COLLOQUIUM MATHEMATICUM

VOL. 84/85 2000 PART 1

RELATIVELY MINIMAL EXTENSIONS OF TOPOLOGICAL FLOWS

BY

MIECZYSŁAW K. MENTZEN (TORUŃ)

Abstract. The concept of relatively minimal (rel. min.) extensions of topological flows is introduced. Several generalizations of properties of minimal extensions are shown. In particular the following extensions are rel. min.: distal point transitive, inverse limits of rel. min., superpositions of rel. min. Any proximal extension of a flow Y with a dense set of almost periodic (a.p.) points contains a unique subflow which is a relatively minimal extension of Y. All proximal and distal factors of a point transitive flow with a dense set of a.p. points are rel. min. In the class of point transitive flows with a dense set of a.p. points, distal open extensions are disjoint from all proximal extensions. An example of a relatively minimal point transitive extension determined by a cocycle which is a coboundary in the measure-theoretic sense is given.

1. Introduction. The theory of minimal flows and their minimal extension has a long history and it enjoys a great collection of useful tools and valuable results. It was developed by J. Auslander, I. U. Bronshteĭn, R. Ellis, H. Furstenberg, E. Glasner and others. However, many natural examples of flows are not minimal, e.g. the full shift. The purpose of the present paper is to relativize the notion of minimality with respect to a given factor. We will study several analogies of the notion of minimality in topological dynamics. The definitions and facts below can be found in [2] and [6].

Throughout we will denote by T some fixed discrete group. If a continuous action of T is defined on a Hausdorff space X, then the pair (X,T) will be called a *flow*. Except in Section 3 we will assume X to be compact. To simplify notation, we will refer to (X,T) as X. If $x\in X$, the *orbit* $\{tx : t\in T\}$ is denoted by $\mathrm{O}(x)$ and the *orbit closure* $\overline{\{tx : t\in T\}}$ by $\overline{\mathrm{O}}(x)$. The flow X is *point transitive* if there is an $x_0\in X$ with dense orbit: $\overline{\mathrm{O}}(x_0)=X$. In this case we will write (X,x_0) instead of (X,T). A nonempty subset $X_0\subset X$ is *minimal* if it is closed, invariant ($tX_0=X_0$, $t\in T$) and contains no proper subset with these properties. In particular, the flow X is minimal iff $\overline{\mathrm{O}}(x)=X$ for each $x\in X$. If $x\in X$ is such that its orbit closure is a minimal set, then we call x an *almost periodic* point. For open

2000 *Mathematics Subject Classification*: Primary 54H20.

Key words and phrases: topological dynamics, flows, factors.

Research partly supported by KBN grant 2 P03A 002 14.

sets $U, V \subset X$ the *dwelling set* $D(U,V) \subset T$ is defined by

$$D(U,V) = \{t \in T : tU \cap V \neq \emptyset\}.$$

By definition, X is *topologically ergodic* if for any nonempty open sets $U, V \subset X$, $D(U,V) \neq \emptyset$. Equivalently, X is topologically ergodic iff each nonempty open invariant subset of X is dense. One can define the dwelling set for an $x \in X$ and an open set $U \subset X$ by

$$D(x,U) = \{t \in T : x \in tU\}.$$

Clearly X is point transitive iff there exists an $x_0 \in X$ such that $D(x_0,U) \neq \emptyset$ for every nonempty open $U \subset X$. Each point transitive flow is topologically ergodic, but not *vice versa*. Both notions coincide for instance on second countable Baire spaces. A point $x \in X$ is almost periodic iff for any open $\emptyset \neq U \subset X$ the dwelling set $D(x,U)$ is syndetic, i.e. there exists a compact subset K of T such that $KD(x,U) = \{kt : k \in K,\ t \in D(x,U)\} = T$.

Given two flows X and Y we say that Y is a *factor* of X, or that X is an *extension* of Y, if there exists a continuous map $\phi : X \to Y$ such that $\phi(X) = Y$ and ϕ is *equivariant*, i.e. $\phi(tx) = t\phi(x)$ for all $x \in X$ and $t \in T$. The function ϕ is called a *factor map* or an *extension*. If the surjectivity assumption is dropped we will speak about *homomorphisms*. If X is point transitive, $\overline{\mathrm{O}}(x_0) = X$, then Y is also point transitive and $y_0 = \phi(x_0)$ has dense orbit in Y. In such a situation we will write $\phi : (X, x_0) \to (Y, y_0)$. Writing $(X, x_0) \to (Y, y_0)$ we will understand that (X, x_0) is an extension of (Y, y_0) via some factor map $\phi : (X, x_0) \to (Y, y_0)$ with $y_0 = \phi(x_0)$.

