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THE GELFAND–NAIMARK–SEGAL CONSTRUCTION FOR UNITARY REPRESENTATIONS OF $\mathbb{Z}_2^n$-GRADED LIE SUPERGROUPS

MOHAMMAD MOHAMMADI

Department of Mathematics, Institute for Advanced Studies in Basic Sciences (IASBS)
No. 444, Prof. Yousef Sobouti Blvd., P. O. Box 45195-1159
Zanjan, Iran
E-mail: moh.mohamady@iasbs.ac.ir

HADI SALMASIAN

Department of Mathematics and Statistics, University of Ottawa
585 King Edward Ave, Ottawa K1N 6N5, ON, Canada
E-mail: hsalmasi@uottawa.ca

Abstract. We establish a Gelfand–Naimark–Segal construction which yields a correspondence between cyclic unitary representations and positive definite superfunctions of a general class of $\mathbb{Z}_2^n$-graded Lie supergroups.

1. Introduction. It is by now well known that unitary representations of Lie supergroups appear in various areas of mathematical physics related to supersymmetry (see [9], [10], [8], [17] as some important examples among numerous references). A mathematical approach to analysis on unitary representations of Lie supergroups was pioneered in [3], where unitary representations are defined based on the notion of the Harish-Chandra pair associated to a Lie supergroup.

More recently, there is growing interest in studying generalized supergeometry, that is, geometry of graded manifolds where the grading group is not $\mathbb{Z}_2$, but $\mathbb{Z}_2^n := \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$. The foundational aspects of the theory of $\mathbb{Z}_2^n$-supermanifolds were recently established in the works of Covolo–Grabowski–Poncin (see [5], [6]) and Covolo–Kwok–Poncin (see [7]).

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In this short note, we make a first attempt to extend the theory of unitary representations to the $\mathbb{Z}_2^n$-graded setting. To this end, we use the concept of a $\mathbb{Z}_2^n$-graded Harish-Chandra pair, which consists of a pair $(G_0, \mathfrak{g})$ where G_0 is a Lie group and $\mathfrak{g}$ is a $\mathbb{Z}_2^n$-graded generalization of a Lie superalgebra, sometimes known as a *Lie color algebra*. Extending one of the main results of [15], in Theorem 5.4 we prove that under a "perfectness" condition on $\mathfrak{g}$, there exists a Gelfand–Naimark–Segal (GNS) construction which yields a correspondence between positive definite smooth super-functions and cyclic unitary representations of $(G_0, \mathfrak{g})$. Theorem 5.4 is applicable to interesting examples, such as $\mathbb{Z}_2^n$-Lie supergroups of classical type, e.g., the Harish-Chandra pair corresponding to $\mathfrak{gl}(V)$ defined in Example 2.5, where V is a $\mathbb{Z}_2^n$-graded vector space.

The key technical tool in the proof of Theorem 5.4 is a Stability Theorem (see Theorem 4.3) which guarantees the existence of a unique unitary representation associated to a weaker structure, called a *pre-representation*. Such a Stability Theorem holds unconditionally when $n = 1$. But for $n > 1$, it is not true in general. We are able to retrieve a variation of the Stability Theorem under the aforementioned extra condition on $\mathfrak{g}$. However, this still leaves the question of a general GNS construction open for further investigation. We defer the latter question, presentation of explicit examples, as well as some proof details, to a future work.

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2. $\mathbb{Z}_2^n$-supergeometry. We begin by reviewing the basic concepts of $\mathbb{Z}_2^n$-graded supergeometry, in the sense of [6]. Let $\Gamma := \mathbb{Z}_2^n := \mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ where $\mathbb{Z}_2 := \{\overline{\mathbf{0}}, \overline{\mathbf{1}}\}$, and let $\mathsf{b} : \Gamma \times \Gamma \to \mathbb{Z}_2$ be a non-degenerate symmetric $\mathbb{Z}_2$-bilinear map. By a result of Albert (see [1, Theorem 6] or [11, Sections 1–10]), if $\mathsf{b}(\cdot,\cdot)$ is of alternate type (i.e., $\mathsf{b}(a,a) = \overline{\mathbf{0}}$ for every $a \in \Gamma$), then $\mathsf{b}(\cdot,\cdot)$ is equivalent to the standard "symplectic" form

$$\mathsf{b}_-(a,b) := \sum_{j=1}^{n} a_{2j-1}b_{2j} + a_{2j}b_{2j-1} \quad \text{for every } a = (a_1,\ldots,a_n) \text{ and } b = (b_1,\ldots,b_n) \in \Gamma,$$

whereas if $\mathsf{b}(a,a) \neq \overline{\mathbf{0}}$ for some $a \in \Gamma$, then $\mathsf{b}(\cdot,\cdot)$ is equivalent to the standard symmetric form

$$\mathsf{b}_+(a,b) := \sum_{j=1}^{m} a_j b_j \quad \text{for every } a = (a_1,\ldots,a_n) \text{ and } b = (b_1,\ldots,b_n) \in \Gamma.$$

Henceforth we assume that $\mathsf{b}(\cdot,\cdot)$ is not of alternate type. Since an equivalence of $\mathsf{b}(\cdot,\cdot)$ and $\mathsf{b}_+(\cdot,\cdot)$ is indeed an automorphism of the finite abelian group Γ, without loss of generality from now on we can assume that $\mathsf{b} = \mathsf{b}_+$. In particular, from now on we represent an element $a \in \Gamma$ by $a := \sum_{j=1}^n a_j \mathsf{e}_j$, where $\{\mathsf{e}_j\}_{j=1}^n$ is an orthonormal basis of Γ with respect to $\mathsf{b}(\cdot,\cdot)$.

We now define $\beta : \Gamma \times \Gamma \to \{\pm 1\}$ by $\beta(a,b) := (-1)^{\mathsf{b}(a,b)}$ for $a, b \in \Gamma$. We equip the category Vec_Γ of Γ-graded complex vector spaces with the symmetry operator

$$\mathbb{S}_{V,W} : V \otimes W \to W \otimes V, \qquad \mathbb{S}_{V,W}(v \otimes w) := \beta(|v|, |w|) w \otimes v, \tag{1}$$

where $|v| \in \Gamma$ denotes the degree of a homogeneous vector $v \in V$.

