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SPHERICAL FUNCTIONS ON SPHERES OF RANK TWO

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Abstract. Under a suitable multiplicity freeness assumption we show that the spherical functions of the pair $(\mathrm{SU}(n+1), \mathrm{SU}(n))$ can be described by families of vector-valued polynomials in two variables. The matrix weight that establishes the orthogonality is supported on the closed unit disc and depends on n. We calculate the matrix weight in a couple of examples using computer algebra.

1. Introduction. Consider the Gelfand pair $(U, L) = (\mathrm{SU}(n+1), \mathrm{SU}(n))$ with $n \geq 2$ and an irreducible representation $\pi^L : L \to \mathrm{GL}(V)$ that satisfies the following multiplicity freeness property: for every irreducible U-representation $\pi^U : U \to \mathrm{GL}(V')$ the restriction $\pi^U|_L$ contains π^L at most once, i.e. $\dim \mathrm{Hom}_L(V, V') \leq 1$. We are interested in the space of functions $F : U \to \mathrm{End}(V)$ such that

$$F(\ell_1 u \ell_2) = \pi^L(\ell_1) F(u) \pi^L(\ell_2) \quad \text{for all } (\ell_1, u, \ell_2) \in L \times U \times L. \tag{1}$$

For example, if π^L is the trivial representation, then the space of algebraic functions on U satisfying (1) can be given the structure of a polynomial algebra generated by two elements. The building blocks of this space are the zonal spherical functions, i.e. certain matrix coefficients of irreducible U-representations that contain an L-invariant vector. These functions form a vector space basis which is moreover orthogonal with respect to integration over U against the Haar measure (Schur orthogonality). Because the double cosets $L\backslash U/L$ can be parametrized by the closed unit disc $\{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 \leq 1\}$ one obtains families of orthogonal polynomials in two variables. The orthogonality measure is given by $(1 - x^2 - y^2)^{n-1}\, dx\, dy$.

The quotient $\mathrm{SU}(n+1)/\mathrm{SU}(n)$ is isomorphic to the sphere S^{2n+1}. The group L acts on S^{2n+1} from the left and there exists a torus $A \subset U$ of dimension two whose image in S^{2n+1} intersects all L-orbits, see Lemma 2.1. Hence we call U/L a sphere of rank two.

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1.1. Results. In this note we show that under an additional assumption on the irreducible representation π^L, the space of functions $F : U \to \mathrm{End}(V)$ that satisfy (1) can be described as a space of vector-valued orthogonal polynomials in two variables. The additional condition amounts to requiring the existence of a face $\mathcal{F}$ of the dominant Weyl chamber of L such that the highest weight of π^L is in $\mathcal{F}$ and the multiplicity freeness condition holds for all irreducible representations of L whose highest weight is contained in $\mathcal{F}$. Every face $\mathcal{F}$ of the dominant Weyl chamber of L of codimension one gives an example for this particular pair (U, L).

The vector-valued polynomials that describe the spherical functions are orthogonal with respect to a matrix weight that is supported on the unit disc. We explain how to calculate the matrix weight in general. Because explicit calculations become rather involved we have used GAP [6] to obtain a couple of formulas.

1.2. Related work. If we take for π^L the trivial representation we obtain families of scalar-valued polynomials that have been investigated for example in [3]. The dependence on the parameter n has also been investigated. For these questions one brings differential operators into the game. The orthogonal polynomials that one obtains are related to Heckman–Opdam theory [20].

The pair (U, L) together with a face $\mathcal{F}$ of the dominant Weyl chamber of L for which the multiplicity freeness condition holds is called a multiplicity free system. These systems have been classified, see e.g. [15]. In the same paper the existence of families of multivariable vector-valued polynomials has been shown. Those families of polynomials, as well as the ones in this paper, have the property that they are determined as simultaneous eigenfunctions of a commutative algebra of differential operators. This is worked out systematically for multiplicity free systems related to compact symmetric pairs in [12].

The theory of multivariable vector-valued polynomials related to matrix coefficients on compact Gelfand pairs is a generalization of the theory of zonal spherical functions on compact symmetric spaces [7, Ch.5] and of vector-valued orthogonal polynomials related to compact Gelfand pairs of rank one [8].

1.3. Future work. The existence of the vector-valued polynomials is based on an *ad hoc* analysis of the spectrum of the induced representation $\mathrm{ind}_L^U \pi^L$. The spectra for the other multiplicity free systems are currently being investigated by means of the theory of spherical varieties and a paper on this topic is in preparation. Once the existence (in the generality of the classification [15]) is established, many questions remain. For example: do shift operators exist and are the families of polynomials associated to different multiplicity free systems related by shift operators? The answers are affirmative for particular classes of examples in the one-variable matrix valued cases [11, 19] and in the multivariable scalar-valued cases (Heckman–Opdam theory).

1.4. Organization. In Section 2 we introduce the zonal spherical functions on U and we show that it is sufficient to study their restrictions to a two-dimensional torus $A \subset U$ for which $LAL = U$. In Section 3 we use the theory of spherical varieties to obtain the decomposition of $\mathrm{ind}_L^U \pi^L$ into irreducible U-representations under the appropriate multiplicity freeness assumption. We show that the spherical functions can be described by multivariable vector-valued orthogonal polynomials. In Section 4 we construct these polynomials and the matrix weight with respect to which they are orthogonal. In Section 5 we calculate a couple of examples using the GAP-code [16].

NOTATION. Lie groups are indicated with Roman capitals $U, L, A, T, G, H, \dots$ and their Lie algebras are denoted by $\mathfrak{u}, \mathfrak{l}, \mathfrak{a}, \mathfrak{t}, \mathfrak{g}, \mathfrak{h}, \dots$.

2. Invariant functions. Before we look at the invariant functions on U we establish some facts about the coset space $L\backslash U/L$.

2.1. Spheres of rank two. Consider $\mathbb{C}^{n+1}$ with the standard basis $(e_1, \dots, e_{n+1})$ equipped with the standard Hermitian inner product $\langle v, w\rangle = \sum_{i=1}^{n+1} \overline{v}_i w_i$, where $v = \sum_{i=1}^{n+1} v_i e_i$ and $w = \sum_{i=1}^{n+1} w_i e_i$. Let $U = \mathrm{SU}(n+1) \subset \mathrm{GL}(n+1, \mathbb{C})$ denote the group of linear transformations with determinant one that leave the standard Hermitian form invariant, i.e. $u \in U$ if and only if $\det(u) = 1$ and for all $v, w \in \mathbb{C}^{n+1}$ the equality $(uv, uw) = (v, w)$ holds.