If $\phi : X \to Y$ is a factor map, we can define a closed invariant equivalence relation $R_\phi \subset X \times X$ by $(x,y) \in R_\phi$ iff $\phi(x) = \phi(y)$. Obviously, the quotient space $X_{R_\phi} = X/R_\phi$ with the quotient topology is a compact Hausdorff space which is isomorphic to Y. This allows us to picture factors of X as invariant, closed equivalence relations on X, also called *factor relations*. Conversely, given such a relation R we can define a factor map $\phi : X \to X_R$ by $\phi(x) = [x]_R$ (here $[x]_R$ denotes the equivalence class of x). Note that if we have two factor relations R_1, R_2 and a factor map $\pi : X_{R_1} \to X_{R_2}$ with $\pi_{R_2} = \pi \circ \pi_{R_1}$, then $R_1 \subset R_2$.

If $\phi : X \to Z$ and $\psi : Y \to Z$ are extensions of the flow Z, then by the *relative product* of ϕ and ψ we understand the subflow $(R_{\phi\psi}, T)$ of $(X \times Y, T)$, where $R_{\phi\psi} = \{(x,y) \in X \times Y :\ \phi(x) = \psi(y)\}$. The relative product of ϕ and ψ is an extension of Z having both ϕ and ψ as factors via the projections π_X, π_Y, i.e. such that $\psi \circ \pi_X = \phi \circ \pi_Y$. In general this extension is not the smallest one with this property.

Two extensions $\phi : X \to Z$ and $\psi : Y \to Z$ are said to be *disjoint*, $\phi \perp \psi$, whenever each extension $\theta : Z^* \to Z$ of which ϕ and ψ are factors also has $R_{\phi\psi}$, the relative product of ϕ and ψ, as a factor. More precisely, if $\phi^* : Z^* \to X$, $\psi^* : Z^* \to Y$ are extensions such that $\phi \circ \phi^* = \psi \circ \psi^*$ then the map

$$Z^* \ni z^* \mapsto (\phi^*(x^*), \psi^*(z^*)) \in R_{\phi\psi}$$

is onto. To illustrate this definition we give the following commutative diagram of factor maps.

$$\begin{tikzcd}
Z^* \arrow[dd] \arrow[dr, "\psi^*"] \arrow[ddd, "\phi^*"] & & \\
& Y \arrow[dr, "\psi"] & \\
R_{\phi\psi} \arrow[ur] \arrow[dr] & & Z \\
& X \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Observe that $\phi \perp \psi$ iff for every closed invariant set $D \subset R_{\phi\psi}$ the conditions $\pi_X(D) = X$, $\pi_Y(D) = Y$ imply $D = R_{\phi\psi}$. If Z is the trivial flow, i.e. $Z = *$ is a one-point set, and $\phi : X \to *$, $\psi : Y \to *$ are disjoint, then we say that the flows X and Y are *disjoint* and write $X \perp Y$. It turns out that if $X \perp Y$ then at least one of X, Y is minimal.

By βT we denote the Čech–Stone compactification of the discrete group T. One can define an action of T on βT by extending the map $T \ni s \mapsto ts \in T \subset \beta T$ to a map $\beta T \ni p \mapsto tp \in \beta T$. Thus βT becomes a T-flow. There is a unique natural structure of a semigroup on βT such that for each $p \in \beta T$ the map $q \mapsto qp$ is continuous. Then βT acts on any flow X and for $x \in X$ we have $\beta T x = \overline{\mathrm{O}}(x)$. If (X, x_0) is a point transitive flow, then $\beta T x_0 = \overline{\mathrm{O}}(x_0) = X$. Thus we can see X as a factor of βT.

Now fix $x_0 \in X$. The (closed) set $\{p \in \beta T : \ px_0 = x_0\}$ contains an idempotent u ($uu = u$) and the principal ideal $\beta T u$ is a closed subset of βT satisfying $\beta T u x_0 = \beta T x_0 = \overline{\mathrm{O}}(x_0)$. The point x_0 is almost periodic iff there exists an idempotent u in some minimal ideal in βT such that $ux_0 = x_0$.

Let X be a flow and $x, y \in X$. We say that the pair (x, y) is *proximal* if there exists a $p \in \beta T$ such that $px = py$. If either $x = y$ or the pair (x, y) is not proximal, we call it *distal*. Assume now that $\phi : X \to Y$ is a factor map. Then ϕ is called *proximal* if each pair (x, y) from the factor relation R_ϕ is proximal. If each $(x, y) \in R_\phi$ is distal, then we call ϕ *distal*.

