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Prescribing growth type of complete Riemannian manifolds of bounded geometry

by Marek Badura (Łódź)

Abstract. We describe certain properties of growth types of nondecreasing sequences. We build a complete, connected Riemannian surface of bounded geometry and of a given growth type provided that the type satisfies some natural conditions.

0. Introduction. Growth of leaves plays an important role in the study of topology and dynamics of foliations. The existence of leaves with nonexponential or polynomial growth has some influence on the structure of foliations. Constructing leaves with neither exponential nor polynomial growth can be pretty difficult (see [CC] and references there). The space of all growth types is very rich and contains many types which cannot be compared with polynomial, fractional or exponential ones. In this article, we show that any growth type ξ (not greater than the exponential one and satisfying simple conditions described in Sections 2, 3) can be realized by the volumes of balls on a suitable complete Riemannian manifold of bounded geometry.

We believe that in the near future we will be able to apply our construction to obtain leaves of a given growth type on some compact foliated manifolds.

1. Growth types. In this section we recall the notion of the growth type of nondecreasing functions and of complete, connected Riemannian manifolds (compare [HH], [E]). Let $\mathcal{I}$ be the set of nonnegative nondecreasing functions on $\mathbb{N}$:

$$\mathcal{I} = \{f : \mathbb{N} \to \mathbb{R}_+ : f(n) \leq f(n+1) \text{ for all } n \in \mathbb{N}\}.$$

Define a preorder $\preceq$ in $\mathcal{I}$. Let $f, h \in \mathcal{I}$. We say that h dominates f (and write $f \preceq h$) if for some $A \in \mathbb{R}_+$ and $B \in \mathbb{N}$,

$$f(n) \leq Ah(Bn) \quad \text{for any } n \in \mathbb{N}.$$

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The preorder $\preceq$ induces an equivalence relation $\simeq$ in $\mathcal{I}$:

$$f \simeq h \;\Leftrightarrow\; f \preceq h \preceq f.$$

The equivalence class of $f \in \mathcal{I}$ is called the *growth type* of f and is denoted by $[f]$.

We denote by $\mathcal{E}$ the set of all equivalence classes in $\mathcal{I}$. Then $\mathcal{E}$ has the partial order $\leq$ induced by the preorder $\preceq$. If $f \preceq h$, then $[f] \leq [h]$ and if $f \preceq h \preceq f$, then f and h have the same growth type and we write $[f] = [h]$. If $f \preceq h$, but $h \not\preceq f$, then we write $[f] < [h]$.

For example, we can easily see that

$$[0] < [1] = [2] < [n] < [n^2] < \ldots < [2^n] = [3^n] < [2^{2^n}].$$

For $k \geq 0$, $[n^k] \in \mathcal{E}$ is called the *polynomial growth* of degree k. For $a > 1$, $[a^n]$ is equal to $[e^n]$ and is called the *exponential growth*. We will denote the exponential growth by $[\exp]$.

Now, consider two functions $f, h \in \mathcal{I}$ defined by $f(n) = n$ and

$$h(n) = (k+2)^{k+2} \quad \text{ if } k^k \leq n < (k+4)^{k+4},\ k = 1, 5, 9, \ldots$$

We can see that the growth types $[f]$ and $[h]$ are incomparable (i.e. for all $A \in \mathbb{R}_+$, $B \in \mathbb{N}$ there exist $m, n \in \mathbb{N}$ such that $f(m) > Ah(Bm)$ and $h(n) > Af(Bn)$). Indeed, for each $A \in \mathbb{R}_+$, $B \in \mathbb{N}$ we can choose $k \in \{1, 5, 9, \ldots\}$ and $k \geq \max\{A, B\}$. Then

$$\begin{aligned} f((k+3)^{k+3}) &= (k+3)^{k+3} > k(k+2)^{k+2} = kh(k(k+3)^{k+3}) \\ &\geq Ah(B(k+3)^{k+3}) \end{aligned}$$

and

$$h(k^k) = (k+2)^{k+2} > k^2 k^k \geq ABk^k = Af(Bk^k).$$

Observation. Note that if $\xi > [0]$, then we may assume that $f(1) \geq 1$ for $f \in \xi$. Moreover in this case there exists $h \in \xi$ such that $h : \mathbb{N} \to \mathbb{N}$.

