COLLOQUIUM MATHEMATICUM

VOL. 83 2000 NO. 1

ON A SEMIGROUP OF MEASURES WITH IRREGULAR DENSITIES

BY

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Dedicated to my Mother

Abstract. We study the densities of the semigroup generated by the operator $-X^2+|Y|$ on the 3-dimensional Heisenberg group. We show that the 7th derivatives of the densities have a jump discontinuity. Outside the plane $x=0$ the densities are C^∞. We give explicit spectral decomposition of images of $-X^2+|Y|$ in representations.

1. Introduction. In [1] P. Głowacki and A. Hulanicki discovered that there exist convolution semigroups $(\mu_t)_{t>0}$ of probability measures on a Lie group G such that all μ_t have densities p_t whose first group derivatives Xp_t are in L^2 but higher derivatives are not. Paper [1] does not say anything about pointwise derivatives of p_t. The aim of the present paper is to clarify this point. Indeed, the densities p_t of the semigroup considered in [1] do have a number of derivatives but at some points the seventh derivative does not exist. In order to obtain the result and clarify the situation, we study the semigroup and its infinitesimal generator in some detail, mainly when transferred by unitary irreducible representations of G. The operators thus obtained are known objects but the information about them needed here is perhaps easier to prove directly than to recover from the fairly complicated general theory. Therefore we include many proofs here. Of course, we do not claim any originality at this point.

By the 3-dimensional Heisenberg group $\mathbb{G}$ we mean the Euclidean space $\mathbb{R}^3$ with multiplication defined by

$$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2)=(x_1+x_2,y_1+y_2,z_1+z_2+x_1y_2).$$

The basis X, Y, Z of the left invariant Lie algebra of $\mathbb{G}$ related to the coordinate system is

$$X=\frac{\partial}{\partial x},\quad Y=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\quad Z=\frac{\partial}{\partial z}.$$

2000 *Mathematics Subject Classification*: Primary 22E30.

Supported by grant 2269/W/IM/98 from Wrocław University.

The operator we are going to study is defined as

$$A = -X^2 + |Y|.$$

For $t > 0$, let δ_t be the automorphism

$$\delta_t(x, y, z) = (t^a x, t^b y, t^c z),$$

where a, $b > 0$, $c = a + b$. If f is a function from the Schwartz class $\mathcal{S}(\mathbb{G})$, then

$$\begin{aligned} X(f \circ \delta_t) &= t^a \cdot (Xf) \circ \delta_t, \\ Y(f \circ \delta_t) &= t^b \cdot (Yf) \circ \delta_t, \\ Z(f \circ \delta_t) &= t^c \cdot (Zf) \circ \delta_t. \end{aligned}$$

To simplify calculations we assume from now on that $b = 2a$, $c = 3a$ and a is an odd, positive number. Then

$$A(f \circ \delta_t) = t^b \cdot (Af) \circ \delta_t.$$

Let us define a family of unitary representations $\pi^\lambda : \mathbb{G} \to \mathcal{U}(L^2(\mathbb{R}))$ for $\lambda \in \mathbb{R}^*$ (where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$):

$$\pi^{\pm 1}_{(x,y,z)}\varphi(u) = e^{\pm i(z+uy)}\varphi(u + x), \quad \text{where } u \in \mathbb{R},\ (x, y, z) \in \mathbb{G},\ \varphi \in L^2(\mathbb{R}),$$

$$\begin{aligned} \pi^\lambda_g &= \pi^1_{\delta_{|\lambda|}(g)} \quad \text{if } \lambda > 0, \\ \pi^\lambda_g &= \pi^{-1}_{\delta_{|\lambda|}(g)} \quad \text{if } \lambda < 0. \end{aligned}$$

We shall write simply π for π^1. The representations π^λ can be extended to the space of bounded measures:

$$\pi^\lambda_\mu = \int_{\mathbb{G}} \pi^\lambda_g \, d\mu(g) \quad \text{if } \mu \in \mathcal{M}(\mathbb{G}).$$

Expanding the formula for π^λ_f, where $f \in L^1(\mathbb{G}) \cap L^2(\mathbb{G})$, we obtain

$$\pi^\lambda_f \varphi(u) = \int_{\mathbb{R}} \bigg(\varphi(v)|\lambda|^{-a} \int_{\mathbb{R}^2} f\bigg(\frac{v-u}{|\lambda|^a}, y, z\bigg) e^{i\,\mathrm{sgn}(\lambda)(|\lambda|^c z + |\lambda|^b uy)} \, dy\, dz\bigg) \, dv.$$

Thus the kernel of the operator π^λ_f is given by

$$k^\lambda_f(u, v) = |\lambda|^{-a} \mathcal{F}_2 \mathcal{F}_3 f\bigg(\frac{v-u}{|\lambda|^a}, -\mathrm{sgn}(\lambda)|\lambda|^b u, -\mathrm{sgn}(\lambda)|\lambda|^c\bigg), \tag{1.1}$$

where $\mathcal{F}_i$, $i = 1, 2, 3$, denotes the Fourier transform with respect to the ith coordinate. The operators $\frac{1}{\sqrt{2\pi}}\mathcal{F}_i$ are isometries, hence we have the Plancherel formula

$$\|f\|^2_{L^2(\mathbb{G})} = \frac{c}{(2\pi)^2} \int_{\mathbb{R}} \|\pi^\lambda_f\|^2_{\mathrm{HS}} |\lambda|^{2c-1} \, d\lambda. \tag{1.2}$$

Applying the inverse Fourier transform to (1.1) one can verify that

$$f(x,y,z)=\frac{c}{(2\pi)^2}\int\limits_{\mathbb{R}^2} k_f^\lambda(|\lambda|^a u,|\lambda|^a(u+x))|\lambda|^{a+2c-1}e^{-i\lambda^c(z+uy)}\,du\,d\lambda. \tag{1.3}$$

We have just defined π_μ^λ for measures. Now we extend this representation over some class of operators. Let T be a (possibly unbounded) operator on $L^2(\mathbb{G})$ such that every function f from the Schwartz class $\mathcal{S}(\mathbb{G})$ belongs to the domain of T and $Tf\in L^1(\mathbb{G})$. If for all $\lambda\in\mathbb{R}^*$, $f\in\mathcal{S}(\mathbb{G})$, $\varphi\in L^2(\mathbb{R})$,

$$\pi_f^\lambda\pi_T^\lambda\varphi=\pi_{Tf}^\lambda\varphi$$

and the operator π_T^λ is closable, then we define the representation of T as the closure of π_T^λ. We denote this closure also by π_T^λ. One can check that

$$\pi_X^\lambda=|\lambda|^a\left(-\frac{\partial}{\partial u}\right),\qquad \pi_Y^\lambda=\operatorname{sgn}(\lambda)|\lambda|^b(-iu),\qquad \pi_Z^\lambda=\operatorname{sgn}(\lambda)|\lambda|^c(-i),$$

$$\pi_A^\lambda=|\lambda|^b\left(-\frac{\partial^2}{\partial u^2}+|u|\right).$$

Acknowledgments. I am very grateful to my tutors, Prof. W. Hebisch, Prof. A. Hulanicki and Prof. T. Pytlik for their teaching and guidance throughout my study. Very special thanks are due to my Mother for her help and encouragement at difficult moments while writing this article.

