T)$$

formally adjoint to $\mathrm{d}^s : C^\infty(S^k\otimes T) \to C^\infty(S^{k+1}\otimes T)$ *is of the form*

$$\mathrm{d}^{s\,*} = -\,\tilde{\mathrm{tr}}\,\nabla_{|C^\infty(S^{k+1}\otimes T)},$$

i.e., for $\Phi \in C^\infty(S^k\otimes T)$ *and* $\Psi \in C^\infty(S^{k+1}\otimes T)$,

$$(\mathrm{d}^s\,\Phi\,|\,\Psi) = (\Phi\,|\,-\,\tilde{\mathrm{tr}}\,\nabla\Psi),$$

if only Φ *or* Ψ *is of compact support.*

Proof. The proof is similar to that of Theorem 3.3. ■

4. The gradient and the divergence. Two differential operators: the gradient and the divergence in the bundles of symmetric forms are the subject of this section. They were introduced and examined in the Ph.D. thesis of the first author [Ka]. The definition is similar to that for analogous operators in the bundle of skew-symmetric forms given by H. Rummler [Rr].

DEFINITION 4.1. The *gradient* is the operator grad : $C^\infty(S^k) \to C^\infty(S^k\otimes T)$ defined by

$$\mathrm{grad} = \mathfrak{a}\,\mathrm{d}^s - \mathrm{d}^s\,\mathfrak{a}. \tag{19}$$

The operator grad has many nice properties. Let us mention here only that grad is a differentiation of the algebra of symmetric forms in the following sense.

PROPOSITION 4.1. *For* $\varphi \in C^\infty(S^k)$ *and* $\psi \in C^\infty(S^l)$ *we have*

$$\mathrm{grad}(\varphi\odot\psi) = \mathrm{grad}\,\varphi\odot\psi + \varphi\odot\mathrm{grad}\,\psi.$$

Proof. See [Ka]. ■

Another important property describes a local shape of grad.

PROPOSITION 4.2. *Let* $e_1,\dots,e_n$ *be a local orthonormal frame of sections of* T, *then locally*

$$\mathrm{grad}\,\varphi = \sum_{i=1}^n \nabla_{e_i}\varphi\otimes e_i, \tag{20}$$

for $\varphi \in C^\infty(S^k)$.

Proof. The proof by the use of the shape (19) of grad and next of the local formulas (5), (6) and (11) or (14) for $\mathfrak{a}$ and d^s, respectively, can be found in [Ka]. ■

Notice, by (20), that for $k = 0$ grad reduces to the usual gradient of functions.

DEFINITION 4.2. The *divergence* is the operator div : $C^\infty(S^k\otimes T)\to C^\infty(S^k)$ defined by

$$\mathrm{div} = \mathrm{tr}\,\mathrm{d}^s - \mathrm{d}^s\,\mathrm{tr}\,. \tag{21}$$

Also the operator of divergence has many nice properties and also in the particular case $k = 0$ it reduces to the usual divergence on vector fields. In analogy to Proposition 4.1 one can prove

Proposition 4.3. *Let* $\varphi \in C^\infty(S^k)$ *and* $\psi \in C^\infty(S^l \otimes T)$. *Then*

$$\operatorname{div}(\varphi \odot \psi) = \varphi \odot \operatorname{div} \psi + \operatorname{grad} \varphi \odot \psi.$$

Proof. See [Ka]. ∎

We will need the following relation between div and ∇:

Proposition 4.4. *For* $\varphi \in C^\infty(S^k)$ *and* $X \in C^\infty(T)$ *we have*

$$\operatorname{div}(\varphi \otimes X) = \nabla_X \varphi + \varphi \operatorname{div} X,$$

where $\operatorname{div} X$ *is defined locally in any local orthonormal basis* $e_1, \dots, e_n$ *by* $\operatorname{div} X = \sum_{j=1}^n \langle e_j, \nabla_{e_j} X\rangle$.

Proof. Let $e_1, \dots, e_n$ be a local orthonormal frame on M and let $e_1^*, \dots, e_n^*$ be the dual frame. Then we have locally

$$\begin{aligned}\operatorname{div}(\varphi \otimes X) &= (\operatorname{tr} \mathrm{d}^s - \mathrm{d}^s \operatorname{tr})(\varphi \otimes X)\\ &= \operatorname{tr}(\mathrm{d}^s \varphi \otimes X + \varphi \odot \nabla X) - \mathrm{d}^s \operatorname{tr}(\varphi \otimes X) = \iota_X(\mathrm{d}^s \varphi) + \operatorname{tr}(\varphi \odot \nabla X) - \mathrm{d}^s \iota_X \varphi.\end{aligned}$$

By Propositions 3.3 and 2.3 we can continue sequentially with

$$\begin{aligned}&\iota_X\Big(\sum_{j=1}^n e_j^* \odot \nabla_{e_j}\varphi\Big) + \operatorname{tr}\Big(\varphi \odot \sum_{j=1}^n e_j^* \otimes \nabla_{e_j} X\Big) - \sum_{j=1}^n e_j^* \odot \nabla_{e_j} \iota_X \varphi\\ &= \sum_{j=1}^n \iota_X e_j^* \odot \nabla_{e_j}\varphi + \sum_{j=1}^n e_j^* \odot \iota_X \nabla_{e_j}\varphi + \sum_{j=1}^n \iota_{\nabla_{e_j} X}(e_j^* \odot \varphi) - \sum_{j=1}^n e_j^* \odot \nabla_{e_j} \iota_X \varphi\\ &= \sum_{j=1}^n \iota_X e_j^* \odot \nabla_{e_j}\varphi + \sum_{j=1}^n e_j^* \odot \iota_X \nabla_{e_j}\varphi + \sum_{j=1}^n \iota_{\nabla_{e_j} X}(e_j^*) \odot \varphi\\ &\quad + \sum_{j=1}^n e_j^* \odot \iota_{\nabla_{e_j} X}\varphi - \sum_{j=1}^n e_j^* \odot \nabla_{e_j} \iota_X \varphi\\ &= \sum_{j=1}^n \iota_X e_j^* \odot \nabla_{e_j}\varphi + \sum_{j=1}^n e_j^* \odot (\iota_X \nabla_{e_j}\varphi + \iota_{\nabla_{e_j} X}\varphi - \nabla_{e_j} \iota_X \varphi) + \sum_{j=1}^n \iota_{\nabla_{e_j} X}(e_j^*) \odot \varphi.\end{aligned}$$

By Proposition 3.5, we can continue with

$$\begin{aligned}&\sum_{j=1}^n \iota_X e_j^* \odot \nabla_{e_j}\varphi + \sum_{j=1}^n \iota_{\nabla_{e_j} X}(e_j^*) \odot \varphi\\ &\qquad = \sum_{j=1}^n X^j \nabla_{e_j}\varphi + \varphi \sum_{j=1}^n \langle e_j, \nabla_{e_j} X\rangle = \nabla_X \varphi + \varphi \operatorname{div} X. \ \blacksquare\end{aligned}$$

We will need also the following property of the covariant derivative.