An extension $\phi : X \to Y$ is called a *group extension* if there is a topological compact Hausdorff group G acting continuously and equivariantly on X as a group of homeomorphisms such that the fibers of ϕ are precisely the G-orbits in X: $\forall x_1, x_2 \in X$ $(\phi(x_1) = \phi(x_2)$ iff $\exists g \in G$ $(x_2 = x_1 g))$

iff $x_1G = x_2G$). Equivalently, $\forall x \in G$ ($\phi^{-1}(\phi(x)) = xG$). Note that group extensions are open (i.e. the factor map is open) and not necessarily distal.

2. Relatively minimal extensions. Assume (X, T) to be a flow. Let R be a factor relation on X. Put

$$\mathcal{A}_R = \{D \subset X : D \text{ is closed, invariant and } \forall x \in X \ ([x]_R \cap D \neq \emptyset)\}.$$

Thus an invariant closed D is in $\mathcal{A}_R$ iff the map $D \ni x \mapsto [x]_R \in X_R$ is onto.

Clearly $\mathcal{A}_R$ is nonempty and is partially ordered by inclusion. If $\{D_\alpha : \alpha \in A\}$ is a chain and $x \in X$, then $\bigcap_{\alpha\in A} D_\alpha \cap [x]_R \neq \emptyset$ by compactness. Now, by the Zorn lemma, each member of $\mathcal{A}_R$ is a superset of some minimal element of this family.

Definition 2.1. The minimal (with respect to inclusion) elements of $\mathcal{A}_R$ will be called *relatively minimal* with respect to R. The family of all subsets of X which are relatively minimal with respect to R will be denoted by $\mathcal{M}(X, R)$ or $\mathcal{M}(R)$. If $\phi : X \to Y$ is a factor map, then we will often write $\mathcal{M}(X, Y)$ instead of $\mathcal{M}(R_\phi)$.

Let us write down the following observations.

Proposition 2.1. (i) *Assume that $X_0 \in \mathcal{M}(X, R)$. Then X_0 is point transitive if and only if X_R is point transitive.*

(ii) *If X_R is point transitive and $\overline{\mathrm{O}}([x_0]_R) = X_R$, then for each $X_0 \in \mathcal{M}(X, R)$ and for each $x \in [x_0]_R \cap X_0$ we have $\overline{\mathrm{O}}(x) = X_0$.*

(iii) *If X_R is minimal then the family $\mathcal{M}(X, R)$ consists of all minimal subsets of X.*

Proof. (i) If X_R is point transitive, then there is an $x_0 \in X_0$ such that $\overline{\mathrm{O}}([x_0]_R) = X_R$. Then $D = \overline{\mathrm{O}}(x_0) \subset X_0$ is a closed invariant set. Since X_R is point transitive, we have $D \in \mathcal{A}_R$. Hence $D = X_0$.

(ii) Set $D = \overline{\mathrm{O}}(x)$. Clearly $D \subset X_0$. If $[y]_R \in X_R$ then $[y]_R = p[x]_R = [px]_R$ for some $p \in \beta T$. Since $px \in D$, the set D meets $[y]_R$. Consequently, $D = X_0$.

(iii) is obvious.

The following proposition indicates a situation in which the almost periodic points are contained in each relatively minimal set.

Proposition 2.2. *If the extension $X \to X_R$ is proximal and Q denotes the closure of the set of all almost periodic points from X, then $Q \subset \bigcap \mathcal{M}(R)$.*

Proof. Let $D \in \mathcal{M}(R)$ and choose any almost periodic point $x \in X$. Then there is an $x' \in [x]_R \cap D$ and, by proximality of the extension $X \to X_R$, there is a $p \in \beta T$ such that $px = px' \in D$. Thus $\overline{\mathrm{O}}(x) \subset D$, so $x \in D$ and the result follows. ∎

Definition 2.2. Call X *relatively minimal* with respect to its factor Y ($\phi : X \to Y$), or equivalently a *relatively minimal extension* of Y, if $\mathcal{M}(Y) = \{X\}$. If this is the case, then we call ϕ a *relatively minimal map*. We then write $\phi : X \to Y$ rel. min. or $X \to Y$ rel. min.

From Proposition 2.1(iii) and Definition 2.2 we have the following corollary.

Corollary 2.1. *If Y is a minimal flow and X is relatively minimal with respect to Y, then X is minimal as well.*

Proposition 2.3. *Assume that we are given the following commuting diagram of factor maps.*

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, "\chi"'] & Y \arrow[d, "\psi"] \\ & Z \end{tikzcd}$$

Then χ is relatively minimal if and only if so are ϕ and ψ.