Let $L \subset U$ be the stabilizer of e_{n+1} for the natural action $U \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1} : (u, v) \mapsto uv$. Then L is isomorphic to $\mathrm{SU}(n)$. The image Ue_{n+1} is isomorphic to $U/L \cong S^{2n+1} = \{v \in \mathbb{C}^{n+1} \mid \langle v, v\rangle = 1\}$. For $s, t \in \mathbb{R}$ define the elements

$$a_1(s) = \begin{pmatrix} e^{i(n-1)s} & 0 & 0 \\ 0 & e^{-2is} I_{n-1} & 0 \\ 0 & 0 & e^{i(n-1)s} \end{pmatrix}, \quad a_2(t) = \begin{pmatrix} \cos(t) & 0 & -\sin(t) \\ 0 & I_{n-1} & 0 \\ \sin(t) & 0 & \cos(t) \end{pmatrix} \tag{2}$$

in U. Furthermore, define

$$A_1 = \big\{a_1(s) \mid s \in [0, \pi) \text{ if } n-1 \text{ even, } s \in [0, 2\pi) \text{ if } n-1 \text{ odd}\big\}, \tag{3}$$

$$A_2 = \{a_2(t) \mid t \in [0, 2\pi)\}, \tag{4}$$

which are one-dimensional tori contained in U. Note that the two-dimensional torus $A = A_1 \cdot A_2$ is isomorphic to $(A_1 \times A_2)/(A_1 \cap A_2)$. Since $A_1 \cap A_2 = \{1\}$ if $n-1$ is even and $A_1 \cap A_2 = \{1, a_2(\pi)\}$ if $n-1$ is odd, we have

$$A = \{a_1(s)a_2(t) \mid (s, t) \in [0, \pi) \times [0, 2\pi)\}.$$

LEMMA 2.1. *The map $L \times A \times L \to U : (\ell, a, \ell') \mapsto \ell a \ell'$ is surjective.*

Proof. First we show that the image of A under the orbit map $U \to S^{2n+1} : u \mapsto ue_{n+1}$ intersects all L-orbits. Indeed, let $v = v_1e_1 + \dots + v_{n+1}e_{n+1} \in S^{2n+1}$ and write $v_1' = \sqrt{|v_1|^2 + \dots + |v_n|^2}$. Use the L-action to rotate v to $v_1'e_1 + v_{n+1}e_{n+1}$. There is a unique $s \in [0, 2\pi/(n-1))$ such that $v_{n+1} = e^{i(n-1)s}|v_{n+1}|$. Use L once more to rotate $v_1'e_1 + v_{n+1}e_{n+1}$ to $e^{i(n-1)s}(-|v_1'|e_1 + |v_{n+1}|e_{n+1})$. There is a unique $t \in [0, \pi/2]$ such that $|v_1'| = \sin(t)$ and $|v_{n+1}| = \cos(t)$. Hence we can use L to rotate v to $a_1(s)a_2(t)e_{n+1}$.

It follows that $L \times A \to U/L$ is surjective from which we deduce that $L \times A \times L \to U$ is surjective. ∎

2.2. Zonal spherical functions. We identify irreducible representations of compact Lie groups with their highest weights according to the Theorem of the Highest Weight, see e.g. [9, Theorem 5.110]. The set of dominant integral weights of U is denoted P_U^+. We have

$$P_U^+ = \mathbb{N}_0\varpi_1 + \dots + \mathbb{N}_0\varpi_n,$$

where the choice and the notation of the fundamental weights is taken from [9]. This means that ϖ_i is a character of the torus of diagonal elements $T_U \subset U$ given by the formula $\varpi_i(\mathrm{diag}(t_1,\ldots,t_{n+1})) = t_1\cdots t_i$. The simple roots of U are denoted by $\alpha_1,\ldots,\alpha_n$ where $\alpha_i(\mathrm{diag}(t_1,\ldots,t_{n+1})) = t_i t_{i+1}^{-1}$.

The elements in P_U^+ are the weights that can occur as the weight of a highest weight vector and under this correspondence P_U^+ classifies all irreducible representations modulo isomorphism. Given an element $\lambda \in P_U^+$ we denote the corresponding representation by $\pi_\lambda^U : U \to \mathrm{GL}(V_\lambda^U)$. We equip every representation space V_λ^U with a Hermitian inner product $\langle\cdot,\cdot\rangle_\lambda$ for which the U-representation is unitary. Note that π_0^U denotes the trivial U-representation.

We employ the same notation for the subgroup L, although we denote the fundamental weights by $\omega_1,\ldots,\omega_{n-1}$. Furthermore, we put $T_L = T_U \cap L$. Note that $\varpi_i|_{T_L} = \omega_i$ for $i = 1,\ldots,n-1$ and $\varpi_n|_{T_L} = 1$.

The set $P_{(U,L)}^+(0) = \{\lambda \in P_U^+ \,|\, [\pi_\lambda^U|_L : \pi_0^L] = 1\}$, also called the 0-well (see Definition 2.7), is a semigroup that is generated by ϖ_1, ϖ_n, see e.g. [13, Table 1]. Consider the left-right-regular representation of $U\times U$ on $L^2(U,du)$, where du is the normalized Haar measure. The space of invariants of this action restricted to $L\times L$ is given by

$$L^2(U,du)^{L\times L} = \widehat{\bigoplus_{\lambda\in P_{(U,L)}^+(0)}} \mathrm{End}(V_\lambda^U)^{L\times L},$$

where the hat indicates the Hilbert completion of the direct sum. The spaces $\mathrm{End}(V_\lambda^U)^{L\times L}$ are all one-dimensional since (U,L) is a Gelfand pair. Let $v \in V_\lambda^U$ be an L-invariant vector of length one, i.e. $||v|| = 1$ and $\pi_\lambda^U(\ell)v = v$ for all $\ell \in L$. The matrix coefficient $u \mapsto \langle v, \pi_\lambda^U(u)v\rangle_\lambda$ is a zonal spherical function which we denote by $\phi_\lambda : U \to \mathbb{C}$. Furthermore, Schur-orthogonality [9, Corollary 4.10] implies that $\int_U \overline{\phi_\lambda(u)}\phi_{\lambda'}(u)\,du = \delta_{\lambda,\lambda'}/\dim(V_\lambda^U)$. This shows that $\{\phi_\lambda \,|\, \lambda \in P_{(U,L)}^+(0)\}$ is an orthogonal basis of $L^2(U,du)^{L\times L}$.

PROPOSITION 2.2. *Let $\lambda = \lambda_1\varpi_1 + \lambda_2\varpi_n$. Then there exists $c \in \mathbb{C}^\times$ with $\phi_\lambda = c\phi_{\varpi_1}^{\lambda_1}\phi_{\varpi_n}^{\lambda_2} +$ l.o.t., where a monomial $\phi_{\varpi_1}^{\lambda_1'}\phi_{\varpi_n}^{\lambda_2'}$ is called of lower order if $\lambda_1' \le \lambda_1$ or $\lambda_2' \le \lambda_2$.*

Proof. Suppose that $\lambda_1 > 1$. The tensor product $V_{\lambda-\varpi_1}^U \otimes V_{\varpi_1}^U$ decomposes into irreducible U-representations $V_{\lambda'}^U$ such that $\lambda - \lambda' = \sum_{i=1}^n c_i\alpha_i$ with $c_i \in \mathbb{N}_0$. Let $\mathrm{ht} : P_U \to \mathbb{Z}$ denote the function defined by $\mathrm{ht}(r_1\varpi_1 + \ldots + r_n\varpi_n) = r_1 + \ldots + r_n$. Since $\mathrm{ht}(\alpha_i) = 0$ if $i = 2,\ldots,n-1$ and $\mathrm{ht}(\alpha_1) = \mathrm{ht}(\alpha_n) = 1$, we see that $\mathrm{ht}(\lambda') \le \mathrm{ht}(\lambda)$.

Suppose that $\lambda' = \ell_1\varpi_1 + \ell_2\varpi_n = \lambda - \sum_{i=1}^n c_i\alpha_i$ and that $V_{\lambda'}^U$ occurs in the decomposition of $V_{\lambda-\varpi_1}^U \otimes V_{\varpi_1}^U$. This implies that $\ell_1 + \ell_2 \le \lambda_1 + \lambda_2$. If we have equality, then $c_1 = c_n = 0$. But then $\ell_1 \ge \lambda_1$ and $\ell_2 \ge \lambda_2$, because $\alpha_i = -\varpi_{i-1} + 2\varpi_i - \varpi_{i+1}$ for $i = 2,\ldots,n-1$. Neither inequality can be strict and it follows that $\ell_1 = \lambda_1$ and $\ell_2 = \lambda_2$.

