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DIFFERENTIABILITY ALONG ONE-PARAMETER SUBGROUPS COMPARED TO DIFFERENTIABILITY ON LIE GROUPS AS MANIFOLDS

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Abstract. Smooth functions $f : G \to E$ from a topological group G to a locally convex space E were considered by Riss (1953), Boseck, Czichowski and Rudolph (1981), Beltiţă and Nicolae (2015), and others, in varying degrees of generality. A notion of $C^{r,s}$-functions on products $G \times H$ of topological groups was introduced by Nikitin (2016). We recall this concept and an exponential law of the form $C^{r,s}(G \times H, E) \cong C^r(G, C^s(H, E))$ (under suitable hypotheses on G and H). Furthermore, we show that in the case where G is a locally exponential Lie group or a certain direct limit Lie group our calculus of C^r-functions coincides with the differential calculus on G as a locally convex manifold.

1. Introduction. Let $f : G \to E$ be a function on an infinite-dimensional Lie group G with values in a locally convex space E. One possible concept of differentiability of such functions goes back to Milnor [10], where G is considered as a differentiable infinite-dimensional manifold and the differential calculus arises from the calculus of functions between locally convex spaces, the so-called Keller's C^k_c-calculus [9] (we will call such functions C^r_{mfd}*-functions*). On the other hand, considering G as a topological group, we can use the differential calculus along one-parameter subgroups, as by Boseck, Czichowski and Rudolph [2], Beltiţă and Nicolae [1], Nikitin [14].

In the first part of this article, we will recall the latter concept of differentiable functions on topological groups (called C^r_{gp}*-functions*) as well as of functions on products of topological groups with different degrees of differentiability (called $C^{r,s}_{\mathrm{gp}}$*-functions*) from [14]. We will further recall the Exponential Law of the form $C^{r,s}(G \times H, E) \cong$

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$C^r(G, C^s(H, E))$ [14, Theorem (B)], which holds (among other situations) if both the topological groups G and H are locally compact or if both are metrizable.

The other part of the article is devoted to the question under which conditions both the concepts of differentiability of vector-valued functions on infinite-dimensional Lie groups coincide. We obtain the result that $C^r_{\mathrm{mfd}}(G, E) = C^r_{\mathrm{gp}}(G, E)$ as topological vector spaces if the Lie group G is locally exponential or $G = \varinjlim G_n$ is the direct limit Lie group of an ascending sequence $G_1 \subseteq G_2 \subseteq \ldots$ of finite-dimensional Lie groups such that the inclusions $G_n \to G_{n+1}$ are continuous (Theorem 4.9).

The interplay of differentiability along one-parameter subgroups and differentiability on a Lie group G as a manifold plays a role, for example, in the study of spaces of smooth vectors, cf [12], [13].

All topological spaces are assumed Hausdorff, all vector spaces are real vector spaces.

2. Differentiability along one-parameter subgroups on products of topological groups and the Exponential Law. In this section, we recall the concept of differentiability along one-parameter subgroups for functions defined on topological groups and on products of topological groups, and the corresponding Exponential Law.

Definition 2.1. Let G be a topological group, a *one-parameter subgroup* γ is a group homomorphism $\gamma : \mathbb{R} \to G$. We denote by $\mathfrak{L}(G)$ the set of all continuous one-parameter subgroups and endow it with the compact-open topology.

Note that the space $\mathfrak{L}(G)$ does not have a topological vector space structure in general.

Definition 2.2. Let $U \subseteq G$, $V \subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $r, s \in \mathbb{N}_0 \cup \{\infty\}$.

(i) We call a continuous function $f : U \to E$ a C^0_{gp}*-function* and write $d^{(0)}_{\mathrm{gp}} f := f$. We call f a C^r_{gp}*-function* if for each $k \in \mathbb{N}$, $k \leq r$ the differential

$$d^{(k)}_{\mathrm{gp}} f : U \times \mathfrak{L}(G)^k \to E, \quad d^{(k)}_{\mathrm{gp}} f(x, \gamma_1, \ldots, \gamma_k) := (D_{\gamma_k} \cdots D_{\gamma_1} f)(x)$$

is defined and continuous, where $d_{\mathrm{gp}} f(x, \gamma) := D_\gamma f(x) := \lim_{t \to 0} \frac{1}{t}(f(x \cdot \gamma(t)) - f(x))$.

(ii) We call a continuous function $f : U \times V \to E$ a $C^{r,s}_{\mathrm{gp}}$*-function* if for all $k, l \in \mathbb{N}_0$ with $k \leq r$, $l \leq s$ the differentials

$$d^{(k,l)}_{\mathrm{gp}} f : U \times V \times \mathfrak{L}(G)^k \times \mathfrak{L}(H)^l \to E$$
$$(x, y, \gamma_1, \ldots, \gamma_k, \eta_1, \ldots, \eta_l) \mapsto (D_{(\gamma_k, 0)} \cdots D_{(\gamma_1, 0)} D_{(0, \eta_l)} \cdots D_{(0, \eta_1)} f)(x, y).$$

are defined and continuous, where $D_{(\gamma,0)} f(x, y) := \lim_{t \to 0} \frac{1}{t}(f(x \cdot \gamma(t), y) - f(x, y))$ and $D_{(0,\eta)} f(x, y) := \lim_{t \to 0} \frac{1}{t}(f(x, y \cdot \eta(t)) - f(x, y))$.

