APPLICATIONES MATHEMATICAE
27,4 (2000), pp. 419–436

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DISCRETE TIME ARBITRAGE UNDER TRANSACTION COSTS

Abstract. Conditions for the absence of arbitrage in discrete time markets with various kinds of transaction costs are shown.

1. Introduction. Let (Ω, F, F_t, P) be a complete probability space endowed with an increasing family $\{F_t : t = 0, 1, \ldots, T\}$ of sub-σ-fields of F. Assume we are given a price of asset process $(S_t)_{t=0}^T$, which is adapted to F_t, and satisfies the following equation:

$$S_{t+1} = (1 + \xi_t)S_t$$

where $(\xi_t)_{t=0}^T$ is a sequence of square integrable, F_{t+1}-measurable random variables representing a random rate of return at time t.

Furthermore, assume that there are two possible investments: in nonrisky assets (bank account with a deterministic rate of return r) and risky assets with price S_t at time t. We are interested in characterizing the absence of the so-called arbitrage opportunity which is equivalent to the possibility of a nonrisky gain.

The problem has been intensively studied in the case without transaction costs (see [1]–[2], [7], [9], [10]), where the equivalence of the absence of arbitrage and the existence of a martingale measure was shown. The case with proportional transaction costs was studied in [5], [6], [8].

In this paper conditions for the absence of arbitrage are given for four kinds of transaction costs:

- proportional costs,
- proportional + fixed costs,
- concave costs,
- concave + fixed costs.

2000 *Mathematics Subject Classification*: Primary 90A12; Secondary 93E20.
Key words and phrases: arbitrage opportunity, martingale measure.

We first study the condition for the absence of arbitrage in one step and then extend our result to the T-step horizon case.

The main result holds under the additional assumption that the random rate of return (ξ_t) is a sequence of independent random variables.

In what follows we denote by (X_t, Y_t) the amounts of money invested at time t (before a possible transaction) in the bank account and in the risky assets respectively.

2. Absence of arbitrage with proportional transaction costs. Consider first the case with proportional transaction costs. Let λ be the transaction cost rate for purchasing the asset and μ be the transaction cost rate for selling the asset.

We say that our portfolio (X_0, Y_0) at time 0 is *equal to zero* if

$$\begin{aligned} X_0 + (1+\lambda_0)Y_0 &= 0 \quad \text{whenever } Y_0 < 0, \\ X_0 + (1-\mu_0)Y_0 &= 0 \quad \text{whenever } Y_0 \geq 0. \end{aligned} \tag{2.1}$$

Similarly we can say that our portfolio (X_0, Y_0) is *nonnegative* when

$$\begin{aligned} X_0 + (1+\lambda_0)Y_0 &\geq 0 \quad \text{whenever } Y_0 < 0, \\ X_0 + (1-\mu_0)Y_0 &\geq 0 \quad \text{whenever } Y_0 \geq 0. \end{aligned}$$

The set of nonnegative portfolios (X_0, Y_0) forms a cone denoted by C_{p}, and illustrated below.

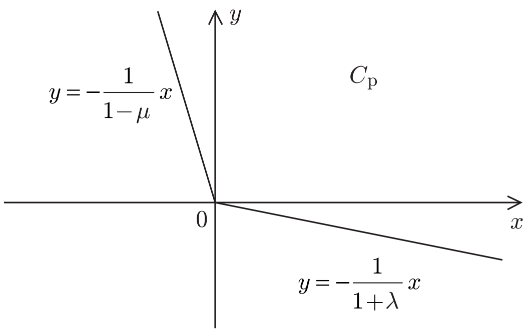

Fig. 1

Let m_t be the stock value of assets sold at time t, and l_t be the stock value of assets purchased at time t. Our portfolio (X_{t+1}, Y_{t+1}) at time $t+1$ can then be described as follows:

$$\begin{aligned} X_{t+1} &= X_t + (1-\mu)m_t - (1+\lambda)l_t, \\ Y_{t+1} &= (1+\xi_t)(Y_t - m_t + l_t). \end{aligned} \tag{2.2}$$

The sequence of nonnegative, F_t-adapted, square integrable pairs (l_t, m_t) will be called a *trading strategy*.

We say that a trading strategy (l_t, m_t) is *self-financing* if possible purchases are financed by sales. Moreover we say that a self-financing trading strategy (l_t, m_t) admits an *arbitrage opportunity* if for $(X_0, Y_0) \in \partial C_{\mathrm{p}}$ we have

$$(X_T, Y_T) \in C_{\mathrm{p}} \quad \text{and} \quad P\{(X_T, Y_T) \in \operatorname{int} C_{\mathrm{p}}\} > 0.$$

In other words we have the absence of arbitrage if for any $(X_0, Y_0) \in \partial C_{\mathrm{p}}$ and self-financing strategy (l_t, m_t) the following implication holds:

$$(X_T, Y_T) \in C_{\mathrm{p}} \Rightarrow (X_T, Y_T) \in \partial C_{\mathrm{p}} \qquad P\text{-a.s.}$$

In what follows we shall use the following spaces of square integrable random variables:

$$\begin{aligned} L_+^0(F_t) &= L^2(\Omega, F_t, K), \qquad L_+^0(F) = L^2(\Omega, F, K), \\ L_i^0(F_t) &= L^2(\Omega, F_t, \partial K) \end{aligned} \tag{2.3}$$

where by K we denote the domain in which the portfolio is nonnegative (in the case of proportional transaction costs $K = C_{\mathrm{p}}$). Moreover, let $T(X, Y)$ and $T_{l,m}(X, Y)$ be the operators defined by

$$\begin{aligned} T(X, Y) &= (X, (1+\xi)Y), \\ T_{l,m}(X, Y) &= (X - \eta_2(l) + \eta_1(m), Y + l - m), \end{aligned} \tag{2.4}$$

where $\eta_1(m)$ is the amount of money obtained from selling assets of stock value m, and $\eta_2(l)$ is the amount of money needed to purchase assets of stock value l. For proportional transaction costs we have $\eta_1(m) = (1-\mu)m$ and $\eta_2(l) = (1+\lambda)l$. Let

$$K_t^0 = \bigcup_{l_t, m_t \in L^2(\Omega, F_t, \mathbb{R}_+^2)} \ \bigcup_{(X,Y) \in L_i^0(F_t)} TT_{l_t, m_t}(X, Y).$$

Denote by $K_t^{0\mathrm{p}}$ the set K_t^0 corresponding to proportional transaction costs.