Proposition 4.5. *For* $X, Y \in C^\infty(T)$

$$(\nabla_Y X)^\flat = \nabla_Y X^\flat.$$

Proof. The proof is a consequence of the fact that $\nabla g = 0$ and that $\flat$ is defined by g. ∎

Let us close the section with the fact relating the two investigated operators.

THEOREM 4.1. *The differential operators* $-\operatorname{grad} : C^\infty(S^k) \to C^\infty(S^k \otimes T)$ *and* $\operatorname{div} : C^\infty(S^k \otimes T) \to C^\infty(S^k)$ *are formally adjoint (each to the other) with respect to the global scalar product* $(\cdot\,|\,\cdot)$.

Proof. Let $\varphi \otimes X \in C^\infty(S^k \otimes T)$ be such that φ or X is of compact support. Let grad^* be the operator formally adjoint to grad then, by Theorems 2.1 and 3.4, we have

$$\begin{aligned}\operatorname{grad}^*(\varphi \otimes X) &= (\mathfrak{a}\,\mathrm{d}^s - \mathrm{d}^s\,\mathfrak{a})^*(\varphi \otimes X)\\ &= (\mathrm{d}^{s\,*}\,\mathfrak{a}^* - \mathfrak{a}^*\,\mathrm{d}^{s\,*})(\varphi \otimes X) = \mathrm{d}^{s\,*}(X^\flat \odot \varphi) - \mathfrak{a}^*(-\tilde{\operatorname{tr}}\,\nabla(\varphi \otimes X)).\end{aligned}$$

Further calculations will be led in a local orthonormal frame $e_1, \dots, e_n$. So, by Theorem 2.1, Theorem 3.3 and Proposition 2.3, we can continue with

$$\begin{aligned}
&-\tilde{\operatorname{tr}}\,\nabla(X^\flat \odot \varphi) + \mathfrak{a}^*\Big(\sum_{j=1}^n \iota_{e_j}\,\nabla_{e_j}(\varphi \otimes X)\Big)\\
&\quad= -\sum_{j=1}^n \iota_{e_j}\,\nabla_{e_j}(X^\flat \odot \varphi) + \mathfrak{a}^*\Big(\sum_{j=1}^n \iota_{e_j}\,\big(\nabla_{e_j}\varphi \otimes X + \varphi \otimes \nabla_{e_j}X\big)\Big)\\
&\quad= -\sum_{j=1}^n \iota_{e_j}\,\nabla_{e_j}(X^\flat \odot \varphi) + \sum_{j=1}^n \mathfrak{a}^*\big(\iota_{e_j}\,\nabla_{e_j}\varphi \otimes X + \iota_{e_j}\,\varphi \otimes \nabla_{e_j}X\big)\\
&\quad= -\sum_{j=1}^n \iota_{e_j}\,(\nabla_{e_j}X^\flat \odot \varphi) - \sum_{j=1}^n \iota_{e_j}\,(X^\flat \odot \nabla_{e_j}\varphi)\\
&\qquad\;+ \sum_{j=1}^n X^\flat \odot \iota_{e_j}\,\nabla_{e_j}\varphi + \sum_{j=1}^n (\nabla_{e_j}X)^\flat \odot \iota_{e_j}\,\varphi\\
&\quad= -\sum_{j=1}^n \iota_{e_j}\,\nabla_{e_j}X^\flat \odot \varphi - \sum_{j=1}^n \nabla_{e_j}X^\flat \odot \iota_{e_j}\,\varphi - \sum_{j=1}^n \iota_{e_j}\,X^\flat \odot \nabla_{e_j}\varphi\\
&\qquad\;- \sum_{j=1}^n X^\flat \odot \iota_{e_j}\,\nabla_{e_j}\varphi + \sum_{j=1}^n X^\flat \odot \iota_{e_j}\,\nabla_{e_j}\varphi + \sum_{j=1}^n (\nabla_{e_j}X)^\flat \odot \iota_{e_j}\,\varphi =
\end{aligned}$$

By Proposition 4.5, we can continue with

$$\begin{aligned}
-\sum_{j=1}^n \iota_{e_j}\,(\nabla_{e_j}X)^\flat \odot \varphi - \sum_{j=1}^n \iota_{e_j}\,X^\flat \odot \nabla_{e_j}\varphi &= -\varphi\sum_{j=1}^n \iota_{e_j}\,(\nabla_{e_j}X)^\flat - \sum_{j=1}^n X^j \nabla_{e_j}\varphi\\
&= -\varphi\sum_{j=1}^n \langle e_j, \nabla_{e_j}X\rangle - \nabla_X\varphi = -\varphi\operatorname{div}X - \nabla_X\varphi.
\end{aligned}$$

Finally, by Proposition 4.4, we get the assertion. ∎

5. The Laplace operator and the Weitzenböck type formula. In analogy to the case of skew-symmetric tensors a Laplace type operator can be defined in the bundle of symmetric tensors. Also here it is a linear combination of operators $\mathrm{d}^{s\,*}\,\mathrm{d}^s$ and $\mathrm{d}^s\,\mathrm{d}^{s\,*}$.

Definition 5.1. The *Laplace* operator $\Delta^s : C^\infty(S^k) \to C^\infty(S^k)$ in the bundle of symmetric forms is defined by

$$\Delta^s = \mathrm{d}^{s\,*}\,\mathrm{d}^s - \mathrm{d}^s\,\mathrm{d}^{s\,*}\,. \tag{22}$$

Δ^s was introduced first by Sampson in [Sn]. It is a second order formally self-adjoint linear differential operator. It is also, like in the skew-symmetric case, strongly elliptic. This is a consequence of, e.g., the Weitzenböck formula and the ellipticity of the operator div grad. The Weitzenböck formula will be mentioned below and the ellipticity of div grad will be proved in the next section.

Let us also mention that Δ^s has been investigated in the category of Lie algebroids in [BaP]. For $k = 1$ in [SM] a similar operator: the Yano rough Laplacian were analysed in a context of its spectral properties. Some elliptic operators in the bundle of symmetric forms were also investigated in [HMS] in a context of so called conformal Killing tensors.

In our case the Weitzenböck formula will relate two differential operators on symmetric forms: $-$ div grad and the Laplace operator defined in (22). The geometric importance of the formula comes from the fact that the difference between these second order operators is an operator of order zero (tensor) and that this zero order operator depends essentially on the curvature of M. A direct proof of our Weitzenböck formula that uses elementary properties of some special compositions of operators tr, $\tilde{\mathrm{tr}}$, $\mathfrak{a}$, d^s and $\mathrm{d}^{s\,*}$ is given in [Ka].