Remark 2.1. As it is customary in supergeometry, equality (1) should be construed as a relation for homogeneous vectors that is subsequently extended by linearity to non-homogeneous vectors. In the rest of the paper we will stick to this convention.

Equipped with $\mathbb{S}$ and the usual Γ-graded tensor product of vector spaces, Vec_Γ is a symmetric monoidal category. This fact was also observed in [2, Proposition 1.5].

Remark 2.2. Note that $\beta : \Gamma \times \Gamma \to \{\pm 1\}$ is a 2-cocycle. In fact it represents the obstruction to the lifting of the map $\Gamma \to \{\pm 1\}$, $(a_1, \dots, a_n) \mapsto (-1)^{\sum_{i=1}^n a_i}$, with respect to the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \{\pm 1, \pm i\} \xrightarrow{t \mapsto t^2} \{\pm 1\} \longrightarrow 1 \ ,$$

where $i := \sqrt{-1}$. More precisely, the lifting obstruction cocycle is naturally represented by

$$\delta : \Gamma \times \Gamma \to \{\pm 1\}, \ (a_1, \dots, a_n) \mapsto (-1)^{\sum_{1 \le i,j \le n} a_i b_j},$$

but $\beta - \delta = d\eta$ where $\eta(a_1, \dots, a_n) := (-1)^{\sum_{1 \le i \ne j \le n} a_i a_j}$. We thank Professor Karl-Hermann Neeb for letting us know about this property of β.

Let $\prec$ denote the lexicographic order on elements of Γ. That is, for $a := \sum_{j=1}^n a_j \mathsf{e}_j$ and $b := \sum_{j=1}^n b_j \mathsf{e}_j$, we set $a \prec b$ if and only if there exists some $1 \le j \le n$ such that $a_j = \overline{\mathbf{0}}$, $b_j = \overline{\mathbf{1}}$, and $a_k = b_k$ for all $k < j$. Thus we can express Γ as $\Gamma = \{\gamma_0 \prec \gamma_1 \prec \dots \prec \gamma_{2^n - 1}\}$, where $\gamma_0 = 0$.

Let $p \in \mathbb{N} \cup \{0\}$ and let $\mathbf{q} := (q_1, \dots, q_{2^n-1})$ be a $(2^n - 1)$-tuple such that $q_j \in \mathbb{N} \cup \{0\}$ for all j. We set $|\mathbf{q}| := \sum_{j=1}^{2^n-1} q_j$. By a Γ-*superdomain* of dimension $p|\mathbf{q}$ we mean a locally ringed space $(U, \mathcal{O}_U)$ such that $U \subset \mathbb{R}^p$ is an open set, and the structure sheaf $\mathcal{O}_U$ is given by

$$\mathcal{O}_U(U') := C^\infty(U'; \mathbb{C})[[\xi_1, \dots, \xi_{|\mathbf{q}|}]] \quad \text{for every open set } U' \subseteq U,$$

where the right hand side denotes the Γ-graded algebra of formal power series in variables $\xi_1, \dots, \xi_{|\mathbf{q}|}$ with coefficients in $C^\infty(U'; \mathbb{C})$, such that $|\xi_r| = \gamma_k$ for $\sum_{j=1}^{k-1} q_j < r \le \sum_{j=1}^k q_j$, subject to the relations

$$\xi_r \xi_s = \beta(|\xi_r|, |\xi_s|) \xi_s \xi_r \quad \text{for every } 1 \le r, s \le |\mathbf{q}|.$$

By a *smooth* Γ-*supermanifold* M of dimension $p|\mathbf{q}$ we mean a locally ringed space $(M, \mathcal{O}_M)$ that is locally isomorphic to a $p|\mathbf{q}$-dimensional Γ-superdomain. From this viewpoint, a Γ-*Lie supergroup* is a group object in the category of smooth Γ-supermanifolds.

As one expects, to a Γ-Lie supergroup one can canonically associate a Γ-*Lie superalgebra*, which is an object of Vec_Γ of the form $\mathfrak{g} = \bigoplus_{a \in \Gamma} \mathfrak{g}_a$, equipped with a Γ-superbracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the following properties:

(i) $[\cdot,\cdot]$ is bilinear, and $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{ab}$ for $a, b \in \Gamma$.
(ii) $[x, y] = -\beta(a, b)[y, x]$ for $x \in \mathfrak{g}_a, y \in \mathfrak{g}_b$, where $a, b \in \Gamma$.
(iii) $[x, [y, z]] = [[x, y], z] + \beta(a, b)\beta(a, c)[y, [z, x]]$ for $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_b$, $z \in \mathfrak{g}_c$, where $a, b, c \in \Gamma$.

Remark 2.3. We remark that a Γ-Lie superalgebra is more commonly known as a *Lie color algebra*. Nevertheless, in order to keep our nomenclature compatible with [6], we use the term Γ-Lie superalgebra instead.

From classical supergeometry (that is, when $n = 1$) one knows that the category of Lie supergroups can be replaced by another category with a more concrete structure, known as the category of *Harish-Chandra pairs* (see [12], [13]). A similar statement holds in the case of Γ-Lie supergroups.

Definition 2.4. A Γ-*Harish-Chandra pair* is a pair $(G_0, \mathfrak{g})$ where G_0 is a Lie group, and $\mathfrak{g} = \bigoplus_{a\in\Gamma} \mathfrak{g}_a$ is a Γ-Lie superalgebra equipped with an action $\mathrm{Ad} : G_0 \times \mathfrak{g} \to \mathfrak{g}$ of G_0 by linear operators that preserve the Γ-grading, which extends the adjoint action of G_0 on $\mathfrak{g}_0 \cong \mathrm{Lie}(G_0)$.

Example 2.5. Let $V := \bigoplus_{a\in\Gamma} V_a$ be a Γ-graded vector space. The Γ-Lie superalgebra $\mathfrak{gl}(V)$ is the vector space of linear transformations on V, with superbracket $[S, T] := ST - \beta(|S|, |T|)TS$. One can also consider the Γ-Harish-Chandra pair $(G_0, \mathfrak{g})$, where $G_0 \cong \prod_{a\in\Gamma} \mathrm{GL}(V_a)$.