2. Eigenfunctions of π_A^λ. In this section we aim to show that all eigenfunctions of π_A^λ arise in a simple way from the Airy function. Since the operator π_A^λ differs from π_A only by a scalar factor, we restrict our attention to π_A. The operator π_A is selfadjoint and positive definite (cf. [4], Theorem X.28), hence

$$\sigma(\pi_A)\subseteq[0,\infty).$$

PROPOSITION 2.1. *The operator $(\pi_A+I)^{-1}$ is compact.*

Proof. It is enough to show that the set

$$K=\{\varphi\in L^2(\mathbb{R}):\|(\pi_A+I)\varphi\|_{L^2(\mathbb{R})}\le 1\}$$

has a finite ε-mesh for every $\varepsilon>0$. Let $\varphi\in K$. Then

$$\begin{aligned}\|\varphi\|_{L^2(\mathbb{R})}&\ge\|(\pi_A+I)\varphi\|_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})}\ge\langle(\pi_A+I)\varphi,\varphi\rangle\\&=\left\|\frac{\partial}{\partial u}\varphi\right\|^2_{L^2(\mathbb{R})}+\||u|^{1/2}\varphi\|^2_{L^2(\mathbb{R})}+\|\varphi\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Hence

$$\|\varphi\|_{L^2(\mathbb{R})}\le 1,\qquad \left\|\frac{\partial}{\partial u}\varphi\right\|_{L^2(\mathbb{R})}\le 1,\qquad \||u|^{1/2}\varphi\|_{L^2(\mathbb{R})}\le 1. \tag{2.1}$$

If $M^{-1/2} < \varepsilon/2$, then the set $K_1 = \{\psi_1 = \varphi \cdot 1_{[-M,M]^c} : \varphi \in K\}$ has a trivial $\varepsilon/2$-mesh which consists of a single point 0. Moreover (2.1) shows that functions belonging to $K_2 = \{\psi_2 = \varphi - \psi_1 = \varphi \cdot 1_{[-M,M]} : \varphi \in K\}$ are uniformly bounded and equicontinuous, which, combined with the Ascoli–Arzelà theorem, completes the proof. ■

COROLLARY 2.2. *There exists an orthogonal system φ_n of normalized eigenfunctions of π_A such that*

$$\pi_A \varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \varphi_n \rangle \varphi_n.$$

The only point of accumulation of the sequence (λ_n) is ∞, so we can assume that (λ_n) is weakly increasing.

Since the potential of π_A, which equals $|u|$, is symmetric and tends to ∞ as $u \to \pm\infty$, we have

FACT 2.3. *Every eigenvalue of π_A is simple and every eigenfunction of π_A is either even or odd.*

Let φ be an eigenfunction of π_A, which means $(-\partial^2/\partial u^2 + |u|)\varphi(u) = \lambda\varphi(u)$. Then the function

$$\psi(u) = \varphi(u + \lambda)$$

satisfies the formulae

$$\left(-\frac{\partial^2}{\partial u^2} + u\right)\psi(u) = 0 \quad \text{if } u \geq -\lambda, \tag{2.2}$$

$$\int_{-\lambda}^{\infty} |\psi(u)|^2 \, du = \int_{0}^{\infty} |\varphi(u)|^2 \, du \leq \|\varphi\|^2_{L^2(\mathbb{R})} < \infty. \tag{2.3}$$

We can extend $\psi|_{[-\lambda,\infty)}$ to the whole line keeping the condition (2.2); after normalization we obtain a function Φ such that

$$\left(-\frac{\partial^2}{\partial u^2} + u\right)\Phi(u) = 0 \quad \text{if } u \in \mathbb{R}, \qquad \int_{0}^{\infty} |\Phi(u)|^2 \, du < \infty, \qquad \Phi(0) = \frac{\Gamma(1/3)}{2\pi 3^{1/6}}.$$

These are exactly the conditions characterizing the Airy function. The following facts about this function can be found in [2], pp. 213–215:

FACT 2.4. *At infinity both $\Phi(u)$ and $-\Phi'(u)$ decrease to 0 faster than any e^{-Mu}, $M > 0$.*

FACT 2.5. *The function Φ has an analytic extension to the whole $\mathbb{C}$, given by the explicit formula*

$$\Phi(u) = \Phi_y(u) = C \int_{-\infty}^{\infty} e^{i(\frac{1}{3}(\xi+iy)^3 + u(\xi+iy))} \, d\xi \qquad \textit{for all } y > 0. \tag{2.4}$$

The extended function $\Phi(z)$ *satisfies the following estimate*: *For all* $k \in \mathbb{N}$, $z = x + iy \in \mathbb{C}$ *and some constants* C_k, *which depend only on* k,

$$|\Phi^{(k)}(z)| \le C_k(|z|+1)^{k/2}\, e^{\sqrt{|z|+1}\,|y|}. \tag{2.5}$$

It follows from Fact 2.3 that $\Phi(-\lambda) = 0$ or $\Phi'(-\lambda) = 0$ whenever λ is an eigenvalue of π_A. It turns out that also the converse implication holds. More precisely, we have

THEOREM 2.6. *The spectrum of* π_A *equals*

$$\sigma(\pi_A) = \{\lambda \in \mathbb{R} : \Phi(-\lambda) = 0 \text{ or } \Phi'(-\lambda) = 0\}. \tag{2.6}$$

