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Hodge numbers of a double octic with non-isolated singularities

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Abstract. If B is a surface in $\mathbb{P}^3$ of degree 8 which is the union of two smooth surfaces intersecting transversally then the double covering of $\mathbb{P}^3$ branched along B has a non-singular model which is a Calabi–Yau manifold. The aim of this note is to compute the Hodge numbers of this manifold.

1. Introduction. Let B be a surface of degree 8 in $\mathbb{P}^3$. Assume that B is the union of two smooth surfaces B_1 and B_2 of degrees d and e respectively intersecting transversally along a smooth curve C. Denote by $\sigma : \widetilde{\mathbb{P}}^3 \to \mathbb{P}^3$ the blow-up of $\mathbb{P}^3$ with center C and consider the double covering $\pi : X \to \widetilde{\mathbb{P}}^3$ of $\widetilde{\mathbb{P}}^3$ branched along the strict transform $\widetilde{B}$ of B.

From [5] it follows that in this situation X is a Calabi–Yau manifold and $e(X) = 8-(d^3-4d^2+6d)-(e^3-4e^2+6e)-8de$. However it is of great interest to calculate not only the Euler characteristic but also the cohomology groups or equivalently the Hodge numbers of X. For a Calabi–Yau variety only two Hodge numbers are interesting: $h^{1,1}$ and $h^{1,2}$—the others are obvious. We have moreover the following formula:

$$e(X) = 2(h^{1,1} - h^{1,2}).$$

These Hodge numbers have deep topological characterizations:

- $h^{1,1}$ is equal to the rank of the Picard group $\operatorname{Pic} X$,
- $h^{1,2}$ is equal to the number of deformations of X.

In general it is very difficult to calculate the Hodge numbers of a double solid. Some methods are known only in very special cases (see [2,6]). In [3] we gave an elementary proof of the Clemens formula for the Hodge numbers of a nodal double solid. We shall apply the method introduced there. This shows

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that it may be of use also in the case of a double octic with non-isolated singularities in the branch locus.

2. Conormal bundle of $\pi^*\widetilde{B}$ in X. Denote by E the exceptional divisor of σ and by $\widetilde{B}_i$ the strict transform of B_i. Clearly $\sigma|\widetilde{B}_i \to B_i$ is an isomorphism. Since $\widetilde{B}$ is an even element of $\mathrm{Pic}(\widetilde{\mathbb{P}}^3)$ we can define the line bundle $\mathcal{L} = \mathcal{O}_{\widetilde{\mathbb{P}}^3}(\frac{1}{2}\widetilde{B})$. The aim of this section is to study the line bundle $\pi^*(\mathcal{O}_{\widetilde{B}} \otimes \mathcal{L}^{-1})$ which is dual to the normal bundle of $\pi^*\widetilde{B}$ in X.

From the definition of $\mathcal{L}$ we have $\mathcal{L}^{-1} = \sigma^*\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}(E)$ and so $H^i(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{B}}) \cong H^i(\mathcal{O}_{\mathbb{P}^3}(e-4) \otimes \mathcal{O}_{B_1}) \oplus H^i(\mathcal{O}_{\mathbb{P}^3}(d-4) \otimes \mathcal{O}_{B_2})$. Using the last formula we easily get

Lemma 2.1.

$$H^0(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{B}}) \cong \begin{cases} \mathbb{C}^{10} & \textit{if } d=1,\ e=7,\\ \mathbb{C}^{9} & \textit{if } d=2,\ e=6,\\ \mathbb{C}^{4} & \textit{if } d=3,\ e=5,\\ \mathbb{C}^{2} & \textit{if } d=4,\ e=4,\end{cases}$$

$$H^1(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{B}}) = 0,$$

$$H^2(\mathcal{L}^{-1} \otimes \mathcal{O}_{\widetilde{B}}) \cong \begin{cases} \mathbb{C}^{84} & \textit{if } d=1,\ e=7,\\ \mathbb{C}^{35} & \textit{if } d=2,\ e=6,\\ \mathbb{C}^{10} & \textit{if } d=3,\ e=5,\\ \mathbb{C}^{2} & \textit{if } d=4,\ e=4.\end{cases}$$

3. Cohomology of $\pi^*\Omega^1_{\widetilde{\mathbb{P}}^3}$

Lemma 3.1.

$$H^i(\Omega^1_{\widetilde{\mathbb{P}}^3}) \cong \begin{cases} 0 & \textit{if } i=0,3,\\ \mathbb{C}^{2} & \textit{if } i=1,\\ \mathbb{C}^{g} & \textit{if } i=2,\end{cases}$$

where the genus g of C is $2de+1$.