Indeed the Γ-Harish-Chandra pairs form a category in a natural way. The morphisms of this category are pairs of maps $(\phi, \varphi) : (G, \mathfrak{g}) \to (H, \mathfrak{h})$, where $\phi : G \to H$ is a Lie group homomorphism and $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a morphism in the category of Γ-Lie superalgebras such that $\varphi|_{\mathfrak{g}_0} = \mathsf{d}\phi$. The following statement plays a key role in the study of Γ-Lie supergroups.

Proposition 2.6. *The category of* Γ-*Lie supergroups is isomorphic to the category of* Γ-*Harish-Chandra pairs.*

Proof. The proof is a straightforward but lengthy modification of the argument for the analogous result in the case of ordinary supergeometry (see [12], [13], or [4]). Therefore we only sketch an outline of the proof. The functor from Γ-Lie supergroups to Γ-Harish-Chandra pairs is easy to describe: a Γ-Lie supergroup $(G_0, \mathcal{O}_{G_0})$ is associated to the Harish-Chandra pair $(G_0, \mathfrak{g})$, where $\mathfrak{g}$ is the Γ-Lie superalgebra of $(G_0, \mathcal{O}_{G_0})$. As in the classical super case, the functor associates a homomorphism of Γ-Lie supergroups $(G_0, \mathcal{O}_{G_0}) \to (H_0, \mathcal{O}_{H_0})$ to the pair of underlying maps $G_0 \to H_0$ and the tangent map at identity $\mathfrak{g} \to \mathfrak{h}$. Conversely, from a Γ-Harish-Chandra pair $(G_0, \mathfrak{g})$ we construct a Lie supergroup $(G_0, \mathcal{O}_{G_0})$ as follows. For every open set $U \subseteq G_0$, we set $\overline{\mathcal{O}}_{G_0}(U) := \mathrm{Hom}_{\mathfrak{g}_0}\big(\mathfrak{U}(\mathfrak{g}_\mathbb{C}), C^\infty(U; \mathbb{C})\big)$, where $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ denotes the universal enveloping algebra of $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. The Γ-superalgebra structure on $\overline{\mathcal{O}}_{G_0}(U)$ is defined using the algebra structure of $C^\infty(U; \mathbb{C})$ and the Γ-coalgebra structure of $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$, exactly as in the classical super case. Using the PBW Theorem for Γ-Lie superalgebras (see [18]) one can see that $(G_0, \overline{\mathcal{O}}_{G_0})$ is indeed a Γ-supermanifold (see [4, Proposition 7.4.9]). The definition of the Γ-Lie supergroup structure of $(G_0, \overline{\mathcal{O}}_{G_0})$ is similar to the classical case as well (see [4, Proposition 7.4.10]). Finally, to show that the two functors are inverse to each other, we need a sheaf isomorphism $\mathcal{O}_{G_0} \cong \overline{\mathcal{O}}_{G_0}$. This sheaf isomorphism is given by the maps

$$\mathcal{O}_{G_0}(U) \to \overline{\mathcal{O}}_{G_0}(U), \quad s \mapsto \big[D \mapsto \beta(|D|,|s|)\widetilde{\mathrm{L}_D s}\big] \text{ for } D \in \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}),$$

where $|D|, |s| \in \Gamma$ are the naturally defined degrees, L_D denotes the left invariant differential operator on $(G_0, \mathcal{O}_{G_0})$ corresponding to D, and $\widetilde{\mathrm{L}_D s}$ means evaluation of the section at points of U. The proof of bijective correspondence of morphisms in the two categories is similar to [4, Proposition 7.4.12]. ■

3. Γ-Hilbert superspaces and unitary representations. In order to define a unitary representation of a Γ-Harish-Chandra pair, one needs to obtain a well-behaved definition of Hilbert spaces in the category Vec_Γ. This is our first goal in this section.

For any $a = \sum_{j=1}^n a_j \mathsf{e}_j \in \Gamma$, set $\mathbf{u}(a) := |\{1 \le j \le n : a_j = \overline{1}\}|$ (for example, $\mathbf{u}(\mathsf{e}_1 + \mathsf{e}_3 + \mathsf{e}_4) = 3$) and define

$$\alpha(a) := e^{(\pi i/2)\mathbf{u}(a)}. \tag{2}$$

DEFINITION 3.1. Let $\mathscr{H} \in \mathsf{Obj}(\mathsf{Vec}_\Gamma)$. We call $\mathscr{H}$ a Γ-*inner product space* if and only if it is equipped with a non-degenerate sesquilinear form $\langle\cdot,\cdot\rangle$ that satisfies the following properties:

(i) $\langle \mathscr{H}_a, \mathscr{H}_b\rangle = 0$, for $a, b \in \Gamma$ such that $a \ne b$.
(ii) $\langle w, v\rangle = \beta(a,a)\overline{\langle v, w\rangle}$, for $v, w \in \mathscr{H}_a$ where $a \in \Gamma$.
(iii) $\alpha(a)\langle v, v\rangle \ge 0$, for every $v \in \mathscr{H}_a$ where $a \in \Gamma$.

REMARK 3.2. The choice of α in (2) is made as follows. The most natural property that one expects from Hilbert spaces is that the tensor product of (pre-)Hilbert spaces is a (pre-)Hilbert space. Thus, we are seeking $\alpha : \Gamma \to \mathbb{C}^\times$ such that the tensor product of two Γ-inner product spaces is also a Γ-inner product space. Clearly after scaling the values of α by positive real numbers we can assume that $\alpha(\Gamma) \subseteq \{\pm 1, \pm i\}$. For two Γ-inner product spaces $\mathscr{H}$ and $\mathscr{K}$, one has

$$(\mathscr{H} \otimes \mathscr{K})_a := \bigoplus_{bc=a} \mathscr{H}_b \otimes \mathscr{H}_c,$$

equipped with the canonically induced sesquilinear form

$$\langle v \otimes w, v' \otimes w'\rangle_{\mathscr{H}\otimes\mathscr{K}} := \beta(|w|,|v'|)\langle v, v'\rangle_{\mathscr{H}}\langle w, w'\rangle_{\mathscr{K}}.$$

Closedness under tensor product implies that

$$\alpha(bc) = \beta(b,c)\alpha(b)\alpha(c) \quad \text{for all } b, c \in \Gamma. \tag{3}$$

In the language of group cohomology, this means that the 2-cocycle β satisfies $\beta = d\alpha$. Consequently, up to twisting by a group homomorphism $\Gamma \to \{\pm 1\}$, there exists a unique α which satisfies the latter relation.