Proof. First assume that χ is relatively minimal. Let $X_0 \subset X$ be an invariant, closed set satisfying $\phi(X_0) = Y$. Then $\chi(X_0) = \psi\phi(X_0) = \psi(Y) = Z$ and therefore $X_0 = X$, so ϕ is relatively minimal. If $Y_0 \subset Y$ is invariant, closed and satisfies $\psi(Y_0) = Z$, then putting $X_0 = \phi^{-1}(Y_0)$ we obtain $\chi(X_0) = \psi\phi(X_0) = \psi(Y_0) = Z$. Thus $X_0 = X$, hence $Y_0 = Y$. This implies that ψ is relatively minimal.

The opposite implication is obvious. ∎

Proposition 2.4. *Let $\phi : X \to Y$ be relatively minimal. Then Y is topologically ergodic iff so is X.*

Proof. Assume that Y is topologically ergodic. Let $U \subset X$ be an open nonempty invariant set. Then $\phi(U^{\mathrm{c}})^{\mathrm{c}}$ is open invariant, hence by the topological ergodicity, dense in Y. Since $\phi(U^{\mathrm{c}})^{\mathrm{c}} \subset \phi(U) \subset \phi(\overline{U})$, it follows that $\phi(\overline{U})$ is dense in Y, thus $\phi(\overline{U}) = Y$. Because ϕ is rel. min., $\overline{U} = X$. ∎

Proposition 2.5. *Let $(A, \leq)$ be a directed set and let $(X, (\phi_\alpha)_{\alpha \in A})$ be the limit of an inverse system $\mathfrak{U} = ((Y_\beta)_{\beta \in A}, (\phi_{\alpha\beta})_{\alpha \leq \beta})$ of flows, where $\phi_{\alpha\beta} : Y_\beta \to Y_\alpha$ for $\alpha \leq \beta$. If all $\phi_{\alpha\beta}$'s are relatively minimal, then all ϕ_α's are relatively minimal as well.*

Proof. Given $\alpha \in A$ we find a relatively minimal extension $X_\alpha \subset X$ of Y_α via ϕ_α. First we show that $\phi_\beta(X_\alpha) = Y_\beta$ for each $\beta \in A$. If $\beta \leq \alpha$, then $\phi_\beta(X_\alpha) = \phi_{\beta\alpha}\phi_\alpha(X_\alpha) = \phi_{\beta\alpha}(Y_\alpha) = Y_\beta$. If $\alpha \leq \beta$, then $\phi_{\alpha\beta}(\phi_\beta(X_\alpha)) = \phi_\alpha(X_\alpha) = Y_\alpha$. Since $\phi_{\alpha\beta}$ is rel. min., $\phi_\beta(X_\alpha) = Y_\beta$. Finally, if β and α

are not comparable, then we find a $\gamma \in A$ such that $\alpha \le \gamma$, $\beta \le \gamma$. Now $\phi_\gamma(X_\alpha) = Y_\gamma$ and therefore $\phi_\beta(X_\alpha) = \phi_{\beta\gamma}\phi_\gamma(X_\alpha) = \phi_{\beta\gamma}(Y_\gamma) = Y_\beta$.

Now we prove that $X_\alpha = X$. Let $x \in X$. For each $\beta \in A$ we have $\phi_\beta(X_\alpha) = Y_\beta$, therefore there is an $x^\beta \in X_\alpha$ such that $\phi_\beta(x^\beta) = \phi_\beta(x)$. We intend to prove that $\lim x^\beta = x$. To do this fix a $\gamma \in A$ and an open $U \subset Y_\alpha$ satisfying $x \in \phi_\gamma^{-1}(U)$, equivalently, $\phi_\gamma(x) \in U$. If $\beta \ge \gamma$ then $\phi_\gamma(x^\beta) = \phi_{\gamma\beta}\phi_\beta(x^\beta) = \phi_{\gamma\beta}\phi_\beta(x) = \phi_\gamma(x) \in U$. Thus $x^\beta \in \phi_\gamma^{-1}(U)$, which proves that $\lim x^\beta = x$. Since X_α is closed, $x \in X_\alpha$.

We have proved that $X_\alpha = X$, hence $\phi_\alpha : X \to Y_\alpha$ is relatively minimal for all $\alpha \in A$. ∎

PROPOSITION 2.6. *Let $\varphi : X \to Y$ be a factor map. Then there exist a flow $\widetilde{X}$, factor maps $\widetilde{\varphi} : X \to \widetilde{X}$, $\psi : \widetilde{X} \to Y$ and a one-to-one homomorphism $\theta : Y \to \widetilde{X}$ such that*:

(i) $\varphi = \psi \circ \widetilde{\varphi}$, *i.e. the diagram*

$$\begin{tikzcd} X \arrow[r, "\widetilde{\varphi}"] \arrow[dr, "\varphi"'] & \widetilde{X} \arrow[d, "\psi"] \\ & Y \end{tikzcd}$$

commutes;