In terms of the Borel–Weil realizations of the representation spaces as sections of equivariant bundles, the Cartan projection $V_{\lambda-\varpi_1}^U \otimes V_{\varpi_1}^U \to V_\lambda^U$ is given by the multiplication of algebraic functions. It follows that the constant $c_\lambda \ne 0$ in the expression $\phi_{\lambda-\varpi_1}\phi_{\varpi_1} = c_\lambda\phi_\lambda + \sum_{(\ell_1,\ell_2)} c_{(\ell_1,\ell_2)}\phi_{\ell_1\varpi_1+\ell_2\varpi_n}$, where the sum is taken over $(\ell_1,\ell_2) \in \mathbb{N}_0^2$ such that $\ell_1 + \ell_2 < \lambda_1 + \lambda_2$.

The analogous statement holds for $V_{\lambda-\varpi_n}^U \otimes V_{\varpi_n}^U$ if $\lambda_2 > 0$ and the proof is completed by induction on $\lambda_1 + \lambda_2$. ■

Suppose that there is an algebraic relation between the functions ϕ_{ϖ_1} and ϕ_{ϖ_n}, i.e. there exists a polynomial $q \in \mathbb{C}[z_1, z_2]$ such that $q(\phi_{\varpi_1}, \phi_{\varpi_n}) = 0$. Using Proposition 2.2 we can rewrite $q(\phi_{\varpi_1}, \phi_{\varpi_n}) = \sum_{(\ell_1,\ell_2)} c_{(\ell_1,\ell_2)} \phi_{\ell_1\varpi_1+\ell_2\varpi_n}$, where the sum is taken over the pairs $(\ell_1, \ell_2) \in \mathbb{N}_0^2$ with $\ell_1 + \ell_2 \leq \deg q$. Since the spherical functions are linearly independent, all coefficients $c_{(\ell_1,\ell_2)}$ are zero and hence $q = 0$. Hence the functions ϕ_{ϖ_1} and ϕ_{ϖ_n} are algebraically independent.

Definition 2.3. Let $\lambda \in P^+_{(G,L)}(0)$. Then $q_\lambda \in \mathbb{C}[z_1, z_2]$ denotes the unique polynomial for which $\phi_\lambda = q_\lambda(\phi_{\varpi_1}, \phi_{\varpi_n})$.

It follows that the space $\mathbb{C}[z_1, z_2]$ embeds into $L^2(U, du)^{L\times L}$ by plugging in ϕ_{ϖ_1} and ϕ_{ϖ_n}. The embedding $\mathbb{C}[z_1, z_2] \to L^2(U, du)^{L\times L}$ provides $\mathbb{C}[z_1, z_2]$ with a Hermitian inner product which is given by

$$\langle q, q'\rangle = \int_U \overline{q(\phi_{\varpi_1}(u), \phi_{\varpi_n}(u))}\, q'(\phi_{\varpi_1}(u), \phi_{\varpi_n}(u))\, du.$$

We want to describe this inner product without integrating over the group U. Let $D = \{\zeta \in \mathbb{C} \mid |\zeta| \leq 1\}$ and consider the map $h : S^{2n+1} \to D : z \mapsto \langle e_{n+1}, z\rangle$. This map is surjective and $h(z) = h(z')$ implies that z and z' are in the same L-orbit on S^{2n+1}. Moreover, h separates the L-orbits on S^{2n+1}. Hence the closed disc D parametrizes the L-orbits on S^{2n+1}.

The spherical function ϕ_{ϖ_1} is calculated as follows. The irreducible U-representation of highest weight ϖ_1 is just the standard representation of U on $\mathbb{C}^{n+1}$. The vector e_{n+1} spans the one-dimensional space that is invariant under L. Hence $\phi_{\varpi_1}(u) = \langle e_{n+1}, u \cdot e_{n+1}\rangle$ and ϕ_{ϖ_1} takes values in D. The irreducible U-representation of highest weight ϖ_n is $(\mathbb{C}^{n+1})^*$ and hence $\phi_{\varpi_n}(u) = \overline{\phi_{\varpi_1}(u)}$. The restrictions of ϕ_{ϖ_1} and ϕ_{ϖ_n} to A are given by

$$\phi_{\varpi_1}(a_1(s)a_2(t)) = e^{-i(n-1)s}\cos(t), \quad \phi_{\varpi_n}(a_1(s)a_2(t)) = e^{i(n-1)s}\cos(t),$$

which follows from plugging in the elements (2).

Let $f : D \to \mathbb{C} : (z, \overline{z}) \mapsto f(z, \overline{z})$ be a continuous function. Define the functions $\widetilde{f} : U \to \mathbb{C} : u \mapsto f(\phi_{\varpi_1}(u), \phi_{\varpi_n}(u))$ and $\overline{\widetilde{f}} : S^{2n+1} \to \mathbb{C} : \xi \mapsto \widetilde{f}(u_\xi)$, where $u_\xi \in U$ satisfies $u_\xi e_{n+1} = \xi$. The latter is well defined and both maps are continuous. We have

$$\int_U \widetilde{f}(u)\, du = \int_{S^{2n+1}} \overline{\widetilde{f}}(\xi)\, d\sigma_{2n+1}(\xi),$$

where $d\sigma_{2n+1}$ is the normalized volume form on S^{2n+1}. Following [3, §3.1] we can rewrite the integral over S^{2n+1} into

$$\int_{S^{2n+1}} \overline{\widetilde{f}}(\xi)\, d\sigma_{2n+1}(\xi) = \frac{n}{\pi}\int_D \overline{\widetilde{f}}(ze_{n+1} + (1-|z|^2)^{1/2}e_1)(1-|z|^2)^{n-1}\, d\lambda(z, \overline{z}),$$

where $d\lambda$ is the normalized Lebesgue measure on D, because $\overline{\widetilde{f}}$ is left L-invariant. Let $u_z \in U$ be an element such that $u_z e_{n+1} = ze_{n+1} + (1-|z|^2)^{1/2}e_1$. Then $\overline{\widetilde{f}}(ze_{n+1} + (1-|z|^2)^{1/2}e_1) = f(z, \overline{z})$ and we obtain the equality

$$\int_U f(\phi_{\varpi_1}(u), \phi_{\varpi_n}(u))\, du = \frac{n}{\pi}\int_D f(z, \overline{z})(1-|z|^2)^{n-1}\, d\lambda(z, \overline{z}), \quad f \in C(D). \tag{5}$$

REMARK 2.4. Let $K \subset U$ be the image of the homomorphism $\mathrm{U}(n) \to \mathrm{SU}(n+1) : k \mapsto \mathrm{diag}(k, \det k^{-1})$. Then (U,K) is a compact Riemannian symmetric pair for which $U = KA_2K$. Harmonic analysis for such pairs greatly benefits from this decomposition and the rest of the well developed structure theory like the existence of restricted root systems. An important subgroup of K in this structure theory is $M = Z_K(A_2)$ and we observe that $M = A_1L_*$, where L_* is the stabilizer of the L-action on S^{2n+1} for the point e_1. Note that L_* is isomorphic to $\mathrm{SU}(n-1)$.

The analysis of zonal spherical functions for a Riemannian symmetric pair like (U,K) restricted to a torus like A_2 is closely related to the analysis of hypergeometric functions associated to root systems.

