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CLASSIFICATIONS AND EXISTENCE OF POSITIVE SOLUTIONS OF A HIGHER ORDER NONLINEAR DIFFERENCE EQUATION

BY

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Abstract. A classification scheme for the eventually positive solutions of a class of higher order nonlinear difference equations is given in terms of their asymptotic magnitudes, and necessary as well as sufficient conditions for the existence of such solutions are provided.

1. Introduction. This paper is concerned with a class of higher order nonlinear difference equations of the form

$$\Delta(r_n \Delta^{m-1} x_n) + f(n, x_n) = 0, \qquad n = K, K+1, \ldots, \tag{1}$$

where K is a fixed integer, m is an integer greater than or equal to 2, $\{r_n\}_{n=K}^{\infty}$ is a positive sequence and $f(n,x)$ is a real-valued function defined on $\{K, K+1, \ldots\} \times \mathbb{R}$ which is continuous in the second variable x and satisfies $f(n,x) > 0$ for $x > 0$. We intend to give a classification scheme for eventually positive solutions of our equations in terms of their asymptotic magnitude and provide necessary conditions as well as sufficient conditions for the existence of such solutions. In order to accomplish our goal, additional conditions will be imposed on the coefficient sequences $\{r_n\}$ and the function f. We will need either one of the following two assumptions for the sequence $\{r_n\}$ so as to include the case where $r_n \equiv 1$:

(H1) $\Delta r_n \geq 0$ for $n \geq K$, and
(H2) $\sum_{n=K}^{\infty} 1/r_n = \infty$.

As for the function f, if for each fixed integer n, $f(n,x)/x$ is nondecreasing in x for $x > 0$, then is called *superlinear*. If for each integer n, $f(n,x)/x$ is nonincreasing in x for $x > 0$, then f is said to be *sublinear*. Superlinear or sublinear functions f will be assumed in later results. Here we note that

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if $0 < a \leq x \leq b$, then

$$f(n,a) \leq f(n,x) \leq f(n,b)$$

if f is superlinear, and

$$\frac{a}{b} f(n,b) \leq f(n,x) \leq \frac{b}{a} f(n,a)$$

if f is sublinear.

Nonlinear difference equations have been studied by a number of authors [1]–[13]. In particular, He [6] (see also [10]) studied equation (1) when $m = 2$, and obtained existence criteria for eventually positive solutions. Zhou and Yan [12] studied the equation

$$\varDelta(r_n(\varDelta^{m-1}x_n)^{1/\delta}) + f(n,x_n) = 0, \tag{2}$$

where δ is a quotient of odd positive integers. In that paper bounded and eventually positive solutions are considered. In particular, it is shown that when (H2) holds, then under the additional assumption that m is even, (2) has a bounded and eventually positive solution if, and only if,

$$\sum_{n=K}^{\infty} n^{m-2} \left\{ \frac{1}{r_n} \sum_{j=n}^{\infty} |f(j,c))| \right\}^{1/\delta} < \infty \tag{3}$$

for some $c \neq 0$.

By a solution of (1), we mean a real sequence $\{x_n\}_{n=K}^{\infty}$ which satisfies it. Since (1) can be written in the form of a recurrence relation, it is clear that given appropriate initial conditions, a solution can be obtained by successive iterations. For the sake of convenience, we will employ the following notations:

$$R_{s,n} = \sum_{i=s}^{n-1} \frac{1}{r_i}, \quad K \leq s \leq n-1,$$

$$R_s = \sum_{i=s}^{\infty} \frac{1}{r_i}, \quad s \geq K.$$

We will need the following results. The first one requires the concept of a set of uniformly Cauchy sequences. Let l^∞ be the Banach space of all bounded real sequences $x = \{x_n\}_{n=K}^{\infty}$ endowed with the usual operations and supremum norm. A subset S of l^∞ is said to be *uniformly Cauchy* if for every $\varepsilon > 0$, there exists an integer M such that whenever $i, j > M$, we have $|x_i - x_j| < \varepsilon$ for any $x \in S$.

LEMMA 1 (Cheng and Patula [4]). *Let $\varOmega$ be a closed bounded convex subset of l^∞. Suppose T is a continuous mapping such that $T(\varOmega)$ is contained in $\varOmega$, and $T(\varOmega)$ is uniformly Cauchy. Then T has a fixed point in $\varOmega$.*

The following theorem of Stolz is a discrete analog of l'Hopital's rule [1, Theorem 1.7.7 and Corollary 1.7.8].

LEMMA 2. *Let* $\{u_k\}$ *and* $\{v_k\}$ *be two real sequences such that* $v_k > 0$ *and* $\Delta v_k > 0$ *for all large* k. *If* $\lim_{k\to\infty} v_k = \infty$ *and* $\lim_{k\to\infty} \Delta u_k/\Delta v_k = c$, *where* c *may be infinite, then* $\lim_{k\to\infty} u_k/v_k = c$.

Given a real function $u(t)$ whose derivative $u^{(r)}(t)$ is sign regular, the intermediate derivatives will also satisfy certain sign conditions. Such results are well known in the theory of ordinary differential equations and their discrete analogs have been reported by a number of authors. Two of such sign regularity conditions are stated by Zafer and Dahiya [11].