Now we show that the order defined in $\mathcal{E}$ is dense.

Lemma. *For any growth types $\xi, \eta \in \mathcal{E}$ such that $[0] < \xi < \eta$ there exists a growth type $\vartheta \in \mathcal{E}$ such that*

$$\xi < \vartheta < \eta. \tag{1.1}$$

Proof. Take $f_1 \in \xi$, $f_2 \in \eta$. We will find a function $h \in \mathcal{I}$ such that $\xi < [h] < \eta$. By assumption, for all $A \in \mathbb{R}_+$ and $B \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $f_2(n) > Af_1(Bn)$. Obviously, we can find an arbitrarily large $n \in \mathbb{N}$ satisfying this inequality. Hence, we can define a sequence $\{m_k\}_{k \in \mathbb{N}}$ of natural numbers as follows.

Put $m_1 = 1$ and choose next elements in such a way that

$$f_2(m_k) > k(k+1)f_1(km_k) \quad \text{and} \quad (k-1)m_{k-1} < m_k, \tag{1.2}$$

where $k = 2, 3, \ldots$ Now, define

$$h(n) = kf_1(km_k) + f_1(n) \quad \text{if } m_k \le n < m_{k+1} \text{ and } k \in \mathbb{N}. \tag{1.3}$$

Obviously $h \in \mathcal{I}$. By the above definition, $[f_1] \le [h]$. For each $A \in \mathbb{R}_+$, $B \in \mathbb{N}$ we can choose $k \ge \max\{A, B\}$. Then

$$Af_1(Bm_k) \le kf_1(km_k) < kf_1(km_k) + f(m_k) = h(m_k).$$

So, we have $[f_1] < [h]$.

By assumption, there exist $A \in \mathbb{R}_+, B \in \mathbb{N}$ such that $f_1(n) \le Af_2(Bn)$ for all $n \in \mathbb{N}$. Since for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $m_k \le n < m_{k+1}$, we have

$$\begin{aligned} h(n) &= kf_1(km_k) + f_1(n) \le f_2(m_k) + Af_2(Bn) \\ &\le f_2(n) + Af_2(Bn) \le (A+1)f_2(Bn). \end{aligned}$$

Hence $[h] \le [f_2]$.

Now, for arbitrary $A \in \mathbb{R}_+, B \in \mathbb{N}$, we take any nonnegative integer $k \ge \max\{A, B\}$. Note that $m_k \le Bm_k \le km_k < m_{k+1}$. From (1.2), (1.3) we now obtain

$$\begin{aligned} Ah(Bm_k) &= Akf_1(km_k) + Af_1(Bm_k) \le k^2 f_1(km_k) + kf_1(km_k) \\ &= k(k+1)f_1(km_k) < f_2(m_k). \end{aligned}$$

This shows that $[h] < [f_2]$.

Putting $\vartheta = [h]$ we see that the condition (1.1) holds. ∎

Let (M, g) be a complete, connected Riemannian manifold. Fix $x \in M$ and define $f_x : \mathbb{N} \to \mathbb{R}_+$, the *growth function* of M at x, by

$$f_x(n) = \mathrm{Vol}(B(x, n)),$$

where $B(x, n)$ is the ball centered at x of radius n on M and Vol is the measure (volume) on M induced by the Riemannian structure g (see [KN]). Obviously f_x belongs to $\mathcal{I}$. If y is another point of M, then $B(y, n) \subset B(x, n + l)$ and $B(x, n) \subset B(y, n + l)$, where $l \ge \mathrm{dist}(x, y)$ and dist is the distance function on (M, g). Therefore, if f_y is the growth function of M at y, then

$$f_x(n) \le f_y((l+1)n) \quad \text{and} \quad f_y(n) \le f_x((l+1)n) \quad \text{for all } n \in \mathbb{N}.$$

Hence $[f_x] = [f_y]$.