REMARK 3.3. Associated to any Γ-inner product space $\mathscr{H}$, there is an inner product (in the ordinary sense) defined by

$$(v, w) := \begin{cases} 0 & \text{if } |v| \ne |w|, \\ \alpha(a)\langle v, w\rangle & \text{if } |v| = |w| = a \text{ where } a \in \Gamma. \end{cases}$$

The vector space $\mathscr{H}$, equipped with $(\cdot,\cdot)$, is indeed a pre-Hilbert space in the usual sense. Thus we can consider the completion of $\mathscr{H}$ with respect to the norm $\|v\| := (v,v)^{1/2}$. By reversing the process of obtaining $(\cdot,\cdot)$ from $\langle\cdot,\cdot\rangle$, we obtain a Γ-inner product on the completion of $\mathscr{H}$.

DEFINITION 3.4. Let $\mathscr{H}$ be a Γ-inner product space, and let $T \in \mathrm{End}_{\mathbb{C}}(\mathscr{H})$. We define the *adjoint* $T^\dagger$ of T as follows. If $T \in \mathrm{End}_{\mathbb{C}}(\mathscr{H})_a$ for some $a \in \Gamma$, we define $T^\dagger$ by

$$\langle v, Tw\rangle = \beta(a,b)\langle T^\dagger v, w\rangle,$$

where $v \in \mathscr{H}_b$. We then extend the assignment $T \mapsto T^\dagger$ to a conjugate-linear map on $\mathrm{End}_{\mathbb{C}}(\mathscr{H})$.

Now let $\mathscr{H}$ be a Γ-inner product space, and let $(\cdot,\cdot)$ be the ordinary inner product associated to $\mathscr{H}$ as in Remark 3.3. Then for a linear map $T : \mathscr{H} \to \mathscr{H}$ we can define an adjoint with respect to $(\cdot,\cdot)$, by the relation

$$(Tv, w) = (v, T^* w) \qquad \text{for every } v, w \in \mathscr{H}.$$

A straightforward calculation yields

$$T^* = \overline{\alpha(|T|)}T^\dagger.$$

It follows that $T^{\dagger\dagger} = T$. Furthermore, $(ST)^\dagger = \beta(|S|,|T|)T^\dagger S^\dagger$.

We are now ready to define unitary representations of Γ-Harish-Chandra pairs. The definition of a unitary representation of a Γ-Harish-Chandra pair is a natural extension of the one for the $\mathbb{Z}_2$-graded case. First recall that for a unitary representation $(\pi, \mathscr{H})$ of a Lie group G on a Hilbert space $\mathscr{H}$, we denote the space of C^∞ vectors by $\mathscr{H}^\infty$. Thus the vector space $\mathscr{H}^\infty$ consists of all vectors $v \in \mathscr{H}$ for which the map $G \to \mathscr{H}$, $g \mapsto \pi(g)v$ is smooth. Furthermore, given $x \in \mathrm{Lie}(G)$ and $v \in \mathscr{H}$, we set

$$\mathsf{d}\pi(x) := \lim_{t\to 0}\frac{1}{t}\big(\pi(\exp(tx))v - v\big),$$

whenever the limit exists. We denote the domain of the unbounded operator $\mathsf{d}\pi(x)$ by $\mathcal{D}(\mathsf{d}\pi(x))$. The unbounded operator $-i\mathsf{d}\pi(x)$ is the self-adjoint generator of the one-parameter unitary representation $t \mapsto \pi(\exp(tx))$. For a comprehensive exposition of the theory of unitary representations and relevant facts from the theory of unbounded operators, see [19].

DEFINITION 3.5. A *smooth unitary representation* of a Γ-Harish-Chandra pair $(G_0, \mathfrak{g})$ is a triple $(\pi, \rho^\pi, \mathscr{H})$ that satisfies the following properties:

(R1) $(\pi, \mathscr{H})$ is a smooth unitary representation of the Lie group G_0 on the Γ-graded Hilbert space $\mathscr{H}$ by operators $\pi(g)$, $g \in G_0$, which preserve the Γ-grading.
(R2) $\rho^\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty)$ is a representation of the Γ-Lie superalgebra $\mathfrak{g}$.
(R3) $\rho^\pi(x) = \mathsf{d}\pi(x)\big|_{\mathscr{H}^\infty}$ for $x \in \mathfrak{g}_0$.
(R4) $\rho^\pi(x)^\dagger = -\rho^\pi(x)$ for $x \in \mathfrak{g}$.
(R5) $\pi(g)\rho^\pi(x)\pi(g)^{-1} = \rho^\pi(\mathrm{Ad}(g)x)$ for $g \in G_0$ and $x \in \mathfrak{g}$.

Unitary representations of a Γ-Harish-Chandra pair form a category $\mathsf{Rep} = \mathsf{Rep}(G_0, \mathfrak{g})$. A morphism in this category from $(\pi, \rho^\pi, \mathscr{H})$ to $(\sigma, \rho^\sigma, \mathscr{K})$ is a bounded linear map $T : \mathscr{H} \to \mathscr{K}$ which respects the Γ-grading and satisfies $T\pi(g) = \sigma(g)T$ for $g \in G_0$ (from which it follows that $T\mathscr{H}^\infty \subseteq \mathscr{K}^\infty$) and $T\rho^\pi(x) = \rho^\sigma(x)T$ for $x \in \mathfrak{g}$.