We are now in a position to define the arbitrage opportunity for one unit of time.

Definition 2.1. A *one-step arbitrage opportunity* exists at time t if

$$\exists (X, Y) \in L_i^0(F_t),\ \exists l_t, m_t \in L^2(\Omega, F_t, \mathbb{R}_+^2), \qquad TT_{l_t, m_t}(X, Y) \in L_+^0(F_{t+1})$$

and

$$P\{\omega : TT_{l_t, m_t}(X, Y)(\omega) \in \operatorname{int} C_{\mathrm{p}}\} > 0.$$

We have:

Theorem 2.1. *Assume that $P[\xi_t = 0 \mid F_t] = 0$ and ξ_t is independent of F_t for a time t. If $K_t^{0\mathrm{p}} \cap L_+^0(F_t) = \{(0,0)\}$ then there exists an equivalent measure $Q \sim P$ called a martingale measure such that*

$$E_Q[\xi_t \mid F_t] = 0.$$

Without the assumption that ξ_t is independent of F_t, if there exists a martingale measure $Q \sim P$ such that

$$E_Q[\xi_t \mid F_t] = 0,$$

then $K_t^{0\mathrm{p}} \cap L_+^0(F_t) = \{(0,0)\}$.

In the proof we shall need the following auxiliary lemma.

LEMMA 2.2. *Assume that ξ_t is independent of F_t. Then $E_P[z\xi_t \mid F_t] \neq 0$ for each bounded $z \in L_+^1(P)$ if and only if*

$$\begin{cases} \xi_t \geq 0 \text{ and } P[\xi_t > 0] > 0, \text{ or} \\ \xi_t \leq 0 \text{ and } P[\xi_t < 0] > 0. \end{cases}$$

Proof. We show the implication $\Rightarrow$ only.

Assume that $E_P[z\xi_t \mid F_t] \neq 0$ for any $z \in L_+^1(P)$ and $P[\xi_t < 0] > 0$ and $P[\xi_t > 0] > 0$. Choosing $z \in L_+^1(P)$ independent of F_t we see that $E_P[z\xi_t] \neq 0$.

Put $E_P[\xi_t^+] = m^+$ and $E_P[\xi_t^-] = m^-$. Assume that $m^+, m^- > 0$. Let

$$z = \begin{cases} 1/m^+ & \text{for } \xi_t > 0, \\ 1 & \text{for } \xi_t = 0, \\ 1/m^- & \text{for } \xi_t < 0. \end{cases}$$

Then z is $\sigma(\xi_t)$-measurable and therefore independent of F_t. Moreover

$$E_P[z\xi_t] = E_P[z\xi_t \mathbf{1}_{\xi_t>0} + z\xi_t \mathbf{1}_{\xi_t<0}] = \frac{1}{m^+} E[\xi_t \mathbf{1}_{\xi_t>0}] + \frac{1}{m^-} E[\xi_t \mathbf{1}_{\xi_t<0}] = 0,$$

which contradicts the fact that $E_P[z\xi_t] \neq 0$. ■

Proof of Theorem 2.1. We first prove that the existence of a martingale measure implies the absence of arbitrage. Namely we shall prove that if $(X,Y) \in \partial C_{\mathrm{p}}$ and $(X_1, Y_1) \in K_1^{0\mathrm{p}} \cap L_+^0(F_1)$, then $(X_1, Y_1) = (0,0)$. Suppose that $E_Q[\xi_t \mid F_t] = 0$ and $P[\xi_t = 0 \mid F_t] = 0$. For simplicity let $t = 1$. Consider the pair $(X,Y) \in \partial C_{\mathrm{p}}$, that is, we have

$$\begin{cases} (1-\mu)Y + X = 0 & \text{when } Y \geq 0, \\ (1+\lambda)Y + X = 0 & \text{when } Y < 0. \end{cases}$$

Assume that $(X_1, Y_1) \in K_1^{0\mathrm{p}} \cap L_+^0(F_1)$, where

$$(X_1, Y_1) = (X + (1-\mu)m_0 - (1+\lambda)l_0, (1+\xi_1)(Y - m_0 + l_0)).$$

To simplify notations put $l_0 = l$ and $m_0 = m$. Notice first that it suffices to consider the cases:

$$(l > 0 \text{ and } m = 0) \quad \text{or} \quad (l = 0 \text{ and } m > 0),$$

which correspond to the fact that there is no simultaneous buying and selling.

In fact, if $m > 0$ and $l > 0$ we obtain:

1) For $m < l$, we put $l' = l - m$ and $m' = 0$. Then

$$(X_1', Y_1') = (X - (1+\lambda)(l-m), (1+\xi_1)(Y + l - m)).$$

It is clear that $Y_1 = Y_1'$ and $X_1' \geq X_1$. From this we have $(X_1, Y_1) \in C_{\mathrm{p}} \Rightarrow (X_1', Y_1') \in C_{\mathrm{p}}$.

2) For $m > l$, we put $m' = m - l$ and $l' = 0$. Then

$$(X_1', Y_1') = (X + (1-\mu)(m-l), (1+\xi_1)(Y - m + l)).$$

It is easily seen that $Y_1 = Y_1'$ and $X_1' \geq X_1$. Hence $(X_1, Y_1) \in C_{\mathrm{p}} \Rightarrow (X_1', Y_1') \in C_{\mathrm{p}}$. Then we get $(X_1', Y_1') \in \partial C_{\mathrm{p}} \Rightarrow (X_1, Y_1) \in \partial C_{\mathrm{p}}$.

We continue the proof.

The proof falls naturally into 5 cases.