Theorem 5.1 (Weitzenböck formula). *The following formula holds*

$$\Delta^s = -\operatorname{div}\operatorname{grad} - \mathfrak{R}.$$

where the Ricci type tensor $\mathfrak{R}$ *is locally defined by*

$$\mathfrak{R} = \sum_{i,j=1}^{n} e_j^* \odot \iota_{e_i} \mathrm{R}_{e_i,e_j}, \tag{23}$$

for a local orthonormal frame $e_1, \dots, e_n$ *of* T *and its dual frame* $e_1^*, \dots, e_n^*$ *of* T^*.

Here the *curvature* operator in (23) is the zero order operator $\mathrm{R}_{X,Y}$ defined by

$$\mathrm{R}_{X,Y} = \nabla^2{}_{X,Y} - \nabla^2{}_{Y,X},$$

for any $X, Y \in C^\infty(T)$.

Let us terminate the section with the remark that the operators $-$ div grad and the Sampson Laplacian Δ^s are related to the Lichnerowicz Laplacian Δ_L defined in [Ln], p. 26, and discussed in the case $k = 2$ also in [Pi]. All the three operators differ (each to any other) by a zero order terms depending on the curvature. This is a simple consequence of Theorem 5.1 and the Weitzenböck type formulas in [Sn], p. 173 and [HMS], Section 6.1.

6. Natural boundary value conditions for div grad **operator.** Now let M be an oriented compact Riemannian manifold of dimension n with a nonempty smooth boundary ∂M. Assume that the orientation of ∂M is induced by the orientation of M.

First of all, derive an important integral formula relating the gradient and the divergence to an integral over the boundary

Proposition 6.1. *For $\varphi \in C^\infty(S^k)$ and $\Psi \in C^\infty(S^k \otimes TM)$*

$$(\operatorname{grad}\varphi, \Psi) = (\varphi, -\operatorname{div}\Psi) - \int_{\partial M} \langle \varphi \otimes \nu, \Psi \rangle \, \Omega_{\partial M}, \tag{24}$$

where ν is the outer unit vector field normal to the boundary.

Let us prove first the following

Lemma 6.1. *For $\varphi \in C^\infty(S^k)$ and $\Psi \in C^\infty(S^k \otimes TM)$ we have*

$$\mathrm{d}\big(\star((\varphi \lrcorner \Psi)^\flat)\big) = \big(\langle \operatorname{grad}\varphi, \Psi\rangle + \langle \varphi, \operatorname{div}\Psi\rangle\big)\Omega. \tag{25}$$

Proof. Without loss of generality we may assume that $\Psi = \psi \otimes X$ for some $\psi \in C^\infty(S^k)$, $X \in C^\infty(TM)$. Let $p \in M$, then, for any local positively oriented orthonormal frame $e_1, \dots, e_n$ in a neighborhood of p, such that $\nabla_{e_i} e_j = 0$ at p, we have

$$\begin{aligned}
\mathrm{d}\big(\star((\varphi \lrcorner \Psi)^\flat)\big)(e_1, \dots, e_n) &= \sum_{i=1}^n (-1)^{i-1} e_i\big((\star(\varphi \lrcorner \Psi)^\flat)(e_1, \dots, \hat{e_i}, \dots, e_n)\big) \\
&= \sum_{i=1}^n (-1)^{i-1} e_i((-1)^{i-1}(\varphi \lrcorner \Psi)^\flat(e_i)) = \sum_{i=1}^n e_i(\langle \varphi \otimes e_i, \Psi\rangle) \\
&= \sum_{i=1}^n \big\langle \nabla_{e_i}\varphi \otimes e_i + \varphi \otimes \nabla_{e_i} e_i, \Psi \big\rangle + \sum_{i=1}^n \big\langle \varphi \otimes e_i, \nabla_{e_i}(\psi \otimes X)\big\rangle \\
&= \sum_{i=1}^n \langle \nabla_{e_i}\varphi \otimes e_i, \Psi\rangle + \sum_{i=1}^n \big\langle \varphi \otimes e_i, \nabla_{e_i}\psi \otimes X + \psi \otimes \nabla_{e_i} X\big\rangle.
\end{aligned}$$

By Propositions 4.2 and 4.4 we can sequentially continue with

$$\begin{aligned}
&\langle \operatorname{grad}\varphi, \Psi\rangle + \sum_{i=1}^n \langle \varphi, \nabla_{e_i}\psi\rangle\langle e_i, X\rangle + \sum_{i=1}^n \langle \varphi, \psi\rangle\langle e_i, \nabla_{e_i} X\rangle \\
&= \langle \operatorname{grad}\varphi, \Psi\rangle + \sum_{i=1}^n \langle \varphi, X^i \nabla_{e_i}\psi\rangle + \langle \varphi, \psi \operatorname{div} X\rangle \\
&= \langle \operatorname{grad}\varphi, \Psi\rangle + \langle \varphi, \nabla_X \psi + \psi \operatorname{div} X\rangle = \big(\langle \operatorname{grad}\varphi, \Psi\rangle + \langle \varphi, \operatorname{div}\Psi\rangle\big)\,\Omega(e_1, \dots, e_n)
\end{aligned}$$

at p. ∎

Proof of Proposition 6.1. Integrating (25) over M and applying the Stokes formula we get (24). ∎

Using Proposition 6.1 twice we obtain

Theorem 6.1. *For $\eta \in C^\infty(S^k)$, $\vartheta \in C^\infty(S^k)$*

$$(\operatorname{div}\operatorname{grad}\eta, \vartheta) - (\eta, \operatorname{div}\operatorname{grad}\vartheta) = \int_{\partial M} (\langle \eta \otimes \nu, \operatorname{grad}\vartheta\rangle - \langle \operatorname{grad}\eta, \vartheta \otimes \nu\rangle)\,\Omega_{\partial M}. \tag{26}$$

Now we are going to introduce the so called *half-geodesic* coordinate system. Our definition is similar to the one in [Gy].

Let $x = (y, r)$ be a local coordinate system on M near ∂M such that $y = (y_1, \dots, y_{n-1})$ is a local coordinate system on ∂M and r is the normal distance to the boundary. We assume $\partial M = \{x : r(x) = 0\}$ and that $\frac{\partial}{\partial r}$ is the inward unit normal.