$[f_x]$ is called the *growth type* of (M, g) and is denoted by $\mathrm{gr}(M)$.

2. Nice growth types. We say that a growth type $\xi \in \mathcal{E}$ is *nice* if there exists a function $f \in \mathcal{I}$ and a positive integer p such that

$$f \in \xi \quad \text{and} \quad f(n+1) \le pf(n) \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

LEMMA. *If $\xi \le [\exp]$, then ξ is a nice growth type.*

P r o o f. Obviously $[0]$ is a nice growth type. Let $\xi > [0]$. Take any $h \in \xi$ such that $h(1) \geq 1$. By assumption, there are positive integers A, B such that

$$h(n) \leq A2^{Bn} \quad \text{for all } n \in \mathbb{N}.$$

Putting, for example, $c = A2^B$ we have

$$h(n) \leq c^n \quad \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Put $p = c^2$. Then for each $n \in \mathbb{N}$ there exists $m > n$ such that

$$\frac{h(m)}{h(n)} \leq p^{m-n}. \tag{2.3}$$

In fact, suppose that $n \in \mathbb{N}$ does not satisfy the above condition. Then putting $m = 2n$ we have

$$\frac{h(m)}{h(n)} > p^{m-n} = c^{2(m-n)} = c^m.$$

But this contradicts (2.2). Hence for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $n < m \leq 2n$ and inequality (2.3) holds.

Now, define

$$Z_h = \{n : h(n+1) > ph(n)\}.$$

We define a function f_1 as follows. If $Z_h = \emptyset$, then $f_1 = h$. If $Z_h \neq \emptyset$, then we put $k_1 = \min Z_h$ and $m_1 = \min\{m > k_1 : h(m) \leq p^{m-k_1} h(k_1)\}$. Finally, we define

$$f_1(n) = \begin{cases} h(n) & \text{if } 1 \leq n < k_1, \\ h(k_1)p^{n-k_1} & \text{if } k_1 \leq n < m_1, \\ h(n) & \text{if } m_1 \leq n. \end{cases}$$

We have $m_1 \leq 2k_1$ and $f_1 \in \mathcal{I}$. Moreover $f_1(n) \leq h(2n)$ and $h(n) \leq f_1(2n)$ for any $n \in \mathbb{N}$. This shows that $f_1 \in \xi$.

Next, we define a function f_2 similarly to f_1. If $Z_{f_1} = \emptyset$, then we put $f_2 = f_1$. Otherwise we put $k_2 = \min Z_{f_1}$, $m_2 = \min\{m > k_2 : f_1(m) \leq p^{m-k_2} f_1(k_2)\}$ and define

$$f_2(n) = \begin{cases} f_1(n) & \text{if } 1 \leq n < k_2, \\ f_1(k_2)p^{n-k_2} & \text{if } k_2 \leq n < m_2, \\ f_1(n) & \text{if } m_2 \leq n. \end{cases}$$

Note that $m_1 \leq k_2 < m_2 \leq 2k_2$ and $f_2 \in \xi$.