For compact Gelfand pairs such a rich structure theory is not available. However, in this particular example, which is close to a Riemannian symmetric pair, we can also parametrize the double cosets by a torus. This is the motivation for stating the following result. Let da denote the normalized Haar measure on A.

PROPOSITION 2.5. *Let* $f \in C(U)^{L\times L}$. *Then* $\int_U f(u)\,du = c\int_A f(a)|\delta_n(a)|\,da$, *where*

$$\delta_n(a_1(s)a_2(t)) = |\sin^2(t)|^{n-1}|\sin(2t)|$$

and $c^{-1} = \int_A |\delta_n(a)|\,da$.

Proof. Let $\mathfrak{a} = \mathrm{Lie}(A)$ and consider $\mathfrak{a} \to D : H \mapsto \phi(\exp(H))$. With respect to the coordinates (s,t) on $\mathfrak{a}$ for which $\exp(s,t) = a_1(s)a_2(t)$ the Jacobian is given by a multiple of $(s,t) \mapsto \sin(2t)$. In view of (5) this implies the claim. ∎

2.3. Spherical functions. Let π^L_μ be an irreducible representation that induces multiplicity free to U, i.e. $\mu \in P^+_L$ is such that for all $\lambda \in P^+_U$ we have $\dim \mathrm{Hom}_L(V^L_\mu, V^U_\lambda) \leq 1$. In this case we call (U,L,μ) a multiplicity free triple.

DEFINITION 2.6. Let F be a face of the dominant Weyl chamber of L such that for all elements $\mu \in P^+_L \cap F$ the representation π^L_μ induces multiplicity free to U. We call the triple (U,L,F) a *multiplicity free system.*

Since (U,L) is a Gelfand pair, $(U,L,\{0\})$ is a multiplicity free system. In Section 3 we investigate the induction of π^L_μ to U in greater detail and we shall see that there are more examples of multiplicity free systems other than only $(U,L,\{0\})$. In this section we investigate the spherical functions not yet requiring this induction. Throughout the rest of this section we fix $\mu \in P^+_L$ such that (U,L,μ) is a multiplicity free triple.

DEFINITION 2.7. The set $P^+_{(U,L)}(\mu) = \{\lambda \in P^+_U \,|\, [\pi^U_\lambda|_L : \pi^L_\mu] = 1\}$ is called the *μ-well.*

Consider the $L\times L$-action on $\mathrm{End}(V^L_\mu)$ via $((\ell_1,\ell_2),x) \mapsto \pi^L_\mu(\ell_1)\circ x \circ \pi^L_\mu(\ell_2)^{-1}$. The inner product $(x,y) \mapsto \mathrm{tr}(x^*y)$ on $\mathrm{End}(V^L_\mu)$, where x^* denotes the Hermitian adjoint with respect to the invariant Hermitian inner product on V^L_μ, is $L\times L$-invariant. Together with the left-right-regular representation of $L\times L$ in $L^2(U,du)$ we consider the tensor product $L^2(U,du)\otimes \mathrm{End}(V^L_\mu)$ as an $L\times L$ representation which is unitary for the Hermitian inner product given by

$$\langle \Phi, \Psi\rangle_\mu = \int_U \mathrm{tr}(\Phi(u)^*\Psi(u))\,du. \tag{6}$$

The space of $L\times L$-invariants decomposes as the Hilbert completion

$$(L^2(U,du)\otimes \mathrm{End}(V^L_\mu))^{L\times L} = \widehat{\bigoplus_{\lambda\in P^+_{(U,L)}(\mu)}} \mathrm{Hom}_L(V^L_\mu,V^U_\lambda)\otimes \mathrm{Hom}_L(V^U_\lambda,V^L_\mu). \tag{7}$$

The spaces $\mathrm{Hom}_L(V^L_\mu,V^U_\lambda)\otimes \mathrm{Hom}_L(V^U_\lambda,V^L_\mu)$ are one-dimensional by assumption. Let $j:V^L_\mu\to V^U_\lambda$ be an isometric L-invariant linear map and let $p:V^U_\lambda\to V^L_\mu$ denote its adjoint. Then

$$\Phi^\mu_\lambda : U\to \mathrm{End}(V^L_\mu) : u\mapsto p\circ \pi^U_\lambda(u)\circ j$$

is called a spherical function of type μ associated to λ. In fact, $\Phi^\mu_\lambda\in \mathrm{Hom}_L(V^L_\mu,V^U_\lambda)\otimes \mathrm{Hom}_L(V^U_\lambda,V^L_\mu)$ in the decomposition (7). Schur orthogonality implies

$$\langle \Phi^\mu_\lambda,\Phi^\mu_{\lambda'}\rangle_\mu = \frac{\dim(V^L_\mu)^2}{\dim(V^U_\lambda)}\,\delta_{\lambda,\lambda'}, \tag{8}$$

see e.g. [14, Proposition 3.3.25]. According to (7) the set $\{\Phi^\mu_\lambda \,|\, \lambda\in P^+_{(U,L)}(\mu)\}$ is an orthogonal basis of $(L^2(U,du)\otimes \mathrm{End}(V^L_\mu))^{L\times L}$.

Note that $\Psi\in (L^2(U,du)\otimes \mathrm{End}(V^L_\mu))^{L\times L}$ satisfies

$$\Psi(\ell_1 u\ell_2) = \pi^L_\mu(\ell_1)\Psi(u)\pi^L_\mu(\ell_2)\quad \text{for all } (\ell_1,\ell_2,u)\in L\times L\times U.$$

Let $\Psi\in (L^2(U,du)\otimes \mathrm{End}(V^L_\mu))^{L\times L}$ be continuous. It follows from the decomposition $U=LAL$ and the transformation behavior of Ψ the values of Ψ on U are determined by the restriction $\Psi|_A$ and the matrices $\pi^L_\mu(\ell)$ for $\ell\in L$.

Let $L_*\cong \mathrm{SU}(n-1)$ be the isotropy group of $e_1\in S^{2n+1}$ for the action of L on S^{2n+1}. Then L_* commutes with A. We see that $\Psi(A)\subset \mathrm{End}_{L_*}(V^L_\mu)$. We want to show that the matrices $\Psi(a)$, $a\in A$, can be diagonalized simultaneously. To see this, let $v\in S^{2n+1}\setminus\{e_{n+1}\}$ and note that the stabilizer L_v is conjugated to L_*. This implies that the complexification of L_* is equal to the generic stabilizer of the action of the complexification $L_{\mathbb{C}}$ of L on the annihilator $\mathfrak{l}^\perp_{\mathbb{C}}=\{f\in \mathfrak{u}^*_{\mathbb{C}}\,|\,\forall X\in \mathfrak{l}_{\mathbb{C}} : f(X)=0\}$ of $\mathfrak{l}_{\mathbb{C}}$ in $\mathfrak{u}^*_{\mathbb{C}}$. The restriction $\pi^L_\mu|_{L_*}$ decomposes multiplicity free, see e.g. [15, Proposition 2.4], and hence $\Psi(a)$, $a\in A$, is diagonal with respect to a basis that consists of basis vectors of the L_*-isotypical constituents of V^L_μ.

For later reference we record the following remark.

Remark 2.8. The integrand in (6) is $L\times L$-invariant. This implies that the inner product (6) can be given using integration over A,

$$\langle\Phi,\Psi\rangle_\mu = c\int_A \mathrm{tr}(\Phi(a)^*\Psi(a))|\delta_n(a)|\,da,$$

where we have used Proposition 2.5.