(ii) $\psi \circ \theta = \mathrm{id}_Y$;

(iii) $\widetilde{\varphi}$ *is relatively minimal.*

Proof. Define $R = R_\varphi = \{(x_1, x_2) \in X \times X : \varphi(x_1) = \varphi(x_2)\}$. Find an $X_0 \in \mathcal{M}(X, Y)$ and define an equivalence relation R_0 on X by setting

$$R_0 = [R \cap (X_0 \times X_0)] \cup \Delta_X.$$

The equivalence classes of R_0 are of the form

$$[x]_{R_0} = \begin{cases} \{x\} & \text{if } x \notin X_0, \\ [x]_R \cap X_0 & \text{if } x \in X_0. \end{cases}$$

Then $\Delta_X \subset R_0 \subset R$. Let $\widetilde{X} = X_{R_0}$, $\widetilde{\varphi} : X \to \widetilde{X}$ and $\psi : \widetilde{X} \to Y$ be the natural quotient maps. Clearly $\varphi = \psi\widetilde{\varphi}$ and $\psi\widetilde{\varphi}(X_0) = Y$. Let $\theta : Y \to \widetilde{X}$ be defined by $\theta([x]_R) = [x]_R \cap X_0 = [x]_{R_0}$, $x \in X_0$. Then $\theta(Y) = \widetilde{\varphi}(X_0)$ and

$$\psi\theta([x]_R) = \psi([x]_R \cap X_0) = [[x]_R \cap X_0]_R = [x]_R = \mathrm{id}_Y([x]_R).$$

Since ψ restricted to $\widetilde{\varphi}(X_0)$ is one-to-one, θ is continuous.

Finally we show that $\widetilde{\varphi}$ is rel. min. Assume that $W \subset X$ is a closed invariant set with $\widetilde{\varphi}(W) = \widetilde{X}$. Since $[x]_{R_0} = \{x\}$ for $x \notin X_0$, we have $X \setminus X_0 \subset W$. Now $\widetilde{\varphi}(W \setminus X_0) = Y \setminus \widetilde{\varphi}(X_0)$ and $\varphi(X_0 \cap W) = \psi\widetilde{\varphi}(X_0 \cap W) = \psi\widetilde{\varphi}(X_0) = \varphi(X_0) = Y$. Thus $X_0 \subset W$. We have shown that $W = X$. ∎

Let (X, x_0) be point transitive. Since the system $(\beta T, e)$ is an extension of (X, x_0), there exists a compact invariant subset $A \subset \beta T$ and an $a \in A$ such that $(A, a) \to (X, x_0)$ rel. min. Actually we have more:

LEMMA 2.1. *For any point transitive flow (X, x_0) there exists an idempotent $u \in \beta T$ such that $(\beta Tu, u) \to (X, x_0)$ rel. min. Moreover, each closed invariant $A \subset \beta T$ satisfying $A \to (X, x_0)$ rel. min. is of the form $A = \beta Tu$ for some idempotent $u \in \beta T$.*

Proof. Take any closed invariant $A \subset \beta T$ such that $(A, a) \to (X, x_0)$ rel. min. Let $B = \{p \in A : px_0 = x_0\}$. The set B is nonempty ($a \in B$), compact and $BB \subset B$. By Lemma 2.9 of [2], there exists an idempotent u in B. Since A is invariant, $\beta Tu \subset A$. On the other hand $\beta Tu(x_0) = \beta T(ux_0) = \beta Tx_0 = X$. This forces $\beta Tu = A$ because the extension $(A, u) \to (X, x_0)$ is relatively minimal. ■

LEMMA 2.2. *Let $\phi : X \to Y$ be a factor map and $Y = \overline{\mathrm{O}}(y_0)$. If $u \in \beta T$ is an idempotent such that $(\beta Tu, u) \to (Y, y_0)$ rel. min., then $\mathcal{M}(X, Y) = \{\beta Tux : x \in \phi^{-1}(y_0)\}$.*

Proof. If $x \in \phi^{-1}(y_0)$, then by Proposition 2.3, $\beta Tux \in \mathcal{M}(X, Y)$. On the other hand, if $X_0 \in \mathcal{M}(X, Y)$, then $X_0 = \overline{\mathrm{O}}(x)$ for some $x \in \phi^{-1}(y_0) \cap X_0$ (Proposition 2.1) and $X_0 = \overline{\mathrm{O}}(x) = \overline{\mathrm{O}}(ux) = \beta Tux$. ■

From Lemma 2.2 and Proposition 2.3 we get the following proposition.