Moreover, *the function*

$$\varphi_n(u) = C_n\Phi(u - \lambda_n) \quad \text{if } u \ge 0, \quad C_n = \Big(2\int_{-\lambda_n}^{\infty} |\Phi(u)|^2\, du\Big)^{-1/2}, \tag{2.7.a}$$

$$\varphi_n(u) = (-1)^{n+1}\varphi_n(-u) \quad \text{if } u \le 0, \tag{2.7.b}$$

is the normalized eigenfunction corresponding to the eigenvalue λ_n. *We have* $\Phi(-\lambda_n) = 0$ *for* n *even and* $\Phi'(-\lambda_n) = 0$ *for* n *odd.*

The next fact gives an estimate of the nth eigenvalue of π_A, and of the difference of two consecutive eigenvalues. The formula (2.8) follows from (2.6), and from the estimate of the zeros of Φ in [2], p. 215. The formula (2.9) holds, since the Airy function Φ oscillates faster than solutions of the differential equation $(-\partial^2/\partial z^2 + \lambda_n)\varphi = 0$ and slower than solutions of $(-\partial^2/\partial z^2 + \lambda_{n+1})\varphi = 0$ on the interval $[-\lambda_{n+1}, -\lambda_n]$ (cf. [3], pp. 311–316), and the above equations are satisfied by $\sin(\sqrt{\lambda_n}\, u + u_0)$ and $\sin(\sqrt{\lambda_{n+1}}\, u + u_1)$.

FACT 2.7. *The eigenvalues* λ_n *satisfy the following estimates*:

$$\lambda_n \sim \Big(\frac{3\pi}{4} n\Big)^{2/3}, \tag{2.8}$$

$$\frac{\pi}{2}\lambda_{n+1}^{-1/2} \le \lambda_{n+1} - \lambda_n \le \frac{\pi}{2}\lambda_n^{-1/2}. \tag{2.9}$$

The greatest zero of the function Φ' is $\lambda_1 = -1.0189\ldots < -1$, hence we obtain the estimate

FACT 2.8. *For all* $f \in D_{\pi_A}$,

$$\|\pi_A f\| \ge \|f\|. \tag{2.10}$$

3. Estimates of $\|P^n f\|_{L^2(\mathbb{G})}$ for $P \in \{X, Y, Z\}$. In this section we show that any operator of multiplication by a polynomial and some differential operators can be estimated by powers of the operator π_A. We use these results to estimate differential operators on the group $\mathbb{G}$ by powers of the operator A.

DEFINITION 3.1. Define V_1 to be the linear space spanned by $(\varphi_n)_{n=1}^{\infty}$, and V_2 to be the linear space spanned by

$$u^k\varphi_n\cdot 1_{\mathbb{R}^+},\quad u^k\frac{\partial}{\partial u}\varphi_n\cdot 1_{\mathbb{R}^+},\quad u^k\varphi_n\cdot 1_{\mathbb{R}^-},\quad u^k\frac{\partial}{\partial u}\varphi_n\cdot 1_{\mathbb{R}^-},\qquad k=0,1,2,\ldots,\ n=1,2,\ldots$$

(Note that by Fact 2.4, V_2 is contained in L^2.)

Most of the proofs in this section consist of checking inequalities for functions from the space V_1. Then the results for L^2-functions follow by the density of V_1 in L^2. Since higher derivatives of functions from V_1 do not necessarily belong to L^2, it is useful to consider also V_2. Notice that V_2 is preserved by the operators of multiplication by u, $|u|$, and also by $\partial/\partial u$ acting on $\mathbb{R}^*$. But it is not preserved by π_A if we consider the elements of this space as functions on $\mathbb{R}$. Throughout this section we use (2.10), which enables us to find explicit constants in the forthcoming inequalities. Some of them already appeared in [1].

PROPOSITION 3.2. *The following estimates hold*:

(i) $$\Big\|\frac{\partial}{\partial u}\varphi\Big\|_{L^2(\mathbb{R})}\le \|\pi_A^{1/2}\varphi\|_{L^2(\mathbb{R})}\qquad \text{if } \varphi\in D_{\pi_A^{1/2}},$$

(ii) $$\|u\varphi\|_{L^2(\mathbb{R})}^2+\Big\|\frac{\partial^2}{\partial u^2}\varphi\Big\|_{L^2(\mathbb{R})}^2\le 3\|\pi_A\varphi\|_{L^2(\mathbb{R})}^2\qquad \text{if } \varphi\in D_{\pi_A},$$

(iii) $$\Big\|\frac{\partial^3}{\partial u^3}\varphi\Big\|_{L^2(\mathbb{R})}\le 2\sqrt{3}\|\pi_A^{3/2}\varphi\|_{L^2(\mathbb{R})}\qquad \text{if } \varphi\in D_{\pi_A^{3/2}},$$

Proof. We have

$$\Big\|\frac{\partial}{\partial u}\varphi\Big\|_{L^2(\mathbb{R})}^2\le\Big\langle\Big(-\frac{\partial^2}{\partial u^2}+|u|\Big)\varphi,\varphi\Big\rangle=\langle\pi_A\varphi,\varphi\rangle=\|\pi_A^{1/2}\varphi\|_{L^2(\mathbb{R})}^2,$$

which proves (i). For (ii),

$$\begin{aligned}\Big\|\frac{\partial^2}{\partial u^2}\varphi\Big\|_{L^2(\mathbb{R})}^2&+\|u\varphi\|_{L^2(\mathbb{R})}^2\\ &=\Big\|\Big(-\frac{\partial^2}{\partial u^2}+|u|\Big)\varphi\Big\|_{L^2(\mathbb{R})}^2-2\operatorname{Re}\Big\langle-\frac{\partial^2}{\partial u^2}\varphi,|u|\varphi\Big\rangle\\ &=\|\pi_A\varphi\|_{L^2(\mathbb{R})}^2-2\operatorname{Re}\Big\langle\frac{\partial}{\partial u}\varphi,\frac{\partial}{\partial u}|u|\varphi\Big\rangle\end{aligned}$$

$$= \|\pi_A\varphi\|^2_{L^2(\mathbb{R})} - 2\,\mathrm{Re}\left\langle \frac{\partial}{\partial u}\varphi, \mathrm{sgn}(u)\varphi\right\rangle - 2\left\||u|^{1/2}\frac{\partial}{\partial u}\varphi\right\|^2_{L^2(\mathbb{R})}$$

$$\le \|\pi_A\varphi\|^2_{L^2(\mathbb{R})} + 2\left\|\frac{\partial}{\partial u}\varphi\right\|_{L^2(\mathbb{R})}\|\varphi\|_{L^2(\mathbb{R})} \le 3\|\pi_A\varphi\|^2_{L^2(\mathbb{R})}.$$