REMARK 3.6. At first glance, it seems that the definition of a unitary representation of a Γ-Harish-Chandra pair depends on the choice of α. Nevertheless, it is not difficult to verify that for two coboundaries α, α' which satisfy (3), the corresponding categories Rep_α and $\mathsf{Rep}_{\alpha'}$ are isomorphic. Indeed if $\chi : \Gamma \to \{\pm 1\}$ is a group homomorphism such that $\alpha' = \chi\alpha$, then we can define a functor $\mathscr{F} : \mathsf{Rep}_\alpha \to \mathsf{Rep}_{\alpha'}$ which maps $(\pi, \rho^\pi, \mathscr{H})$ to $(\pi', \rho^{\pi'}, \mathscr{H}')$ where $\mathscr{H} := \mathscr{H}'$ (but the Γ-inner product of $\mathscr{H}'$ is defined by $\langle v, v\rangle_{\mathscr{H}'} = \chi(a)\langle v, v\rangle_{\mathscr{H}}$ for $v \in \mathscr{H}_a$ where $a \in \Gamma$), $\pi' := \pi$, and $\rho^{\pi'}(x) := \chi(a)\rho^\pi(x)$ for $x \in \mathfrak{g}_a$ and $a \in \Gamma$. The functor $\mathscr{F}$ is defined to be identity on morphisms. That is, for a morphism $T : (\pi, \rho^\pi, \mathscr{H}) \to (\sigma, \rho^\sigma, \mathscr{K})$ we set $\mathscr{F}(T) := T$. It is straightforward to see that with $(\pi', \rho^{\pi'}, \mathscr{H}')$ and $(\sigma', \rho^{\sigma'}, \mathscr{K}')$ defined as above, the map $T : (\pi', \rho^{\pi'}, \mathscr{H}') \to (\sigma', \rho^{\sigma'}, \mathscr{K}')$ is still an intertwining map. The main point is that $\rho^{\pi'}$ and $\rho^{\sigma'}$ are obtained from ρ^π and ρ^σ via scaling by the same scalar. The inverse of $\mathscr{F}$ is defined similarly.

4. The stability theorem.

We now proceed towards the statment and proof of the Stability Theorem. In what follows, we will need the following technical definition (see [14]).

DEFINITION 4.1. Let $(G_0, \mathfrak{g})$ be a Γ-Harish-Chandra pair. By a *pre-representation* of $(G_0, \mathfrak{g})$, we mean a 4-tuple $(\pi, \mathscr{H}, \mathscr{B}, \rho^{\mathscr{B}})$ that satisfies the following conditions:

(PR1) $(\pi, \mathscr{H})$ is a smooth unitary representation of the Lie group G_0 on the Γ-graded Hilbert space $\mathscr{H}$ by operators $\pi(g)$, $g \in G_0$, which preserve the Γ-grading.
(PR2) $\mathscr{B}$ is a dense, G_0-invariant, and Γ-graded subspace of $\mathscr{H}$ that is contained in $\bigcap_{x\in\mathfrak{g}_0} \mathcal{D}(\mathsf{d}\pi(x))$.
(PR3) $\rho^{\mathscr{B}} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{B})$ is a representation of the Γ-Lie superalgebra $\mathfrak{g}$.
(PR4) $\rho^{\mathscr{B}}(x) = \mathsf{d}\pi(x)\big|_{\mathscr{B}}$ and $\rho^{\mathscr{B}}(x)$ is essentially skew adjoint for $x \in \mathfrak{g}_0$.
(PR5) $\rho^{\mathscr{B}}(x)^\dagger = -\rho^{\mathscr{B}}(x)$ for $x \in \mathfrak{g}$.
(PR6) $\pi(g)\rho^{\mathscr{B}}(x)\pi(g)^{-1} = \rho^{\mathscr{B}}(\mathrm{Ad}(g)x)$ for $g \in G_0$ and $x \in \mathfrak{g}$.

REMARK 4.2. It is shown in [14, Remark 6.5] that (PR2) and (PR3) imply $\mathscr{B} \subseteq \mathscr{H}^\infty$.

Set

$$\Gamma_{\overline{\mathbf{0}}} := \{a \in \Gamma : \mathsf{b}(a, a) = \overline{\mathbf{0}}\} \text{ and } \Gamma_{\overline{\mathbf{1}}} := \{a \in \Gamma : \mathsf{b}(a, a) = \overline{\mathbf{1}}\}.$$

THEOREM 4.3 (Stability Theorem). *Let $(\pi, \mathscr{H}, \mathscr{B}, \rho^{\mathscr{B}})$ be a pre-representation of a Γ-Harish-Chandra pair $(G_0, \mathfrak{g})$. Assume that for every $a \in \Gamma_{\overline{\mathbf{0}}} \setminus \{0\}$*

$$\mathfrak{g}_a = \sum_{b,c\in\Gamma_{\overline{\mathbf{1}}}, bc=a} [\mathfrak{g}_b, \mathfrak{g}_c].$$

Then there exists a unique extension of $\rho^{\mathscr{B}}$ to a linear map $\rho^\pi : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty)$ such that $(\pi, \rho^\pi, \mathscr{H})$ is a smooth unitary representation of $(G_0, \mathfrak{g})$.

Proof. For every $a \in \Gamma_{\overline{\mathbf{1}}}$ the direct sum $\mathfrak{g}_0 \oplus \mathfrak{g}_a$ is a Lie superalgebra (in the $\mathbb{Z}_2$-graded sense). We define a $\mathbb{Z}_2$-grading of $\mathscr{H}$ by

$$\mathscr{H}_{\overline{\mathbf{0}}} := \bigoplus_{\mathsf{b}(a,b)=\overline{\mathbf{0}}} \mathscr{H}_b \text{ and } \mathscr{H}_{\overline{\mathbf{1}}} := \bigoplus_{\mathsf{b}(a,b)=\overline{\mathbf{1}}} \mathscr{H}_b,$$

and using the Stability Theorem in the $\mathbb{Z}_2$-graded case (see [14, Theorem 6.14]) we deduce that there exists a unique Γ-Lie superalgebra homomorphism $\rho^{\pi,a} : \mathfrak{g}_0 \oplus \mathfrak{g}_a \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty)$, such that $(\pi, \rho^{\pi,a}, \mathscr{H})$ is a unitary representation of the ($\mathbb{Z}_2$-graded) Harish-Chandra pair $(G_0, \mathfrak{g}_0 \oplus \mathfrak{g}_a)$. Since the maps $\rho^{\pi,a}$ agree on $\mathfrak{g}_0$, they give rise to a linear map

$$\mathfrak{g}_0 \oplus \Big(\bigoplus_{a \in \Gamma_{\overline{1}}} \mathfrak{g}_a \Big) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H}^\infty).$$

It remains to obtain a suitable extension of the latter map to $\mathfrak{g}_a$ for every $a \in \Gamma_{\overline{0}} \setminus \{0\}$.