LEMMA 3. *Let* N *be a positive integer. Let* $\{y_n\}$ *be a real sequence such that* $\{y_n\}$ *and* $\{\Delta^N y_n\}$ *are of constant sign. Suppose further that* $\{\Delta^N y_n\}$ *does not vanish identically for all large* n *and that* $y_n \Delta^N y_n \le 0$ *for* $n \ge 0$. *Then*

(i) *for each* $j \in \{1, \dots, N-1\}$, *the sequence* $\{\Delta^j y_n\}$ *is of constant sign for all large* n; *and*

(ii) *there is an integer* $k \in \{0, 1, \dots, N-1\}$ *such that* $(-1)^{N-k-1} = 1$ *and for each* $j \in \{0, 1, \dots, k\}$, $y_n \Delta^j y_n > 0$ *for all large* n, *while for each* $j \in \{k+1, \dots, N-1\}$, $(-1)^{j-k} y_n \Delta^j y_n > 0$ *for all large* n.

LEMMA 4. *Let* N *be a positive integer. Let* $\{y_n\}$ *be a real sequence such that for each* $j \in \{0, 1, \dots, N-1\}$, $\{\Delta^j y_n\}$ *is of constant sign for all large* n. *Suppose further that* $y_n \Delta^N y_n \ge 0$ *for all large* n. *Then either*

(i) *for each* $j \in \{1, \dots, N\}$, $y_n \Delta^j y_n \ge 0$ *for all large* n; *or*

(ii) *there is an integer* $k \in \{0, 1, \dots, N-2\}$ *such that* $(-1)^{N-k} = 1$, *and for each* $j \in \{1, \dots, k\}$, $y_n \Delta^j y_n > 0$ *for all large* n, *while for each* $j \in \{k+1, \dots, N-2\}$, $(-1)^{j-k} y_n \Delta^j y_n > 0$ *for all large* n.

2. A classification scheme. Let $\{x_n\}$ be an eventually positive solution of (1). Then $\Delta(r_n \Delta^{m-1} x_n) = -f(n, x_n) < 0$ for all large n. The sequence $\{\Delta^{m-1} x_n\}$ is therefore strictly decreasing for all large n. We further assert that $\{\Delta^{m-1} x_n\}$ is eventually positive.

LEMMA 5. *Suppose the condition* (H2) *holds. Let* $\{x_n\}$ *be an eventually positive solution of* (1). *Then the sequence* $\{\Delta^{m-1} x_n\}$ *is eventually positive.*

Proof. Assume without loss of generality that $x_n > 0$ for $n \ge K$. Then in view of (1),

$$r_{n+1} \Delta^{m-1} x_{n+1} < r_n \Delta^{m-1} x_n, \quad n \ge K.$$

If it were the case that $\Delta^{m-1} x_N < 0$ for some $N \ge K$, then

$$r_n \Delta^{m-1} x_n < \ldots < r_N \Delta^{m-1} x_N, \quad n \ge N+1,$$

which implies that

$$\Delta^{m-2}x_n - \Delta^{m-2}x_N = \sum_{i=N}^{n-1} \Delta^{m-1}x_i < \sum_{i=N}^{n-1} \frac{r_N}{r_i}\Delta^{m-1}x_N = R_{N,n}r_N\Delta^{m-1}x_N.$$

Since (H2) implies $\lim_{n\to\infty} R_{N,n} = \infty$, we see that the right hand side tends to $-\infty$. Thus $\lim_{n\to\infty} \Delta^{m-2}x_n = -\infty$, which implies that $\{x_n\}$ is eventually negative. This is a contradiction.

Under the additional hypothesis (H1), more can be said.

LEMMA 6. *Suppose the conditions* (H1) *and* (H2) *hold. Let* $\{x_n\}$ *be an eventually positive solution of* (1). *Then the sequence* $\{\Delta^m x_n\}$ *is eventually negative.*

Proof. By Lemma 5, $\{\Delta^{m-1}x_n\}$ is eventually positive. Furthermore, in view of (1) and our assumption on $\{r_n\}$, we see that

$$r_n\Delta^m x_n = -\Delta r_n \Delta^{m-1}x_n - f(n, x_n) < 0$$

as required.

Under the conditions (H1) and (H2), it is clear that Lemma 3 provides a classification scheme for eventually positive solutions of (1). Such a scheme is crude, however. We will propose an auxiliary classification scheme for eventually positive solutions of (1). For the sake of convenience, we use the following notations:

$$E_j(\infty, *) = \left\{ \{x_n\}_{n=K}^{\infty} \,\middle|\, \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty,\ \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} \in \mathbb{R}\setminus\{0\} \right\},$$

$$E_j(\infty, 0) = \left\{ \{x_n\}_{n=K}^{\infty} \,\middle|\, \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty,\ \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0 \right\},$$

$$E_j(*, 0) = \left\{ \{x_n\}_{n=K}^{\infty} \,\middle|\, \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} \in \mathbb{R}\setminus\{0\},\ \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0 \right\},$$

$$O_j(\infty, *) = \left\{ \{x_n\}_{n=K}^{\infty} \,\middle|\, \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = \infty,\ \lim_{n\to\infty} \frac{x_n}{n^{2j}} \in \mathbb{R}\setminus\{0\} \right\},$$

$$O_j(\infty, 0) = \left\{ \{x_n\}_{n=K}^{\infty} \,\middle|\, \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = \infty,\ \lim_{n\to\infty} \frac{x_n}{n^{2j}} = 0 \right\},$$

$$O_j(*, 0) = \left\{ \{x_n\}_{n=K}^{\infty} \,\middle|\, \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} \in \mathbb{R}\setminus\{0\},\ \lim_{n\to\infty} \frac{x_n}{n^{2j}} = 0 \right\},$$

where j is some integer to be specified.