Continuing this procedure, we obtain a sequence $\{f_j\}_{j\in\mathbb{N}}$. It is easy to see that for all $j \in \mathbb{N}$ we have $f_j \in \mathcal{I}$, $f_j \in \xi$ and

$$f_j(n+1) \leq pf_j(n) \quad \text{for } n < m_j. \tag{2.4}$$

For any $n \in \mathbb{N}$ we can take $i \in \mathbb{N}$ such that

$$f_j(n) = f_i(n) \quad \text{for all } j \geq i.$$

Hence
$$f = \lim_{j\to\infty} f_j$$
is well defined and belongs to $\mathcal{I}$. Now we show that $f \in \xi$. Take any $n_0 \in \mathbb{N}$ such that $h(n_0) \neq f(n_0)$. There exists $i \in \mathbb{N}$ such that $k_i \le n_0 < m_i$ and $f(n_0) = f_i(n_0)$. By definition of $\{f_j\}_{j\in\mathbb{N}}$, for any $j \in \mathbb{N}$,
$$f_j(2n) = h(2n)$$
when $k_j \le n < m_j$. Hence
$$f(n_0) = f_i(n_0) \le f_i(2n_0) = h(2n_0)$$
and
$$h(n_0) \le h(m_i) = f_i(m_i) = f(m_i) \le f(2n_0).$$
So, $f(n) \le h(2n)$ and $h(n) \le f(2n)$ for all $n \in \mathbb{N}$. Therefore $[f] = [h]$, which implies that $f \in \xi$.

Finally we show that
$$f(n+1) \le pf(n) \quad \text{for all } n \in \mathbb{N}. \tag{2.5}$$
Fix $n \in \mathbb{N}$ and take $j \in \mathbb{N}$ satisfying $n < m_j$. Then $f(n) = f_j(n)$. Hence from (2.4), $f(n+1) = f_j(n+1)f(n+1) = f_j(n+1) \le pf_j(n) = pf(n)$. This shows (2.5). Consequently, $f \in \mathcal{I}$ and $p \in \mathbb{N}$ satisfy (2.1). ■

Observation. Note that if we take $h : \mathbb{N} \to \mathbb{N}$ in the above proof, then f will also be integer-valued.

3. Derived and primitive growth type. Let $\xi, \eta \in \mathcal{E}$. We say that η is the *derived growth type* of ξ if
$$[\Sigma f] = \xi \quad \text{for any } f \in \eta,$$
where $\Sigma f(n) = \sum_{k=1}^{n} f(k)$. We then also say that ξ is the *primitive growth type* of η.

For example,
$$[\Sigma 1] = [n], \quad [\Sigma n] = [n^2], \quad [\Sigma n^k] = [n^{k+1}], \quad [\Sigma \exp] = [\exp].$$

We show that the above definitions are correct.

Lemma. *Let $f, h, F, H \in \mathcal{I}$ and $F = \Sigma f$, $H = \Sigma h$. Then*
$$[F] = [H] \quad \text{if and only if} \quad [f] = [h].$$

Proof. If $[F] = [H]$, then there are positive integers A, B such that for all $n \in \mathbb{N}$,
$$H(n) \le AF(Bn), \tag{3.1}$$
$$F(n) \le AH(Bn). \tag{3.2}$$
Put $a = 2AB$ and $b = 2B$.

Suppose that $h(n) > af(bn)$ for some $n \in \mathbb{N}$. Then

$$h(n) > 2ABf(2Bn) \geq A \sum_{k=2Bn-2B+1}^{2Bn} f(k).$$

Since $h(2n) \geq h(2n-1) \geq \ldots \geq h(n+1) \geq h(n)$, we have

$$\sum_{k=n+1}^{2n} h(k) \geq nh(n) > nA \sum_{k=2Bn-2B+1}^{2Bn} f(k) \geq A \sum_{k=1}^{2Bn} f(k).$$

So,

$$H(2n) = \sum_{k=1}^{2n} h(k) > A \sum_{k=1}^{2Bn} f(k) = AF(2Bn).$$

But this contradicts (3.1). Consequently,

$$h(n) \leq af(bn) \quad \text{for all } n \in \mathbb{N}.$$

Analogously we can show that

$$f(n) \leq ah(bn) \quad \text{for all } n \in \mathbb{N}.$$

Hence $[f] = [h]$.