1. Assume that $m > 0$ and $(1-\mu)Y + X = 0$. Then

$$(X_1, Y_1) \in K_1^{0\mathrm{p}} \cap L_+^0(F_1)$$
$$\Leftrightarrow \begin{cases} (1-\mu)(1+\xi_1)(Y-m) + X + (1-\mu)m \geq 0, \\ (1+\lambda)(1+\xi_1)(Y-m) + X + (1-\mu)m \geq 0. \end{cases}$$

To simplify notation we write

$$Z_1 = (1-\mu)(1+\xi_1)(Y-m) + X + (1-\mu)m,$$
$$Z_2 = (1+\lambda)(1+\xi_1)(Y-m) + X + (1-\mu)m.$$

Since $E_Q[\xi_1 \,|\, F_1] = 0$, taking the conditional expectation with respect to F_1 we get

$$E_Q[Z_1 \,|\, F_1] = (1-\mu)Y - (1-\mu)m + X + (1-\mu)m \overset{(1-\mu)Y+X=0}{=} 0.$$

As $Z_1 \geq 0$ and $E_Q[Z_1 \,|\, F_1] = 0$ we get $Z_1 = 0$. Since $Z_1 = (1-\mu)\xi_1(Y-m)$ we conclude that either $\xi_1 = 0$ or $Y = m$. In view of the assumption $P[\xi_1 = 0 \,|\, F_1] = 0$ we have $Y = m$. Therefore $Z_2 = X + (1-\mu)Y = 0$, and finally $(X_1, Y_1) = (0, 0)$.

2. Assume that $m > 0$ and $(1+\lambda)Y + X = 0$. Then

$$(X_1, Y_1) \in K_1^{0\mathrm{p}} \cap L_+^0(F_1)$$
$$\Leftrightarrow \begin{cases} (1-\mu)(1+\xi_1)(Y-m) + X + (1-\mu)m \geq 0, \\ (1+\lambda)(1+\xi_1)(Y-m) + X + (1-\mu)m \geq 0. \end{cases}$$

We define Z_1 and Z_2 as in case 1. Since $E_Q[\xi_1 \,|\, F_1] = 0$, taking the expectation with respect to F_1 we get $E_Q[Z_2 \,|\, F_1] = -m(\mu + \lambda)$. From $Z_2 \geq 0$ it follows that $-m(\mu+\lambda) \geq 0$. Hence $m \leq 0$, but $m > 0$ by assumption, a contradiction.

3. Assume that $l > 0$ and $(1-\mu)Y + X = 0$. The same arguments as in the proof of case 2 yield a contradiction.

4. Assume that $l > 0$ and $(1+\lambda)Y + X = 0$. Similarly to the proof in case 1 we obtain $(Z_1, Z_2) = (0, 0)$, which implies that $(X_1, Y_1) = (0, 0)$.

5. Assume that $m = l = 0$. Then

$$(X_1, Y_1) \in K_1^{0\mathrm{p}} \cap L^0_+(F_1) \Leftrightarrow \begin{cases} (1-\mu)(1+\xi_1)Y + X \geq 0, \\ (1+\lambda)(1+\xi_1)Y + X \geq 0. \end{cases}$$

Taking the conditional expectation with respect to F_1 we get

$$\begin{cases} E[(1-\mu)(1+\xi_1)Y + X \,|\, F_1] = (1-\mu)Y + X, \\ E[(1+\lambda)(1+\xi_1)Y + X \,|\, F_1] = (1+\lambda)Y + X. \end{cases}$$

a) For $(1-\mu)Y + X = 0$ we obtain $E[(1-\mu)(1+\xi_1)Y + X \,|\, F_1] = 0$. This implies that $(1-\mu)(1+\xi_1)Y + X = 0$, thus $Y = 0$, and consequently $X = 0$ and we get $(1+\lambda)(1+\xi_1)Y + X = 0$. Finally, $(X_1, Y_1) = (0, 0)$.

b) For $(1+\lambda)Y + X = 0$, similarly to case a), we deduce that $(X_1, Y_1) = (0, 0)$.

Now, we prove that the absence of arbitrage implies the existence of a martingale measure. The proof is by contradiction. Assume that there are no martingale measures, which means that

$$\forall Q \sim P \qquad E_Q[\xi_t \,|\, F_t] \neq 0.$$

By Bayes' formula we have

$$E_Q[\xi_t \,|\, F_t] = \frac{E_P[z\xi_t \,|\, F_t]}{E_P[z \,|\, F_t]} \quad \text{for } z = \frac{dQ}{dP}. \tag{2.6}$$

Lemma 2.2 shows that the fact that $E_P[z\xi_t \,|\, F_t] \neq 0$ for any bounded $z \in L^1_+(P)$ can be written equivalently in the form

$$\begin{aligned} &\xi_t \geq 0 \quad \text{and} \quad P[\xi_t > 0] > 0, \quad \text{or} \\ &\xi_t \leq 0 \quad \text{and} \quad P[\xi_t < 0] > 0. \end{aligned} \tag{2.7}$$

Consequently,

- if $\xi_t \geq 0$ and $P[\xi_t > 0] > 0$ we have an arbitrage opportunity for $Y \geq 0$.
- if $\xi_t \leq 0$ and $P[\xi_t < 0] > 0$ an arbitrage opportunity exists for $Y \leq 0$. ∎

In the case of proportional transaction costs we proved that the existence of a martingale measure is equivalent to the absence of arbitrage in one unit of time. Now, we formulate the theorem which shows the equivalence of the existence of a martingale measure and the absence of arbitrage in any time period $[0, T]$. The transaction costs in the proof of this theorem will be described using general functions $\eta_1(x)$ and $\eta_2(x)$. Let $\eta_1 : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ and $\eta_2 : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ satisfy the following conditions:

1) $\eta_1(x)$ and $\eta_2(x)$ are increasing functions,
2) $\eta_1(0) = 0$ and $\eta_2(0) = 0$.

The function $\eta_1(x)$ is the amount of money obtained for selling assets of stock value x and $\eta_2(x)$ is the amount of money needed to purchase assets of stock value x. Following the arguments of [9] we can show the following theorem with general transaction costs.

THEOREM 2.3. *If the random variables ξ_t are i.i.d., the following conditions are equivalent*:

(i) *there exists an arbitrage opportunity in T steps*,
(ii) *there exists an arbitrage opportunity in one step.*

Proof. (ii)$\Rightarrow$(i). If an arbitrage opportunity exists in one step at a time t, then we liquidate the stock account, put all money to the bank and do nothing. We obtain an arbitrage in any time T.