3. Multiplicity free induction. Let $i\in\{1,\ldots,n-1\}$ and let $\mu\in P^+_L$ be of the form

$$\mu=\sum_{j=1}^{n-1}\mu_j\omega_j,\quad \text{with}\quad \mu_i=0. \tag{9}$$

We want to describe the irreducible U-representations that contain π^L_μ upon restriction to L. Instead of the classical branching rules adapted to our situation we apply the theory of spherical varieties, in particular the extended weight semigroup (see e.g. [2]). To explain this object we consider the complexifications $G = \mathrm{SL}(n+1,\mathbb{C})$ and $H = \mathrm{SL}(n,\mathbb{C})$ of the compact groups U and L respectively. The roots and choices for positivity are essentially the same for U and G and for L and H. Using Weyl's unitary trick we employ the identifications $P_U^+ = P_G^+$ and $P_L^+ = P_H^+$. In the same spirit we define and identify $P^+_{(G,H)}(\mu) = P^+_{(U,L)}(\mu)$ for all $\mu \in P_H^+$.

Let $P_{\alpha_i} \subset H$ be the standard parabolic subgroup of H associated to the positive root α_i. This means that the semisimple part of P_{α_i} is isomorphic to $\mathrm{SL}(2,\mathbb{C})$ with positive root α_i. The character $\mu : T_L \to \mathbb{C}^\times$ can be extended to a character $\mu : P_{\alpha_i} \to \mathbb{C}^\times$. Let $H \times^{P_{\alpha_i}} \mathbb{C}$ denote the associated line bundle over H/P_{α_i}. The representation π^H_μ can be realized in the space of sections of the vector bundle $H \times^{P_{\alpha_i}} \mathbb{C}$ over H/P_{α_i}.

It is known (see [15, Theorem 2.1]) that $P_{\alpha_i} \subset G$ is a spherical subgroup, i.e. G contains a Borel subgroup that admits an open orbit in the quotient G/P_{α_i}. This is equivalent to the fact that the spaces of sections of all G-equivariant line bundles over G/P_{α_i} decompose multiplicity free into direct sums of irreducible G-representations, see [21, Theorem 25.1]. Let $X^*(P_{\alpha_i})$ denote the group of characters of P_{α_i}.

Definition 3.1. The extended weight semigroup $\widehat{\Lambda}(G, P_{\alpha_i}) \subset P_U^+ \times X^*(P_{\alpha_i})$ consists of the pairs (λ, χ) such that V_λ^G contains a line on which P_{α_i} acts with character $-\chi$.

It turns out that $\widehat{\Lambda}(G, P_{\alpha_i})$ is a finitely generated free semigroup, see [2, Theorem 2]. Using induction in stages and Frobenius reciprocity one shows: $\lambda \in P^+_{(G,H)}(\mu)$ if and only if $(\lambda^*, \mu) \in \widehat{\Lambda}(G, P_{\alpha_i})$, where λ^* is the highest weight of the representation dual to V_λ^G. To formulate the following lemma we set $\omega_n = \omega_0 = 0 \in P_L^+$.

Lemma 3.2. *Let $i \in \{1, \ldots, n-1\}$. The extended weight semigroup $\widehat{\Lambda}(G, P_{\alpha_i})$ is freely generated by the $2n-2$ elements*

$$(\varpi_j^*, \omega_j) \quad \text{with } 1 \le j \le n \text{ and } j \ne i, \tag{10}$$

$$(\varpi_j^*, \omega_{j-1}) \quad \text{with } 1 \le j \le n \text{ and } j \ne i+1. \tag{11}$$

Proof. Let $\mathrm{GL}(n,\mathbb{C}) \subset G$ denote the image of the map $x \mapsto \mathrm{diag}(x, \det x^{-1})$. The standard Borel subgroup $B_{\mathrm{GL}(n,\mathbb{C})} \subset \mathrm{GL}(n,\mathbb{C})$ consisting of upper-triangular matrices remains spherical in G. This makes $B_{\mathrm{GL}(n,\mathbb{C})} \subset \mathrm{GL}(n,\mathbb{C})$ a strongly solvable spherical subgroup and for these there exists a general procedure to calculate the extended weight semigroups, see [1]. Alternatively, the extended weight semigroup $\widehat{\Lambda}(G, B_{\mathrm{GL}(n,\mathbb{C})})$ has been calculated in [17, Lemma 2.2]. Its generators are

$$(\varpi_j^*, \varpi_j), \quad (\varpi_j^*, \varpi_{j-1} - \varpi_n), \quad j = 1, \ldots, n. \tag{12}$$

An element of $\widehat{\Lambda}(G, P_{\alpha_i})$ can be viewed as an element of $\widehat{\Lambda}(G, B_{\mathrm{GL}(n,\mathbb{C})})$ by extending the second component to the center of $\mathrm{GL}(n,\mathbb{C})$ trivially. As such it can be written as a linear combination of the elements (12) where the coefficients are in $\mathbb{N}_0$. Certainly the coefficients of (ϖ_i^*, ϖ_i) and $(\varpi_{i+1}^*, \varpi_i - \varpi_n)$ are zero. This shows that the elements (10), (11) generate $\widehat{\Lambda}(G, P_{\alpha_i})$. Since these elements are also linearly independent, the result follows. ∎

Write $\lambda = \sum_{j=1}^{n} \lambda_j \varpi_j$. Then $(\lambda^*, \mu) \in \widehat{\Lambda}(G, P_{\alpi_i})$ if and only if there are $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ with

- $r_j + s_j = \lambda_j$ for $j = 1, \ldots, n$,
- $r_j + s_{j+1} = \mu_j$ for $j = 1, \ldots, n-1$,

i.e. $(\lambda^*, \mu) = \sum_{k=1}^{n} \big(r_k(\varpi_k^*, \omega_k) + s_k(\varpi_k^*, \omega_{k-1}) \big)$. Note that $r_i = 0, s_{i+1} = 0$ automatically. We say that the pair $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$ determines the pair (λ^*, μ) and vice versa.

If $\lambda \in P^+_{(G,H)}(\mu)$ so that (λ^*, μ) is determined by the pair $(r, s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$, then $\lambda - r'_n \varpi_n - s'_1 \varpi_1 \in P^+_{(G,H)}(\mu)$ for $(r'_n, s'_1) \in \mathbb{Z}^2$ with $r'_n \le r_n$ and $s'_1 \le s_1$. This inspires the following definition:

$$B(\mu) := \{\lambda \in P^+_{(G,H)}(\mu) \,|\, r_n = s_1 = 0\},$$

called the *bottom of the μ-well*. Moreover, we define the μ-degree $d_\mu(\lambda) := (r_n, s_1)$ and the total μ-degree $|d_\mu(\lambda)| = r_n + s_1$. We put $\nu_\lambda = \lambda - r_n \varpi_n - s_1 \varpi_1 \in B(\mu)$.