PROPOSITION 2.7. *Let (Y, y_0) be a point transitive flow. Assume that $u \in \beta T$ is an idempotent such that the extension $(\beta Tu, u) \to (Y, y_0)$ is relatively minimal. Then for any extension $\phi : X \to Y$, ϕ is relatively minimal iff there exists an $x_0 \in \phi^{-1}(y_0)$ such that $\beta Tux_0 = X$.*

PROPOSITION 2.8. *If X is point transitive and $\phi : X \to Y$ is distal then ϕ is relatively minimal.*

Proof. Let $Y = \overline{\mathrm{O}}(y_0)$ and fix $X_0 \in \mathcal{M}(X, Y)$. Choose $x_0 \in \phi^{-1}(y_0) \cap X_0$, where $\phi : X \to Y$ is distal. Choose an idempotent $u \in \beta T$ such that $(\beta Tu, u) \to (Y, y_0)$ rel. min. Then $\phi(ux_0) = u\phi(x_0) = uy_0 = y_0 = \phi(x_0)$ and, by distality of ϕ, $ux_0 = x_0$. We have $X = \overline{\mathrm{O}}(x_0) = \beta Tx_0 = \beta T(ux_0) = \beta Tux_0$ and by Lemma 2.2, $\phi : X \to Y$ is relatively minimal. ■

PROPOSITION 2.9. *Let $\phi : X \to Y$ be a group extension and $X_0 \subset X$ be a point transitive subflow such that $\phi(X_0) = Y$. Then the extension $\phi : X_0 \to Y$ is relatively minimal. In particular, each point transitive group extension is rel. min.*

Proof. Let $\phi : X \to Y$ be a G-extension and $X_0 = \overline{\mathrm{O}}(x_0)$. Then $y_0 = \phi(x_0)$ has dense orbit in Y. Let $u \in \beta T$ be an idempotent such that $\beta Tu \to (Y, y_0)$ is rel. min. Then the extension $(\beta Tux_0, ux_0) \to (Y, y_0)$ is also rel. min.

Because $uy_0 = y_0$, we have $\phi(ux_0) = y_0$ and therefore there is a $g \in G$ such that $ux_0 = x_0g$. We also have $(ux_0)g = u(x_0g) = u(ux_0) = ux_0$, so $(ux_0)g = x_0g$. Since g acts on X as a homeomorphism, $ux_0 = x_0$, which finishes the proof. ■

DEFINITION 2.3. Let (X, T) be a flow. A closed invariant set $\emptyset \neq D \subset X$ will be called *almost minimal* if the set of almost periodic points of D is dense in D. The flow (X, T) is an *almost minimal flow* if X is an almost minimal set.

Usually the above property is referred to by writing that D or X is a B-set. However there is also the notion of B-flows, which is different (stronger) than the notion of B-set and may be ambiguous. To omit difficulties in formulating statements we will use the new name.

Clearly, each factor of an almost minimal flow is again almost minimal. The converse is not true, but we have the following lemma.

LEMMA 2.3. *If X_R is almost minimal and $X_0 \in \mathcal{M}(X, R)$, then X_0 is also almost minimal.*

Proof. Denote by ϕ the factor map $X \to X_R$ and by Q the set of all almost periodic points of X_0. Then $\phi(Q)$ is the set of all almost periodic points of X_R. Also $X_R = \overline{\phi(Q)} = \phi(\overline{Q})$ because X_R is almost minimal. Since $\overline{Q}$ is invariant and projects via ϕ onto X_R, we get $\overline{Q} = X_0$. ■

As a conclusion we have

COROLLARY 2.2. *A relatively minimal extension of an almost minimal flow is almost minimal.*

It is well known that any proximal extension of a minimal flow contains a unique minimal subset. The relative version of this theorem in the class of almost minimal flows is given by the following proposition.

PROPOSITION 2.10. *Assume that $\phi : X \to Y$ is a proximal extension and Y is an almost minimal flow. Then $\mathcal{M}(X, Y)$ consists of exactly one element. More precisely, if Q denotes the closure of the set of all almost periodic points from X, then $\mathcal{M}(Y) = \{Q\}$.*

Proof. Since the set of all almost periodic points in Y is dense, we have $\phi(Q) = Y$. Now, take any $X_0 \in \mathcal{M}(Y)$ and any almost periodic point $x \in X$. There is an almost periodic $x_0 \in X_0$ such that $\phi(x) = \phi(x_0)$. We can find a $p \in \beta T$ (ϕ is proximal) such that $px = px_0$. Then $px \in \overline{\mathrm{O}}(x_0)$ and, because x is almost periodic, $\overline{\mathrm{O}}(x) = \overline{\mathrm{O}}(x_0)$. Therefore $x \in X_0$ and $X_0 = Q$. ■

From Proposition 2.10 we get the following fact.