Conversely, if $[f] = [h]$, then for some $A, B \in \mathbb{N}$,

$$f(n) \leq Ah(Bn) \quad \text{and} \quad h(n) \leq Af(Bn)$$

for all $n \in \mathbb{N}$. So, for any $n \in \mathbb{N}$ we have

$$F(n) = \sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{n} Ah(Bk) \leq A \sum_{k=1}^{Bn} h(k) = AH(Bn)$$

and analogously $H(n) \leq AF(Bn)$. This shows that $[H] = [F]$. ∎

Observation. It is easy to see that $[F] < [H]$ if and only if $[f] < [h]$. Note also that $[F]$, $[H]$ are incomparable if and only if $[f]$, $[h]$ are.

4. p-pants and gluing. Let (M, g) be a smooth connected Riemannian surface with boundary ∂M and $p \in \mathbb{N} \cup \{0\}$. Assume that ∂M has $p+1$ components, i.e.

$$\partial M = \bigcup_{i=0}^{p} \partial_i M$$

and each component $\partial_i M$ has a collar neighborhood $N_i \subset M$, which is diffeomorphically isometric to $S^1(r) \times [0, \varepsilon)$, where $S^1(r)$ is the circle of radius $r > 0$ on $\mathbb{R}^2$ and $\varepsilon > 0$. Such a surface (M, g) will be called *p-pants* here.

Recall that a Riemannian surface has *bounded geometry* when its curvature is bounded and for any $r > 0$ there exists $v_0 > 0$ such that $\mathrm{Vol}(B(x,r)) > v_0$ for any point x on this surface.

Now, let (M_1, g_1), (M_2, g_2) be p_1-pants and p_2-pants, respectively, with bounded geometry and assume that $\mathrm{diam}(M_1) = d_1$, $\mathrm{diam}(M_2) = d_2$, $\mathrm{Vol}(M_1) = v_1$, $\mathrm{Vol}(M_2) = v_2$. Take the boundary components $\partial_0 M_1$, $\partial_0 M_2$. Let $\varphi : \partial_0 M_1 \to \partial_0 M_2$ be an isometry. One forms a surface

$$M = M_1 \cup_\varphi M_2$$

from the disjoint union $M_1 \cup M_2$ by identifying $x \equiv \varphi(x)$ for each $x \in \partial_0 M_1$. So, we obtain a smooth Riemannian manifold (M, g) by gluing M_1 to M_2 via φ, where g is the Riemannian structure such that $g|M_1 = g_1$ and $g|M_2 = g_2$. Moreover (M, g) has bounded geometry,

$$\mathrm{diam}(M) \le d_1 + d_2 \quad \text{and} \quad \mathrm{Vol}(M) = v_1 + v_2.$$

5. Realization of growth types. We say that a growth type $\xi \in \mathcal{E}$ is *realizable* if there exists a complete connected Riemanian manifold (M, g) with bounded geometry such that $\mathrm{gr}(M) = \xi$.

THEOREM. *If a growth type $\xi \le [\exp]$ has a derived growth type η, then ξ is realizable.*

Proof. From the assumption and Lemma in Section 2 we see that η is a nice growth type. So, there are $f \in \eta$ and $p \in \mathbb{N}$ such that

$$f : \mathbb{N} \to \mathbb{N} \quad \text{and} \quad f(n+1) \le p f(n) \quad \text{for all } n \in \mathbb{N}. \tag{5.1}$$

Let M, M_0, M_1 be p-pants, 0-pants and $(f(1) - 1)$-pants, respectively. We may assume that they have bounded geometry and there are $r, \varepsilon > 0$ such that each boundary component of these surfaces has a collar neighborhood isometric to $S^1(r) \times [0, \varepsilon)$. Moreover, suppose that

$$\mathrm{diam}(M) = 1, \quad \mathrm{diam}(M_0) \le 1, \quad \mathrm{diam}(M_1) \le 1, \tag{5.2}$$

$$\mathrm{Vol}(M) = v, \quad \mathrm{Vol}(M_0) \le v/p, \quad \mathrm{Vol}(M_1) \le v \tag{5.3}$$

and assume that for the components of ∂M, we have the inequalities

$$1/2 \le \mathrm{dist}(\partial_0 M, \partial_k M) \le 1, \quad k = 1, \ldots, p. \tag{5.4}$$

Next, we build a complete connected Riemannian surface L with bounded geometry as follows.