THEOREM 3.3. *Let $i \in \{1, \ldots, n-1\}$ and let $\mu \in P_H^+$ satisfy* (9). *Then*

$$P^+_{(G,H)}(\mu) = B_{(G,H)}(\mu) + P^+_{(G,H)}(0).$$

Proof. Immediate from the preceding discussion by writing $\lambda = \lambda - r_n \varpi_n - s_1 \varpi_1 + (r_n \varpi_n + s_1 \varpi_1)$. ■

Let H_* denote the complexification of L_*. Let $T_{H_*} \subset H_*$ denote the maximal torus of H_* consisting of diagonal elements. Then $A_{\mathbb{C}} T_{H_*} \subset G$ is a maximal torus of G, where $A_{\mathbb{C}}$ is the complexification of A. Fix a Weyl chamber for this torus that projects to the positive Weyl chamber of H_* determined by the upper triangular matrices. Let $\widetilde{\lambda}$ be a dominant integral weight and suppose that the corresponding irreducible representation $\pi^G_{\widetilde{\lambda}}$ contains π^H_μ upon restriction to H. The restriction $\widetilde{\lambda}_*$ of $\widetilde{\lambda}$ to T_{H_*} is also dominant and integral. The corresponding irreducible representation $\pi^{H_*}_{\widetilde{\lambda}_*}$ of H_* occurs with multiplicity one in the restriction $\pi^H_\mu|_{H_*}$ by [15, Theorem 3.1]. This sets up a bijection

$$B(\mu) \to \{\sigma \in P^+_{H_*} \mid \dim \mathrm{Hom}_{H_*}(V^{H_*}_\sigma, V^H_\mu) \ge 1\}.$$

The torus $A_{\mathbb{C}} T_{H_*}$ is conjugated to T_G. The projection in $\mathfrak{a} \oplus \mathfrak{t}_{H_*} \to \mathfrak{t}_{H_*}$ along $\mathfrak{a}_{\mathbb{C}}$ corresponds to the projection $\mathfrak{t}_G \to \mathfrak{t}_{H_*}$ along ϖ_1 and ϖ_n.

DEFINITION 3.4. The elements of $B(\mu)$ can be regarded as the highest weights of the irreducible constituents of $\pi^H_\mu|_{H_*}$. As such, the element $\nu \in B(\mu)$ is denoted by ν_*.

PROPOSITION 3.5. *Let $i \in \{1, \ldots, n-1\}$ and let $\mu \in P_H^+$ satisfy* (9). *Let $\lambda, \lambda' \in P^+_{(G,H)}(\mu)$. If* $\dim \mathrm{Hom}_G(V^G_{\lambda'}, V^G_\lambda \otimes V^G_{\varpi_j}) \ge 1$ *for $j = 1$ or $j = n$, then $|d_\mu(\lambda')| \le |d_\mu(\lambda + \varpi_j)|$.*

Proof. The weights that occur in the tensor product $V^G_\lambda \otimes V^G_{\varpi_j}$ are of the form $\lambda' = \lambda + \varpi_j - \sum_{k=1}^{n} c_k \alpha_k$ for some $c_k \in \mathbb{N}_0$. We claim that $|d_\mu(\lambda')| \le |d_\mu(\lambda + \varpi_j)|$ whenever $\lambda' \in P^+_{(G,H)}(\mu)$. If $\lambda' \in P^+_{(G,H)}(\mu)$, then $((\lambda')^*, \mu) \in \widehat{\Lambda}(G, P_{\alpha_i})$. Note that

- $(\alpha_1, 0) = (\varpi_1, \varpi_1) + (\varpi_1, 0) - (\varpi_2, \varpi_1)$,
- $(\alpha_k, 0) = -(\varpi_{k-1}, \varpi_{k-1}) + (\varpi_k, \varpi_k) + (\varpi_k, \varpi_{k-1}) - (\varpi_{k+1}, \varpi_k)$, for $k = 2, \ldots, n-1$,
- $(\alpha_n, 0) = -(\varpi_{n-1}, \varpi_{n-1}) + (\varpi_n, \varpi_{n-1}) + (\varpi_n, 0)$.

Since $((\lambda')^*,\mu) = ((\lambda+\varpi_j)^*,\mu) - \sum_{k=1}^n c_k(\alpha_k^*,0)$, we have $|d_\mu(\lambda')| = |d_\mu(\lambda+\varpi_j)| - c_1 - c_n$ and the claim follows. The claim holds in particular when λ' is moreover the weight of a highest weight vector in $V_\lambda^G \otimes V_{\varpi_j}^G$. ∎

COROLLARY 3.6. *Let $i \in \{1,\dots,n-1\}$, let $\mu \in P_H^+$ satisfy* (9) *and let $\lambda \in P_{(G,H)}^+(\mu)$ be associated to $(r,s) \in \mathbb{N}_0^n \times \mathbb{N}_0^n$. Then there exist polynomials $q_{\lambda,\nu}^\mu \in \mathbb{C}[\phi_{\varpi_1},\phi_{\varpi_n}]$, $\nu \in B(\mu)$, such that*

$$\Phi_\lambda^\mu(g) = \sum_{\nu \in B(\mu)} q_{\lambda,\nu}^\mu\big(\phi_{\varpi_1}(g),\phi_{\varpi_n}(g)\big)\Phi_\nu^\mu(g). \tag{13}$$

The polynomials are uniquely determined by Φ_λ^μ. Moreover, the total degree of $q_{\lambda,\nu_\lambda}^\mu$ is $r_n + s_1$. The total degrees of all other polynomial coefficients in (13) *are not greater than $r_n + s_1$. If the degree (r_n', s_1') of a polynomial coefficient $q_{\lambda,\nu}^\mu$ in* (13) *satisfies $r_n + s_1 = r_n' + s_1'$ then $\nu + r_n\varpi_n + s_1\varpi_1 \le \lambda$ in the usual ordering.*

Proof. The proof is with induction on $r_n + s_1$, similar to the proof of Proposition 2.2. In view of Theorem 3.3 and Proposition 3.5 we only have to show that the coefficient c_i in $\phi_{\varpi_i}\Phi_\lambda^\mu = c_i\Phi_{\lambda+\varpi_i}^\mu + \text{l.o.t.}$ is non-zero for $i = 1, n$. This follows from surjectivity of the Cartan projection $V_\lambda^U \otimes V_{\varpi_i}^U \to V_{\lambda+\varpi_i}^U$. ∎

4. Vector-valued orthogonal polynomials. Let $i \in \{1,\dots,n-1\}$ and let $\mu \in P_H^+$ satisfy (9). Fix a total ordering on $B(\mu) = \{\nu_1,\dots,\nu_N\}$ that is compatible with the usual partial ordering on P_G^+. The space of L_*-equivariant endomorphisms $\mathrm{End}_{L_*}(V_\mu^L)$ consists of block matrices, the blocks being multiples of the identity that correspond to the L_*-intertwiner of the L_*-isotypical type $\nu_{i,*}$. These multiples are collected in a vector in $\mathbb{C}^N$. The linear map $i : \mathrm{End}_{L_*}(V_\mu^L) \to \mathbb{C}^N$ that we obtain in this way is unitary for the Hermitian inner products $(x,y) \mapsto \mathrm{tr}(x^*y)$ and $(z,\zeta) \mapsto z^*Y^\mu\zeta$, where Y^μ is the diagonal matrix whose entries are the dimensions of $V_{\nu_{i,*}}^{L_*}$. We identify the space of invariant functions (7) restricted to A with $\mathbb{C}^N$-valued functions on A. In particular we put

$$\Psi_\lambda^\mu(a) = i(\Phi_\lambda^\mu(a)) \quad \text{for } a \in A.$$

DEFINITION 4.1.

(1) Let $\Psi_0^\mu : A \to \mathrm{Mat}(\mathbb{C}, N\times N)$ be the matrix-valued function on A whose i-th column in $a \in A$ is given by $\Psi_{\nu_i}^\mu(a)$.

(2) Let $\widetilde{W}_{\mathrm{pol}}^\mu : A \to \mathrm{Mat}(\mathbb{C}, N\times N)$ be the matrix-valued function on A defined by $\widetilde{W}_{\mathrm{pol}}^\mu(a) := \Psi_0^\mu(a)^*Y^\mu\Psi_0^\mu(a)$.