Let $e_1 = \frac{\partial}{\partial x_1}, \ldots, e_{n-1} = \frac{\partial}{\partial x_{n-1}}, e_n = \frac{\partial}{\partial r}$ and $e_1^* = dx_1, \ldots, e_{n-1}^* = dx_{n-1}, e_n^* = dr$. We further normalize the choice of coordinate by requiring the curves $x(r) = (y_0, r)$ for r being small enough are unit speed geodesics for any $y_o \in \partial M$. Since M is compact, the inward geodesic flow identifies a neighborhood of ∂M in M with the collar $\partial M \times [0, \delta)$ for some $\delta > 0$. The collaring gives a splitting of $TM = T\partial M \oplus T\mathbb{R}$ and a dual splitting $T^*M = T^*\partial M \oplus T^*\mathbb{R}$. To reflect this splitting we let for $\xi \in T^*M$ that $\xi = (\zeta, z)$ where $\zeta \in T^*\partial M$, $z \in T^*\mathbb{R}$.

Now, similarly to [Nn], introduce the notions of symbol and of ellipticity for a differential operator.

Let E and F be two vector bundles over M and $P : C^\infty(E) \to C^\infty(F)$ be a linear differential operator of order m.

Let $p \in M$ and $\xi \in T_p^*$. Let $\mathfrak{m}_p$ denote the ring of germs of functions vanishing at p. Let $f \in \mathfrak{m}_p$ be a function defining ξ, i.e. $\xi = (\mathrm{d}\, f)(p)$. Let $e \in E_p$, where E_p is the fiber of E at p, and $\varphi \in C^\infty(E)$ be so that $\varphi(p) = e$.

The *symbol* of P at p is the map $\sigma_P : E_p \times T_p^* \to F_p$ defined by

$$\sigma_P(e, \xi) = P(f^m \varphi)(p). \tag{27}$$

The right hand side of (27) is independent of the choice of f and φ.

A linear differential operator P is called *elliptic at* p if the map

$$E_p \ni e \mapsto \sigma_P(e, \xi) \in F_p$$

is injective for every $\xi \in T_p^*$, $\xi \neq 0$.

We say that P is *elliptic* if it is elliptic at every p.

It is worth to mention here that the ellipticity of P is a not negotiable demand in the process of investigating of so called elliptic boundary conditions (see (50)).

First we are going to construct and test a system of natural boundary conditions in the sense of Branson and Pierzchalski for the operator div grad. The construction was described in their manuscript [BP] from 2004 and next repeated in the introduction of the Kozłowski–Pierzchalski paper [KP]. To state the system in our case, split—at the boundary—the bundle of symmetric k-tensors into summands that are irreducible with respect to the action of $\mathrm{SO}(n-1)$. We treat here $\mathrm{SO}(n-1)$ as a subgroup of $\mathrm{SO}(n)$ keeping ν invariant. The branching rule (cf. [Zo]) says that there are $k+1$ summands in the splitting. We will determine them exactly.

Let $\nu^* = -dr$. Then $\nu = -\frac{\partial}{\partial r}$. For any $\varphi \in C^\infty(S^k)$ we have at the boundary the following unique decomposition:

$$\varphi = \varphi_0 \odot (\nu^*)^k + \varphi_1 \odot (\nu^*)^{k-1} + \ldots + \varphi_{k-1} \odot (\nu^*)^1 + \varphi_k \odot (\nu^*)^0, \tag{28}$$

where $\varphi_j \in C^\infty(S^j)$, $j \in \{0, 1, \ldots, k\}$, do not contain ν^* in the expansion.

The decomposition (28) defines just the announced splitting of the bundle S^k at the boundary.

By taking $\varphi_j = \sum_{i_1,\ldots,i_j=1}^{n-1} f^j_{i_1,\ldots,i_j} dx_{i_1} \odot \ldots \odot dx_{i_j}$ in (28) we get

$$\varphi = \sum_{j=0}^{k} \sum_{i_1,\ldots,i_j=1}^{n-1} f^j_{i_1,\ldots,i_j} dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j}. \tag{29}$$

Now we are ready to construct a system of boundary conditions.

First our aim is to construct such boundary conditions that each of them will assure self-adjointness of div grad in the subspace of forms satisfying the condition, i.e. that the integral on the right hand side of (26) will vanish. In practice, we will demand more: all individual terms under this integral must vanish.

Notice that at the boundary the bundle $S^k \otimes TM$ splits into the direct sum of two bundles

$$(S^k \otimes T\partial M) \oplus (S^k \otimes (T\partial M)^\perp). \tag{30}$$

So, according to this splitting, we have for $\varphi \in C^\infty(S^k)$

$$\operatorname{grad}\varphi = \operatorname{grad}\varphi^T + \operatorname{grad}\varphi^N, \tag{31}$$

where $\operatorname{grad}\varphi^T$ and $\operatorname{grad}\varphi^N$ are the tangent and the normal parts of $\operatorname{grad}\varphi$, respectively. Recall that $\operatorname{grad}\varphi$ is a symmetric form with values in the tangent bundle. So the splitting of this tangent bundle at the boundary defines the splitting (31). Since ν is orthogonal to the boundary it implies that for $\eta \in C^\infty(S^k)$, $\vartheta \in C^\infty(S^k)$

$$\langle \eta \otimes \nu, \operatorname{grad}\vartheta\rangle = \langle \eta \otimes \nu, (\operatorname{grad}\vartheta)^N\rangle.$$

Since $(\operatorname{grad}\vartheta)^N$ defines uniquely a section $\operatorname{grad}^1\vartheta$ of S^k over ∂M such that

$$(\operatorname{grad}\vartheta)^N = \operatorname{grad}^1\vartheta \otimes \nu, \tag{32}$$

we have

$$\langle \eta \otimes \nu, \operatorname{grad}\vartheta\rangle = \langle \eta, \operatorname{grad}^1\vartheta\rangle \tag{33}$$

on ∂M.

According to (28), a symmetric k-form φ may be expressed as a sum of $(k+1)$ summands:

$$\varphi = \pi_0\varphi + \ldots + \pi_k\varphi,$$

where π_j, $j = 0, \ldots, k$ are projections on sequent, in fact mutually orthogonal subbundles. More exactly, for φ of the form (28),

$$\pi_j(\varphi) = \varphi_j \odot (\nu^*)^{k-j}.$$

The scalar product (33) has then the following expression:

$$\langle \eta, \operatorname{grad}^1\vartheta\rangle = +\langle \pi_0\eta, \pi_0(\operatorname{grad}^1\vartheta)\rangle + \langle \pi_1\eta, \pi_1(\operatorname{grad}^1\vartheta)\rangle + \ldots + \langle \pi_k\eta, \pi_k(\operatorname{grad}^1\vartheta)\rangle. \tag{34}$$

The right hand side of (34) can symbolically be written in a matrix form:

$$\begin{bmatrix} \pi_0\eta & \pi_0(\operatorname{grad}^1\vartheta) \\ \pi_1\eta & \pi_1(\operatorname{grad}^1\vartheta) \\ \vdots & \vdots \\ \pi_k\eta & \pi_k(\operatorname{grad}^1\vartheta) \end{bmatrix}.$$

Now, let us demand that exactly one of the terms in each row vanishes. Each such particular demand defines a single natural homogeneous boundary condition. All the possible demands define altogether 2^{k+1} different boundary conditions. This is in contrast to the case of skew-symmetric forms where (for analogous differential operators) there are always four such conditions independently of the degree k of these forms (cf. [KP]).