(i)$\Rightarrow$(ii). Assume that an arbitrage opportunity exists in a period $[0,T]$. We first show that there must exist a time t at which an arbitrage exists. An arbitrage at time T means that

$$X_T + \eta_1(Y_T^+) - \eta_2(Y_T^-) \ge 0 \quad \text{a.s.},$$
$$P[X_T + \eta_1(Y_T^+) - \eta_2(Y_T^-) > 0] > 0,$$

where Y_T^+, Y_T^- denote the positive and negative parts of the random variable Y_T. Assume that our initial capital is 0. Let

$$t = \inf\{n : X_n + \eta_1(Y_n^+) - \eta_2(Y_n^-) \ge 0 \text{ a.s. and } P[X_n + \eta_1(Y_n^+) - \eta_2(Y_n^-) > 0] > 0\}.$$

Clearly $t \ge 1$. Consider two cases:

1) $X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-) = 0$ a.s.
2) $P[X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-) < 0] > 0$.

If 1) holds, then

$$X_t + \eta_1(Y_t^+) - \eta_2(Y_t^-) - (X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-)) = X_t + \eta_1(Y_t^+) - \eta_2(Y_t^-) \ge 0 \quad \text{a.s.}$$

and

$$P[X_t + \eta_1(Y_t^+) - \eta_2(Y_t^-) - (X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-)) > 0] = P[X_t + \eta_1(Y_t^+) - \eta_2(Y_t^-) > 0] > 0,$$

which means that an arbitrage exists at time t.

Consider now case 2). Let

$$A \equiv \{X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-) < 0\}$$

be such that $P(A) > 0$. On the set A we have

$$X_t + \eta_1(Y_t^+) - \eta_2(Y_t^-) - (X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-)) \ge -(X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-)) > 0 \quad \text{a.s.}$$

and

$$P[X_t + \eta_1(Y_t^+) - \eta_2(Y_t^-) - (X_{t-1} + \eta_1(Y_{t-1}^+) - \eta_2(Y_{t-1}^-)) > 0] = P(A) > 0.$$

Therefore we also have an arbitrage at time t.

Consequently, if an arbitrage exists in the time interval $[0, T]$, there must exist a time t defined as above. By Theorem 2.1 there is no martingale measure at time t. Consequently, by Lemma 2.2 we have

$$\begin{aligned} &\xi_t \geq 0 \quad \text{and} \quad P[\xi_t > 0] > 0, \ \text{or} \\ &\xi_t \leq 0 \quad \text{and} \quad P[\xi_t < 0] > 0. \end{aligned} \tag{2.8}$$

Since the random variables ξ_t are i.i.d., for any $t = 1, \ldots, T$ we have

$$(\xi_t \geq 0 \text{ and } P[\xi_t > 0] > 0) \quad \text{or} \quad (\xi_t \leq 0 \text{ and } P[\xi_t < 0] > 0)$$

and a martingale measure does not exist at any time $t = 1, \ldots, T$. Therefore we have an arbitrage at any time $t = 1, \ldots, T$. ■

3. No arbitrage with concave transaction costs. We define two functions: $c : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ and $d : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$, which satisfy the following conditions:

1) $c(x)$ is a convex increasing function,
2) $c(0) = 0$ and $(1 - \mu)x \leq c(x) \leq x$,
3) $\lim_{x\to\infty} c'(x) = 1$, $c'(x) < 1$ and $c'(0) = 1 - \mu$,
4) $d(x)$ is a concave increasing function,
5) $d(0) = 0$ and $x \leq d(x) \leq (1 + \lambda)x$,
6) $\lim_{x\to\infty} d'(x) = 1$, $d'(x) > 1$ and $d'(0) = 1 + \lambda$.

The graphs of c and d are shown in Figure 2.

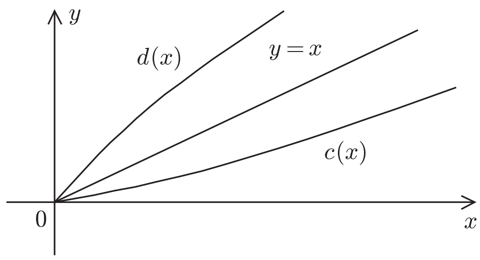

Fig. 2

The function $c(x)$ characterizes the amount of money obtained by selling assets of value x, and $d(x)$ the amount of money needed to purchase assets of value x. It will be convenient to introduce the inverse functions $C(x) = c^{-1}(x)$ and $D(x) = d^{-1}(x)$. Consider two cases:

1) For $Y_t \geq 0$ and $X_t \leq 0$ we have $X_t + c(Y_t) \geq 0$. From this

$$Y_t \geq C(-X_t). \tag{3.1}$$

2) For $Y_t \leq 0$ and $X_t \geq 0$ we have $X_t - d(-Y_t) \geq 0$. Hence

$$Y_t \geq -D(X_t). \tag{3.2}$$

This is illustrated in Figure 2.

Our portfolio evolves according to

$$\begin{cases} X_{t+1} = X_t + c(m_t) - d(l_t), \\ Y_{t+1} = (1+\xi_t)(Y_t - m_t + l_t). \end{cases} \tag{3.3}$$

Letting $c(m_t) = M_t$ and $d(l_t) = L_t$ we obtain

$$\begin{cases} X_{t+1} = X_t + M_t - L_t, \\ Y_{t+1} = (1+\xi_t)(Y_t - C(M_t) + D(L_t)). \end{cases} \tag{3.4}$$

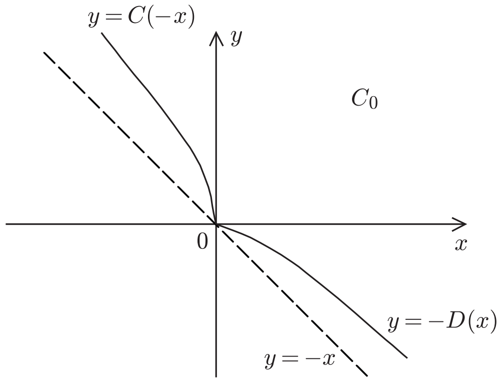

Fig. 3

Denote by C_0 the domain of nonnegative portfolios (see Figure 3), i.e. the set of pairs (X, Y) such that

$$\begin{cases} Y + D(X) \geq 0 & \text{for } X \geq 0, \\ Y - C(-X) \geq 0 & \text{for } X < 0. \end{cases} \tag{3.5}$$

A self-financing trading strategy (l_t, m_t) admits an arbitrage opportunity if for $(X_0, Y_0) \in \partial C_0$ we have

$$(X_T, Y_T) \in C_0 \quad \text{and} \quad P\{(X_T, Y_T) \in \operatorname{int} C_0\} > 0.$$