THEOREM 1. *Suppose the conditions* (H1) *and* (H2) *hold. If* m *is even, then for each eventually positive solution* $\{x_n\}$ *of* (1), *there is some integer* j *in* $\{1, \ldots, m/2\}$ *such that* $\{x_n\}$ *belongs to one of the classes* $E_j(\infty, *)$, $E_j(\infty, 0)$ *or* $E_j(*, 0)$. *If* m *is odd, then for each eventually positive solution*

$\{x_n\}$ *of* (1), *either there is an integer* $i \in \{1, \ldots, (m-1)/2\}$ *such that* $\{x_n\}$ *belongs to one of the classes* $O_i(\infty, *), O_i(\infty, 0), O_i(*, 0)$, *or else it converges.*

Proof. First of all, we infer from Lemma 6 that $\{\Delta^m x_n\}$ is eventually negative. Suppose m is even. By Lemma 3, there is an integer $t = 2j-1$, where $j \in \{1, \ldots, m/2\}$, such that for each $k \in \{0, 1, \ldots, t-1\}$, $\Delta^k x_n > 0$ for all large n, and for each $k \in \{t, t+1, \ldots m-1\}$, $(-1)^{k+1}\Delta^k x_n > 0$ for all large n. In particular, $\Delta^{2j-2} x_n > 0$, $\Delta^{2j-1} x_n > 0$ and $\Delta^{2j} x_n < 0$ for all large n. Therefore the limits

$$\lim_{n\to\infty} \Delta^{2j-1} x_n = \lambda_{2j-1} \quad \text{and} \quad \lim_{n\to\infty} \Delta^{2j-2} x_n = \lambda_{2j-2}$$

satisfy $0 \le \lambda_{2j-1} < \infty$ and $0 < \lambda_{2j-2} \le \infty$. If $\lambda_{2j-1} > 0$, then by the theorem of Stolz, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{x_n}{n^{2j-1}} &= \lim_{n\to\infty} \frac{\Delta x_n}{(2j-1)n^{2j-2}} = \ldots = \lim_{n\to\infty} \frac{\Delta^{2j-1} x_n}{(2j-1)!} \\ &= \lim_{n\to\infty} \frac{\lambda_{2j-1}}{(2j-1)!} \neq 0.\end{aligned}$$

It follows that $\lim_{n\to\infty} x_n/n^{2j-2} = \infty$. That is, $\{x_n\}$ belongs to $E(\infty, *)$.

If $\lambda_{2j-1} = 0$ and $\lambda_{2j-2} = \infty$, then by the theorem of Stolz again, it is easy to see that

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty.$$

That is, $\{x_n\}$ belongs to $E(\infty, 0)$. Finally, if $\lambda_{2j-1} = 0$ and $0 < \lambda_{2j-2} < \infty$, then by the theorem of Stolz,

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \frac{\lambda_{2j-2}}{(2j-2)!} \neq 0.$$

It follows that $\lim_{n\to\infty} x_n/n^{2j-1} = 0$, and hence $\{x_n\}$ belongs to $E(*, 0)$.

When the integer m is odd, in view of Lemma 3, there is an even integer $t \in \{0, 1, \ldots, m-1\}$ such that for each $k \in \{0, 1, \ldots, t\}$, $\Delta^k x_n > 0$ for all large n, and for each $k \in \{t+1, \ldots, m-1\}$, $(-1)^{k-t}\Delta^k x_n > 0$ for all large n. In case $t \in \{1, \ldots, m-1\}$, the proof is similar to that given above. If $t = 0$, then $x_n > 0$, $\Delta x_n < 0$ and $\Delta^2 x_n > 0$ for all large n. It follows that $\{x_n\}$ converges to some nonnegative constant. The proof is complete.

3. Existence criteria. Under the conditions (H1) and (H2), eventually positive solutions can be classified according to Theorem 1. To justify our classification scheme, we need to find sufficient conditions for the existence of positive solutions in various subsets. We remark that there is an uncertainty involved, namely, the integer j which is needed in the definitions of the subsets E and O. We first deal with the case where m is even.

Theorem 2. *Suppose that m is even, and that* (H1) *and* (H2) *hold. Suppose further that f is either superlinear or sublinear. If there is a constant $c > 0$ and $j \in \{1, \ldots, m/2-1\}$ such that*

$$\sum_{n=K}^{\infty} n^{m-2j-1}\left\{\frac{1}{r_n}\sum_{k=n}^{\infty}|f(k, ck^{2j-1})|\right\} < \infty, \tag{4}$$

then (1) *has an eventually positive solution in $E_j(\infty, *)$. The converse is also true.*

Proof. Let $a = c/2$ if f is superlinear and $a = c$ if f is sublinear. Set $\Gamma(n) = n^{2j-1}$. In view of (4), we may choose an integer N so large that

$$\sum_{i=N}^{\infty}\frac{(i-N+1)\ldots(i-N+m-2j-1)}{(2j-1)!(m-2j-1)!}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty} f(k, ck^{2j-1})\right\} < \frac{a}{2}. \tag{5}$$

Let us introduce the linear space X of all real sequences $x = \{x_n\}_{n=N}^{\infty}$ such that

$$\sup_{n\ge N} |x_n|/\Gamma(n) < \infty.$$

It is not difficult to verify that X endowed with the norm

$$\|x\| = \sup_{n\ge N}|x_n|/\Gamma(n)$$

is a Banach space. Define a subset Ω of X as follows:

$$\Omega = \{\{x_n\}_{n=N}^{\infty} \in X \mid a\Gamma(n) \le x_n \le 2a\Gamma(n),\ n \ge N\}.$$