Coming back to our case, let us write the 2^{k+1} matrices obtained by the described demands:

$$\begin{bmatrix} 0 & * \\ 0 & * \\ \vdots & \vdots \\ 0 & * \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} 0 & * \\ * & 0 \\ \vdots & \vdots \\ 0 & * \end{bmatrix}, \quad \ldots, \quad \begin{bmatrix} * & 0 \\ * & 0 \\ \vdots & \vdots \\ * & 0 \end{bmatrix}. \tag{35}$$

The first matrix defines the homogeneous Dirichlet boundary condition which for $\varphi \in C^\infty(S^k)$ reads:

$$\pi_0\varphi = 0, \quad \pi_1\varphi = 0, \quad \ldots, \quad \pi_k\varphi = 0 \quad \text{ on } \partial M$$

or, equivalently, $\varphi = 0$ on ∂M.

The last matrix defines the homogeneous Neumann boundary condition which for $\varphi \in C^\infty(S^k)$ reads:

$$\pi_0(\operatorname{grad}^1 \varphi) = 0, \quad \pi_1(\operatorname{grad}^1 \varphi) = 0, \quad \ldots, \quad \pi_k(\operatorname{grad}^1 \varphi) = 0 \quad \text{ on } \partial M$$

or, equivalently, $(\operatorname{grad} \varphi)^N = 0$ on ∂M.

These two peripheral conditions together with all other $2^{k+1} - 2$ ones form a complete set of natural homogeneous boundary conditions.

Under each of them the operator div grad is self-adjoint with respect to the global scalar product. More exactly:

Proposition 6.2. *For $\eta \in C^\infty(S^k)$, $\vartheta \in C^\infty(S^k)$*

$$(\operatorname{div}\operatorname{grad} \eta, \vartheta) = (\eta, \operatorname{div}\operatorname{grad} \vartheta) \tag{36}$$

if only both η and ϑ satisfy one of just defined homogeneous boundary condition i.e., one of the defined by matrices (35).

Proof. It is just a direct consequence of Theorem 6.1. ■

Our nearest aim is to discuss so called *ellipticity at the boundary* of boundary conditions suitably defined by the matrices (35). The exact definition will be given soon.

First, let us evaluate the symbols of considered operators. Denote by $\sharp : T^* \to T$ the isomorphism inverse to isomorphism $\flat : T \to T^*$ defined by (7).

Proposition 6.3. *Let $p \in M$, $\xi \in T^*_p$, $\xi \neq 0$, $v \in T_p$ and $e \in S^k_p$. The symbols of* d^s, grad *and* div *at p are given by*

$$\sigma_{\mathrm{d}^s}(e, \xi) = e \odot \xi, \tag{37}$$

$$\sigma_{\operatorname{grad}}(e, \xi) = e \otimes \xi^\sharp, \tag{38}$$

$$\sigma_{\operatorname{div}}(e \otimes v, \xi) = (\iota_v \xi) e, \tag{39}$$

respectively.

Proof. Let $p \in M$ and let $f : M \to \mathbb{R}$ be a function such that $f(p) = 0$ and $\mathrm{d} f = \xi$ at p. Let φ be such a section of S^k that $\varphi(p) = e$ and X such a vector field that $X(p) = v$. By the definition of symbol and Proposition 3.4 we have

$$\begin{aligned}\sigma_{\mathrm{d}^s}(e,\xi) = \mathrm{d}^s(f\varphi)(p) &= (\mathrm{d}^s f \odot \varphi) + f\,\mathrm{d}^s \varphi)(p)\\ &= (\varphi \odot (\mathrm{d}\, f))(p) = e \odot \xi.\end{aligned}$$

Similarly, by Proposition 4.1,

$$\begin{aligned}\sigma_{\mathrm{grad}}(e,\xi) = \mathrm{grad}(f\varphi)(p) &= (\varphi \otimes (\mathrm{d}\, f)^\sharp + f\,\mathrm{grad}\,\varphi)(p)\\ &= (\varphi \otimes (\mathrm{d}\, f)^\sharp)(p) = e \otimes \xi^\sharp.\end{aligned}$$

Finally, by Proposition 4.4,

$$\begin{aligned}\sigma_{\mathrm{div}}(e\otimes v,\xi) &= (\mathrm{div}(f\varphi \otimes X))(p) = (\nabla_X(f\varphi) + f\varphi\,\mathrm{div}\,X)(p)\\ &= ((\nabla_X f)\varphi + f\nabla_X\varphi)(p) = ((\nabla_X f)\varphi)(p)\\ &= ((\mathrm{d}\, f)(X)\varphi)(p) = (\iota_X(\mathrm{d}\, f)\varphi)(p) = (\iota_v \xi)e. \ \blacksquare\end{aligned}$$

Corollary 6.1. *Let $p \in M$, $\xi \in T_p^*$, $v \in T_p$ and $\xi \neq 0$ and $e \in S_p^k$. The symbol of* div grad *at p is given by*

$$\sigma_{\mathrm{div\,grad}}(e,\xi) = \|\xi\|^2 e. \tag{40}$$

Proof. Since the symbol of a composition of differential operators is the composition of their symbols (cf. [Nn]), (40) is a consequence of (38) and (39). ■

Notice that by (37) and (38) the actions of σ_{d^s} and σ_{grad} are the symmetric tensor multiplication by the covector ξ and the vector $\xi^\sharp$, respectively, so the both symbols are injective for $\xi \neq 0$ and we obtain

Proposition 6.4. d^s *and* grad *are elliptic operators of the first order.*

Similarly, since by (40), the symbol $\sigma_{\mathrm{div\,grad}}$ is positively defined we get

Proposition 6.5. div grad *is a strongly elliptic operator of the second order.*

Recall the definition of the second fundamental form of ∂M (cf. [KN]). Notice that ν is a unit vector field on ∂M (and in fact in a collar of ∂M) orthogonal to the boundary. Let $p \in \partial M$. The formula

$$A(X) = -\nabla_X \nu, \quad X \in T_p\partial M$$

defines an endomorphism of the tangent space $T_p\partial M$.

The *second fundamental form* h of ∂M at p is a bilinear form dual to A, i.e. the form h defined by

$$h(X,Y) = \langle A(X), Y\rangle, \quad X,Y \in T_p\partial M. \tag{41}$$

In a consequence h is a symmetric bilinear form represented in our half-geodesic system by

$$h_{\alpha,\beta} = \left\langle -\nabla_{\partial/\partial x_\alpha}\nu, \frac{\partial}{\partial x_\beta}\right\rangle, \quad \alpha,\beta = 1,\dots,n-1. \tag{42}$$

The boundary ∂M is said to be *totally geodesic* if the second fundamental form h of ∂M is identically equal to zero, and *totally umbilical* if h is proportional to g.