Finally, (iii) follows from (i) and (ii); indeed,

$$\left\|\frac{\partial^3}{\partial u^3}\varphi\right\|_{L^2(\mathbb{R})} \le \sqrt{3}\left\|\pi_A\frac{\partial}{\partial u}\varphi\right\|_{L^2(\mathbb{R})} = \sqrt{3}\left\|\frac{\partial}{\partial u}\pi_A\varphi - \mathrm{sgn}(u)\varphi\right\|_{L^2(\mathbb{R})}$$

$$\le \sqrt{3}(\|\pi_A^{3/2}\varphi\|_{L^2(\mathbb{R})} + \|\varphi\|_{L^2(\mathbb{R})}) \le 2\sqrt{3}\|\pi_A^{3/2}\varphi\|_{L^2(\mathbb{R})}. \ ■$$

REMARK 3.3. Modifying slightly the proof of (ii) of the last proposition we obtain, for $\varphi \in V_2$,

$$\left\|\frac{\partial^2}{\partial u^2}\varphi\right\|_{L^2(\mathbb{R}^*)} \le \left(\left\|\left(-\frac{\partial^2}{\partial u^2} + |u|\right)\varphi\right\|^2_{L^2(\mathbb{R}^*)} + 2\left\|\frac{\partial}{\partial u}\varphi\right\|_{L^2(\mathbb{R}^*)}\|\varphi\|_{L^2(\mathbb{R}^*)}\right)^{1/2}$$

$$\le \left\|\left(-\frac{\partial^2}{\partial u^2} + |u|\right)\varphi\right\|_{L^2(\mathbb{R}^*)} + \left\|\frac{\partial}{\partial u}\varphi\right\|_{L^2(\mathbb{R}^*)} + \|\varphi\|_{L^2(\mathbb{R}^*)}.$$

PROPOSITION 3.4. *If* $\varphi \in D_{\pi_A^k}$ *for some* $k \in \mathbb{N}$, *then*

$$\|u^k\varphi\|_{L^2(\mathbb{R})} \le 2^k(k+1)!\|\pi_A^k\varphi\|_{L^2(\mathbb{R})}.$$

Proof. Denote by C_k, $k \in \mathbb{N}$, and D_k, $k \in \mathbb{N}\setminus\{0\}$, the smallest numbers such that

$$\|u^k\varphi\|_{L^2(\mathbb{R})} \le C_k\|\pi_A^k\varphi\|_{L^2(\mathbb{R})}, \qquad \|\pi_A u^{k-1}\varphi\|_{L^2(\mathbb{R})} \le D_k\|\pi_A^k\varphi\|_{L^2(\mathbb{R})}.$$

Put also $D_0 = 1/2$. What we have to show is the estimate $C_k \le 2^k(k+1)!$. Notice that

$$\|u^k\varphi\|_{L^2(\mathbb{R})} \le C_1\|\pi_A u^{k-1}\varphi\|_{L^2(\mathbb{R})} \le C_1 D_k\|\pi_A^k\varphi\|_{L^2(\mathbb{R})},$$

hence

$$C_k \le C_1 D_k \le \sqrt{3}D_k \le 2D_k \quad \text{ if } k \ge 1.$$

The inequality $C_k \le 2D_k$ is also valid for $k = 0$, thus it is enough to verify that $D_k \le 2^{k-1}(k+1)!$. We prove this by induction.

For $k = 0, 1$ the inequality is obvious. For $k \ge 2$,

$$\pi_A u^{k-1}\varphi = u^{k-1}\pi_A\varphi - 2(k-1)\frac{\partial}{\partial u}(u^{k-2}\varphi) + (k-1)(k-2)u^{k-3}\varphi,$$

therefore

$$\|\pi_A u^{k-1}\varphi\|_{L^2(\mathbb{R})} \le C_{k-1}\|\pi_A^k\varphi\|_{L^2(\mathbb{R})} + 2(k-1)D_{k-1}\|\pi_A^{k-1}\varphi\|_{L^2(\mathbb{R})}$$
$$+ \ (k-1)(k-2)C_{k-3}\|\pi_A^{k-3}\varphi\|_{L^2(\mathbb{R})},$$
$$D_k \le C_{k-1} + 2(k-1)D_{k-1} + (k-1)(k-2)C_{k-3},$$

and using the inductive hypothesis we obtain $D_k \le 2^{k-1}(k+1)!$. ■

THEOREM 3.5. *If* $\varphi \in D_{\pi_A^{k/2}}$ *for some* $k \in \mathbb{N}$, *then*

$$\left\| \frac{\partial^k}{\partial u^k} \varphi \right\|_{L^2(\mathbb{R}^*)} \le C_k \|\pi_{A^{k/2}} \varphi\|_{L^2(\mathbb{R})}.$$

(*We denote by the same letters functions on* $\mathbb{R}$ *and their restrictions to* $\mathbb{R}^*$.)

Proof. The proof is by induction on k. The formula is obvious for $k = 0$, and for $k = 1, 2, 3$ it follows from Proposition 3.2. Let now $k \ge 4$. If $\varphi \in V_1$, then $\frac{\partial^k}{\partial u^k}\varphi \in V_2$, hence, by Remark 3.3,

$$\begin{aligned}
\left\| \frac{\partial^k}{\partial u^k} \varphi \right\|_{L^2(\mathbb{R}^*)} &\le \left\| \left(-\frac{\partial^2}{\partial u^2} + |u| \right) \frac{\partial^{k-2}}{\partial u^{k-2}} \varphi \right\|_{L^2(\mathbb{R}^*)} \\
&\quad + \left\| \frac{\partial^{k-1}}{\partial u^{k-1}} \varphi \right\|_{L^2(\mathbb{R}^*)} + \left\| \frac{\partial^{k-2}}{\partial u^{k-2}} \varphi \right\|_{L^2(\mathbb{R}^*)} \\
&\le \left\| \left(\frac{\partial^{k-2}}{\partial u^{k-2}} \pi_A - (k-2)\operatorname{sgn}(u) \frac{\partial^{k-3}}{\partial u^{k-3}} \right) \varphi \right\|_{L^2(\mathbb{R}^*)} \\
&\quad + C_{k-1} \|\pi_A^{(k-1)/2} \varphi\|_{L^2(\mathbb{R})} + C_{k-2} \|\pi_A^{(k-2)/2} \varphi\|_{L^2(\mathbb{R})} \\
&\le C_{k-2} \|\pi_A^{k/2} \varphi\|_{L^2(\mathbb{R})} + (k-2) C_{k-3} \|\pi_A^{(k-3)/2} \varphi\|_{L^2(\mathbb{R})} \\
&\quad + C_{k-1} \|\pi_A^{(k-1)/2} \varphi\|_{L^2(\mathbb{R})} + C_{k-2} \|\pi_A^{(k-2)/2} \varphi\|_{L^2(\mathbb{R})} \\
&\le (C_{k-1} + 2C_{k-2} + (k-2)C_{k-3}) \|\pi_A^{k/2} \varphi\|_{L^2(\mathbb{R})}. \ \blacksquare
\end{aligned}$$