(iii) We denote the space of all C^r_{gp}-functions $f : U \to E$ by $C^r_{\mathrm{gp}}(U, E)$ and endow it with the initial topology with respect to the family $(d^{(k)}_{\mathrm{gp}})_{k \in \mathbb{N}_0, k \leq r}$ of maps

$$d^{(k)}_{\mathrm{gp}} : C^r_{\mathrm{gp}}(U, E) \to C(U \times \mathfrak{L}(G)^k, E)_{\mathrm{c.o}}, \quad f \mapsto d^{(k)}_{\mathrm{gp}} f$$

(the right-hand side is equipped with the compact-open topology) turning $C^r_{\mathrm{gp}}(U, E)$ into a Hausdorff locally convex space. (This topology is known as the *compact-open* C^r_{gp}*-topology*.) The space of all $C^{r,s}_{\mathrm{gp}}$-functions $f : U \times V \to E$ will be denoted by $C^{r,s}_{\mathrm{gp}}(U \times V, E)$

and we endow it with the Hausdorff locally convex initial topology with respect to the family $(d_{\mathrm{gp}}^{(k,l)})_{k,l\in\mathbb{N}_0,k\le r,l\le s}$ of maps

$$d_{\mathrm{gp}}^{(k,l)} : C_{\mathrm{gp}}^{r,s}(U\times V,E)\to C(U\times V\times\mathfrak{L}(G)^k\times\mathfrak{L}(H)^l,E)_{\mathrm{c.o}},\quad f\mapsto d_{\mathrm{gp}}^{(k,l)}f,$$

where the right-hand side is equipped with the compact-open topology. (The so obtained topology on $C_{\mathrm{gp}}^{r,s}(U\times V,E)$ is called the *compact-open $C_{\mathrm{gp}}^{r,s}$-topology.*)

THEOREM 2.3 (Exponential Law, [14, Theorem 3.3 and Theorem (B)]). *Let $U\subseteq G$, $V\subseteq H$ be open subsets of topological groups G and H, let E be a locally convex space and $r,s\in\mathbb{N}_0\cup\{\infty\}$. If $f:U\times V\to E$ is a $C_{\mathrm{gp}}^{r,s}$-map, then $f^\vee(x):=f(x,\bullet)\in C_{\mathrm{gp}}^{s}(V,E)$ for each $x\in U$ and $f^\vee\in C_{\mathrm{gp}}^{r}(U,C_{\mathrm{gp}}^{s}(V,E))$.*

Further, the map

$$\Phi : C_{\mathrm{gp}}^{r,s}(U\times V,E)\to C_{\mathrm{gp}}^{r}(U,C_{\mathrm{gp}}^{s}(V,E)),\quad f\mapsto f^\vee$$

is a linear topological embedding. If $U\times V\times\mathfrak{L}(G)^k\times\mathfrak{L}(H)^l$ is a $k_{\mathbb{R}}$-space[1] *for all $k,l\in\mathbb{N}_0$ with $k\le r$, $l\le s$, then Φ is an isomorphism of topological vector spaces.*

COROLLARY 2.4 ([14, Corollary 3.5]). *The function Φ in the Exponential Law* 2.3 *is an isomorphism of topological vector spaces if at least one of the following conditions is satisfied*:

(i) *G and H are metrizable,*
(ii) *G and H are locally compact.*

3. Differentiability on locally convex spaces and manifolds. Now, let us recall the concepts of differentiability for functions between locally convex spaces and functions between manifolds modeled on locally convex spaces.

DEFINITION 3.1.

(i) Let E, F be locally convex spaces and $f:U\to F$ be a continuous function on an open subset $U\subseteq E$. For $r\in\mathbb{N}$, we call f a C_{lcx}^{r}-*function* if for each $1\le k\le r$ the differential

$$d_{\mathrm{lcx}}^{(k)}f : U\times E^k\to F,\quad d_{\mathrm{lcx}}^{(k)}f(x,y_1,\dots,y_k):=(D_{y_k}\cdots D_{y_1}f)(x)$$

is defined and continuous, where $d_{\mathrm{lcx}}f(x,y):=D_yf(x):=\lim_{t\to 0}\frac{1}{t}(f(x+ty)-f(x))$. If f is C_{lcx}^{r} for each $r\in\mathbb{N}$, then f is called $C_{\mathrm{lcx}}^{\infty}$. We put $d_{\mathrm{lcx}}^{(0)}f:=f$.

(This concept can be understood as a special case of the concept in Definition 2.2 (i), as E is, in particular, a topological group and $E\cong\mathfrak{L}(E)$ via $y\mapsto\gamma_y$, where γ_y denotes the one-parameter subgroup of E of the form $t\mapsto ty$.)

In the case $E=\mathbb{R}$, we write $f':U\to F$, $f'(s):=\lim_{t\to 0}\frac{1}{t}(f(s+t)-f(s))$.

(ii) Let M, N be manifolds modeled on locally convex spaces and $f:M\to N$ be a continuous function. For $r\in\mathbb{N}\cup\{\infty\}$, we call f a C_{mfd}^{r}-*function* if for each $x\in M$ there are charts φ for M around x and ψ for N around $f(x)$ such that the composition

[1] A Hausdorff topological space X is called a $k_{\mathbb{R}}$-*space* if real-valued functions $f:X\to\mathbb{R}$ are continuous if and only if the restrictions $f\big|_K$ are continuous for all compact subsets $K\subseteq X$.

$\psi \circ f \circ \varphi^{-1}$ is C^r_{lcx}. In this case, if N is a locally convex space, then we write

$$d_{\mathrm{mfd}} f : TM \to N, \quad [\gamma] \mapsto (f \circ \gamma)'(0)$$

for the second component of the tangent map $Tf : TM \to N \times N$. Further, we define $f' : \mathbb{R} \to TN$, $f'(t) := Tf(t,1)$ for a C^1_{mfd}-curve $f : \mathbb{R} \to N$ into a manifold N.