Then Ω is a bounded, convex closed subset of X. Let us further define an operator $T : \Omega \to X$ as follows:

$$(Tx)_n = \frac{3a}{2}\Gamma(n) + \sum_{i_{m-1}=N}^{n-1}\sum_{i_{m-2}=N}^{i_{m-1}-1}\cdots\sum_{i_{m-2j+1}=N}^{i_{m-2j+2}-1} H(i_{m-2j+1}), \qquad n \ge N,$$

where

$$H(n) = \sum_{i=n}^{\infty}\frac{(i-n+1)\ldots(i-n+m-2j-1)}{(m-2j-1)!}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty} f(k, x_k)\right\}.$$

The mapping T has the following properties. First of all, T maps Ω into Ω. Indeed, if $x = \{x_n\}_{n=N}^{\infty}$ belongs to Ω, then

$$(Tx)_n \ge \frac{3a}{2}\Gamma(n) \ge a\Gamma(n), \qquad n \ge N.$$

Furthermore, by (5), we also have

$$(Tx)_n \le \frac{3a}{2}\Gamma(n)$$
$$+\frac{(n-N)^{2j-1}}{(2j-1)!}\sum_{i=N}^{\infty}\frac{(i-N+1)\dots(i-N+m-2j-1)}{(m-2j-1)!}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty}f(k,x_k)\right\}$$
$$\le \frac{3a}{2}\Gamma(n)+\frac{a}{2}\Gamma(n)=2a\Gamma(n).$$

Next, we show that T is continuous. To see this, let $\varepsilon > 0$. Choose $M \ge N$ so large that

$$\sum_{i=M}^{\infty}\frac{(i-M+1)\dots(i-M+m-2j-1)}{(m-2j-1)!}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty}f(k,ck^{2j-1})\right\}<\varepsilon,$$

and

$$\sum_{k=M}^{\infty}f(k,ck^{2j-1})<\varepsilon.$$

Let $\{x^{(v)}\}$ be a sequence in Ω such that $x^{(v)} \to x$. Since Ω is closed, $x \in \Omega$. Furthermore, for all large v,

$$\left|\sum_{k=n}^{\infty}f(k,x_k^{(v)})-\sum_{k=n}^{\infty}f(k,x_k)\right| \le \left|\sum_{k=n}^{M-1}f(k,x_k^{(v)})-\sum_{k=n}^{M-1}f(k,x_k)\right|$$
$$+\left|\sum_{k=M}^{\infty}f(k,x_k^{(v)})\right|+\left|\sum_{k=M}^{\infty}f(k,x_k)\right| \le 3\delta\varepsilon,$$

where $\delta = 1$ if f is superlinear, and $\delta = 1/2$ if f is sublinear. In view of the definition of T,

$$|(Tx^{(v)})_n-(Tx)_n|$$
$$\le \Gamma(n)\sum_{i=n}^{M-1}\frac{(i-n+1)\dots(i-n+m-2j-1)}{(m-2j-1)!}$$
$$\times\frac{1}{r_i}\left|\sum_{k=i}^{\infty}f(k,x_k^{(v)})-\sum_{k=i}^{\infty}f(k,x_k)\right|$$
$$+\Gamma(n)\left|\sum_{i=M}^{\infty}\frac{(i-M+1)\dots(i-M+m-2j-1)}{(m-2j-1)!}\cdot\frac{1}{r_i}\sum_{k=i}^{\infty}f(k,x_k^{(v)})\right|$$
$$+\Gamma(n)\left|\sum_{i=M}^{\infty}\frac{(i-M+1)\dots(i-M+m-2j-1)}{(m-2j-1)!}\cdot\frac{1}{r_i}\sum_{k=i}^{\infty}f(k,x_k)\right|$$
$$\le \Gamma(n)\cdot 3\delta\varepsilon.$$

This shows that $\|Tx^{(v)}-Tx\|$ tends to zero, i.e., T is continuous.

Finally, note that when $t > n \geq m$,

$$
\begin{aligned}
&|(Tx)_t - (Tx)_n| \\
&\quad \leq \Gamma(n)\Bigg|\sum_{i=n}^{\infty} \frac{(i-n+1)\ldots(i-n+m-2j-1)}{(m-2j-1)!} \cdot \frac{1}{r_i} \sum_{k=i}^{\infty} f(k, x_k)\Bigg| \\
&\qquad + \Gamma(n)\Bigg|\sum_{i=t}^{\infty} \frac{(i-t+1)\ldots(i-t+m-2j-1)}{(m-2j-1)!} \cdot \frac{1}{r_i} \sum_{k=i}^{\infty} f(k, x_k)\Bigg| \\
&\quad \leq 2\Gamma(n)\Bigg|\sum_{i=n}^{\infty} \frac{(i-n+1)\ldots(i-n+m-2j-1)}{(m-2j-1)!} \cdot \frac{1}{r_i} \sum_{k=i}^{\infty} f(k, x_k)\Bigg| \\
&\quad \leq 2\Gamma(n)\delta\varepsilon.
\end{aligned}
$$

Therefore, $T(\Omega)$ is uniformly Cauchy.