Fix $x \in \mathfrak{g}_a$, $a \in \Gamma_{\overline{0}} \setminus \{0\}$. Then we can write $x = \sum_j [y_j, z_j]$ such that $y_j \in \mathfrak{g}_{b_j}$, $z_j \in \mathfrak{g}_{c_j}$, where $b_j, c_j \in \Gamma_{\overline{1}}$, and $b_j c_j = a$. For every $v \in \mathscr{H}^\infty$, we define

$$\rho^\pi(x)v := \sum_j \big(\rho^{\pi,b_j}(y_j)\rho^{\pi,c_j}(z_j)v - \beta(b_j, c_j)\rho^{\pi,c_j}(z_j)\rho^{\pi,b_j}(y_j)v \big). \tag{4}$$

Let us first verify that $\rho^\pi(x)v$ is well-defined. To this end, it suffices to show that if $\sum_j [y_j, z_j] = 0$, then the right hand side of (4) vanishes. To verify the latter assertion, observe that for every $w \in \mathscr{B}_s$, $s \in \Gamma$, we have

$$\begin{aligned} \langle w, \rho^\pi(x)v \rangle &= \sum_j \beta(b_j, s)\beta(c_j, sb_j)\big\langle \rho^{\pi,c_j}(z_j)^\dagger \rho^{\pi,b_j}(y_j)^\dagger w, v \big\rangle \\ &\quad - \sum_j \beta(c_j, s)\beta(b_j, sc_j)\beta(b_j, c_j)\big\langle \rho^{\pi,b_j}(y_j)^\dagger \rho^{\pi,c_j}(z_j)^\dagger w, v \big\rangle \\ &= \beta(a, s)\Big\langle \sum_j \beta(b_j, c_j)\rho^{\pi,c_j}(z_j)^\dagger \rho^{\pi,b_j}(y_j)^\dagger w - \sum_j \rho^{\pi,b_j}(y_j)^\dagger \rho^{\pi,c_j}(b_j)^\dagger w, v \Big\rangle \\ &= -\beta(a, s)\Big\langle \sum_j \rho^{\mathscr{B}}([y_j, z_j])w, v \Big\rangle = 0. \end{aligned}$$

The assertion now follows from density of $\mathscr{B}$ in $\mathscr{H}$.

It remains to show that given any $x_a \in \mathfrak{g}_a$ and $y_b \in \mathfrak{g}_b$, where $a, b \in \Gamma$, we have

$$\rho^\pi([x_a, y_b])v = \big(\rho^\pi(x_a)\rho^\pi(y_b) - \beta(a, b)\rho^\pi(y_b)\rho^\pi(x_a)\big)v \quad \text{for } v \in \mathscr{H}^\infty. \tag{5}$$

To verify the last equality note that for every $w \in \mathscr{B}_s$ where $s \in \Gamma$,

$$\begin{aligned} &\big\langle w, \rho^\pi(x_a)\rho^\pi(y_b)v - \beta(a, b)\rho^\pi(y_b)\rho^\pi(x_a)v \big\rangle \\ &\quad = \beta(a, s)\beta(as, b)\big\langle \rho^\pi(y_b)^\dagger \rho^\pi(x_a)^\dagger w, v \big\rangle - \beta(b, s)\beta(s, a)\big\langle \rho^\pi(x_a)^\dagger \rho^\pi(y_b)^\dagger w, v \big\rangle \\ &\quad = -\beta(a, s)\beta(b, s)\big\langle \rho^{\mathscr{B}}(x_a)\rho^{\mathscr{B}}(y_b)w - \beta(a, b)\rho^{\mathscr{B}}(y_b)\rho^{\mathscr{B}}(x_a)w, v \big\rangle \\ &\quad = \beta(a, s)\beta(b, s)\big\langle \rho^{\mathscr{B}}\big([x_a, y_b]\big)^\dagger w, v \big\rangle = \big\langle w, \rho^\pi\big([x_a, y_b]\big)v \big\rangle. \end{aligned}$$

Again density of $\mathscr{B}$ in $\mathscr{H}$ implies that both sides of (5) are equal. Finally, uniqueness of $\rho^\pi(x)$ can be proved by the same technique. ∎

Example 4.4. For $n > 1$, one cannot expect the Stability Theorem to hold without any condition on $\mathfrak{g}$. For example, let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$, let G_0 be the trivial group, and let $\mathfrak{g} = \mathfrak{g}_{\overline{00}} \oplus \mathfrak{g}_{\overline{01}} \oplus \mathfrak{g}_{\overline{10}} \oplus \mathfrak{g}_{\overline{11}}$ where

$$\mathfrak{g}_{\overline{11}} := \mathbb{R} \text{ and } \mathfrak{g}_{\overline{00}} := \mathfrak{g}_{\overline{01}} := \mathfrak{g}_{\overline{10}} := \{0\}.$$

Then a pre-representation of $(G_0, \mathfrak{g})$ is the same (up to scaling) as a symmetric operator defined on a dense subspace of a Hilbert space, whereas a unitary representation of $(G_0, \mathfrak{g})$ is the same (up to scaling) as a *bounded* self-adjoint operator. Therefore a pre-representation does not necessarily extend to a unitary representation.