PROPOSITION 2.11. *If X is almost minimal and $X \to X_R$ is proximal then it is relatively minimal.*

From Propositions 2.8 and 2.11 we get the following conclusion.

COROLLARY 2.3. *If X is a point transitive almost minimal flow, then all proximal factors and all distal factors of X are relatively minimal.*

In the class of minimal flows two extensions ϕ and ψ are disjoint iff the relative product $R_{\phi\psi}$ is minimal. This is not true in the general situation. Nevertheless we have the following obvious fact.

PROPOSITION 2.12. *Let $\phi : X \to Z$ and $\psi : Y \to Z$ be two extensions such that at least one of the projections $\pi_X : R_{\phi\psi} \to X$, $\pi_Y : R_{\phi\psi} \to Y$ is relatively minimal. Then $\phi \perp \psi$.*

If we add some natural assumptions then the converse is true.

THEOREM 1. *Assume that $\phi : X \to Z$ and $\psi : Y \to Z$ are factor maps of point transitive flows and ϕ is relatively minimal. Then $\phi \perp \psi$ if and only if the projection $\pi_Y : R_{\phi\psi} \to Y$ is relatively minimal.*

Proof. The sufficiency is always true (Proposition 2.12). To prove the necessity assume that $\phi \perp \psi$ and choose $A \in \mathcal{M}(R_{\phi\psi}, X)$, $B \in \mathcal{M}(R_{\phi\psi}, Y)$. Then $\pi_X(A \cup B) = X$, $\pi_Y(A \cup B) = Y$. From $\phi \perp \psi$ we get $A \cup B = R_{\phi\psi}$. Now, because ϕ is rel. min., we can find $x_0 \in X$, $y_0 \in Y$ such that $X = \overline{\mathrm{O}}(x_0)$, $Y = \overline{\mathrm{O}}(y_0)$, $\phi(x_0) = \phi(y_0)$ (x_0 is an arbitrary element of $\phi^{-1}(\psi(y_0))$, where $\overline{\mathrm{O}}(y_0) = Y$). Thus $(x_0, y_0) \in A \cup B$. Assume that $(x_0, y_0) \in A$. This forces $\pi_Y(A) = Y$ and the disjointness of ϕ and ψ gives $A = R_{\phi\psi}$, so $\pi_X : R_{\phi\psi} \to X$ rel. min. The case $(x_0, y_0) \in B$ is similar and gives $\pi_Y : R_{\phi\psi} \to Y$ rel. min. We have proved that at least one of π_X, π_Y is rel. min. If π_X is rel. min., then, by Proposition 2.3, $\phi \circ \pi_X$ is rel. min., hence $\psi \circ \pi_Y$ is rel. min. and applying again Proposition 2.3 we conclude that ψ and π_Y are rel. min. ∎

We note that in general π_X is not rel. min. (see Example 2.1).

EXAMPLE 2.1. Let $X = \{0,1\}^{\mathbb{Z}}$, $Y = \{0,1,2\}^{\mathbb{Z}}$ be full shifts. Set $Z = X$ and let $\phi : X \to X$ be the identity map, $\phi = \mathrm{id}_X$. Define $\psi : Y \to X$ by the code $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 1$. Then

$$\begin{tikzcd}
& R_{\phi\psi} \arrow[dl, "\pi_X", "\text{not rel. min.}"'] \arrow[dr, "\pi_Y"', "\text{rel. min.}"] & \\
X \arrow[dr, "\phi", "\text{rel. min.}"'] & & Y \arrow[dl, "\psi"', "\text{not rel. min.}"] \\
& X &
\end{tikzcd}$$

Proof. Clearly ϕ is rel. min., and ψ is not rel. min. because $X \hookrightarrow Y$ as a subshift and ψ restricted to X is equal to $\phi = \mathrm{id}_X$. Also π_X is not rel. min.

since the set $D = \{(x,x) : x \in X\} \subset R_{\phi\psi}$ is closed, invariant and projects onto the whole X via π_X. To end the proof observe that $R_{\phi\psi} \subset X \times Y$ is just the graph of $\psi : Y \to X$, so π_Y is rel. min. ∎

In the class of minimal flows each distal extension is necessarily open. This is not true in general, even for point transitive almost minimal extensions. Therefore in the next theorem we assume that the distal extension ϕ is open.