Evaluate the gradient in a local coordinate system near the boundary.

Lemma 6.2. *Let (cf. (29)) in our half-geodesic system*

$$\varphi = \sum_{j=0}^{k} \sum_{i_1,\ldots,i_j=1}^{n-1} f^j_{i_1,\ldots,i_j} dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j}.$$

Then, at ∂M,

$$\begin{aligned} -\operatorname{grad}^1 \varphi = \sum_{j=0}^{k} \sum_{i_1,\ldots,i_j=1}^{n-1} \Big(\frac{\partial}{\partial r}(f^j_{i_1,\ldots,i_j}) dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j} \\ + \sum_{l=1}^{j} \sum_{\alpha=1}^{n-1} f^j_{i_1,\ldots,i_j} h_{i_l\alpha} dx_{i_1} \odot \ldots \odot dx_\alpha \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j} \Big). \end{aligned} \tag{43}$$

Proof. According to Proposition 4.2 and (31) we have

$$\begin{aligned} (\operatorname{grad} \varphi)^N &= \nabla_{\partial/\partial r}\varphi \otimes \frac{\partial}{\partial r} \\ &= \sum_{j=0}^{k} \sum_{i_1,\ldots,i_j=1}^{n-1} \nabla_{\partial/\partial r}\big(f^j_{i_1,\ldots,i_j} dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j}\big) \otimes \frac{\partial}{\partial r}. \end{aligned}$$

So, by (32) and $\frac{\partial}{\partial r} = -\nu$,

$$-\operatorname{grad}^1 \varphi = \sum_{j=0}^{k} \sum_{i_1,\ldots,i_j=1}^{n-1} \nabla_{\partial/\partial r}\big(f^j_{i_1,\ldots,i_j} dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j}\big).$$

By the definition of half-geodesic system we have $\nabla_{\partial/\partial r}\frac{\partial}{\partial r} = 0$, so

$$\begin{aligned} -\operatorname{grad}^1 \varphi = \sum_{j=0}^{k} \sum_{i_1,\ldots,i_j=1}^{n-1} \Big(\nabla_{\partial/\partial r}(f^j_{i_1,\ldots,i_j}) dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j} \\ + \sum_{l=1}^{j} f^j_{i_1,\ldots,i_j} dx_{i_1} \odot \ldots \nabla_{\partial/\partial r}(dx_{i_l}) \odot \ldots dx_{i_j} \odot (\nu^*)^{k-j} \Big). \end{aligned}$$

According to (42) we get

$$\begin{aligned} -\operatorname{grad}^1 \varphi = \sum_{j=0}^{k} \sum_{i_1,\ldots,i_j=1}^{n-1} \Big(\frac{\partial}{\partial r}(f^j_{i_1,\ldots,i_j}) dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j} \\ + \sum_{l=1}^{j} \sum_{\alpha=1}^{n-1} f^j_{i_1,\ldots,i_j} dx_{i_1} \odot \ldots h_{i_l\alpha} dx_\alpha \odot \ldots dx_{i_j} \odot (\nu^*)^{k-j} \Big). \quad \blacksquare \end{aligned}$$

Let us come back to discussion of the ellipticity of our boundary conditions. We will follow the method described in Section 1.9 of [Gy].

Let $W = W_0 \oplus W_1$ be the bundle of Cauchy data, i.e. each W_a, $a = 0, 1$, is just S^k restricted to ∂M. Define the operator

$$\gamma : C^\infty(S^k) \to C^\infty(W)$$

by

$$\gamma(\varphi) = \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix}$$

where, for $a = 0, 1$,

$$\varphi^a = \frac{\partial^a \varphi}{\partial r^a}\Big|_{\partial M}. \tag{44}$$

Here, for φ of the form (29), $\frac{\partial^a \varphi}{\partial r^a}$ is defined by

$$\frac{\partial^a \varphi}{\partial r^a} = \sum_{j=0}^{k} \frac{\partial^a \varphi_j}{\partial r^a} \odot (\nu^*)^{k-j} = \sum_{j=0}^{k} \sum_{i_1,\dots,i_j=1}^{n-1} \frac{\partial^a (f^j_{i_1,\dots,i_j})}{\partial r^a} dx_{i_1} \odot \ldots \odot dx_{i_j} \odot (\nu^*)^{k-j}.$$

Let

$$W' = W'_0 \oplus W'_1 \oplus \ldots \oplus W'_k$$

be an auxiliary graded vector bundle over ∂M. So W' is the bundle $S^k \oplus S^k$ restricted to ∂M and its splitting (gradation) is the orthogonal sum of sub-bundles defined by the splitting (28).

Define the *tangential differential operator* $\tilde{\mathrm{B}} : C^\infty(W) \to C^\infty(W')$ over ∂M. According to the above splittings of W and W', $\tilde{\mathrm{B}}$ decomposes into

$$\tilde{\mathrm{B}}^a_j : C^\infty(W_a) \to C^\infty(W'_j) \qquad a = 0, 1,\ j = 0, 1, \ldots, k.$$

Here

$$\tilde{\mathrm{B}}^0_j(\varphi) = \varphi^0_j \odot (\nu^*)^{k-j} \quad \text{and} \quad \tilde{\mathrm{B}}^1_j(\varphi) = \tilde{\varphi}^1_j \odot (\nu^*)^{k-j}$$

and for $\varphi \in C^\infty(S^k)$ the forms φ^0_j are defined by the splitting (28) and the forms $\tilde{\varphi}^1_j$ are defined as follows. Since, near ∂M, $-\operatorname{grad}^1 \varphi$ is a section of S^k we have

$$-\operatorname{grad}^1 \varphi = \tilde{\varphi}^1_0 \odot (\nu^*)^k + \tilde{\varphi}^1_1 \odot (\nu^*)^{k-1} + \ldots + \tilde{\varphi}^1_k \odot (\nu^*)^0 \tag{45}$$

and the forms $\tilde{\varphi}^1_j$ are uniquely defined by the splitting.

By Lemma 6.2 and (44), $\tilde{\mathrm{B}}$ can be represented by a matrix of block form

$$\begin{bmatrix} I & 0 \\ H & I \end{bmatrix}, \tag{46}$$

where the terms of H depend on the coefficients of the second fundamental form $h = (h_{\alpha,\beta})_{\alpha,\beta=1,\dots,n-1}$ of ∂M. Since $\tilde{\mathrm{B}}$ is a linear operator $\sigma(\tilde{\mathrm{B}}) = \tilde{\mathrm{B}}$, so, the equation

$$\sigma(\tilde{\mathrm{B}}) \begin{bmatrix} \varphi^0 \\ \varphi^1 \end{bmatrix} = \begin{bmatrix} \varphi^0 \\ \tilde{\varphi}^1 \end{bmatrix} \tag{47}$$

expresses symbolically the action of $\sigma(\tilde{\mathrm{B}})$.