5. The GNS representation. Our goal in this section is to extend the GNS construction of [15] to the setting of Γ-Harish-Chandra pairs. We begin by outlining some generalities about this construction. Let $(G_0, \mathfrak{g})$ be a Γ-Harish-Chandra pair, and let $\mathcal{G} := (G_0, \mathcal{O}_{G_0})$ denote the Γ-Lie supergroup corresponding to $(G_0, \mathfrak{g})$. Our main goal, to be established in Theorem 5.4, is to construct a correspondence between unitary representations of $(G_0, \mathfrak{g})$ with a cyclic vector, and smooth positive definite functions on $\mathcal{G}$. The suitable definition of a unitary representation with a cyclic vector is given in Definition 5.3. The main subtlety is to define positive definite functions on $\mathcal{G}$. To this end, we use the method developed in [15] for the classical super case, which we describe below. Recall that the Γ-superalgebra $C^\infty(\mathcal{G})$ of smooth functions on $\mathcal{G}$ is isomorphic to

$$\mathrm{Hom}_{\mathfrak{g}_0}\big(\mathfrak{U}(\mathfrak{g}_\mathbb{C}), C^\infty(G_0;\mathbb{C})\big),$$

where $\mathfrak{g}_0$ acts on $C^\infty(G_0;\mathbb{C})$ by left invariant differential operators. We realize elements of the latter algebra as functions on a semigroup $\mathcal{S}$ which is equipped with an involution. Then we use the abstract definition of a positive definite function on a semigroup with an involution (see Definition 5.1).

Given a unitary representation of $(G_0, \mathfrak{g})$ with a cyclic vector, the matrix coefficient of that cyclic vector will be a positive definite element of $C^\infty(\mathcal{G})$. Conversely, from a positive definite $f \in C^\infty(\mathcal{G})$, we construct a unitary representation of $(G_0, \mathfrak{g})$ by considering the reproducing kernel Hilbert space associated to f, now realized as a function on $\mathcal{S}$. The Lie group G_0 has a canonical action on the reproducing kernel Hilbert space. However, the Γ-Lie superalgebra $\mathfrak{g}$ acts on a dense subspace of the latter Hilbert space, which is in general strictly smaller than the space of smooth vectors for the G_0-action. The main part of the proof of Theorem 5.4 is to show that the action of $\mathfrak{g}$ is well defined on the entire space of smooth vectors. For this we need Theorem 4.3, which is the reason for presence of the condition (8) in Theorem 5.4.

Set $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. Let $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ be the universal enveloping algebra of $\mathfrak{g}_\mathbb{C}$, that is, the quotient $T(\mathfrak{g}_\mathbb{C})/I$, where $T(\mathfrak{g}_\mathbb{C})$ denotes the tensor algebra of $\mathfrak{g}_\mathbb{C}$ in the category Vec_Γ, and I denotes the two-sided ideal of $T(\mathfrak{g}_\mathbb{C})$ generated by elements of the form

$$x \otimes y - \beta(|x|, |y|) y \otimes x - [x, y] \quad \text{for homogeneous } x, y \in \mathfrak{g}_\mathbb{C}.$$

See [16, Section 2.1] for further details.

Let $x \mapsto x^*$ be the (unique) conjugate-linear map on $\mathfrak{g}_\mathbb{C}$, defined by the relation

$$x^* := -\overline{\alpha(a)}x \quad \text{for every } x \in \mathfrak{g}_a,\ a \in \Gamma. \tag{6}$$

We extend the map $x \mapsto x^*$ to a conjugate-linear anti-automorphism of the algebra $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$. Thus $(D_1 D_2)^* = D_2^* D_1^*$ for every $D_1, D_2 \in \mathfrak{U}(\mathfrak{g}_\mathbb{C})$. Such an extension is possible because of $*$-invariance of I. We now define a monoid

$$\mathcal{S} := G_0 \ltimes \mathfrak{U}(\mathfrak{g}_\mathbb{C}),$$

with a multiplication given by

$$(g_1, D_1)(g_2, D_2) := \big(g_1 g_2, (\mathrm{Ad}(g_2^{-1})(D_1)) D_2\big).$$

The neutral element of $\mathcal{S}$ is $1_\mathcal{S} := (1_{G_0}, 1_{\mathfrak{U}(\mathfrak{g}_\mathbb{C})})$. The map

$$\mathcal{S} \to \mathcal{S}, \quad (g, D) \mapsto (g, D)^* := \big(g^{-1}, \mathrm{Ad}(g)(D^*)\big)$$

is an involution of $\mathcal{S}$.

The proof of the latter assertion is similar to the one in the $\mathbb{Z}_2$-graded case (see [13], [12], [15, Theorem 5.5.2]).

Next, for every $f \in C^\infty(\mathcal{G})$, we define a map

$$\check{f} : \mathcal{S} \to \mathbb{C}, \quad (g, D) \mapsto f(D)(g).$$

Also, for any $a \in \Gamma$ we set $\mathcal{S}_a := \{(g, D) \in \mathcal{S} : |D| = a\}$.

Definition 5.1. An $f \in C^\infty(\mathcal{G})$ is called *positive definite* if it satisfies the following two conditions:

(i) $\check{f}(g, D) = 0$ unless $(g, D) \in \mathcal{S}_0$.
(ii) $\sum_{1 \le i,j \le n} \overline{c_i} c_j \check{f}(s_i^* s_j) \ge 0$ for all $n \ge 1$, $c_1, \dots, c_n \in \mathbb{C}$, $s_1, \dots, s_n \in \mathcal{S}$.

Given a unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of G, for any two vectors $v, w \in \mathscr{H}$ we define the matrix coefficient $\varphi_{v,w}$ to be the map

$$\varphi_{v,w} : \mathcal{S} \to \mathbb{C}, \quad \varphi_{v,w}(g, D) := (\pi(g)\rho^\pi(D)v, w).$$

Proposition 5.2. *Let $(\pi, \rho^\pi, \mathscr{H})$ be a smooth unitary representation of $(G_0, \mathfrak{g})$, and let $v, w \in \mathscr{H}^\infty$ be homogeneous vectors such that $|v| = |w|$. Then there exists an $f \in C^\infty(\mathcal{G})$ such that $\check{f} = \varphi_{v,w}$. Furthermore, $\check{f}(s) = 0$ unless $s \in \mathcal{S}_0$. If $v = w$ then $\check{f}$ is positive definite.*

Proof. Similar to [15, Proposition 6.5.2]. ∎

We are now ready to describe the GNS construction for unitary representations of Γ-Harish-Chandra pairs. It is an extension of the one given in [15, Section 6] in the $\mathbb{Z}_2$-graded case, and therefore we will skip the proof details.