From this we see that there exists no arbitrage opportunity if for any $(X_0, Y_0) \in \partial C_0$ and self-financing strategy (l_t, m_t) we have

$$(X_T, Y_T) \in C_0 \Rightarrow (X_T, Y_T) \in \partial C_0 \qquad P\text{-a.s.}$$

Following the notations of Section 2 for $K = C_0$ we define the operator $T^0_{l,m}(X,Y)$ as $T_{l,m}(X,Y)$ with $\eta_1(m) = c(m)$ and $\eta_2(l) = d(l)$. Let

$$K^{0c}_t = \bigcup_{l_t,m_t\in L^2(\Omega,F_t,\mathbb{R}^2_+)} \bigcup_{(X,Y)\in L^0_i(F_t)} TT^0_{l_t,m_t}(X,Y). \tag{3.6}$$

THEOREM 3.1. *Assume that for any time t, ξ_t is independent of F_t. If $K^{0c}_t \cap L^0_+(F_t) = \{(0,0)\}$, then there exists a martingale measure $Q \sim P$ such that $E_Q[\xi_t \mid F_t] = 0$, and without the assumption that ξ_t is independent of F_t, if there exists a martingale measure $Q \sim P$ such that $E_Q[\xi_t \mid F_t] = 0$, then $K^{0c}_t \cap L^0_+(F_t) = \{(0,0)\}$.*

Proof. For simplicity we let $t = 1$. As in the proof of Theorem 2.1 we need only consider the cases

$$(l > 0 \text{ and } m = 0) \quad \text{or} \quad (l = 0 \text{ and } m > 0).$$

In fact, for $m > 0$, $l > 0$, $m < l$ we put $l' = l - m$ and $m' = 0$. Then

$$(X'_1, Y'_1) = (X - d(l-m), (1+\xi_1)(Y - m + l)).$$

It is clear that $Y'_1 = Y_1$. Since by Lagrange's theorem for any $l > m$ there exists $\theta \in (l - m, l)$ such that

$$d(l) - d(l-m) = d'(\theta)m > m > c(m) \tag{3.7}$$

(because $d'(x) > 1$), we obtain $X'_1 > X_1$, so that $(X_1, Y_1) \in C_0 \Rightarrow (X'_1, Y'_1) \in C_0$.

For $m > 0$, $l > 0$, $l < m$ we put $m' = m - l$ and $l' = 0$. Thus

$$(X'_1, Y'_1) = (X + c(m-l), (1+\xi_1)(Y - m + l)).$$

It is easily seen that $Y'_1 = Y_1$. Since by Lagrange's theorem there exists $\theta \in (m - l, m)$ such that

$$c(m) - c(m-l) = c'(\theta)l < l < d(l) \tag{3.8}$$

(because $c'(x) < 1$), we obtain $X'_1 > X_1$ and it follows that $(X'_1, Y'_1) \in \partial C_0 \Rightarrow (X_1, Y_1) \in \partial C_0$.

We return to the proof of our theorem.

The proof that the absence of arbitrage implies the existence of a martingale measure is the same as in the proof of Theorem 2.1, because it does not depend on the kind of transaction costs considered.

Now we prove that the existence of a martingale measure implies the absence of arbitrage. Assume that $E_Q[\xi_t \mid F_t] = 0$ and $(X,Y) \in \partial C_0$.

1) Let $X \geq 0$ and $Y + D(X) = 0$.

a) We consider the case when $0 < L < X$. By the assumption $Y =$

$-D(X)$ we have

$$\begin{aligned}(X_1, Y_1) &= (X - L, (1+\xi_1)(Y + D(L)))\\ &= (X - L, (1+\xi_1)(-D(X) + D(L))).\end{aligned}$$

Notice first that

$$D(X) - D(L) > D(X - L). \tag{3.9}$$

In fact, if $L > X/2$, then by Lagrange's theorem there exist $\theta \in (L, X)$ and $\theta' \in (0, X - L)$, $\theta' < \theta$, such that

$$\begin{aligned}D(X) - D(L) = D'(\theta)(X - L) &> D'(\theta')(X - L)\\ &= D(X - L) - D(0) = D(X - L).\end{aligned}$$

If $L \le X/2$ we consider the inequality

$$D(L) < D(X) - D(X - L). \tag{3.10}$$

By Lagrange's theorem there exist $\theta_1 \in (0, L)$ and $\theta_2 \in (X - L, L)$, $\theta_1 < \theta_2$, such that

$$D(L) = D(L) - D(0) = D'(\theta_1)L < D'(\theta_2)L = D(X) - D(X - L).$$

Consequently, the inequality (3.9) holds for any $L < X$.

Since $E_Q[Y_1 + D(X_1) \mid F_1] = -D(X) + D(L) + D(X - L)$ and $-D(X) + D(L) < -D(X - L)$ we get $(X_1, Y_1) \notin \operatorname{int} C_0$ with a positive probability.

b) Assume that $L > X$. Notice first that

$$-D(X) + D(L) < L - X. \tag{3.11}$$

In fact by Lagrange's theorem there exists $\theta \in (X, L)$ such that

$$D(L) - D(X) = D'(\theta)(L - X) < L - X.$$

Since $E_Q[Y_1 + C(-X_1) \mid F_1] = -D(X) + D(L) + C(-(X - L))$ and $L - X < C(L - X)$ and $-D(X) + D(L) < L - X$ we get $(X_1, Y_1) \notin \operatorname{int} C_0$ with a positive probability.

c) For $L = X$ we obtain $(X_1, Y_1) = (0, 0)$.

d) For $M > 0$ we have

$$(X_1, Y_1) = (X + M, (1+\xi_1)(Y - C(M))).$$

Notice first that

$$D(X + M) - D(X) < C(M). \tag{3.12}$$

In fact by Lagrange's theorem there exists $\theta \in (X, X + M)$ such that

$$D(X + M) - D(X) = D'(\theta)M < M < C(M).$$

Since $E_Q[Y_1 + D(X_1) \mid F_1] = -D(X) - C(M) + D(X + M)$ and $D(X) + C(M) > D(X + M)$ we obtain $(X_1, Y_1) \notin \operatorname{int} C_0$ with a positive probability.