LEMMA 3.6. *Let* ω *be a Hilbert–Schmidt operator on* $L^2(\mathbb{R})$, *and put* $B_0 = 1$, $B_1 = 1$, $B_2 = \sqrt{3}$, $B_3 = 2\sqrt{3}$, $C_k = 2^k (k+1)!$. *Then*

(i) $\|\pi^\lambda_{X^k} \omega\|_{\mathrm{HS}} \le B_k \|(\pi^\lambda_A)^{k/2} \omega\|_{\mathrm{HS}}$ *if* $k = 0, 1, 2, 3$,

(ii) $\|\pi^\lambda_{Y^k Z^l} \omega\|_{\mathrm{HS}} \le C_k \|(\pi^\lambda_A)^{k+3l/2} \omega\|_{\mathrm{HS}}$ *if* $k, l \in \mathbb{N}$.

Proof. We only prove (ii) using Proposition 3.4. The statement (i) follows from Proposition 3.2 in an analogous way.

Let $(e_n)_{n=1}^\infty$ be an orthonormal basis of $L^2(\mathbb{R})$. Then

$$\begin{aligned}
\|\pi^\lambda_{Y^k Z^l} \omega\|^2_{\mathrm{HS}} &= \sum_{n=1}^\infty \|\pi^\lambda_{Y^k Z^l} \omega e_n\|^2_{L^2(\mathbb{R})} = \sum_{n=1}^\infty \||\lambda|^{bk} u^k |\lambda|^{cl} \omega e_n\|^2_{L^2(\mathbb{R})} \\
&\le \sum_{n=1}^\infty C_k^2 \|(\pi^\lambda_A)^{k+3l/2} \omega e_n\|^2_{L^2(\mathbb{R})} = C_k^2 \|(\pi^\lambda_A)^{k+3l/2} \omega\|^2_{\mathrm{HS}}. \ \blacksquare
\end{aligned}$$

THEOREM 3.7. *Let the constants* B_k, C_k *be as in the previous lemma. Then*

(i) $\|X^k f\|_{L^2(\mathbb{G})} \le B_k \|A^{k/2} f\|_{L^2(\mathbb{G})}$ *if* $f \in D_{A^{k/2}}$, $k = 0, 1, 2, 3$,

(ii) $\|Y^k Z^l f\|_{L^2(\mathbb{G})} \le C_k \|A^{k+3l/2} f\|_{L^2(\mathbb{G})}$ *if* $f \in D_{A^{k+3l/2}}$, $k, l \in \mathbb{N}$.

Proof. Using the Plancherel formula (1.2), and the last lemma, we obtain

$$\begin{aligned}\|X^k f\|^2_{L^2(\mathbb{G})} &= C\int_{\mathbb{R}} \|\pi^\lambda_{X^k}\pi^\lambda_f\|^2_{\mathrm{HS}}|\lambda|^{2c-1}\,d\lambda\\ &\le C\int_{\mathbb{R}} B_k^2\|(\pi^\lambda_A)^{k/2}\pi^\lambda_f\|^2_{\mathrm{HS}}|\lambda|^{2c-1}\,d\lambda = B_k^2\|A^{k/2}f\|^2_{L^2(\mathbb{G})}.\end{aligned}$$

The inequality (ii) can be proved in a similar way. ∎

4. Regularity of the semigroup generated by π_A. For each $t>0$, $e^{-t\pi_A}$ is an integral operator with kernel

$$p_t(u,v)=\sum_{n=1}^{\infty} e^{-t\lambda_n}\varphi_n(u)\varphi_n(v). \tag{4.1}$$

PROPOSITION 4.1. *The function $p_t(u,v)$ has an extension to the set $K=\{(t,u,v):\operatorname{Re}t>0,\ u,v\in\mathbb{C}\}$ which is analytic (in each variable) at every point $(t,u,v)\in K$ such that $\operatorname{Re}u\ne0$, $\operatorname{Re}v\ne0$.*

Proof. Consider the analytic functions

$$\begin{aligned}&\varphi_n^+(z)=C_n\Phi(z-\lambda_n),\\ &\varphi_n^-(z)=(-1)^{n+1}C_n\Phi(-z-\lambda_n)\quad\text{ for } z\in\mathbb{C}.\end{aligned}$$

Then, by Theorem 2.6, the function φ_n defined as

$$\varphi_n(z)=\begin{cases}\varphi_n^+(z) & \text{if } \operatorname{Re}z\ge0,\\ \varphi_n^-(z) & \text{if } \operatorname{Re}z<0,\end{cases}$$

coincides for $z\in\mathbb{R}$ with the φ_n introduced in Corollary 2.2. The functions $\varphi_n(z)$ are analytic for $\operatorname{Re}z\ne0$ (for $\operatorname{Re}z=0$ they can even be discontinuous). Therefore it is enough to notice that by (2.5) and (2.8) the series $\sum_{n=1}^{\infty} e^{-t\lambda_n}\varphi_n^{\pm}(u)\varphi_n^{\pm}(v)$ are uniformly convergent on every compact subset of K. Thus they define analytic functions. ∎

PROPOSITION 4.2. *If $0\le k,l\le3$, $t>0$, then*

$$\frac{\partial^k}{\partial u^k}\frac{\partial^l}{\partial v^l}p_t(u,v)\in L^2(\mathbb{R}^2).$$