Let $(\pi, \rho^\pi, \mathscr{H})$ be a smooth unitary representation of $(G_0, \mathfrak{g})$. One can construct a $*$-*representation* $\widetilde{\rho^\pi}$ of the monoid $\mathcal{S}$ by setting

$$\widetilde{\rho^\pi} : \mathcal{S} \to \mathrm{End}_\mathbb{C}(\mathscr{H}^\infty), \quad (g, D) \mapsto \pi(g)\rho^\pi(D).$$

Being a $*$-representation means that $\widetilde{\rho^\pi}(s^*) = \widetilde{\rho^\pi}(s)^*$ for every $s \in \mathcal{S}$, and in particular

$$(\widetilde{\rho^\pi}(s)v, w) = (v, \widetilde{\rho^\pi}(s^*)w). \tag{7}$$

By Proposition 5.2, for every matrix coefficient $\varphi_{v,v}$, where $v \in \mathscr{H}_0$, there exists a positive definite $f \in C^\infty(\mathcal{G})$ such that $\varphi_{v,v} = \check{f}$.

Conversely, given a positive definite function $f \in C^\infty(\mathcal{G})$, one can associate a $*$-representation of $\mathcal{S}$ to f as follows. Set $\psi := \check{f}$, and for every $s \in \mathcal{S}$ let $\psi_s : \mathcal{S} \to \mathbb{C}$ be the map given by $\psi_s(t) := \psi(ts)$. We also set

$$\mathfrak{D}_\psi := \mathrm{Span}_\mathbb{C} \{\psi_s : s \in \mathcal{S}\}.$$

Note that $\mathfrak{D}_\psi$ is a Γ-graded vector space of complex-valued functions on $\mathcal{S}$, where the homogeneous parts of the Γ-grading are defined by

$$\mathfrak{D}_{\psi,a} := \{h \in \mathfrak{D}_\psi : h(s) = 0 \text{ unless } s \in \mathcal{S}_a\}.$$

The space $\mathfrak{D}_\psi$ can be equipped with a sesquilinear form that is uniquely defined by the relation $(\psi_t, \psi_s) := \psi(s^* t)$. The completion $\mathscr{H}_\psi$ of the resulting pre-Hilbert space is the *reproducing kernel Hilbert space* that corresponds to the kernel

$$K : \mathcal{S} \times \mathcal{S} \to \mathbb{C}, \quad (t, s) \mapsto \psi(ts^*).$$

In other words, with respect to the inner product $(\cdot,\cdot)$ on $\mathscr{H}_\psi$, we have

$$h(s) = (h, K_s) \quad \text{for } h \in \mathscr{H}_\psi,\ s \in \mathcal{S}.$$

There is a natural $*$-representation $\widetilde{\rho_\psi} : \mathcal{S} \to \mathrm{End}_{\mathbb{C}}(\mathfrak{D}_\psi)$ by right translation, given by

$$(\widetilde{\rho_\psi}(s)h)(t) := h(ts) \quad \text{for } s,t \in S,\ h \in \mathfrak{D}_\psi.$$

If $s \in \mathcal{S}$ satisfies $ss^* = s^*s = 1_{\mathcal{S}}$, then $\widetilde{\rho_\psi}(s) : \mathfrak{D}_\psi \to \mathfrak{D}_\psi$ is an isometry and extends uniquely to a unitary operator on $\mathscr{H}_\psi$. Using the latter fact for elements of the form $(g, 1_{\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})})$ where $g \in G_0$, one obtains a unitary representation of G_0 on $\mathscr{H}_\psi$ (in fact the vectors K_s, $s \in \mathcal{S}$, have smooth G_0-orbits). Setting

$$\mathscr{B} := \mathfrak{D}_\psi,\ \mathscr{H} := \mathscr{H}_\psi,\ \rho^{\mathscr{B}}(x) := \widetilde{\rho_\psi}(1_{G_0}, x) \text{ for } x \in \mathfrak{g}, \text{ and } \pi(g) := \widetilde{\rho_\psi}(g, 0),$$

we obtain a pre-representation of $(G_0, \mathfrak{g})$. Theorem 4.3 implies that this pre-representation corresponds to a unique unitary representation of $(G_0, \mathfrak{g})$.

DEFINITION 5.3. By a *cyclic vector* in a unitary representation $(\pi, \rho^\pi, \mathscr{H})$ we mean a vector $v \in \mathscr{H}$ such that $\widetilde{\rho^\pi}(\mathcal{S})v$ is dense in $\mathscr{H}$.

The above construction results in Theorem 5.4 below.

THEOREM 5.4. *Let $(G_0, \mathfrak{g})$ be a Γ-Harish-Chandra pair such that*

$$\mathfrak{g}_a = \sum_{b,c \in \Gamma_{\bar{1}}, bc = a} [\mathfrak{g}_b, \mathfrak{g}_c]. \tag{8}$$

Also, let $f \in C^\infty(\mathcal{G})$ be positive definite.

(i) *There exists a unitary representation $(\pi, \rho^\pi, \mathscr{H})$ of $(G_0, \mathfrak{g})$ with a cyclic vector $v_0 \in \mathscr{H}_0$ such that $\check{f} = \varphi_{v_0, v_0}$.*

(ii) *Let $(\sigma, \rho^\sigma, \mathscr{K})$ be another unitary representation of $(G_0, \mathfrak{g})$ with a cyclic vector $w_0 \in \mathscr{K}_0$ such that $\check{f} = \varphi_{w_0, w_0}$. Then $(\pi, \rho^\pi, \mathscr{H})$ and $(\sigma, \rho^\sigma, \mathscr{K})$ are unitarily equivalent via an intertwining operator that maps v_0 to w_0.*

Proof. The proof is an extension of the argument of [15, Theorem 6.7.5]. ∎

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