Proof. Consider the following long exact sequence:

$$0 \to \sigma^*\Omega^1_{\mathbb{P}^3} \to \Omega^1_{\widetilde{\mathbb{P}}^3} \to \Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3} \to 0. \tag{1}$$

Following [7, Thm. II.8.24] we can identify E with the projectivization $\mathbb{P}(\mathcal{N}^\vee_{C|\mathbb{P}^3})$ of the conormal bundle $\mathcal{N}^\vee_{C|\mathbb{P}^3}$ of C in $\mathbb{P}^3$. Since in this situation $\Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3} \cong \Omega^1_{E/C}$ and (by [7, Ex. III.8.4]) $\Omega^1_{E/C} \cong \sigma^*(\bigwedge^2 \mathcal{N}^\vee_{C|\mathbb{P}^3}) \otimes \mathcal{O}_E(-2)$, using the projection formula and again [7, Ex. III.8.4] we get

$$\sigma_*\Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3} \cong \sigma_*\Omega^1_{E/C} \cong (\textstyle\bigwedge^2 \mathcal{N}^\vee_{C|\mathbb{P}^3}) \otimes \sigma_*\mathcal{O}_E(-2) = 0$$

and

$$\begin{aligned} R^1\sigma_*\Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3} &\cong (\textstyle\bigwedge^2 \mathcal{N}^\vee_{C|\mathbb{P}^3}) \otimes R^1\sigma_*\mathcal{O}_E(-2) \\ &\cong (\textstyle\bigwedge^2 \mathcal{N}^\vee_{C|\mathbb{P}^3}) \otimes (\sigma_*\mathcal{O}_E)^\vee \otimes (\textstyle\bigwedge^2 \mathcal{N}^\vee_{C|\mathbb{P}^3})^\vee \\ &\cong (\sigma_*\mathcal{O}_E)^\vee \cong \mathcal{O}_C. \end{aligned}$$

The direct image functor applied to the short exact sequence (1) yields

$$\sigma_*\Omega^1_{\widetilde{\mathbb{P}}^3} \cong \Omega^1_{\mathbb{P}^3} \quad \text{and} \quad R^1\sigma_*\Omega^1_{\widetilde{\mathbb{P}}^3} \cong \mathcal{O}_C.$$

The Leray spectral sequence $H^p(R^q(\sigma_*\Omega^1_{\widetilde{\mathbb{P}}^3}))$ has the following terms:

$$\begin{array}{c|cccc} q & & & & \\ & \mathbb{C} & \mathbb{C}^g & 0 & 0 \\ & 0 & \mathbb{C} & 0 & 0 \\ \hline & & & & p \end{array}$$

where $g = 2de + 1$ is the genus of C. The above sequence degenerates and the lemma follows. ∎

LEMMA 3.2.

$$H^0(\Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{L}^{-1}) = 0,$$

$$H^1(\Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{L}^{-1}) \cong \begin{cases} \mathbb{C}^{10} & \textit{if } d = 1,\ e = 7, \\ \mathbb{C}^9 & \textit{if } d = 2,\ e = 6, \\ \mathbb{C}^4 & \textit{if } d = 3,\ e = 5, \\ \mathbb{C}^2 & \textit{if } d = 4,\ e = 4. \end{cases}$$

P r o o f. Tensoring the exact sequence (1) with $\mathcal{L}^{-1}$ we get

$$0 \to \sigma^*\Omega^1_{\mathbb{P}^3} \otimes \mathcal{L}^{-1} \to \Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{L}^{-1} \to \Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3} \otimes \mathcal{L}^{-1} \to 0. \tag{2}$$