In view of Lemma 1, we see that there is an $x^* \in \Omega$ such that $Tx^* = x^*$. It is easy to check that x^* is an eventually positive solution of (1). Furthermore, by the theorem of Stolz,

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^{2j-1}} \sum_{i_{m-1}=n_1}^{n-1} \sum_{i_{m-2}=n_1}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+1}=n_1}^{i_{m-2j+2}-1} H(i_{m-2j+1}) \\
&\qquad = \ldots = \lim_{n\to\infty} \frac{1}{(2j-1)!} \sum_{i=n}^{\infty} \frac{(i-n+1)\ldots(i-n+m-2j-1)}{(m-2j-1)!} \\
&\qquad\qquad \times \frac{1}{r_i} \sum_{k=i}^{\infty} f(k, x_k^*) = 0.
\end{aligned}
$$

Thus

$$
\lim_{n\to\infty} \frac{x_n^*}{n^{2j-1}} = \lim_{n\to\infty} \frac{(Tx^*)_n}{n^{2j-1}} = \frac{3a}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{x_n^*}{n^{2j-2}} = \infty.
$$

That is to say, x^* belongs to $E_j(\infty, *)$.

We now show that the converse holds. Let $\{x_n\}$ be an eventually positive solution of (1) which belongs to $E_j(\infty, *)$. In view of Lemmas 3 and 6, we see that $\Delta^{m-1} x_n > 0$ and $\Delta^m x_n < 0$ for n greater than or equal to some integer n_1, and $\{\Delta^k x_n\}$ is eventually monotonic for each $k \in \{1, \ldots, m-1\}$. Since $\lim_{n\to\infty} x_n/n^{2j-1} = a > 0$, there exists an integer $n_2 \geq n_1$ such that

$$
\frac{a}{2} n^{2j-1} \leq x_n \leq \frac{3a}{2} n^{2j-1}, \quad n \geq n_2,
$$

so that

$$
f(k, x_k) \geq f\left(k, \frac{a}{2} k^{2j-1}\right), \quad k \geq n_2,
$$

if f is superlinear, and

$$f(k,x_k) \geq 3f\left(k, \frac{a}{2}k^{2j-1}\right), \qquad k \geq n_2,$$

if f is sublinear. We assert that

$$\lim_{n\to\infty} \Delta^{2j-1}x_n = (2j-1)!a.$$

In fact, by the theorem of Stolz,

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = \lim_{n\to\infty} \frac{\Delta x_n}{(2j-1)n^{2j-2}} = \ldots = \lim_{n\to\infty} \frac{\Delta^{2j-1}x_n}{(2j-1)!} = a.$$

In case $j < m/2$, we see further that

$$\lim_{n\to\infty} \Delta^{2j}x_n = \lim_{n\to\infty} \Delta^{2j+1}x_n = \ldots = \lim_{n\to\infty} \Delta^{m-1}x_n = 0 \tag{6}$$

since $\{\Delta^i x_n\}$ is eventually monotonic for $i = 2j, 2j+1, \ldots, m-1$.

By (1),

$$r_s\Delta^{m-1}x_s + \sum_{j=n}^{s-1} f(j,x_j) = r_n\Delta^{m-1}x_n, \qquad s \geq n+1 \geq n_2,$$

so that

$$\Delta^{m-1}x_n > \frac{1}{r_n}\sum_{j=n}^{\infty} f(j,x_j), \qquad n \geq n_2.$$

Summing the above inequalities successively, and invoking (6) if necessary, we see that

$$-\Delta^{2j}x_n > \sum_{i_{m-2j-2}=n}^{\infty} \ \sum_{i_{m-2j-3}=i_{m-2j-2}}^{\infty} \cdots \sum_{i=i_1}^{\infty} \frac{1}{r_i} \sum_{k=i}^{\infty} f(k,x_k) = H(n), \qquad n \geq n_2.$$

Summing the above inequalities one more time, we then obtain

$$\begin{aligned}
\infty > \Delta^{2j-1}x_{n_2} > \Delta^{2j-1}x_{n_2} - (2j-1)!a &> \sum_{i=n_2}^{\infty} H(i) \\
&\geq \frac{1}{(m-2j-1)!2^{m-2j-1}} \sum_{i=2n_2}^{\infty} i^{m-2j-1}\frac{1}{r_i}\sum_{k=i}^{\infty} f(k,x_k) \\
&\geq C\sum_{i=2n_2}^{\infty} i^{m-2j-1}\frac{1}{r_i}\sum_{k=i}^{\infty} f(k,ck^{2j-1}),
\end{aligned}$$

for some appropriate constants C and c. The proof is complete.

In the above result, it would be nice to include the case where $j = m/2$. Indeed, we can obtain a similar result provided $r_n = r > 0$ for $n = K$,

$K+1,\ldots$ The proof is not much different from that of Theorem 2 and is therefore omitted.

THEOREM 2′. *Suppose that* m *is even, that* $r_n = r > 0$ *for* $n \geq K$, *and that* (H1) *and* (H2) *hold. Suppose further that* f *is either superlinear or sublinear. If there a constant* $c > 0$ *such that*

$$\sum_{n=K}^{\infty} |f(k, ck^{m-1})| < \infty,$$

then (1) *has an eventually positive solution in* $E_{m/2}(\infty, *)$. *The converse is also true.*

We now turn to eventually positive solutions in $E_j(*, 0)$. By means of the same reasoning used in the proof of Theorem 2, it is not difficult to show that the converse part of the following result holds.