Let us come back to the boundary conditions defined by matrices (35). To each such matrix we can associate a boundary condition (not necessarily homogeneous this time). To this aim notice that a choice of the matrix is equivalent to the choice of a pair (J^0, J^1) of increasing subsequences of $(0, 1, \ldots, k)$, $J^0 = (j^0_0, j^0_1, \ldots, j^0_{s-1})$, and $J^1 = (j^1_0, j^1_1, \ldots, j^1_{k-s})$, for some $s \in \{0, \ldots, k+1\}$, satisfying

$$\{j^0_0, j^0_1, \ldots, j^0_{s-1}\} \cap \{j^1_0, j^1_1, \ldots, j^1_{k-s}\} = \emptyset$$
$$\{j^0_0, j^0_1, \ldots, j^0_{s-1}\} \cup \{j^1_0, j^1_1, \ldots, j^1_{k-s}\} = \{0, 1, \ldots, k\}.$$

We accept also the situation when one of the sequences J^0, J^1 is empty (this holds for $s = 0$ or $s = k+1$, respectively). Then the other sequence is $(0, \ldots, k)$. The terms of J^0 point out the places of zeros in the first column of matrices (35) and the terms of J^1 point out the places of zeros in the second column. In fact, the sequence J^1 is uniquely determined by the choice of J^0. One can easily see that there are 2^{k+1} such pairs. Any pair (J^0, J^1) defines the projection

$$\pi_{(J^0,J^1)} : W' \to W'_{\mathrm{B}_{(J^0,J^1)}}$$

of the form

$$\begin{bmatrix} a_0^0 \\ \vdots \\ a_k^0 \\ a_0^1 \\ \vdots \\ a_k^1 \end{bmatrix} \overset{\pi_{(J^0,J^1)}}{\longmapsto} \begin{bmatrix} a^0_{j^0_0} \\ \vdots \\ a^0_{j^0_{s-1}} \\ a^1_{j^1_0} \\ \vdots \\ a^1_{j^1_{k-s}} \end{bmatrix}.$$

Notice that $W'_{\mathrm{B}_{(J^0,J^1)}} = \pi_{(J^0,J^1)}(W')$ can be treated as a subspace of W' and that

$$\dim W'_{\mathrm{B}_{(J^0,J^1)}} = \frac{1}{2} \dim W' = \frac{1}{2} \dim W.$$

Let

$$\tilde{\mathrm{B}}_{(J^0,J^1)} = \pi_{(J^0,J^1)} \circ \tilde{\mathrm{B}}\,.$$

For a given pair (J^0, J^1) define the boundary condition $\mathrm{B}_{(J^0,J^1)}$ by

$$C^\infty(S^k) \ni \varphi \overset{\mathrm{B}_{(J^0,J^1)}}{\longmapsto} \begin{bmatrix} \varphi^0_{j^0_0} \odot (\nu^*)^{k-j^0_0} \\ \vdots \\ \varphi^0_{j^0_{s-1}} \odot (\nu^*)^{k-j^0_{s-1}} \\ \tilde{\varphi}^1_{j^1_0} \odot (\nu^*)^{k-j^1_0} \\ \vdots \\ \tilde{\varphi}^1_{j^1_{k-s}} \odot (\nu^*)^{k-j^1_{k-s}} \end{bmatrix} \in W'_{\mathrm{B}_{(J^0,J^1)}}. \tag{48}$$

Notice that the vector on the right hand side of (48), though defined in a given half-geodesic system, is—in fact—independent of a choice of the system. Notice also that

$$\mathrm{B}_{(J^0,J^1)} = \tilde{\mathrm{B}}_{(J^0,J^1)} \circ \gamma.$$

Fix (J^0, J^1). We will write B instead of $\mathrm{B}_{(J^0,J^1)}$.

To define the notion of ellipticity of a boundary condition for a linear differential operator of second order $P : C^\infty(E) \to C^\infty(E)$, we consider the ordinary differential equation (cf. [Gy]):

$$\sigma_P(\zeta, D_r) f(r) = -\lambda f(r) \quad \text{with} \quad \lim_{r\to\infty} f(r) = 0 \tag{49}$$

where

$$(0,0) \neq (\zeta, \lambda) \in T^*\partial M \times \mathbb{C} \setminus \{\mathbb{R}_+ \cup \mathbb{R}_-\}$$

and where D_r is the derivation in the r direction.

Definition 6.1. A boundary condition B is said to be *elliptic* with respect to $\mathbb{C} \setminus \{\mathbb{R}_+ \cup \mathbb{R}_-\}$ if

$$\det(\sigma_P(x,\xi) - \lambda I) \neq 0 \tag{50}$$

in the interior of M for all $(\xi, \lambda) \neq (0,0) \in T^*M \times \mathbb{C} \setminus \{\mathbb{R}_+ \cup \mathbb{R}_-\}$ and if on the boundary there always exists a unique solution to ordinary differential equation (49) such that

$$\sigma(\tilde{B})\gamma(f) = f'$$

for any prescribed $f' \in W'_{\mathrm{B}}$.

Now we are ready to investigate the problem of ellipticity of our boundary conditions.

Notice first that for our $P = \operatorname{div}\operatorname{grad}$ the condition (50) is satisfied automatically by the ellipticity in the interior, i.e. by (40).

Consider equation (49) where $\sigma_P = \sigma_{\operatorname{div}\operatorname{grad}}$ is the symbol of div grad and $\lambda \in \mathbb{C} \setminus \{\mathbb{R}_+ \cup \mathbb{R}_-\}$. We have

$$\sigma_{\operatorname{div}\operatorname{grad}}(\xi, e) = \|\xi\|^2 e \quad \text{and} \quad \|\xi\|^2 = \|\xi^T\|^2 + \|\xi^N\|^2$$

so our equation is of the form

$$\frac{\partial^2 f}{\partial r^2} = -\mu f, \tag{51}$$

where $\mu = \lambda + \|\xi^T\|^2$.

Equation (51) is a homogeneous linear second order differential equation with constant complex coefficients. Let $z_0 = a + \sqrt{-1}\, b$, $b > 0$, be one of the square roots of μ. Taking into account the expected behavior when r tends to ∞ we see that

$$f_C(r) = Ce^{z_0 r\sqrt{-1}}$$

are the searched solutions.