Summarizing, if we start from $Y = -D(X)$ we have $(X_1, Y_1) \notin \operatorname{int} C_0$ with a positive probability.

2) Let $X \le 0$ and $Y = C(-X)$.

a) Consider $0 < m < Y$. Then

$$(X_1, Y_1) = (X + c(m), (1 + \xi_1)(Y - m)).$$

Notice first that

$$C(-X) - C(-X - c(m)) < m. \tag{3.13}$$

We show that

$$\forall X < Z \in \mathbb{R}^- \quad C(-(X - Z)) > C(-X) - C(-Z). \tag{3.14}$$

For $Z < X/2$, by Lagrange's theorem, there exist $\theta \in (X - Z, 0)$ and $\theta' \in (X, Z)$, $\theta > \theta'$, such that

$$\begin{aligned} C(-(X - Z)) &= C'(-\theta) \cdot (X - Z) \\ &> C'(-\theta') \cdot (X - Z) = C(-X) - C(-Z). \end{aligned}$$

For $Z \ge X/2$ we show the equivalent inequality

$$C(-Z) > C(-X) - C(-(X - Z)). \tag{3.15}$$

Again by Lagrange's theorem there exist $\theta_1 \in (Z, 0)$ and $\theta_2 \in (X, X - Z)$, $\theta_1 > \theta_2$, such that

$$\begin{aligned} C(-Z) &= C(-Z) - C(0) = C'(-\theta_1)Z > C'(-\theta_2)Z \\ &= C(-X) - C(-(X - Z)). \end{aligned}$$

Thus, for any $X < Z \in \mathbb{R}^-$ the inequality (3.14) holds. Consequently,

$$\begin{aligned} C(-X) - C(-X - c(m)) &= C(-X) - C(-(X + c(m))) \\ &< C(-(X - X - c(m))) = C(c(m)) = m. \end{aligned}$$

Since $E_Q[Y_1 - C(-X_1) \,|\, F_1] = C(-X) - m - C(-(X + c(m)))$ and $C(-X) - m < C(-(X + c(m)))$ we get $(X_1, Y_1) \notin \operatorname{int} C_0$ with a positive probability.

b) Let $m > Y$. Notice first that since the function $y = c(x)$ increases more slowly than $y = x$, and $m > Y$ and $Y = C(-X)$, we have

$$m - Y > c(m) - c(C(-X)) = c(m) + X. \tag{3.16}$$

Therefore taking into account that $D(x) + c(m)) < X + c(m)$ we obtain $E_Q[Y_1 + D(X_1) \,|\, F_1] = Y - m + D(X + c(m)) < 0$ so that $(X_1, Y_1) \notin \operatorname{int} C_0$ with a positive probability.

c) For $m = Y$ we have $(X_1, Y_1) = (0, 0)$.

d) For $l > 0$ we get

$$(X_1, Y_1) = (X - d(l), (1 + \xi_1)(Y + l)).$$

Notice that by Lagrange's theorem there exists $\theta \in (X - d(l), X)$ such that

$$C(X) - C(-(X - d(l))) = C'(-\theta)d(l) < -d(l) < -l. \tag{3.17}$$

Since $E_Q[Y_1 - C(-X_1) \mid F_1] = C(-X) + l - C(-(X - d(l)))$ we have $(X_1, Y_1) \notin \operatorname{int} C_0$ with a positive probability. ■

Using Theorem 2.3 we now have the equivalence of the absence of arbitrage and the existence of a martingale measure Q at any time t.

4. No arbitrage with fixed + proportional transaction costs. Let c be fixed costs of selling assets, d fixed costs of purchasing assets and $c, d > 0$, $d > c$. Then C_{cp} is the domain where our nonnegative portfolio is characterized by the system of inequalities:

$$\begin{cases} \text{without limits} & \text{for } X \ge 0,\ Y \ge 0, \\ X + (1-\mu)Y - c \ge 0 & \text{for } Y \ge 0,\ X < 0, \\ X + (1+\lambda)Y - d \ge 0 & \text{for } Y < 0,\ X \ge 0. \end{cases} \tag{4.1}$$

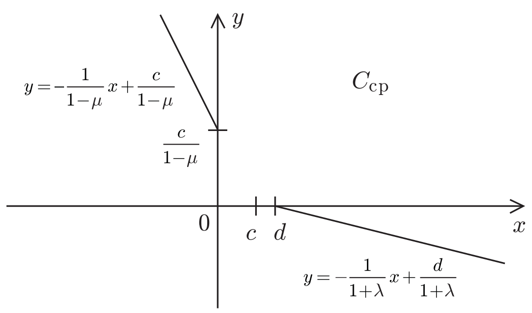

Fig. 4

The boundary ∂C_{cp}, that is, the set of portfolios with zero value consists of the point $(0,0)$ and halflines $l_1 : x + (1-\mu)y - c = 0$ and $l_2 : x + (1+\lambda)y - d = 0$. The portfolio evolves according to

$$\begin{cases} X_{t+1} = X_t + (1-\mu)m_t - (1+\lambda)l_t - \mathbf{1}_{m_t>0}c - \mathbf{1}_{l_t>0}d, \\ Y_{t+1} = (1+\xi_t)(Y_t - m_t + l_t). \end{cases} \tag{4.2}$$

A self-financing trading strategy (l_t, m_t) admits an arbitrage opportunity if for $(X_0, Y_0) \in \partial C_{\mathrm{cp}}$ we have

$$(X_T, Y_T) \in C_{\mathrm{cp}} \quad \text{and} \quad P\{(X_T, Y_T) \in \operatorname{int} C_{\mathrm{cp}}\} > 0.$$

Consequently, we have no arbitrage opportunity if for any $(X_0, Y_0) \in \partial C_{\mathrm{cp}}$ and self-financing strategy (l_t, m_t) the following implication holds:

$$(X_T, Y_T) \in C_{\mathrm{cp}} \Rightarrow (X_T, Y_T) \in \partial C_{\mathrm{cp}} \quad P\text{-a.s.}$$

Similarly to Section 2 define $K_t^{0\mathrm{cp}}$ by the formula (2.5) with $K = C_{\mathrm{cp}}$ and the operator $T_{l,m}(X, Y)$ defined with $\eta_1(m) = (1-\mu)m - \mathbf{1}_{m>0}c$ and $\eta_2(l) = (1+\lambda)l + \mathbf{1}_{l>0}d$.