Proof. Using Proposition 3.2 we obtain

$$\begin{aligned}\sum_{n=1}^{\infty}\|e^{-t\lambda_n}\varphi_n^{(k)}(u)\varphi_n^{(l)}(v)\|_{L^2(\mathbb{R}^2)} &\le \sum_{n=1}^{\infty} e^{-t\lambda_n}\|\varphi_n^{(k)}(u)\|_{L^2(\mathbb{R})}\|\varphi_n^{(l)}(v)\|_{L^2(\mathbb{R})}\\ &\le B_kB_l\sum_{n=1}^{\infty} e^{-t\lambda_n}\lambda_n^{(k+l)/2}<\infty,\end{aligned}$$

thus

$$\frac{\partial^k}{\partial u^k}\frac{\partial^l}{\partial v^l}p_t(u,v)=\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n^{(k)}(u)\varphi_n^{(l)}(v). \ \blacksquare$$

LEMMA 4.3. *Set*

$$r(h)=p_t(hu_0,hv_0),$$

where u_0, v_0 *are fixed real numbers, not simultaneously* 0. *Then the limits*

$$g_1=\lim_{h\nearrow 0}r^{(3)}(h),\qquad g_2=\lim_{h\searrow 0}r^{(3)}(h)$$

exist and $g_1=-g_2\neq 0$. (*In particular* $r^{(3)}$ *is discontinuous at* 0.)

Proof. By the definition of the function r,

$$r^{(3)}(h)=\Big(\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n(\cdot\,u_0)\varphi_n(\cdot\,v_0)\Big)^{(3)}(h) \tag{4.2}$$

$$=\sum_{k=0}^{3}\bigg(\binom{3}{k}u_0^{3-k}v_0^k\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n^{(3-k)}(hu_0)\varphi_n^{(k)}(hv_0)\bigg).$$

(If $u_0=0$, then we only consider terms with $k=3$, and if $v_0=0$, then only those with $k=0$.)

For every $n\in\mathbb{N}$ we have $\varphi_n(0)=0$ or $\varphi_n'(0)=0$, hence

$$\begin{aligned}\varphi_n'(hu_0)\varphi_n''(hv_0)&=\varphi_n'(0)\varphi_n''(0)+o(h)=-\lambda_n\varphi_n'(0)\varphi_n(0)+o(h)=o(h),\\ \varphi_n(hu_0)\varphi_n'''(hv_0)&=\varphi_n(hu_0)(\operatorname{sgn}(hv_0)\varphi_n(hv_0)+(|hv_0|-\lambda_n)\varphi_n'(hv_0))\\ &=\operatorname{sgn}(hv_0)|\varphi_n(0)|^2+o(h).\end{aligned}$$

Each of the functions

$$\mathbb{R}^+\ni h\mapsto\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n^{(3-k)}(hu_0)\varphi_n^{(k)}(hv_0),$$

$$\mathbb{R}^-\ni h\mapsto\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n^{(3-k)}(hu_0)\varphi_n^{(k)}(hv_0)$$

has an analytic extension uniformly converging on compacta to the series

$$\sum_{n=1}^{\infty}e^{-t\lambda_n}(\varphi_n^{\pm})^{(3-k)}(hu_0)(\varphi_n^{\pm})^{(k)}(hv_0)$$

(with $\varphi_n^{\pm}$ defined as in the proof of Proposition 4.1). Hence we can pass to the limits $\lim_{h\nearrow 0}$ and $\lim_{h\searrow 0}$ under the sum sign in (4.2):

$$r^{(3)}(h)=\binom{3}{0}u_0^3\sum_{n=1}^{\infty}e^{-t\lambda_n}\operatorname{sgn}(hu_0)|\varphi_n(0)|^2$$

$$+\binom{3}{3}v_0^3\sum_{n=1}^{\infty}e^{-t\lambda_n}\operatorname{sgn}(hv_0)|\varphi_n(0)|^2+o(h)$$
$$=\operatorname{sgn}(h)(|u_0|^3+|v_0|^3)\sum_{n=1}^{\infty}e^{-t\lambda_n}|\varphi_n(0)|^2+o(h),$$

which means that

$$g_1=-(|u_0|^3+|v_0|^3)\sum_{n=1}^{\infty}e^{-t\lambda_n}|\varphi_n(0)|^2\neq 0,\qquad g_2=-g_1.\ \blacksquare$$

Taking $(u_0,v_0)=(0,1)$ and $(u_0,v_0)=(1,0)$ we obtain

COROLLARY 4.4. *The derivatives* $\frac{\partial^4}{\partial u^4}p_t(u,v)$ *and* $\frac{\partial^4}{\partial v^4}p_t(u,v)$ *do not belong to* L^2.

THEOREM 4.5. *The pointwise left and right derivatives*

$$\left(\frac{\partial^3}{\partial u^3}\right)^{-}p_t(0,v),\quad \left(\frac{\partial^3}{\partial u^3}\right)^{+}p_t(0,v),\quad \left(\frac{\partial^3}{\partial v^3}\right)^{-}p_t(u,0),\quad \left(\frac{\partial^3}{\partial v^3}\right)^{+}p_t(u,0)$$

exist for all v, u. *But outside a discrete set of* v *and* u *respectively we have*

$$\left(\frac{\partial^3}{\partial u^3}\right)^{-}p_t(0,v)\neq\left(\frac{\partial^3}{\partial u^3}\right)^{+}p_t(0,v),\quad \left(\frac{\partial^3}{\partial v^3}\right)^{-}p_t(u,0)\neq\left(\frac{\partial^3}{\partial v^3}\right)^{+}p_t(u,0).$$

In particular $\frac{\partial^3}{\partial u^3}p_t(0,v)$ *and* $\frac{\partial^3}{\partial v^3}p_t(u,0)$ *do not exist at these points.*

Proof. By symmetry we can only consider $\partial^3/\partial u^3$. We see that

$$\frac{\partial^2}{\partial u^2}p_t(u,v)=\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n''(u)\varphi_n(v)=\sum_{n=1}^{\infty}e^{-t\lambda_n}(|u|-\lambda_n)\varphi_n(u)\varphi_n(v),$$
$$\left(\frac{\partial^3}{\partial u^3}\right)^{\pm}p_t(0,v)=-\sum_{n=1}^{\infty}e^{-t\lambda_n}\lambda_n\varphi_n'(0)\varphi_n(v)\pm\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n(0)\varphi_n(v).$$