THEOREM 3. *Suppose that* m *is even, and that* (H1) *and* (H2) *hold. Suppose further that* f *is either superlinear or sublinear. If there is a constant* $c > 0$ *and* $j \in \{1, \ldots, m/2\}$ *such that*

$$\sum_{n=K}^{\infty} n^{m-2j} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, ck^{2j-2})| \right\} < \infty, \tag{7}$$

then (1) *has an eventually positive solution in* $E_j(*, 0)$. *The converse is also true.*

The proof of the sufficiency part is also similar to that of Theorem 2 and is therefore only sketched as follows. Let $a = c/2$ if f is superlinear and $a = c$ if f is sublinear. Set

$$\Gamma(n) = n^{2j-2}, \quad n \geq K.$$

Then as in the proof of Theorem 2, we see that there exists an integer $n_1 \geq K$ and a sequence $\{x_n^*\}$ such that

$$a\Gamma(n) \leq x_n^* \leq 2a\Gamma(n), \quad n \geq n_1,$$

and

$$x_n^* = \frac{3a}{2}\Gamma(n) + \sum_{i_{m-1}=n_1}^{n-1} \sum_{i_{m-2}=n_1}^{i_{m-1}-1} \cdots \sum_{i_{m-2j+2}=n_1}^{i_{m-2j+3}-1} G(i_{m-2j+1}), \quad n \geq n_1,$$

where

$$G(n) = \sum_{i=n}^{\infty} \frac{(i-n+1)\ldots(i-n+m-2j)}{(m-2j)!} \left\{ \frac{1}{r_i} \sum_{k=i}^{\infty} f(k, x_k) \right\}.$$

Then by means of the theorem of Stolz, we may show that

$$\lim_{n\to\infty} \frac{x_n^*}{n^{2j-2}} = \frac{3a}{2} + \beta$$

where β is a constant satisfying

$$0 < \beta \le \sum_{n=n_1}^{\infty} \frac{(n-n_1+1)\dots(n-n_1+m-2j-1)}{(m-2j-1)!} \times \left\{ \frac{1}{r_i} \sum_{k=n}^{\infty} 2f(k, ck^{2j-2}) \right\}.$$

It follows that

$$\lim_{n\to\infty} \frac{x_n^*}{n^{2j-1}} = 0.$$

These show that $\{x_n^*\}$ is an eventually positive solution in $E_j(*, 0)$.

Next we provide an existence criterion for eventually positive solutions in $E_j(\infty, 0)$.

THEOREM 4. *Suppose that m is even, and that* (H1) *and* (H2) *hold. Suppose further that for each $n \ge K$, $f(n, x)$ is nonincreasing in x. If there is some integer $j \in \{1, 2, \dots, m/2-1\}$ such that*

$$\sum_{n=K}^{\infty} n^{m-2j-1} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, ak^{2j-2})| \right\} < \infty \tag{8}$$

for some $a > 0$, and

$$\sum_{n=K}^{\infty} n^{m-2j} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, bk^{2j-2})| \right\} = \infty \tag{9}$$

for every $b > 0$, then equation (1) *has a positive solution in $E_j(\infty, 0)$. Conversely, if* (1) *has a positive solution $x \in E_j(\infty, 0)$, then* (9) *holds for every $b > 0$ and*

$$\sum_{n=K}^{\infty} n^{m-2j-1} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, ck^{2j-1})| \right\} < \infty \tag{10}$$

for every $c > 0$.

Proof. The proof is similar to that of Theorem 2 and is sketched as follows. Replace $\Gamma(n)$ in the proof of Theorem 2 by

$$\Gamma(n) = n^{2j-2}, \quad n \ge K,$$

and also modify the mapping T in an appropriate manner so as to yield a fixed point $\{x_n^*\}$ such that

$$a\Gamma(n) \le x_n^* \le 2a\Gamma(n), \quad n \ge n_1 \le K,$$

and

$$x_n^* = \frac{3a}{2}\Gamma(n) + \sum_{i_{m-1}=n_1}^{n-1} \sum_{i_{m-2}=n_1}^{i_{m-1}-1} \dots \sum_{i_{m-2j+3}}^{i_{m-2j+4}-1} G(i_{m-2j+3}), \quad n \ge n_1,$$

where

$$G(n) = \sum_{i=n}^{\infty} \frac{(i-n+1)\dots(i-n+m-2j-1)}{(m-2j-1)!} \left\{ \frac{1}{r_i} \sum_{k=i}^{\infty} f(k, x_k) \right\}.$$

By means of the theorem of Stolz, we may show that

$$\lim_{n\to\infty} \frac{x_n^*}{n^{2j-2}} = \frac{3a}{2} + \lim_{n\to\infty} \frac{1}{(2j-2)!} \sum_{k=n_1}^{n-1} G(k)$$

and

$$\lim_{n\to\infty} \frac{x_n^*}{n^{2j-1}} = \lim_{n\to\infty} \frac{1}{(2j-1)!} G(n) = 0.$$

Since $\Delta^{2j-2} x_n^* > 0$ and $\Delta^{2j-1} x_n^* > 0$, the sequence $\{\Delta^{2j-2} x_n^*\}$ is positive and increasing, thus it either converges to some positive limit or diverges to ∞. If the former holds, then $\lim_{n\to\infty} x^*/n^{2j-2}$ is a positive constant, so that $x^* \in E_j(*, 0)$. But then by Theorem 3, the condition (7) holds for some positive constant c, contrary to the assumption that (9) holds for every $b > 0$. Thus we may now conclude that $\lim_{n\to\infty} x^*/n^{2j-2} = \infty$, so that $x^* \in E_j(\infty, 0)$.