In this situation

$$(\sigma^*\Omega^1_{\mathbb{P}^3}) \otimes \mathcal{L}^{-1} \cong (\sigma^*\Omega^1_{\mathbb{P}^3}(-4)) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}(E)$$

and

$$\begin{aligned} \Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3} \otimes \mathcal{L}^{-1} &\cong \Omega^1_{E/C} \otimes \sigma^*\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \\ &\cong \sigma^*(\textstyle\bigwedge^2 \mathcal{N}^\vee_{C|\mathbb{P}^3}) \otimes \mathcal{O}_E(-2) \otimes \sigma^*\mathcal{O}_{\mathbb{P}^3}(-4) \otimes \mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \otimes \mathcal{O}_E \\ &\cong \sigma^*(\textstyle\bigwedge^2 \mathcal{N}^\vee_{C|\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(-4)) \otimes \mathcal{O}_E(-3) \end{aligned}$$

because $\mathcal{O}_{\widetilde{\mathbb{P}}^3}(E) \otimes \mathcal{O}_E \cong \mathcal{N}_{E|\widetilde{\mathbb{P}}^3} \cong \mathcal{O}_E(-1)$ by [7, Thm. II.8.24].

By the projection formula,

$$\sigma_*((\sigma^*\Omega^1_{\mathbb{P}^3})\otimes\mathcal{L}^{-1})\cong\Omega^1_{\mathbb{P}^3}(-4)\otimes\sigma_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(E)\cong\Omega^1_{\mathbb{P}^3}(-4),$$
$$R^1\sigma_*((\sigma^*\Omega^1_{\mathbb{P}^3})\otimes\mathcal{L}^{-1})\cong\Omega^1_{\mathbb{P}^3}(-4)\otimes R^1\sigma_*\mathcal{O}_{\widetilde{\mathbb{P}}^3}(E)=0.$$

Using again [7, Ex. III.8.4] and the projection formula we obtain

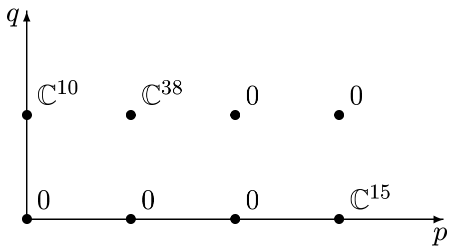
$$\sigma_*((\Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3})\otimes\mathcal{L}^{-1})\cong\bigwedge^2\mathcal{N}^\vee_{C|\mathbb{P}^3}\otimes\mathcal{O}_{\mathbb{P}^3}(-4)\otimes\sigma_*\mathcal{O}_E(-3)=0,$$

$$\begin{aligned}R^1\sigma_*((\Omega^1_{\widetilde{\mathbb{P}}^3/\mathbb{P}^3})\otimes\mathcal{L}^{-1})&\\ &\cong\bigwedge^2\mathcal{N}^\vee_{C|\mathbb{P}^3}\otimes\mathcal{O}_{\mathbb{P}^3}(-4)\otimes R^1\sigma_*\mathcal{O}_E(-3)\\ &\cong\bigwedge^2\mathcal{N}^\vee_{C|\mathbb{P}^3}\otimes\mathcal{O}_{\mathbb{P}^3}(-4)\otimes(\sigma_*\mathcal{O}_E(1))^\vee\otimes(\bigwedge^2\mathcal{N}^\vee_{C|\mathbb{P}^3})^\vee\\ &\cong\mathcal{O}_{\mathbb{P}^3}(-4)\otimes(\mathcal{O}_{\mathbb{P}^3}(d)\oplus\mathcal{O}_{\mathbb{P}^3}(e))\otimes\mathcal{O}_C\\ &\cong(\mathcal{O}_{\mathbb{P}^3}(d-4)\oplus\mathcal{O}_{\mathbb{P}^3}(e-4))\otimes\mathcal{O}_C.\end{aligned}$$

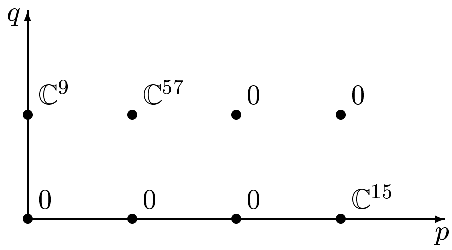
The exact sequence (2) yields therefore

$$\sigma_*(\Omega^1_{\widetilde{\mathbb{P}}^3}\otimes\mathcal{L}^{-1})\cong\Omega^1_{\mathbb{P}^3}(-4),$$
$$R^1\sigma_*(\Omega^1_{\widetilde{\mathbb{P}}^3}\otimes\mathcal{L}^{-1})\cong(\mathcal{O}_{\mathbb{P}^3}(d-4)\oplus\mathcal{O}_{\mathbb{P}^3}(e-4))\otimes\mathcal{O}_C.$$