We denote by $C^r_{\mathrm{mfd}}(M,E)$ the space of all C^r_{mfd}-functions $f : M \to E$ (where E is a locally convex space) and endow this space with the initial Hausdorff locally convex topology with respect to the family $(d^{(k)}_{\mathrm{lcx}})_{k\in\mathbb{N}_0, k\le r}$ of mappings

$$d^{(k)}_{\mathrm{lcx}} : C^r_{\mathrm{mfd}}(M,E) \to C(V_\varphi \times F^k, E)_{\mathrm{c.o}}, \quad f \mapsto d^{(k)}_{\mathrm{lcx}}(f \circ \varphi^{-1}),$$

for charts $\varphi : U_\varphi \to V_\varphi$ of the maximal atlas of M, where F is the modeling space of M.

We will often use the following facts without further mention:

Remark 3.2.

(i) Assume that $\mathfrak{L}(G)$ carries a topological vector space structure. If a function $f : G \to E$ is C^1_{gp} and the differential $d_{\mathrm{gp}} f$ is C^{r-1}_{gp} on the topological group $G \times \mathfrak{L}(G)$, then f is C^r_{gp} with derivatives

$$d^{(k)}_{\mathrm{gp}} f(x,\gamma_1,\ldots,\gamma_k) = d^{(k-1)}_{\mathrm{gp}}(d_{\mathrm{gp}} f)((x,\gamma_1),(\gamma_2,\bar{\gamma}_0),\ldots,(\gamma_k,\bar{\gamma}_0)), \tag{1}$$

where $\bar{\gamma}_0 \in \mathfrak{L}(\mathfrak{L}(G))$ denotes the one-parameter subgroup $t \mapsto \gamma_0$, where $\gamma_0 \in \mathfrak{L}(G)$ is the trivial one-parameter subgroup of G.

On the other hand, if f is C^r_{gp} and $d_{\mathrm{gp}} f$ is linear in the second argument then $d_{\mathrm{gp}} f$ is C^{r-1}_{gp} with derivatives

$$\begin{aligned} &d^{(k)}_{\mathrm{gp}}(d_{\mathrm{gp}} f)((x,\alpha),(\gamma_1,\eta_1),\ldots,(\gamma_k,\eta_k)) \\ &\quad = d^{(k+1)}_{\mathrm{gp}} f(x,\alpha,\gamma_1,\ldots,\gamma_k) + \sum_{j=1}^{k} d^{(k)}_{\mathrm{gp}} f(x,\eta_j(1),\gamma_1,\ldots,\gamma_{j-1},\gamma_{j+1},\ldots,\gamma_k). \end{aligned} \tag{2}$$

A function $f : E \to F$ is C^r_{lcx} if and only if f is C^1_{lcx} and $d_{\mathrm{lcx}} f : E \times E \to F$ is C^{r-1}_{lcx}. If $f : M \to E$ is C^r_{mfd}, then $d_{\mathrm{mfd}} f : TM \to E$ is C^{r-1}_{mfd}.

(ii) Compositions of composable C^r_{mfd}- (resp. C^r_{lcx}-) functions are C^r_{mfd} (resp. C^r_{lcx}). Each continuous linear function between locally convex spaces is C^∞_{lcx}.

(iii) A function between locally convex spaces is C^r_{mfd} if and only if it is C^r_{lcx}.

4. Differentiability on locally exponential Lie groups and on direct limit Lie groups. In the following, we will always assume that E is a locally convex topological vector space and that G is a smooth Lie group modeled on a locally convex space F. We will prove that the concepts of differentiability on G as a topological group and as a manifold coincide if G is locally exponential or if G is a direct limit Lie group $G = \varinjlim G_n$ of certain Lie groups G_n. We denote by $\mathfrak{g} := T_e G$ the Lie algebra of G, where $T_e G$ is the tangent space of G at the identity element e. We write $\sigma : G \times TG \to TG, (x,v) \mapsto x.v := T\lambda_x(v)$ for the C^∞_{mfd}-multiplication in the tangent group TG, where $T\lambda_x$ is the tangent map of the left translation $\lambda_x : y \mapsto x \cdot y$ on G.

Remark 4.1.

(i) Recall that G is called *locally exponential* if G has a C^∞_{mfd}-exponential function $\exp : \mathfrak{g} \to G$ and there exists an open 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\exp\big|_U$ is a diffeomorphism onto an open e-neighborhood $V \subseteq G$; we denote its inverse by $\log : V \to U$. In this case, if γ is a continuous one-parameter subgroup, then γ is C^∞_{mfd} and there exists a unique $v \in \mathfrak{g}$ such that $\gamma(t) = \exp(tv) =: \gamma_v(t)$. Moreover, the function $\Gamma : \mathfrak{g} \to \mathfrak{L}(G)$, $v \mapsto \gamma_v$, is a homeomorphism with the inverse $\Gamma^{-1} : \mathfrak{L}(G) \to \mathfrak{g}$, $\gamma \mapsto [\gamma] = \gamma'(0)$. We equip $\mathfrak{L}(G)$ with the locally convex topological vector space structure making Γ an isomorphism of locally convex spaces (hence C^∞_{lcx}). (Details on locally exponential Lie groups can be found, for example, in [11] or [7].)