For the converse, let $\{x_n\}$ be an eventually positive solution in $E_j(\infty, 0)$ which satisfies

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \infty, \qquad \lim_{n\to\infty} \frac{x_n}{n^{2j-1}} = 0.$$

For any $a > 0$ and $b > 0$, there exists $n_1 \geq K$ such that

$$bn^{2j-2} \leq x_n \leq an^{2j-1}, \qquad n \geq n_1.$$

Since f is nonincreasing, we see that

$$f(n, an^{2j-1}) \leq f(n, x_n) \leq f(n, bn^{2j-2}), \qquad n \geq n_1.$$

We may now proceed as in the proof Theorem 2 to conclude that (10) holds for every $c > 0$. Next, by the theorem of Stolz,

$$\lim_{n\to\infty} \frac{x_n}{n^{2j-2}} = \dots = \lim_{n\to\infty} \frac{\Delta^{2j-2} x_n}{(2j-2)!} = \infty,$$

and

$$\lim_{n\to\infty} \Delta^{2j-1} x_n = \dots = \lim_{n\to\infty} \Delta^{m-1} x_n = 0. \tag{11}$$

In view of (1),

$$r_s \Delta^{m-1} x_s + \sum_{i=n}^{s-1} f(i, x_i) = r_n \Delta^{m-1} x_n, \qquad s \geq n+1 \geq n_1,$$

so that

$$\Delta^{m-1}x_n = \frac{1}{r_n}\sum_{i=n}^{\infty} f(i,x_i), \qquad n \ge n_1.$$

Summing the above equalities successively, and utilizing (11) if necessary, we see that

$$\Delta^{2j-1}x_n = \sum_{l=n}^{\infty}\sum_{i=l}^{\infty} H(i), \qquad n \ge n_1,$$

where $H(n)$ has been defined in Theorem 2. Again by summing the above equalities from n_1 to $n-1$, we see further that

$$\Delta^{2j-2}x_n = \Delta^{2j-2}x_{n_1} + \sum_{k=n_1}^{n-1}\sum_{l=k}^{\infty}\sum_{i=l}^{\infty} H(i).$$

Since

$$\begin{aligned}\sum_{k=n_1}^{n-1}\sum_{l=k}^{\infty}\sum_{i=l}^{\infty} H(i) &\le \sum_{l=n_1}^{n-1}\sum_{i=n_1}^{\infty} i^{m-2j-1}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty} f(k,x_k)\right\}\\ &\le \sum_{l=n_1}^{\infty}\sum_{i=n_1}^{\infty} i^{m-2j-1}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty} f(k,bk^{2j-2})\right\}\\ &\le \sum_{i=n_1}^{\infty} i^{m-2j}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty} f(k,bk^{2j-2})\right\},\end{aligned}$$

and since $\lim_{n\to\infty}\Delta^{2j-2}x_n = \infty$, (9) holds for every $b > 0$. The proof is complete.

Again, when $j = m/2$, we can establish the following by parallel arguments.

THEOREM 4′. *Suppose that m is even, that $r_n = r > 0$ for $n \ge K$, and that* (H1) *and* (H2) *hold. Suppose further that for each $n \ge K$, $f(n,x)$ is nonincreasing in x. If*

$$\sum_{n=K}^{\infty} |f(n,an^{m-2})| < \infty$$

for some $a > 0$, and

$$\sum_{n=K}^{\infty}\sum_{k=n}^{\infty} |f(k,bk^{m-2})| = \sum_{k=K}^{\infty}(k-K)|f(k,bk^{m-2})| = \infty \tag{12}$$

for every $b > 0$, then equation (1) *has a positive solution in $E_{m/2}(\infty,0)$. Conversely, if* (1) *has a positive solution $x \in E_{m/2}(\infty,0)$, then* (12) *holds* *for every* $b > 0$ *and*

$$\sum_{n=K}^{\infty} |f(n, cn^{m-1})| < \infty$$

for every $c > 0$.

We remark that the first equality in (12) is true (see e.g. [12]).

We now turn our attention to the case where m is odd.

Theorem 5. *Suppose that m is odd, and that* (H1) *and* (H2) *hold. Suppose further that f is either superlinear or sublinear. If there is a constant $c > 0$ and $j \in \{1, \ldots, (m-3)/2\}$ such that*

$$\sum_{n=K}^{\infty} n^{m-2j-2} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, ck^{2j})| \right\} < \infty,$$

then (1) *has an eventually positive solution in $O_j(\infty, *)$. The converse is also true.*

The proof is similar to that of Theorem 2, we only need to note that the sequence $\Gamma(n)$ there should be replaced by $\Gamma(n) = n^{2j}$ and the mapping T should be modified so that

$$(Tx)_n = \frac{3a}{2}\Gamma(n) + \sum_{i_{m-1}=N}^{n-1} \sum_{i_{m-2}=N}^{i_{m-1}-1} \cdots \sum_{i_{m-2j}=N}^{i_{m-2j+1}-1} H(i_{m-2j}), \qquad n \geq N,$$

where

$$H(n) = \sum_{i=n}^{\infty} \frac{(i-n+1)\ldots(i-n+m-2j-2)}{(m-2j-2)!} \left\{ \frac{1}{r_i} \sum_{k=i}^{\infty} f(k, x_k) \right\}.$$

In case $j = (m-1)/2$, we have the following result which can be proved by parallel arguments.