STEP 1. We obtain a surface L_1 by gluing $f(1)$ copies of M to M_1 via gluing isometries $\varphi_i : \partial_0 M \to \partial_k M_1$, where $k = 0, \ldots, f(1) - 1$. Note that L_1 has $pf(1)$ boundary components. Moreover from (5.3),

$$v f(1) \le \mathrm{Vol}(L_1) \le 2 v f(1).$$

Step 2. We obtain a surface L_2 by gluing $f(2)$ ($\le pf(1)$) copies of M and $pf(1) - f(2)$ copies of M_0 to L_1. This gluing is via isometries

$$\varphi_k : \partial_0 M \to \partial_k L_1, \quad k = 0, \dots, f(2) - 1,$$
$$\varphi_k : \partial M_0 \to \partial_k L_1, \quad k = f(2), \dots, pf(1).$$

L_2 has $pf(2)$ boundary components and again from (5.3),

$$v(f(1) + f(2)) \le \mathrm{Vol}(L_2) \le 2v(f(1) + f(2)).$$

Analogously we have

Step n. The surface L_{n-1} has $pf(n–1)$ boundary components. Now from (5.1) we obtain a surface L_n by gluing $f(n)$ copies of M and $pf(n-1)-f(n)$ copies of M_0 to L_{n-1}. We glue each copy of M to L_{n-1} always via an isometry of $\partial_0 M$ onto a component of ∂L_{n-1}. The surface L_n has $pf(n)$ boundary components and

$$v\sum_{k=1}^{n} f(k) \le \mathrm{Vol}(L_n) \le 2v\sum_{k=1}^{n} f(k). \tag{5.5}$$

Continuing this procedure we obtain a surface L. By the definition of p-pants, from the assumption about the surfaces M, M_0, M_1 and the above construction, we see that L is a complete, connected Riemannian surface and has bounded geometry.

Now we show that $\mathrm{gr}(L) = \xi$. Fix $x \in M_1$ and consider the growth function f_x of L at x,

$$f_x(n) = \mathrm{Vol}(B(x, n)).$$

Take any $n \in \mathbb{N}$. From (5.4) and the construction of L we have

$$\mathrm{dist}(x, y) > n \quad \text{for all } y \in L \setminus L_{2n}.$$

This shows that $B(x, n) \subset L_{2n}$. Therefore from (5.5) we obtain

$$f_x(n) = \mathrm{Vol}(B(x, n)) \le \mathrm{Vol}(L_{2n}) \le 2v\sum_{k=1}^{2n} f(k) = 2v\Sigma f(2n). \tag{5.6}$$

Obviously, $L_n \subset B(x, 2n)$. So, again from (5.4) we have

$$v\Sigma f(n) = v\sum_{k=1}^{n} f(k) \le \mathrm{Vol}(L_n) \le \mathrm{Vol}(B(x, 2n)) = f_x(2n). \tag{5.7}$$

Inequalities (5.6), (5.7) imply $[f_x] = [\Sigma f]$. Since $[\Sigma f] = \xi$, we have $[f_x] = \xi$. So, by the definitions in Section 1 we obtain $\mathrm{gr}(L) = \xi$. ■

Note that if we take the surface L obtained in the above proof and a compact n-manifold N then the product $L \times N$ is an $(n + 2)$-manifold and $\mathrm{gr}(L \times N) = \xi$.

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Department of Mathematics
Łódź University
Banacha 22
90-238 Łódź, Poland
E-mail: marekbad@math.uni.lodz.pl

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