The two third derivatives are equal only in the case

$$\sum_{n=1}^{\infty}e^{-t\lambda_n}\varphi_n(0)\varphi_n(v)=0.$$

The left-hand side is real analytic for $v\geq 0$ and for $v\leq 0$, and moreover, its value is $\sum_{n=1}^{\infty}e^{-t\lambda_n}|\varphi_n(0)|^2\neq 0$ for $v=0$, thus it is 0 on a discrete set. $\blacksquare$

5. Regularity of the semigroup generated by A. In this section we prove that the semigroup e^{-tA} consists of convolution operators with L^2 functions. Then we discuss singularities of these functions. Applying (4.1) we get

$$\frac{c}{(2\pi)^2}\int_{\mathbb{R}} \|e^{-t\pi_A^\lambda}\|_{\mathrm{HS}}^2 |\lambda|^{2c-1}\,d\lambda = \frac{c}{(2\pi)^2}\int_{\mathbb{R}}\sum_{n=1}^{\infty} e^{-2t\lambda_n|\lambda|^b}|\lambda|^{2c-1}\,d\lambda$$
$$= \biggl(\frac{3}{16\pi^2}\sum_{n=1}^{\infty}\lambda_n^{-3}\biggr)t^{-3}.$$

According to (2.8),

$$\sum_{n=1}^{\infty}\lambda_n^{-3} \le \sum_{n=1}^{\infty} D(n^{2/3})^{-3} < \infty.$$

Hence by the Plancherel formula (1.2) and (1.3) we find that e^{-tA} is a convolution operator with a function $P_t \in L^2(G)$ such that

$$(5.1)\qquad P_t(x,y,z) = C\int_{\mathbb{R}}\int_{\mathbb{R}} p_t^\lambda(|\lambda|^a u, |\lambda|^a(u+x))|\lambda|^{a+2c-1}e^{-i\lambda^c(z+uy)}\,du\,d\lambda$$
$$= C\int_{\mathbb{R}}\int_{\mathbb{R}}\sum_{n=1}^{\infty} e^{-|\lambda|^b\lambda_n t}\varphi_n(|\lambda|^a u)\varphi_n(|\lambda|^a(u+x))$$
$$\times |\lambda|^{a+2c-1}e^{-i\lambda^c(z+uy)}\,du\,d\lambda.$$

Using Theorem 3.7(ii) and standard arguments we obtain

THEOREM 5.1. *For every $k, l \in \mathbb{N}$ and every (x,y,z) the pointwise derivative*

$$\frac{\partial^k}{\partial y^k}\frac{\partial^l}{\partial z^l}P_t(x,y,z)$$

exists, and the function $\frac{\partial^k}{\partial y^k}\frac{\partial^l}{\partial z^l}P_t$ is continuous on the whole $\mathbb{G}$.

THEOREM 5.2. *The derivative*

$$\frac{\partial^k}{\partial x^k}P_t(x,y,z)$$

exists for every $k \in \mathbb{N}$ and every point (x,y,z) such that $x \neq 0$.

Proof. Fix $x_0 > 0$. Let $\chi_1, \chi_2 \in C^\infty(\mathbb{R})$ be functions such that

$$\chi_1(x) = 0 \quad \text{if } x \le -\tfrac{2}{3}x_0, \qquad \chi_1(x) = 1 \quad \text{if } x \ge -\tfrac{1}{3}x_0,$$
$$0 \le \chi_1 \le 1, \qquad \chi_1(x) + \chi_2(x+x_0) = 1.$$

First we estimate the integral with respect to u in (5.1). For x close to x_0,

$$\frac{\partial^k}{\partial x^k}\int_{\mathbb{R}}\varphi_n(|\lambda|^a u)\varphi_n(|\lambda|^a(u+x))e^{-i\lambda^c uy}\,du$$
$$= |\lambda|^{ak}\int_{\operatorname{supp}\chi_1}\varphi_n(|\lambda|^a u)\varphi_n^{(k)}(|\lambda|^a(u+x))e^{-i\lambda^c uy}\chi_1(u)\,du$$

$$+ \int\limits_{u+x_0-x\in\operatorname{supp}\chi_2} \sum_{l+m+r=k} \binom{k}{lmr} (-|\lambda|^a)^l (i\lambda^c y)^m (-1)^r$$
$$\times \varphi_n^{(l)}(|\lambda|^a(u-x))\varphi_n(|\lambda|^a u) e^{-i\lambda^c(u-x)y} \chi_2^{(r)}(u+x_0-x)\,du.$$

We estimate these integrals using Theorem 3.5:

$$\Big|\frac{\partial^k}{\partial x^k} \int\limits_{\mathbb{R}} \varphi_n(|\lambda|^a u)\varphi_n(|\lambda|^a(u+x)) e^{-i\lambda^c uy}\,du\Big|$$
$$\le |\lambda|^{ak} \int\limits_{\mathbb{R}\setminus\{-x\}} |\varphi_n(|\lambda|^a u)\varphi_n^{(k)}(|\lambda|^a(u+x))|\,du$$
$$+ \sum_{l+m+r=k} \binom{k}{lmr} |\lambda|^{al+cm}|y|^m$$
$$\times \int\limits_{\mathbb{R}\setminus\{x\}} |\varphi_n^{(l)}(|\lambda|^a(u-x))\varphi_n(|\lambda|^a u)|\,du\,\|\chi_2^{(r)}\|_\infty$$
$$\le |\lambda|^{ak-a}\|\varphi_n\|_{L^2(\mathbb{R})}\|\varphi_n^{(k)}\|_{L^2(\mathbb{R}^*)}$$
$$+ C\sum_{l+m\le k} |\lambda|^{al+cm-a}|y|^m \|\varphi_n^{(l)}\|_{L^2(\mathbb{R}^*)}\|\varphi_n\|_{L^2(\mathbb{R})}$$
$$\le C|\lambda|^{ak-a}\lambda_n^{k/2} + C\sum_{l+m\le k}|\lambda|^{al+cm-a}|y|^m\lambda_n^{l/2} \le C\sum_{l+m\le k}|\lambda|^{a(l+3m-1)}\lambda_n^{l/2}.$$

We now show that

$$\frac{\partial^k}{\partial x^k}P_t(x,y,z) = C\int\limits_{\mathbb{R}}\sum_{n=1}^{\infty} e^{-|\lambda|^b\lambda_n t} \times \frac{\partial^k}{\partial x^k}\Big(\int\limits_{\mathbb{R}} \varphi_n(|\lambda|^a u)\varphi_n(|\lambda|^a(u+x))e^{-i\lambda^c uy}\,du\Big) \times |\lambda|^{a+2c-1}e^{-i\lambda^c z}\,d\lambda \tag{5.2}$$

and that the integral is absolutely convergent.