Theorem 5′. *Suppose that m is odd, that $r_n = r > 0$ for $n \geq K$, and that* (H1) *and* (H2) *hold. Suppose further that f is either superlinear or sublinear. If there is a constant $c > 0$ such that*

$$\sum_{n=K}^{\infty} |f(n, cn^{m-1})| < \infty,$$

then (1) *has an eventually positive solution in $O_{m/2}(\infty, *)$. The converse is also true.*

Theorem 6. *Suppose that m is odd, and that* (H1) *and* (H2) *hold. Suppose further that f is either superlinear or sublinear. If there is a constant*

$c > 0$ *and* $j \in \{1, \dots, (m-1)/2\}$ *such that*

$$\sum_{n=K}^{\infty} n^{m-2j-1} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, ck^{2j-1})| \right\} < \infty,$$

then (1) *has an eventually positive solution in* $O_j(*, 0)$. *The converse is also true.*

THEOREM 7. *Suppose that* m *is odd, and that* (H1) *and* (H2) *hold. Suppose further that for each* $n \geq K$, $f(n, x)$ *is nonincreasing in* x. *If there is some integer* $j \in \{1, \dots, (m-3)/2\}$ *such that*

$$\sum_{n=K}^{\infty} n^{m-2j-2} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, ak^{2j-1})| \right\} < \infty \tag{13}$$

for some $a > 0$ *and*

$$\sum_{n=K}^{\infty} n^{m-2j-1} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, bk^{2j-1})| \right\} = \infty \tag{14}$$

for every $b > 0$, *then* (1) *has a positive solution in* $O_j(\infty, 0)$. *Conversely, if* (1) *has a positive solution in* $O_j(\infty, 0)$, *then* (14) *holds for every* $b > 0$ *and*

$$\sum_{n=K}^{\infty} n^{m-2j-2} \left\{ \frac{1}{r_n} \sum_{k=n}^{\infty} |f(k, ck^{2j})| \right\} < \infty$$

for every $c > 0$.

THEOREM 7′. *Suppose that* m *is odd, that* $r_n = r > 0$ *for* $n \geq K$, *and that* (H1) *and* (H2) *hold. Suppose further that for each* $n \geq K$, $f(n, x)$ *is nonincreasing in* x. *If*

$$\sum_{n=K}^{\infty} |f(n, an^{m-2})| < \infty$$

for some $a > 0$ *and*

$$\sum_{n=K}^{\infty} \sum_{k=n}^{\infty} |f(k, bk^{m-2})| = \sum_{k=K}^{\infty} (k-K)|f(k, bk^{m-2})| = \infty \tag{15}$$

for every $b > 0$, *then* (1) *has a positive solution in* $O_{(m-1)/2}(\infty, 0)$. *Conversely, if* (1) *has a positive solution in* $O_{(m-1)/2}(\infty, 0)$, *then* (15) *holds for every* $b > 0$ *and*

$$\sum_{n=K}^{\infty} |f(n, an^{m-1})| < \infty$$

for every $c > 0$.

THEOREM 8. *Suppose that m is odd, and that* (H1) *and* (H2) *hold. Suppose further that f is either superlinear or sublinear. If there is a constant $c > 0$ and $j \in \{1, \ldots, (m-1)/2\}$ such that*

$$\sum_{n=K}^{\infty} n^{m-2}\left\{\frac{1}{r_n}\sum_{k=n}^{\infty}|f(k,c)|\right\} < \infty,$$

then (1) *has an eventually positive solution which converges to a positive constant. The converse is also true.*

The proof is similar to that of Theorem 2, we only need to note that the sequence $\Gamma(n)$ should now be replaced by $\Gamma(n) = 1$ and the mapping T should be modified so that

$$(Tx)_n = \frac{3a}{2} + \sum_{i=n}^{\infty}\frac{(i-n+1)\ldots(i-n+m-2)}{(m-2)!}\left\{\frac{1}{r_n}\sum_{k=i}^{\infty} f(k,x_k)\right\}, \quad n \ge N.$$

THEOREM 9. *Suppose that m is odd, and that* (H1) *and* (H2) *hold. Suppose further that f is nondecreasing in x. Then* (1) *has an eventually positive solution $\{x_n\}$ which converges to zero if*

$$n\sum_{i=n}^{\infty}\frac{(i-n+1)\ldots(i-n+m-2)}{(m-2)!}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty}\left|f\left(k,\frac{1}{k}\right)\right|\right\} \le 1 \tag{16}$$

holds for $n \ge N \ge m+2$.

Proof. Let X be the partially ordered Banach space of all bounded real sequences $x = \{x_n\}_{n=N}^{\infty}$ endowed with the usual supremum norm and termwise ordering. Define a subset Ω of X by

$$\Omega = \{\{x_n\} \in X : 0 \le x_n \le 1,\ n \ge N\}.$$

For any subset M of Ω, it is clear that $\inf M \in \Omega$ and $\sup M \in \Omega$. Define an operator T on Ω as follows:

$$(Tx)_n = n\sum_{i=n}^{\infty}\frac{(i-n+1)\ldots(i-n+m-2)}{(m-2)!}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty}\left|f\left(k,\frac{x_k}{k}\right)\right|\right\}, \quad n \ge N.$$

By (16), we see that $T(\Omega) \subseteq \Omega$. Furthermore, it is clear that T is an increasing mapping. By means of the Knaster–Tarski fixed point theorem (see e.g. [5, Theorem 1.7.3]), there exists a sequence $w = \{w_n\} \in \Omega$ such that $Tw = w$. If we let

$$u_n = w_n/n, \qquad n \ge N,$$

then

$$u_n = \sum_{i=n}^{\infty}\frac{(i-n+1)\ldots(i-n+m-2)}{(m-2)!}\left\{\frac{1}{r_i}\sum_{k=i}^{\infty}|f(k,u_k)|\right\}.$$

By taking differences on both sides of the above equality, we may easily verify that $u = \{u_n\}$ is a solution of (1) for all large n. Since u is eventually positive and converges to zero, we have found the desired solution. The proof is complete.

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