LEMMA 4.2. *The function $\widetilde{W}_{\mathrm{pol}}^\mu$ is positive definite almost everywhere on A.*

Proof. Since Y^μ has real entries, $\widetilde{W}_{\mathrm{pol}}^\mu$ is certainly self-adjoint. By [15, Lemma 6.1], Ψ_0^μ is invertible on a dense open subset of A, and the result follows. ∎

We define $Q_\lambda^\mu \in \mathbb{C}^N[x,y]$ by $\Psi_0^\mu Q_\lambda^\mu(\phi_{\varpi_1},\phi_{\varpi_n}) = \Psi_\lambda^\mu$, which makes sense because Ψ_0^μ is invertible almost everywhere.

LEMMA 4.3. *The entries of the function $\widetilde{W}_{\mathrm{pol}}^\mu$ are polynomials in $\phi_{\varpi_1},\phi_{\varpi_n}$.*

Proof. The (i,j)-entry of $\widetilde{W}^{\mu}_{\mathrm{pol}}(a)$ is given by $\mathrm{tr}(\Phi^{\mu}_{\nu_i}(a)^*\Phi^{\mu}_{\nu_j}(a))$, which is the restriction of an $L\times L$ invariant sum of matrix coefficients on U to A, hence it can be written as a polynomial in $\phi_{\varpi_1},\phi_{\varpi_n}$. ■

DEFINITION 4.4. Let $W^{\mu}_{\mathrm{pol}}\in \mathrm{Mat}(\mathbb{C},N\times N)[z,\overline{z}]$ be defined by $W^{\mu}_{\mathrm{pol}}(\phi_{\varpi_1}(a),\phi_{\varpi_n}(a)) = \widetilde{W}^{\mu}_{\mathrm{pol}}(a)$ and $w_n\in C(D)$ by $w_n(z,\overline{z})=(1-|z|^2)^{n-1}$.

Let $Q_1,Q_2\in\mathbb{C}^N[z,\overline{z}]$ and consider the pairing

$$(Q_1,Q_2)\mapsto \langle Q_1,Q_2\rangle_{\mu} = \int_D Q^{\mu}_1(z,\overline{z})^* W^{\mu}_{\mathrm{pol}}(z,\overline{z})Q^{\mu}_2(z,\overline{z})w_n(z,\overline{z})\,d\lambda(z).$$

This provides $\mathbb{C}^N[z,\overline{z}]$ with a Hermitian inner product. Let $\overline{\mathbb{C}^N[z,\overline{z}]}^{\mu}$ denote its completion.

THEOREM 4.5. *The map $\Phi^{\mu}_{\lambda}\mapsto Q^{\mu}_{\lambda}$ extends to an isometry*

$$\big(L^2(U,du)\otimes \mathrm{End}(V^L_{\mu})\big)^{L\times L}\to \overline{\mathbb{C}^N[x,y]}^{\mu}.$$

Proof. This is clear from the preceding discussion. ■

It follows that the space spanned by the spherical functions of type μ can be identified with a space of vector-valued orthogonal polynomials in two variables as long as μ satisfies the multiplicity freeness assumption. The spherical functions are also determined as simultaneous eigenfunctions of a commutative subquotient of $U(\mathfrak{g})$, see [12, Lemma 2.2]. This implies that the vector-valued orthogonal polynomials are also determined as simultaneous eigenfunctions of a commutative algebra of differential operators. In the case $\mu=0$ we have control over this algebra by means of the Harish-Chandra homomorphism for spherical varieties, see [10]. In the general case this algebra is not yet fully understood.

5. Examples. In this section we are concerned with explicit expressions of the functions $\Psi^{\mu}_0 : A\to \mathrm{Mat}(\mathbb{C},N\times N)$, as these functions determine the polynomial part of the matrix weight. The entries of Ψ^{μ}_0 are matrix coefficients of U-representations that are restricted to A. More precisely, the (i,j)-th entry is the matrix coefficient

$$m^{\nu_j}_{v_{\nu_{i,*}},v_{\nu_{i,*}}} : U\to\mathbb{C} : u\mapsto \big\langle v_{\nu_{i,*}},\pi^L_{\nu_j}(u)v_{\nu_{i,*}}\big\rangle_{\nu_j},$$

restricted to A, where $v_{\nu_{i,*}}\in V^L_{\mu}\subset V^U_{\nu_j}$ is a highest weight vector of L_* of weight $\nu_{i,*}$. These calculations can be done systematically by computer algebra. Indeed, in [18, §4.1] such an algorithm is provided for a different pair of groups. We have adapted this algorithm to our situation. First we show that we only have to calculate the restriction $(\Psi^{\mu}_{\nu_i})|_{A_1}$ because of the transformation behavior of $\Psi^{\mu}_{\nu_i}$ under A_2.

LEMMA 5.1. *There exist integers $m_{i,j}, 1\le i,j\le N$ such that*

$$\Psi^{\mu}_{\nu_i}(a_2(s)a) = \mathrm{diag}(e^{im_{i,1}s},\dots,e^{im_{i,N}s})\Psi^{\mu}_{\nu_i}(a)$$

for $a_2(s)\in A_2$ and $a\in A$.

Proof. Let $\lambda\in P^+_{(U,L)}(\mu)$ be associated to the pair $(r,s)\in\mathbb{N}^n_0\times\mathbb{N}^n_0$. The pair (r,s) corresponds to an element $(\lambda^*,\widetilde{\mu})\in\widehat{\Lambda}(G,B_{\mathrm{GL}(n,\mathbb{C})})$. The $\mathrm{GL}(n,\mathbb{C})$-module of highest weight $\widetilde{\mu}$ restricted to $L\subset\mathrm{GL}(n,\mathbb{C})$ is just V^L_{μ}. Let $M=L_*\cdot A_2$. The $M_{\mathbb{C}}$-subrepresentations of $\pi^{\mathrm{GL}(n,\mathbb{C})}_{\widetilde{\mu}}$ restricted to L_* are clearly A_2-stable. Hence A_2 acts with a character on the L_*-subrepresentations of V^L_μ. ∎

For the actual computations we have restricted ourselves to a couple of examples in a small class of representations of L, namely those of highest weight $k\omega_{n-1}$ with $k \in \mathbb{N}_0$. The reason is that this greatly simplifies the determination of the highest weight vectors of the irreducible L_*-representations of V^L_μ, which is step 7 of the algorithm in [19, §4.1]. Recall that $\mathfrak{g} = \{X \in \mathrm{End}(n+1,\mathbb{C}) \,|\, \mathrm{tr}X = 0\}$. Let $E_{i,j} \in \mathfrak{g}$ denote the element with zeros everywhere except for a one in the entry (i,j). We denote the fundamental weights of H_* by $\eta_1,\dots,\eta_{n-2}$, where $\eta_i = \omega_{i+1}|_{T_{H_*}}$ for $i = 1,\dots,n-2$.

Lemma 5.2. *Let $\mu = k\omega_{n-1}$. Let $v_\mu \in V^L_\mu$ be a highest weight vector. Then*

$$\pi^L_\mu|_{L_*} = \sum_{j=0}^{k} \pi^{L_*}_{j\eta_{n-2}}$$

and the highest weight vectors are given by $E^j_{n,1}v_\mu$. The dimension of the representation of L_* of highest weight $j\eta_{n-2}$ is given by $\binom{n+j-2}{j}$.