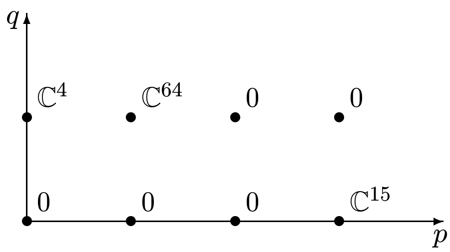
Calculating cohomologies of the right-hand sides of the above equations we can write the Leray spectral sequence:



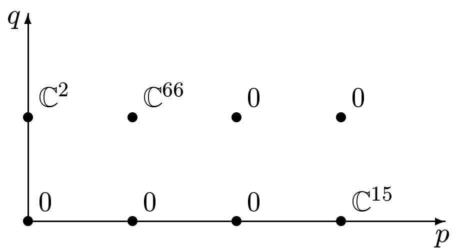
$(d=1,\ e=7)$



$(d=2,\ e=6)$



$(d=3,\ e=5)$



$(d=4,\ e=4)$

We can calculate H^0 and H^1 even if the sequence does not degenerate. This proves the lemma. ∎

We end this section with the following proposition:

PROPOSITION 3.3.

$$H^0(\pi^*\Omega^1_{\widetilde{\mathbb{P}}^3}) = 0,$$

$$H^1(\pi^*\Omega^1_{\widetilde{\mathbb{P}}^3}) \cong \begin{cases} \mathbb{C}^{12} & \textit{if } d=1,\ e=7, \\ \mathbb{C}^{11} & \textit{if } d=2,\ e=6, \\ \mathbb{C}^{6} & \textit{if } d=3,\ e=5, \\ \mathbb{C}^{4} & \textit{if } d=4,\ e=4. \end{cases}$$

Proof. Since π is a double covering,

$$\pi_*\mathcal{O}_X \cong \mathcal{O}_{\widetilde{\mathbb{P}}^3} \oplus \mathcal{L}^{-1} \quad \text{and} \quad H^i(\pi^*\Omega^1_{\widetilde{\mathbb{P}}^3}) \cong H^i(\pi_*(\pi^*\Omega^1_{\widetilde{\mathbb{P}}^3})).$$

By the projection formula $\pi_*\pi^*\Omega^1_{\widetilde{\mathbb{P}}^3} \cong \Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \pi_*\mathcal{O}_X \cong \Omega^1_{\widetilde{\mathbb{P}}^3} \oplus \Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{L}^{-1}$ and consequently

$$H^i(\pi^*\Omega^1_{\widetilde{\mathbb{P}}^3}) \cong H^i(\Omega^1_{\widetilde{\mathbb{P}}^3}) \oplus H^i(\Omega^1_{\widetilde{\mathbb{P}}^3} \otimes \mathcal{L}^{-1}).$$

The proposition now follows from Lemmas 3.1 and 3.2. ∎

4. Main result. Now we can formulate and prove our main result.

THEOREM 4.1.

$$h^{1,1}(X) = 2,$$

$$h^{1,2}(X) = \begin{cases} 122 & \textit{if } d=1,\ e=7, \\ 102 & \textit{if } d=2,\ e=6, \\ 90 & \textit{if } d=3,\ e=5, \\ 86 & \textit{if } d=4,\ e=4. \end{cases}$$

The proof of this theorem is based on the following proposition:

PROPOSITION 4.2 ([3]). *The following sequence of $\mathcal{O}_{\widetilde{X}}$-modules is exact*:

$$0 \to \pi^*\Omega^1_{\widetilde{\mathbb{P}}^3} \to \Omega^1_X \to \pi^*(\mathcal{O}_{\widetilde{B}} \otimes \mathcal{L}^{-1}) \to 0. \tag{3}$$

Proof of Theorem 4.1. By Lemma 2.1 the group $H^1(\mathcal{O}_{\widetilde{B}} \otimes \mathcal{L}^{-1})$ vanishes. Since X is a Calabi–Yau manifold, $H^0(\Omega^1(X)) = 0$. Consequently, the long exact sequence derived from the short sequence (3) splits and its first part together with Lemma 2.1 and Proposition 3.3 gives $h^{1,1} = 2$.

From the relation $e(X) = 2(h^{1,1} - h^{1,2})$ and the formula for $e(X)$ we compute $h^{1,2}(X)$. ∎

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