(ii) Consider an ascending sequence of finite-dimensional Lie groups $G_1 \subseteq G_2 \subseteq \dots$ such that the inclusions $G_n \to G_{n+1}$ are continuous (hence C^∞_{mfd}, being homomorphisms between finite-dimensional Lie groups). Then $G := \bigcup_{n\in\mathbb{N}} G_n$ admits a Lie group structure such that $G = \varinjlim G_n$ in the category of Lie groups modeled on locally convex vector spaces (we call G a *direct limit Lie group*), for the Lie algebra $\mathfrak{g}$ of G we have $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ in the category of topological Lie algebras, where each $\mathfrak{g}_n$ denotes the Lie algebra of G_n (by [4, Theorem 4.3(a)]). (Note that G always has a C^∞_{mfd}-exponential function, but is not necessarily locally exponential [3, Example 5.5].) Further, in this case we have $\mathfrak{L}(G) = \bigcup_{n\in\mathbb{N}} \mathfrak{L}(G_n)$ ([3, 5.3]), that is, each $\gamma \in \mathfrak{L}(G)$ is a continuous one-parameter subgroup of some G_n, hence C^∞_{mfd}; moreover, each of the functions $\Gamma_n : \mathfrak{g}_n \to \mathfrak{L}(G_n)$ (as defined above) is a homeomorphism (each G_n being locally exponential). Hence so is the function $\varinjlim \Gamma_n : \mathfrak{g} \to \mathfrak{L}(G)$ (where $\mathfrak{L}(G) = \varinjlim \mathfrak{L}(G_n)$ is the direct limit in the category of topological spaces). But by [5, Theorem 4.4] the function $\Gamma : \mathfrak{g} \to \mathfrak{L}(G)$ is a homeomorphism (where $\mathfrak{L}(G)$ is equipped with the compact-open topology), thus the direct limit topology and the compact-open topology on $\mathfrak{L}(G)$ coincide. Since each finite-dimensional G_n and each $\mathfrak{L}(G_n) \cong \mathfrak{g}_n \cong \mathbb{R}^{dim(G_n)}$ is locally compact, we conclude that $G \times \mathfrak{L}(G)^k = \varinjlim (G_n \times \mathfrak{L}(G_n)^k)$, for each $k \in \mathbb{N}$ (see [6, Proposition 3.2], [8, Theorem 4.1]). As in (i), we can equip $\mathfrak{L}(G)$ with the locally convex topological vector space structure making Γ an isomorphism of locally convex spaces.

The following properties of differentiable functions on direct limit Lie groups will enable us to reduce the case of direct limit Lie groups to the case of locally exponential Lie groups.

Lemma 4.2. *Let G be a direct limit Lie group as in Remark* 4.1(ii) *and $f : G \to E$ be a continuous function. For $r \in \mathbb{N} \cup \{\infty\}$ the following holds*:

(i) *f is C^r_{gp} if and only if $f\big|_{G_n} : G_n \to E$ is C^r_{gp} for each $n \in \mathbb{N}$,*
(ii) *f is C^r_{mfd} if and only if $f\big|_{G_n} : G_n \to E$ is C^r_{mfd} for each $n \in \mathbb{N}$.*

Proof. (i) First, assume that f is C^r_{gp}. Fix $n \in \mathbb{N}$, $k \leq r$ and let $x \in G_n$, $\gamma_1, \dots, \gamma_k \in \mathfrak{L}(G_n)$. Then $d^{(k)}_{\mathrm{gp}} f(x, \gamma_1, \dots, \gamma_k)$ exists, whence $d^{(k)}_{\mathrm{gp}}(f\big|_{G_n}) : G_n \times \mathfrak{L}(G_n)^k \to E$ is defined. Further, $d^{(k)}_{\mathrm{gp}} f : G \times \mathfrak{L}(G)^k \to E$ is continuous. We have $G \times \mathfrak{L}(G)^k = \varinjlim (G_n \times \mathfrak{L}(G_n)^k)$, hence $d^{(k)}_{\mathrm{gp}}(f\big|_{G_n}) = d^{(k)}_{\mathrm{gp}} f\big|_{G_n \times \mathfrak{L}(G_n)^k}$ is continuous, whence $f\big|_{G_n}$ is C^r_{gp}.

Conversely, let $x \in G$ and $\gamma_1, \dots, \gamma_k \in \mathfrak{L}(G)$ for some $k \leq r$. Since G and $\mathfrak{L}(G)$ are ascending unions of G_n and $\mathfrak{L}(G_n)$, respectively, there exists some $N \in \mathbb{N}$ such that $(x, \gamma_1, \dots, \gamma_k) \in G_N \times \mathfrak{L}(G_N)^k$. Hence $d_{\mathrm{gp}}^{(k)} f : G \times \mathfrak{L}(G)^k \to E$ is defined (with $d_{\mathrm{gp}}^{(k)} f(x, \gamma_1, \dots, \gamma_k) := d_{\mathrm{gp}}^{(k)}(f\big|_{G_N})(x, \gamma_1, \dots, \gamma_k))$. This differential is continuous if and only if $d_{\mathrm{gp}}^{(k)} f\big|_{G_n \times \mathfrak{L}(G_n)^k} = d_{\mathrm{gp}}^{(k)}(f\big|_{G_n})$ is continuous (see above) and this is satisfied by the assumption.

(ii) See [3, Proposition 4.2]. ∎

Proposition 4.3. *If G is a locally exponential Lie group and $f : G \to E$ is a C^r_{mfd}-map for some $r \in \mathbb{N}_0 \cup \{\infty\}$, then f is C^r_{gp}.*

Proof. For $r = 0$ the assertion is clear. For $r \geq 1$, we may assume that $r < \infty$ and proceed by induction.