Proof. Since $E_{n,1}$ commutes with the root vectors of the positive roots of H_* the element $E^j_{n,1}v_\mu$ is a highest weight vector of H_* of weight $(k-j)\omega_{n-2}$ unless it is zero. To see that these vectors are non-zero, let $\mathfrak{s}$ be the Lie subalgebra of $\mathfrak{g}$ generated by $E_{n,1}, E_{1,n}$ and $E_{1,1} - E_{n,n} = [E_{n,1}, E_{1,n}]$. Then $\mathfrak{s} \cong \mathfrak{sl}(2,\mathbb{C})$ and v_μ is a highest weight vector of weight $\mu(E_{1,1}-E_{n,n}) = k$, so the vectors $E^j_{n,1}v_\mu$ are non-zero for $j = 0,\dots,k$. The representation space of $\pi^{H_*}_{j\eta_{n-2}}$ is isomorphic to $S^j(\mathbb{C}^{n-1})$ which is of dimension $\binom{n-2+j}{j}$. ∎

The set $B(k\omega_{n-1})$ is given by the tuples $(r,s) \in \mathbb{N}^n_0 \times \mathbb{N}^n_0$ such that $r_j + s_{j+1} = 0$ for $j = 1,\dots,n-1$, $r_{n-1} + s_n = k$ and $r_n + s_1 = 0$. We obtain

$$B(k\omega_{n-1}) = \big\{\nu_j = (k-j)\varpi_{n-1} + j\varpi_n \,|\, j = 0,\dots,k\big\}.$$

To find $\Psi^{k\omega_{n-1}}_0(a)$ with $a \in A$ we invoke Lemma 5.1 and [17, Proposition 2.3]. Let $(\nu_j, k\omega_{n-1})$ be determined by $(r,s) \in \mathbb{N}^n_0 \times \mathbb{N}^n_0$. This pair determines an element $(\nu_j, \widetilde{\mu}^j) \in \widehat{\Lambda}(G, B_{\mathrm{GL}(n,\mathbb{C})})$, with $\widetilde{\mu}^j = k\varpi_{n-1} - s_n\varpi_n$. The bottom of the $\widetilde{\mu}^j$-well for the pair $(\mathrm{SL}(n+1,\mathbb{C}), \mathrm{GL}(n,\mathbb{C}))$ is given by

$$\begin{aligned} B(\widetilde{\mu}^j) = \{-p\varpi_1 + (k-j-p)\varpi_{n-1} + (j+p)\varpi_n \,|\, p = -j,\dots,-1\} \\ \cup \{(k-j-p)\varpi_{n-1} + (j+2p)\varpi_n \,|\, p = 0,\dots,k-j\}, \end{aligned}$$

see e.g. [17, Proposition 2.3]. This implies that the $M_{\mathbb{C}}$-types (see Remark 2.4) are given by $\frac{j+2p}{2}(\varpi_1 - \varpi_n) + (k-j-p)\eta_{n-2}$. This observation allows us to refine Lemma 5.2.

Lemma 5.3. *Let $\mu = k\omega_{n-1}$. Then there exist $m_1,\dots,m_{k+1} \in \mathbb{Z}$ such that*

$$\begin{aligned} &\Psi^{k\omega_{n-1}}_0(a_1(t)a_2(s)) \\ &\qquad = \mathrm{diag}(e^{im_1 s},\dots,e^{im_{k+1}s})\Psi^{k\omega_{n-1}}_0(a_1(t))\,\mathrm{diag}(1, e^{i(n-1)s},\dots,e^{i(n-1)ks}). \end{aligned}$$

Proof. The considerations preceding the lemma indicate what the $M_{\mathbb{C}}$-types are and hence with which characters the torus A_2 acts on the L_*-isotypical components of V^L_μ. This allows us to calculate the integers of Lemma 5.2 and the result follows. ∎

Our algorithm [16] calculates $\Psi_0^\mu|_{A_1}$ for $\mu = k\omega_{n-1}$. For the values $(n,k) = (3,2)$, $(4,2)$ and $(5,2)$ we find

$$\Psi_0^{2\omega_{n-1}}(a_1(t)) = \begin{pmatrix} \cos^2(t) & \cos(t) & 1 \\ \cos(t) & \frac{1}{2}(1+\cos^2(t)) & \cos(t) \\ 1 & \cos(t) & \cos^2(t) \end{pmatrix}. \tag{14}$$

The diagonal matrix $Y^{2\omega_{n-1}}$ that is needed to calculate the weight matrix is given by $\mathrm{diag}(\frac{1}{2}n(n-1), n-1, 1)$, the dimensions of the irreducible L_*-representations that occur in $V^L_{k\omega_{n-1}}$. For $n = 3,4,5$ we obtain

$$W^{2\omega_{n-1}}_{\mathrm{pol}}(z,\overline{z}) = \begin{pmatrix} 1+\frac{1}{2}(n-1)(2|z|^2+n|z|^4) & \frac{1}{2}z(1+(n-1)|z|^2) & \frac{1}{2}n(n+1)z^2 \\ \frac{1}{2}\overline{z}(1+(n-1)|z|^2) & \frac{1}{4}(n-1+2(1+n^2)|z|^2+(n-1)|z|^4) & \frac{1}{2}z(n-1+|z|^2) \\ \frac{1}{2}n(n+1)\overline{z}^2 & \frac{1}{2}\overline{z}(n-1+|z|^2) & \frac{1}{2}n(n-1)+(n-1)|z|^2+|z|^4 \end{pmatrix}$$

and $w_n(z,\overline{z}) = (1-|z|^2)^{n-1}$. Using computer algebra we calculate

$$\int_D W^{2\omega_{n-1}}_{\mathrm{pol}}(z,\overline{z}) w_n(z,\overline{z})\, d\lambda(z,\overline{z}) = \begin{pmatrix} \frac{3n}{n+2} & 0 & 0 \\ 0 & \frac{3n(n+1)}{4(n+2)} & 0 \\ 0 & 0 & \frac{n^2(n+1)}{2(n+2)} \end{pmatrix} \tag{15}$$

for $n = 2,\dots,23$. The diagonal entries are equal to $(\dim V^L_{2\omega_1})^2$ times the reciprocals of the dimensions of $V^U_{\nu_0}, V^U_{\nu_1}$ and $V^U_{\nu_2}$, in agreement with (8) (use Weyl's dimension formula to obtain $\dim V^L_{2\omega_{n-1}} = \binom{n+1}{2}$, $\dim V^U_{\nu_0} = n(n+1)^2(n+2)/12$, $\dim V^U_{\nu_1} = n(n+1)(n+2)/3$ and $\dim V^U_{\nu_2} = \binom{n+2}{2}$). This supports the conjecture that $\Psi_0^{2\omega_{n-1}}$ is given by (14) for all $n \in \mathbb{N}$ with $n \geq 2$.

Remark 5.4.

(1) Note that we can actually vary $n \in (1,\infty)$ in W because for all such n the matrix W remains positive definite almost everywhere. At this point it would be interesting to bring the algebra of differential operators that have the spherical functions as simultaneous eigenfunctions into the game, see e.g. [12, §3]. Indeed, one could calculate the expressions of some of these differential operators to investigate the n-dependency. This example would then be a candidate to produce examples of multi-variable matrix valued orthogonal polynomials that allow for a shift, see [19].

(2) To obtain more examples of matrix weights the algorithm needs to be extended so that more general L-representations can be dealt with. However, the representation spaces in which we have to perform calculations soon become too large to deal with for the computer. A possible way to circumvent this issue is to use the approximated spherical functions from [17].

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