It is enough to estimate

$$\int\limits_{\mathbb{R}}\sum_{n=1}^{\infty} e^{-|\lambda|^b\lambda_n t}|\lambda|^{a(l+3m-1)}\lambda_n^{l/2}|\lambda|^{a+2c-1}\,d\lambda$$
$$= \sum_{n=1}^{\infty}\Big(\int\limits_{\mathbb{R}} e^{-|\lambda|^{2a}\lambda_n t}|\lambda|^{a(l+3m+6)-1}\,d\lambda\Big)\lambda_n^{l/2} = C\sum_{n=1}^{\infty}\lambda_n^{-(l+3m+6)/2}\lambda_n^{l/2}$$
$$= C\sum_{n=1}^{\infty}\lambda_n^{-(3m+6)/2} \le C\sum_{n=1}^{\infty} n^{-2-m} < \infty.$$

The integral on the right-hand side of (5.2) is a continuous function of (x,y,z), hence in fact it is equal to $\frac{\partial^k}{\partial x^k}P_t(x,y,z)$.

For $x_0<0$ the calculations are the same. ∎

COROLLARY 5.3. *The kernels P_t are C^∞ functions outside the plane $\{(x,y,z): x=0\}$.*

THEOREM 5.4. *The derivatives*
$$\frac{\partial^6}{\partial x^6}P_t(0,y,z),\qquad \Big(\frac{\partial^7}{\partial x^7}\Big)^+P_t(0,y,z),\qquad \Big(\frac{\partial^7}{\partial x^7}\Big)^-P_t(0,y,z)$$
exist for every y,z. Moreover, there exists an open interval B containing 0 *such that for all $z\in B$ and $y\in\mathbb{R}$,*
$$\Big(\frac{\partial^7}{\partial x^7}\Big)^+P_t(0,y,z)\neq\Big(\frac{\partial^7}{\partial x^7}\Big)^-P_t(0,y,z).$$
In particular $\frac{\partial^7}{\partial x^7}P_t(0,y,z)$ does not exist at these points.

Proof. The sixth derivative with respect to x of the inner integral in (5.1) is equal to
$$\begin{aligned}\frac{\partial^6}{\partial x^6}\int_{\mathbb{R}}&\varphi_n(|\lambda|^a u)\varphi_n(|\lambda|^a(u+x))e^{-i\lambda^c uy}\,du\\ &=|\lambda|^{3a}\frac{\partial^3}{\partial x^3}\int_{\mathbb{R}}\varphi_n(|\lambda|^a(u-x))\varphi_n^{(3)}(|\lambda|^a u)e^{-i\lambda^c(u-x)y}\,du\\ &=\sum_{l=0}^{3}\binom{3}{l}(-1)^{3-l}|\lambda|^{(6-l)a}(i\lambda^c y)^l\\ &\quad\times\int_{\mathbb{R}}\varphi_n^{(3-l)}(|\lambda|^a(u-x))\varphi_n^{(3)}(|\lambda|^a u)e^{-i\lambda^c(u-x)y}\,du.\end{aligned}$$
The integral
$$\begin{aligned}C\int_{\mathbb{R}}\sum_{n=1}^{\infty}&e^{-|\lambda|^b\lambda_n t}\\ &\times\frac{\partial^6}{\partial x^6}\Big(\int_{\mathbb{R}}\varphi_n(|\lambda|^a u)\varphi_n(|\lambda|^a(u+x))e^{-i\lambda^c uy}\,du\Big)|\lambda|^{a+2c-1}e^{-i\lambda^c z}\,d\lambda\end{aligned}$$
is absolutely convergent and defines a continuous function, hence, by (5.1), it is equal to $\frac{\partial^6}{\partial x^6}P_t(0,y,z)$. The terms with $l=1,2,3$ are differentiable. Moreover $\varphi_n^{(3)}(v)=\operatorname{sgn}(v)\varphi_n(v)+(|v|-\lambda_n)\varphi_n'(v)$. Observe that $(|v|-\lambda_n)\varphi_n'(v)$ has a derivative in L^2. Thus we only have to discuss the behavior of the

integral

$$- |\lambda|^{6a} \int_{\mathbb{R}} \operatorname{sgn}(u-x)\varphi_n(|\lambda|^a(u-x)) \operatorname{sgn}(u)\varphi_n(|\lambda|^a u) e^{-i\lambda^c(u-x)y}\, du$$
$$= -|\lambda|^{6a} \int_{\mathbb{R}} \operatorname{sgn}(u)\varphi_n(|\lambda|^a u) \operatorname{sgn}(u+x)\varphi_n(|\lambda|^a(u+x)) e^{-i\lambda^c uy}\, du.$$

The difference

$$\left(\left(\frac{\partial}{\partial x}\right)^+ - \left(\frac{\partial}{\partial x}\right)^-\right)\Big(-|\lambda|^{6a} \int_{\mathbb{R}} \operatorname{sgn}(u)\varphi_n(|\lambda|^a u) \times \operatorname{sgn}(u+x)\varphi_n(|\lambda|^a(u+x)) e^{-i\lambda^c uy}\, du\Big)$$

is 0 for $x \neq 0$, and $4|\lambda|^{6a}|\varphi_n(0)|^2$ for $x = 0$, hence

$$\left(\left(\frac{\partial^7}{\partial x^7}\right)^+ - \left(\frac{\partial^7}{\partial x^7}\right)^-\right) P_t(0, y, z) = C \int_{\mathbb{R}} \sum_{n=1}^{\infty} e^{-|\lambda|^b \lambda_n t} 4|\lambda|^{6a}|\varphi_n(0)|^2 |\lambda|^{a+2c-1} e^{-i\lambda^c z}\, d\lambda. \tag{5.3}$$

This integral is continuous, does not depend on y, and for $z = 0$ equals

$$4C \int_{\mathbb{R}} \sum_{n=1}^{\infty} e^{-|\lambda|^b \lambda_n t} |\lambda|^{6a} |\varphi_n(0)|^2 |\lambda|^{a+2c-1}\, d\lambda > 0.$$

Therefore (5.3) is not zero for z in a neighborhood of 0 and for all $y \in \mathbb{R}$. The theorem is proved. ∎

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Received 31 March 1998;
revised 6 August 1999 (3567)