Induction start: For $x \in G$ we put $\lambda_x : y \mapsto x \cdot y$, which is a C^∞_{mfd}-function. For $\gamma \in \mathfrak{L}(G)$ and $t \neq 0$ we have

$$\frac{f(x \cdot \gamma(t)) - f(x)}{t} = \frac{(f \circ \lambda_x \circ \gamma)(t) - (f \circ \lambda_x \circ \gamma)(0)}{t} \to (f \circ \lambda_x \circ \gamma)'(0),$$

as $t \to 0$, because the composition $f \circ \lambda_x \circ \gamma : \mathbb{R} \to E$ is a C^1_{lcx}-curve. We rewrite

$$d_{\mathrm{gp}} f(x, \gamma) = (f \circ \lambda_x \circ \gamma)'(0) = d_{\mathrm{mfd}} f(\sigma(x, \Gamma^{-1}(\gamma))) \tag{3}$$

and see that the differential $d_{\mathrm{gp}} f := d_{\mathrm{mfd}} f \circ \sigma \circ (\mathrm{id}_G \times \Gamma^{-1}) : G \times \mathfrak{L}(G) \to E$ is continuous, hence the function f is C^1_{gp}.

Induction step: Assume that f is C^r_{mfd} for $r \geq 2$. Then f is C^1_{gp}, by the induction start. Using (3), we see that the differential $d_{\mathrm{gp}} f$ can be written as a composition of C^{r-1}_{mfd}-functions, hence it is C^{r-1}_{mfd} on the locally exponential Lie group $G \times \mathfrak{L}(G)$. Therefore, the differential is C^{r-1}_{gp}, by the induction hypothesis, whence f is C^r_{gp}. ∎

Lemma 4.4. *Let G be a locally exponential Lie group, $f : G \to E$ be a C^1_{gp}-map and $\gamma : \mathbb{R} \to G$ be a C^1_{mfd}-curve. Then $(f \circ \gamma)'(0) = d_{\mathrm{gp}} f(\gamma(0), \kappa)$ for some one-parameter subgroup $\kappa \in \mathfrak{L}(G)$.*

Proof. First, we recall from [14, Lemma A.3] that the function $f^{[1]} : G \times \mathfrak{L}(G) \times \mathbb{R} \to E$ such that $f^{[1]}(x, \eta, t) = \frac{1}{t}(f(x \cdot \eta(t)) - f(x))$, for $t \neq 0$, is continuous on $G \times \mathfrak{L}(G) \times \mathbb{R}$, since f is assumed C^1_{gp}, and we have $d_{\mathrm{gp}} f(x, \eta) = f^{[1]}(x, \eta, 0)$. Now, for $\varepsilon > 0$ small enough consider the continuous curve

$$\eta : \left]-\varepsilon, \varepsilon\right[\to \mathfrak{g}, \quad \eta(t) := \begin{cases} \frac{1}{t} \log(\gamma(0)^{-1} \cdot \gamma(t)) & \text{if } t \neq 0 \\ \gamma(0)^{-1} . \gamma'(0) & \text{if } t = 0. \end{cases}$$

Note that the continuity of η in $t = 0$ follows from

$$\begin{aligned} \lim_{t \to 0} \frac{\log(\gamma(0)^{-1} \cdot \gamma(t))}{t} &= \lim_{t \to 0} \frac{\log(\gamma(0)^{-1} \cdot \gamma(t)) - \log(\gamma(0)^{-1} \cdot \gamma(0))}{t} \\ &= (\log \circ \lambda_{\gamma(0)^{-1}} \circ \gamma)'(0) = (d_{\mathrm{mfd}}(\log) \circ T\lambda_{\gamma(0)^{-1}})(\gamma'(0)) = \gamma(0)^{-1} . \gamma'(0), \end{aligned}$$

since $d_{\mathrm{mfd}}(\log)$ is a restriction of $\mathrm{id}_{\mathfrak{g}}$ in this case. Now, for $0 \neq t \in \left]-\varepsilon, \varepsilon\right[$ we have

$$\frac{f(\gamma(t)) - f(\gamma(0))}{t} = \frac{f(\gamma(0) \cdot \kappa_t(t)) - f(\gamma(0))}{t} = f^{[1]}(\gamma(0), \kappa_t, t)$$

with the one-parameter subgroup $\kappa_t : \mathbb{R} \to G, \kappa_t(s) := \exp(s\eta(t))$. Then

$$(f \circ \gamma)'(0) = \lim_{t \to 0} f^{[1]}(\gamma(0), \kappa_t, t) = f^{[1]}(\gamma(0), \kappa_0, 0) = d_{\mathrm{gp}} f(\gamma(0), \kappa_0),$$

and the proof is finished. ∎

Proposition 4.5. *If G is a locally exponential Lie group and $f : G \to E$ is a C^r_{gp}-map for some $r \in \mathbb{N}_0 \cup \{\infty\}$, then f is C^r_{mfd}.*

Proof. For $r = 0$ the assertion is clearly true. Now, we assume that $1 \leq r < \infty$ and prove the assertion by induction.