Theorem 4.1. *Assume that for any time t, the random variables ξ_t are independent of F_t. If $K_t^{0\mathrm{cp}} \cap L_+^0(F_t) = \{(0,0)\}$, then there exists a martingale measure $Q \sim P$ such that $E_Q[\xi_t \mid F_t] = 0$. Without the assumption that ξ_t is independent of F_t, if there exists a martingale measure $Q \sim P$ such that $E_Q[\xi_t \mid F_t] = 0$, then $K_t^{0\mathrm{cp}} \cap L_+^0(F_t) = \{(0,0)\}$.*

Proof. The proof of the implication that the absence of arbitrage implies the existence of a martingale measure is the same as in the proof of Theorem 2.1.

Now we prove that the existence of a martingale measure implies the absence of arbitrage. For simplicity we put $t = 1$. Similarly to the proof of Theorem 2.1 we can show that it suffices to restrict ourselves to the cases

$$(l > 0 \text{ and } m = 0) \quad \text{or} \quad (l = 0 \text{ and } m > 0).$$

We want to prove that if $(X,Y) \in \partial C_{\mathrm{cp}}$ and $(X_1, Y_1) \in K_1^{0\mathrm{cp}} \cap L_+^0(F_1)$ and $E_Q[\xi_1 \mid F_1] = 0$, then $(X_1, Y_1) \in \partial C_{\mathrm{cp}}$ and moreover $(X_1, Y_1) = (0,0)$.

By considerations similar to those in the proof of Theorem 2.1 notice that if $(X,Y) \in \partial C_{\mathrm{cp}}$ and $(1-\mu)Y + X - c = 0$ the only situation when $(X_1, Y_1) \in C_{\mathrm{cp}}$ is when $m = Y$ and then $(X_1, Y_1) = (0,0)$. Similarly, if $(X,Y) \in \partial C_{\mathrm{cp}}$ and $X - d + (1+\lambda)Y = 0$, we have $(X_1, Y_1) \in C_{\mathrm{cp}}$ only when $l = (X-d)/(1+\lambda)$ and in this case $(X_1, Y_1) = (0,0)$. ∎

Using Theorem 2.3 we have the equivalence between the absence of arbitrage in T steps and the existence of a martingale measure Q.

5. No arbitrage with concave + fixed transaction costs. Let functions $c(x)$ and $d(x)$ be defined as in Section 3. Let $\widetilde{c}$ be fixed costs for selling assets, $\widetilde{d}$ be fixed costs for purchasing assets, and $\widetilde{c}, \widetilde{d} > 0$ and $\widetilde{c} < \widetilde{d}$. Similarly to Section 3 we introduce the inverse functions $C(x) = c^{-1}(x)$ and $D(x) = d^{-1}(x)$. The domain where our capital is nonnegative is described by the inequalities

$$\left\{\begin{array}{ll} X_t + c(Y_t) - \widetilde{c} \geq 0 & \text{for } X_t < 0,\ Y_t > 0, \\ X_t - d(-Y_t) - \widetilde{d} \geq 0 & \text{for } X_t > 0,\ Y_t < 0, \\ \text{no limits} & \text{for } X_t \geq 0,\ Y_t \geq 0. \end{array}\right. \tag{5.1}$$

Using inverse functions we get

$$\left\{\begin{array}{ll} Y_t \geq C(-X_t + \widetilde{c}) & \text{for } X_t < 0,\ Y_t > 0, \\ Y_t \geq -D(X_t - \widetilde{d}) & \text{for } X_t > 0,\ Y_t < 0, \\ \text{no limits} & \text{for } X_t \geq 0,\ Y_t \geq 0. \end{array}\right. \tag{5.2}$$

The domain of the nonnegative portfolios will be denoted by C_{c0}.

Our portfolio evolves according to

$$\left\{\begin{array}{l} X_{t+1} = X_t + c(m_t) - d(l_t) - \mathbf{1}_{m_t>0}\widetilde{c} - \mathbf{1}_{l_t>0}\widetilde{d}, \\ Y_{t+1} = (1+\xi_t)(Y_t - m_t + l_t). \end{array}\right. \tag{5.3}$$

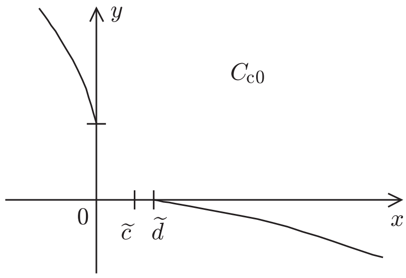

Fig. 5

We put $c(m_t) = M_t$ and $d(l_t) = L_t$. Then

$$\begin{cases} X_{t+1} = X_t + M_t - L_t - \mathbf{1}_{M_t>0}\widetilde{c} - \mathbf{1}_{L_t>0}\widetilde{d}, \\ Y_{t+1} = (1+\xi_t)(Y_t - C(M_t) + D(L_t)). \end{cases} \tag{5.4}$$

A self-financing trading strategy (l_t, m_t) admits an arbitrage opportunity if for $(X_0, Y_0) \in \partial C_{c0}$, where as before the boundary is considered as the set of portfolios with zero value, we have

$$(X_T, Y_T) \in C_{c0} \quad \text{and} \quad P\{(X_T, Y_T) \in \operatorname{int} C_{c0}\} > 0.$$

From this we obtain the absence of arbitrage if for any $(X_0, Y_0) \in \partial C_{c0}$ and self-financing strategy (l_t, m_t) we have

$$(X_T, Y_T) \in C_{c0} \Rightarrow (X_T, Y_T) \in \partial C_{c0} \quad P\text{-a.s.}$$

Following the notations of Section 2 define K_t^{0cc} as in (2.5) with $K = C_{c0}$ and the operator $T_{l,m}(X, Y)$ with $\eta_1(m) = c(m) - \mathbf{1}_{m>0}\widetilde{c}$ and $\eta_2(l) = d(l) + \mathbf{1}_{l>0}\widetilde{d}$. We have

THEOREM 5.1. *Assume that for any time t, the random variables ξ_t are independent of F_t. If $K_t^{0cc} \cap L_+^0(F_t) = \{(0,0)\}$ then there exists a martingale measure $Q \sim P$ such that $E_Q[\xi_t \mid F_t] = 0$, and without the assumption that ξ_t is independent of F_t, if there exists a martingale measure $Q \sim P$ such that $E_Q[\xi_t \mid F_t] = 0$, then $K_t^{0cc} \cap L_+^0(F_t) = \{(0,0)\}$.*

Proof. The proof of the implication that the absence of arbitrage implies the existence of a martingale measure is the same as in the proof of Theorem 2.1.