For the ellipticity at the boundary of any particular boundary condition B, notice that the equation (49) written in coordinates means in practice that the system of linear equations of the form (51) will be satisfied: each equation for each coefficient of φ. A given boundary condition means that the suitable coordinates of the vector on the right hand side of (47) will be prescribed. Since the matrix $\sigma(\tilde{\mathrm{B}})$ is invertible we can uniquely solve the equation. Since $\sigma(\tilde{\mathrm{B}})$ is an isomorphism, the dimension of the inverse image of the subspace generated by the prescribed coordinates will be exactly the same as that of generated by these coordinates. And this implies the uniqueness of solutions to the considered system at this boundary condition and—in a consequence—to (49).

That way we have proved the main theorem of the paper:

THEOREM 6.2. *Each of the* 2^{k+1} *boundary conditions* B *defined by* (48) *is elliptic for the elliptic operator* div grad.

EXAMPLE 6.1. If $k = 1$ then any form $\varphi \in C^\infty(S^1)$ can be written as

$$\varphi = f^0(\nu^*)^1 + \Big(\sum_{\alpha=1}^{n-1} f^1_\alpha \, dx_\alpha\Big)(\nu^*)^0,$$

so over the boundary φ is represented by the column vector

$$\begin{bmatrix} f^0 \\ f^1_1 \\ \vdots \\ f^1_{n-1} \end{bmatrix}.$$

Similarly, by Lemma 6.2, $-\operatorname{grad}^1 \varphi$ can be represented by the column vector

$$\begin{bmatrix} \frac{\partial f^0}{\partial r} \\ \frac{\partial f^1_1}{\partial r} + \sum_{\beta=1}^{n-1} f^1_\beta h_{\beta 1} \\ \vdots \\ \frac{\partial f^1_{n-1}}{\partial r} + \sum_{\beta=1}^{n-1} f^1_\beta h_{\beta n-1} \end{bmatrix}.$$

So $\sigma(\tilde{\mathrm{B}})$ gives a one to one correspondence between the following column vectors:

$$\begin{bmatrix} f^0 \\ f^1_1 \\ \vdots \\ f^1_{n-1} \\ \frac{\partial f^0}{\partial r} \\ \frac{\partial f^1_1}{\partial r} \\ \vdots \\ \frac{\partial f^1_{n-1}}{\partial r} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} f^0 \\ f^1_1 \\ \vdots \\ f^1_{n-1} \\ \frac{\partial f^0}{\partial r} \\ \frac{\partial f^1_1}{\partial r} + \sum_{\beta=1}^{n-1} f^1_\beta h_{\beta 1} \\ \vdots \\ \frac{\partial f^1_{n-1}}{\partial r} + \sum_{\beta=1}^{n-1} f^1_\beta h_{\beta n-1} \end{bmatrix}.$$

First of all, we can easily see that the matrix representing $\sigma(\tilde{\mathrm{B}}) = \tilde{\mathrm{B}}$ is indeed an invertible matrix of the form (46). Next we see that each of the four boundary conditions is an elliptic boundary condition:

- the first (Dirichlet)

$$\mathrm{B}_{((0,1),\emptyset)} = \begin{bmatrix} f^0 \\ f^1_1 \\ \vdots \\ f^1_{n-1} \end{bmatrix}, \tag{52}$$

- the second

$$\mathrm{B}_{((0),(1))} = \begin{bmatrix} f^0 \\ \frac{\partial f^1_1}{\partial r} + \sum_{\beta=1}^{n-1} f^1_\beta h_{\beta 1} \\ \vdots \\ \frac{\partial f^1_{n-1}}{\partial r} + \sum_{\beta=1}^{n-1} f^1_\beta h_{\beta n-1} \end{bmatrix}, \tag{53}$$

- the third

$$\mathrm{B}_{((1),(0))} = \begin{bmatrix} f^1_1 \\ \vdots \\ f^1_{n-1} \\ \frac{\partial f^0}{\partial r} \end{bmatrix}, \tag{54}$$

- the fourth (Neumann)

$$\mathrm{B}_{(\emptyset,(0,1))} = \begin{bmatrix} \frac{\partial f^0}{\partial r} \\ \frac{\partial f_1^1}{\partial r} + \sum_{\beta=1}^{n-1} f_\beta^1 h_{\beta 1} \\ \vdots \\ \frac{\partial f_{n-1}^1}{\partial r} + \sum_{\beta=1}^{n-1} f_\beta^1 h_{\beta n-1} \end{bmatrix}. \tag{55}$$

The situation simplifies essentially when ∂M is totally umbilical, i.e. when $h = cg$ for some $c \in C^\infty(\partial M)$. The conditions (52) and (54) determine the solution to (51) uniquely what is obvious while (53) and (55) reduce to

$$\begin{bmatrix} f^0 \\ \frac{\partial f_1^1}{\partial r} + c f_1^1 \\ \vdots \\ \frac{\partial f_{n-1}^1}{\partial r} + c f_{n-1}^1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \frac{\partial f^0}{\partial r} \\ \frac{\partial f_1^1}{\partial r} + c f_1^1 \\ \vdots \\ \frac{\partial f_{n-1}^1}{\partial r} + c f_{n-1}^1 \end{bmatrix},$$

respectively, and again one can easily see that both the conditions imply uniqueness.

In the case if ∂M is totally geodesic the situation is even more simple, since the matrix representing $\sigma(\tilde{\mathrm{B}})$ is just the identity matrix.

Let us come back to the general case. A consequence of the ellipticity of boundary conditions is the following important fact:

PROPOSITION 6.6. *For each of all considered 2^{k+1} boundary conditions there exists a sequence (ϑ_n), $n = 1, \dots,$ of smooth sections of the bundle of symmetric k-tensors on M such that*:

a) *(ϑ_n) is a complete orthonormal system in L^2 of eigenvectors:* $\operatorname{div}\operatorname{grad}\vartheta_n = \lambda_n \vartheta_n$*,*
b) *the forms ϑ_n satisfy the boundary condition,*
c) *the eigenvalues λ_n are real and* $\lim_{n\to\infty} \lambda_n = -\infty$*.*

Proof. The properties a) and b) are a direct consequence of Lemma 1.9.1 in [Gy].

The property c) is additionally a consequence of the fact that div grad is negatively defined in the sense that

$$(\operatorname{div}\operatorname{grad}\vartheta, \vartheta) \leq 0 \tag{56}$$

for any $\vartheta \in C^\infty(S^k)$ satisfying one of the considered boundary conditions, which implies that the eigenvalues λ_n are all nonpositive. Inequality (56) is a consequence of Proposition 6.1, where φ should be replaced by ϑ and Ψ by $\operatorname{grad}\vartheta$ and where the integral on the right hand side of (24) vanishes by the boundary condition. ■

Acknowledgements. The authors would like to express their gratitude to the referee for pointing out references to the Lichnerowicz Laplacian.

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