Induction start: Fix $g \in G$ and let $\varphi : U_\varphi \to V_\varphi \subseteq F$ be a chart for G around g, where F be the modeling space of G. To show that f is C^1_{mfd} we need to prove that $f \circ \varphi^{-1} : V_\varphi \to E$ is C^1_{lcx}. To this end, let $x \in V_\varphi$, $y \in F$ and define the C^1_{mfd}-curve $\gamma :]{-\varepsilon}, \varepsilon[\to G$, $t \mapsto \varphi^{-1}(x + ty)$ for suitable $\varepsilon > 0$. (Note that $T\varphi^{-1}(x, y) = [t \mapsto \varphi^{-1}(x + ty)] = [\gamma] = \gamma'(0)$.) Then we have

$$d_{\mathrm{lcx}}(f \circ \varphi^{-1})(x, y) = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} = (f \circ \gamma)'(0) = d_{\mathrm{gp}} f(\gamma(0), \kappa)$$

with the one-parameter subgroup $\kappa := \kappa_0 : \mathbb{R} \to G$, $t \mapsto \exp(t(\gamma(0)^{-1}.\gamma'(0)))$ from Lemma 4.4. Using the C^∞_{mfd}-function $\omega : TG \to \mathfrak{g}$, $v \mapsto \pi_{TG}(v)^{-1}.v$ (where π_{TG} denotes the bundle projection, which is C^∞_{mfd}) we rewrite

$$d_{\mathrm{lcx}}(f \circ \varphi^{-1})(x, y) = d_{\mathrm{gp}} f\left(\pi_{TG}([\gamma]), (\Gamma \circ \omega)([\gamma])\right),$$

hence the differential

$$d_{\mathrm{lcx}}(f \circ \varphi^{-1}) := d_{\mathrm{gp}} f \circ (\pi_{TG}, \Gamma \circ \omega) \circ T\varphi^{-1} : V_\varphi \times F \to E \tag{4}$$

is continuous and consequently $f \circ \varphi^{-1}$ is a C^1_{lcx}-function, as required. Note that we found

$$d_{\mathrm{mfd}} f([\gamma]) = d_{\mathrm{gp}} f\left(\pi_{TG}([\gamma]), (\Gamma \circ \omega)([\gamma])\right), \tag{5}$$

for each $[\gamma] \in TG$.

Induction step: Now, let f be a C^r_{gp}-map for $r \geq 2$ and φ be as above. By the induction start, we know that f is C^1_{mfd} (that is, $f \circ \varphi^{-1}$ is C^1_{lcx}). To show that $d_{\mathrm{lcx}}(f \circ \varphi^{-1})$ is C^{r-1}_{lcx}, consider the formula in (4). The function $(\pi_{TG}, \Gamma \circ \omega) \circ T\varphi^{-1} : V_\varphi \times F \to G \times \mathfrak{L}(G)$ is C^∞_{mfd}. Further, using (3) we see that $d_{\mathrm{gp}} f$ is linear in the second argument, hence C^{r-1}_{gp} (see Remark 3.2), Thus $d_{\mathrm{gp}} f$ is C^{r-1}_{mfd} on the locally exponential Lie group $G \times \mathfrak{L}(G)$, by the induction hypothesis. Therefore, the differential $d_{\mathrm{lcx}}(f \circ \varphi^{-1})$ is C^{r-1}_{lcx}, being C^{r-1}_{mfd}. ∎

Proposition 4.6. *Let G be a direct limit Lie group as in Remark* 4.1(ii)*, then a function $f : G \to E$ is C^r_{gp} if and only if f is C^r_{mfd}, for each $r \in \mathbb{N}_0 \cup \{\infty\}$.*

Proof. Since each Lie group G_n is locally exponential (being finite dimensional), each of the restrictions $f\big|_{G_n}$ is C^r_{gp} if and only if it is C^r_{mfd}, by Propositions 4.3 and 4.5. The remainder follows from Lemma 4.2. ∎

Using the fact that the differential $d_{\mathrm{gp}}f$ of each C^r_{gp}-function f defined on a locally exponential Lie group G or on a direct limit Lie group G (as in Remark 4.1) is C^{r-1}_{gp}, we show that in these cases the compact-open C^r_{gp}-topology on $C^r_{\mathrm{gp}}(G,E)$ can be described in the following way (for finite r):

LEMMA 4.7. *The compact-open C^r_{gp}-topology $\mathcal{O}_1$ on the function space $C^r_{\mathrm{gp}}(G,E)$ coincides with the initial topology $\mathcal{O}_2$ with respect to the functions*

$$i : C^r_{\mathrm{gp}}(G,E) \to C(G,E)_{\mathrm{c.o}}, \quad f \mapsto f,$$
$$D_{\mathrm{gp}} : C^r_{\mathrm{gp}}(G,E) \to (C^{r-1}_{\mathrm{gp}}(G \times \mathfrak{L}(G), E), \mathcal{O}_1), \quad f \mapsto d_{\mathrm{gp}}f,$$

for each $r \in \mathbb{N}$ if G is a Lie group as in Remark 4.1.

Proof. First, we show that $\mathcal{O}_2 \subseteq \mathcal{O}_1$. This will hold if both the functions

$$i : (C^r_{\mathrm{gp}}(G,E), \mathcal{O}_1) \to C(G,E)_{\mathrm{c.o}}, \quad f \mapsto f, \tag{6}$$
$$D_{\mathrm{gp}} : (C^r_{\mathrm{gp}}(G,E), \mathcal{O}_1) \to (C^{r-1}_{\mathrm{gp}}(G \times \mathfrak{L}(G), E), \mathcal{O}_1), \quad f \mapsto d_{\mathrm{gp}}f, \tag{7}$$

are continuous. The first function (6) is continuous by the definition of the topology $\mathcal{O}_1$, since we have $i = d^{(0)}_{\mathrm{gp}}$. The continuity of the second function (7) will follow from the continuity of the compositions

$$\begin{aligned} d^{(k)}_{\mathrm{gp}} \circ D_{\mathrm{gp}} : (C^r_{\mathrm{gp}}(G,E), \mathcal{O}_1) &\to C(G \times \mathfrak{L}(G) \times \mathfrak{L}(G \times \mathfrak{L}(G))^k, E)_{\mathrm{c.o}}, \\ f &\mapsto d^{(k)}_{\mathrm{gp}}(d_{\mathrm{gp}}f), \end{aligned} \tag{8}$$