We now show that the existence of a martingale measure implies the absence of arbitrage. For simplicity we put $t = 1$. As in the proof of Theorem 2.1 we can restrict ourselves to the cases

$$(l > 0 \text{ and } m = 0) \quad \text{or} \quad (l = 0 \text{ and } m > 0).$$

We prove below that if $(X,Y) \in \partial C_{c0}$ and $(X_1,Y_1) \in K_1^{0cc} \cap L_+^0(F_1)$ and $E_Q[\xi_1 \,|\, F_1] = 0$, then $(X_1,Y_1) \in \partial C_{c0}$, and moreover $(X_1,Y_1) = (0,0)$. We consider the following two cases:

1. Assume that $Y + D(X - \widetilde{d}) = 0$.

a) For $0 < L + \widetilde{d} < X$ we have

$$\begin{aligned}(X_1,Y_1) &= (X - L - \widetilde{d}, (1+\xi_1)(Y + D(L)))\\ &= (X - L - \widetilde{d}, (1+\xi_1)(-D(X-\widetilde{d}) + D(L))).\end{aligned}$$

Notice that from (3.9), taking into account that D is increasing we have

$$D(X - L - 2\widetilde{d}) < D(X - \widetilde{d}) - D(L). \tag{5.5}$$

Since

$$E_Q[Y_1 + D(X_1 - \widetilde{d}) \,|\, F_1] = -D(X - \widetilde{d}) + D(L) + D(X - L - 2\widetilde{d}) < 0$$

we obtain $(X_1,Y_1) \notin \partial C_{c0}$ with a positive probability.

b) Let $L > X - \widetilde{d}$. Notice that from (3.11),

$$D(L) - D(X - \widetilde{d}) < -X + L + \widetilde{d} + \widetilde{c}. \tag{5.6}$$

Since

$$E_Q[Y_1 - C(-(X_1 + \widetilde{c})) \,|\, F_1] = -D(X - \widetilde{d}) + D(L) - C(-X + L + \widetilde{d} + \widetilde{c})$$

and $C(-X + L + \widetilde{d} + \widetilde{c}) > -X + L + \widetilde{d} + \widetilde{c}$, we have $(X_1,Y_1) \notin \partial C_{c0}$ with a positive probability.

c) For $L = X - \widetilde{d}$ we have $(X_1,Y_1) = (0,0)$.

d) For $M > 0$ we have

$$\begin{aligned}(X_1,Y_1) &= (X + M - \widetilde{c}, (1+\xi_1)(Y - C(M)))\\ &= (X + M - \widetilde{c}, (1+\xi_1)(-D(X - \widetilde{d}) - C(M))).\end{aligned}$$

Notice that

$$D(X + M - \widetilde{d} - \widetilde{c}) - D(X - \widetilde{d}) < C(M) \tag{5.7}$$

since when $M < \widetilde{c}$ the left side is negative, and for $M > \widetilde{c}$ we use (3.12) together with the fact that C is nondecreasing.

Therefore

$$E_Q[Y_1 + D(X_1 - \widetilde{d}) \,|\, F_1] = -D(X - \widetilde{d}) - C(M) + D(X + M - \widetilde{d} - \widetilde{c}) < 0$$

and we get $(X_1,Y_1) \notin \partial C_{c0}$ with a positive probability.

2. Assume that $Y = C(-X + \widetilde{c})$.

a) Consider $0 < m < Y$. Then

$$\begin{aligned}(X_1,Y_1) &= (X + c(m) - \widetilde{c}, (1+\xi_1)(Y - m))\\ &= (X + c(m) - \widetilde{c}, (1+\xi_1)(C(-X + \widetilde{c}) - m)).\end{aligned}$$

Notice that using (3.14) we obtain

$$C(-X+\widetilde{c})-C(-X-c(m)+2\widetilde{c})<m. \tag{5.8}$$

Since

$$E_Q[Y_1-C(-X_1+\widetilde{c})\,|\,F_1]=C(-X+\widetilde{c})-m-C(-X-c(m)+\widetilde{c}+\widetilde{c})<0$$

we have $(X_1,Y_1)\notin\partial C_{c0}$ with a positive probability.

b) Let $m>Y$. Notice that from (3.16) we have

$$C(-X+\widetilde{c})+X-\widetilde{c}<m-c(m)+\widetilde{d}. \tag{5.9}$$

Since $E_Q[Y_1+D(X_1-\widetilde{d})\,|\,F_1]=C(-X+\widetilde{c})-m+D(X+c(m)-\widetilde{c}-\widetilde{d})$ and $D(X+c(m)-\widetilde{c}-\widetilde{d})<X+c(m)-\widetilde{c}-\widetilde{d}$, we have $(X_1,Y_1)\notin\partial C_{c0}$ with a positive probability.

c) For $m=Y$ we have $(X_1,Y_1)=(0,0)$.

d) For $l>0$ we get

$$\begin{aligned}(X_1,Y_1)&=(X-d(l)-\widetilde{d},(1+\xi_1)(Y+l))\\&=(X-d(l)-\widetilde{d},(1+\xi_1)(C(-X+\widetilde{c})+l)).\end{aligned}$$

Notice that by (3.17),

$$C(-X+d(l)+\widetilde{d}+\widetilde{c})-C(-X+\widetilde{c})>l. \tag{5.10}$$

Therefore

$$E_Q[Y_1-C(-X_1+\widetilde{c})\,|\,F_1]=C(-X+\widetilde{c})+l-C(-X+d(l)+\widetilde{c}+\widetilde{d})<0$$

and $(X_1,Y_1)\notin\partial C_{c0}$ with a positive probability. ∎

By Theorem 2.3, as in the previous sections we have the equivalence between the absence of arbitrage in T steps and the existence of a martingale measure Q.

Acknowledgments. I would like to thank Professor Łukasz Stettner for his valuable remarks and his help in the preparation of this paper.

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Received on 15.10.1999;
revised version on 6.1.2000