where $d^{(k)}_{\mathrm{gp}} : (C^{r-1}_{\mathrm{gp}}(G \times \mathfrak{L}(G), E), \mathcal{O}_1) \to C(G \times \mathfrak{L}(G) \times \mathfrak{L}(G \times \mathfrak{L}(G))^k, E)_{\mathrm{c.o}}$ for all $0 \le k \le r-1$. Using two continuous functions

$$\rho_k : G \times \mathfrak{L}(G) \times \mathfrak{L}(G \times \mathfrak{L}(G))^k \to G \times \mathfrak{L}(G)^{k+1},$$
$$(x, \alpha, (\gamma_1, \eta_1), \dots, (\gamma_k, \eta_k)) \mapsto (x, \alpha, \gamma_1, \dots, \gamma_k),$$
$$\rho_{j,k} : G \times \mathfrak{L}(G) \times \mathfrak{L}(G \times \mathfrak{L}(G))^k \to G \times \mathfrak{L}(G)^k,$$
$$(x, \alpha, (\gamma_1, \eta_1), \dots, (\gamma_k, \eta_k)) \mapsto (x, \eta_j(1), \gamma_1, \dots, \gamma_{j-1}, \gamma_{j+1}, \dots, \gamma_k),$$

for $1 \le j \le k$, and equation (2) from Remark 3.2(i), we obtain

$$d^{(k)}_{\mathrm{gp}}(d_{\mathrm{gp}}f) = (d^{(k+1)}_{\mathrm{gp}} f \circ \rho_k) + \sum_{j=1}^{k} (d^{(k)}_{\mathrm{gp}} f \circ \rho_{j,k})$$

for each $f \in C^r_{\mathrm{gp}}(G,E)$. Hence, using the continuous pullbacks $\rho_k^* : g \mapsto g \circ \rho_k$ and $\rho_{j,k}^* : g \mapsto g \circ \rho_{j,k}$ we can write each of the maps from (8) as

$$d^{(k)}_{\mathrm{gp}} \circ D_{\mathrm{gp}} = (\rho_k^* \circ d^{(k+1)}_{\mathrm{gp}}) + \sum_{j=1}^{k} (\rho_{j,k}^* \circ d^{(k)}_{\mathrm{gp}}).$$

(Note that the functions $d^{(k+1)}_{\mathrm{gp}}$, $d^{(k)}_{\mathrm{gp}}$ on the right-hand side are the differential operators on $(C^r_{\mathrm{gp}}(G,E), \mathcal{O}_1)$.) From the definition of the compact-open C^r_{gp}-topology $\mathcal{O}_1$ we conclude that the composition is continuous, as required.

Now, we show that $\mathcal{O}_1 \subseteq \mathcal{O}_2$, which will be the case if for all $0 \le k \le r$ the functions

$$d^{(k)}_{\mathrm{gp}} : (C^r_{\mathrm{gp}}(G,E), \mathcal{O}_2) \to C(G \times \mathfrak{L}(G)^k, E)_{\mathrm{c.o}}, \quad f \mapsto d^{(k)}_{\mathrm{gp}} f, \tag{9}$$

are continuous. For $k=0$ we have $d_{\mathrm{gp}}^{(0)}=i$, hence the continuity follows from the definition of the topology $\mathcal{O}_2$. Now, using the continuous functions

$$\xi_k : G\times\mathfrak{L}(G)^k \to G\times\mathfrak{L}(G)\times\mathfrak{L}(G\times\mathfrak{L}(G))^{k-1},$$
$$(x,\gamma_1,\ldots,\gamma_k)\mapsto((x,\gamma_1),(\gamma_2,\bar\gamma_0),\ldots,(\gamma_k,\bar\gamma_0)),$$

where $\bar\gamma_0\in\mathfrak{L}(\mathfrak{L}(G))$ denotes the one-parameter subgroup $t\mapsto\gamma_0$, where $\gamma_0\in\mathfrak{L}(G)$ is the trivial one-parameter subgroup of G, and equation (1) from Remark 3.2(i) we can express the functions in (9) as

$$d_{\mathrm{gp}}^{(k)}=\xi_k^*\circ d_{\mathrm{gp}}^{(k-1)}\circ D_{\mathrm{gp}},$$

with the continuous differential operators $d_{\mathrm{gp}}^{(k-1)}$ on $(C_{\mathrm{gp}}^{r-1}(G\times\mathfrak{L}(G),E),\mathcal{O}_1)$ on the right-hand side and the continuous pullbacks $\xi_k^*: g\mapsto g\circ\xi_k$. From the definition of the topology $\mathcal{O}_2$ on $C_{\mathrm{gp}}^r(G,E)$ we conclude that the composition above is continuous for each k, as required. ∎

Analogously, we can prove:

LEMMA 4.8. *The compact-open C_{mfd}^r-topology $\mathcal{O}_1$ on the function space $C_{\mathrm{mfd}}^r(G,E)$ coincides with the initial topology $\mathcal{O}_2$ with respect to the functions*

$$i: C_{\mathrm{mfd}}^r(G,E)\to C(G,E)_{\mathrm{c.o}},\quad f\mapsto f,$$
$$d_{\mathrm{mfd}}: C_{\mathrm{mfd}}^r(G,E)\to(C_{\mathrm{mfd}}^{r-1}(TG,E),\mathcal{O}_1),\quad f\mapsto d_{\mathrm{mfd}}f,$$

for each $r\in\mathbb{N}$ if G is a Lie group as in Remark 4.1.

Using these descriptions of the topologies on the function spaces, we finally get the main result:

THEOREM 4.9. *If G is a locally exponential Lie group or a direct limit Lie group* (*as in Remark* 4.1), *E is a locally convex space and $r\in\mathbb{N}_0\cup\{\infty\}$, then $C_{\mathrm{gp}}^r(G,E)=C_{\mathrm{mfd}}^r(G,E)$ as topological vector spaces.*

Proof. From Propositions 4.3, 4.5 and 4.6 it follows that the function spaces coincide as sets, it remains to show that also the topologies coincide.

The topologies on $C_{\mathrm{gp}}^\infty(G,E)$ and $C_{\mathrm{mfd}}^\infty(G,E)$ are initial with respect to the inclusion maps $C_{\mathrm{gp}}^\infty(G,E)\to C_{\mathrm{gp}}^r(G,E)$ and $C_{\mathrm{mfd}}^\infty(G,E)\to C_{\mathrm{mfd}}^r(G,E)$ for $r\in\mathbb{N}_0$, respectively (this is easy to verify using the definitions). Hence it suffices to prove the continuity of both inclusion maps $\mathrm{incl}_r: C_{\mathrm{gp}}^r(G,E)\to C_{\mathrm{mfd}}^r(G,E)$ and $\mathrm{incl}^r: C_{\mathrm{mfd}}^r(G,E)\to C_{\mathrm{gp}}^r(G,E)$ by induction on r.

Induction start: The inclusion maps incl_0 and incl^0 coincide with the functions i from Lemma 4.7 and Lemma 4.8, respectively, hence they are continuous.

Induction step: By Lemma 4.8, the continuity of the inclusion map incl_r will follow from the continuity of the compositions

$$i\circ\mathrm{incl}_r: C_{\mathrm{gp}}^r(G,E)\to C(G,E)_{\mathrm{c.o}},\quad f\mapsto f,\tag{10}$$
$$d_{\mathrm{mfd}}\circ\mathrm{incl}_r: C_{\mathrm{gp}}^r(G,E)\to C_{\mathrm{mfd}}^{r-1}(TG,E),\quad f\mapsto d_{\mathrm{mfd}}f.\tag{11}$$

The first composition (10) is continuous, by Lemma 4.7. Now, for $f \in C^r_{\mathrm{gp}}(G,E)$ and $v \in TG$ we have

$$d_{\mathrm{mfd}} f(v) = d_{\mathrm{gp}} f(\pi_{TG}(v), (\Gamma \circ \omega)(v)),$$

using (5). Recall from Lemma 4.7 that $D_{\mathrm{gp}} : C^r_{\mathrm{gp}}(G,E) \to C^{r-1}_{\mathrm{gp}}(G \times \mathfrak{L}(G), E)$, $f \mapsto d_{\mathrm{gp}} f$, is continuous. Using the induction hypothesis, we conclude that D_{gp} is continuous as a function to $C^{r-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E)$. Further, the operator

$$(\pi_{TG}, \Gamma \circ \omega)^* : C^{r-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E) \to C^{r-1}_{\mathrm{mfd}}(TG, E), \quad f \mapsto f \circ (\pi_{TG}, \Gamma \circ \omega)$$

is continuous (see [7]) and we have

$$d_{\mathrm{mfd}} = (\pi_{TG}, \Gamma \circ \omega)^* \circ D_{\mathrm{gp}},$$

by the above. Therefore, also the composition in (11) is continuous and the first assertion is proved.

Now, by Lemma 4.7 the continuity of the inclusion map incl^r will follow from the continuity of the functions

$$i \circ \mathrm{incl}^r : C^r_{\mathrm{mfd}}(G,E) \to C(G,E)_{\mathrm{c.o}}, \quad f \mapsto f \tag{12}$$

$$D_{\mathrm{gp}} \circ \mathrm{incl}^r : C^r_{\mathrm{mfd}}(G,E) \to C^{r-1}_{\mathrm{gp}}(G \times \mathfrak{L}(G), E), \quad f \mapsto d_{\mathrm{gp}} f. \tag{13}$$

The first composition (12) is continuous by Lemma 4.8. Further, for $f \in C^r_{\mathrm{mfd}}(G,E)$ we have

$$d_{\mathrm{gp}} f = d_{\mathrm{mfd}} f \circ \sigma \circ (\mathrm{id}_G \times \Gamma^{-1}),$$

by (3). The function $d_{\mathrm{mfd}} : C^r_{\mathrm{mfd}}(G,E) \to C^{r-1}_{\mathrm{mfd}}(TG,E)$ is continuous by Lemma 4.8, and also the operator

$$(\sigma \circ (\mathrm{id}_G \times \Gamma^{-1}))^* : C^{r-1}_{\mathrm{mfd}}(TG,E) \to C^{r-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E), \quad f \mapsto f \circ \sigma \circ (\mathrm{id}_G \times \Gamma^{-1})$$

is continuous (see [7]), hence so is the composition

$$D_{\mathrm{gp}} = (\sigma \circ (\mathrm{id}_G \times \Gamma^{-1}))^* \circ d_{\mathrm{mfd}} : C^r_{\mathrm{mfd}}(G,E) \to C^{r-1}_{\mathrm{mfd}}(G \times \mathfrak{L}(G), E).$$

But by the induction hypothesis, it is continuous as a function into $C^{r-1}_{\mathrm{gp}}(G \times \mathfrak{L}(G), E)$, hence the composition in (13) is continuous and the proof is finished. ∎

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