THEOREM 2. *Let X, Y be two almost minimal point transitive flows. Assume that $\phi : X \to Z$ is distal and open, and $\psi : Y \to Z$ is proximal. Then ϕ and ψ are disjoint.*

Proof. Define $R_{\phi\psi} = \{(x,y) \in X \times Y : \phi(x) = \psi(y)\}$. We have the diagram

$$\begin{tikzcd} & R_{\phi\psi} \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & Z & \end{tikzcd}$$

Clearly, the projection $\pi_X : R_{\phi\psi} \to X$ is proximal. By Theorem 1, it suffices to show that $\pi_X : R_{\phi\psi} \to X$ is rel. min. To do this we will show that the flow $R_{\phi\psi}$ is almost minimal.

Since ψ is proximal, $\pi_X : R_{\phi\psi} \to X$ is proximal, and since ϕ is distal, so is $\pi_Y : R_{\phi\psi} \to Y$. Now let $U \subset X$, $V \subset Y$ be open sets such that $(U \times V) \cap R_{\phi\psi} \neq \emptyset$. Then $\psi^{-1}\phi(U) \subset Y$ is open and $(U \times V) \cap R_{\phi\psi} = U \times (\psi^{-1}\phi(U) \cap V) \cap R_{\phi\psi}$. Changing V to $\psi^{-1}\phi(U) \cap V$ we can assume that $\psi(V) \subset \phi(U)$. Because Y is almost minimal, there is an almost periodic $y \in V$. Then $\psi(y) \in \phi(U)$ and we can find an $x \in U$ such that $\phi(x) = \psi(y)$. Since y is almost periodic, so is $\phi(x) = \psi(y)$. If u is an idempotent from some minimal ideal in βT such that $uy = y$, then $u\psi(y) = \psi(uy) = \psi(y)$ and $\phi(x) = \psi(y) = u\psi(y) = u\phi(x) = \phi(ux)$. The distality of ϕ gives $ux = x$. Thus $u(x,y) = (x,y)$ and (x,y) is almost periodic. We have proved that $R_{\phi\psi}$ is almost minimal.

Now we can apply Proposition 2.11 to find that $\pi_X : R_{\phi\psi} \to X$ is relatively minimal. Theorem 1 gives $\phi \perp \psi$. ∎

COROLLARY 2.4. *All point transitive distal group extensions of an almost minimal flow Z are disjoint from all proximal point transitive almost minimal extensions of Z.*

Unlike the case of minimal extensions, in general a distal extension need not be open. An example is given below.

EXAMPLE 2.2. Let $X = \{0,1\}^{\mathbb{Z}}$ be a full shift. Denote by $\overline{0}$ and $\overline{1}$ the fixed sequences from X: $\overline{0}[n] = 0$, $\overline{1}[n] = 1$, $n \in \mathbb{Z}$. Denote by Δ_X the diagonal in the Cartesian square $X \times X$, i.e. $\Delta_X = \{(x,x) : x \in X\}$. Let $D = \{\overline{0}, \overline{1}\}$ and let Y be the factor of X given by the relation $\Delta_X \cup (D \times D)$. Then X is a point transitive almost minimal flow, the extension $X \to Y$ is distal and not open, because the image of the set $[0] = \{x \in X : x[0] = 0\}$ is not open in Y.

3. Flows defined by $\mathbb{Z}$-cocycles. Let X be a compact Hausdorff space and $\tau : X \to X$ a homeomorphism; then we define a flow $(X, \mathbb{Z})$, where the action of $\mathbb{Z}$ on X is given by $(x,n) \mapsto \tau^n x$. In what follows, for a $\mathbb{Z}$-flow given by a homeomorphism τ, we will write (X,τ) rather than $(X,\mathbb{Z})$. The more traditional notation has an obvious advantage: it describes the action of the group $\mathbb{Z}$ on X.

Let G be a topological group with unit element e. For a given continuous map $\varphi : X \to G$ one can define a $\mathbb{Z}$-cocycle $\varphi^{(n)}$ by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(\tau^{n-1}x)\varphi(\tau^{n-2}x)\ldots\varphi(\tau x)\varphi(x), & n \geq 1, \\ e, & n = 0, \\ \varphi(\tau^{-n}x)^{-1}\varphi(\tau^{-n+1}x)^{-1}\ldots\varphi(\tau^{-1}x)^{-1}, & n \leq -1. \end{cases}$$

Then the cocycle condition $\varphi^{(n+k)}(x) = \varphi^{(n)}(\tau^k x)\varphi^{(k)}(x)$ is satisfied. Thus a continuous map φ defines a $\mathbb{Z}$-cocycle $\varphi^{(n)}$. Conversely, each $\mathbb{Z}$-cocycle $\Psi : \mathbb{Z} \times X \to G$ is of the form $\Psi(n,x) = \varphi^{(n)}(x)$, where $\varphi(x) = \Psi(1,x)$. Therefore we will call a continuous function $\varphi : X \to G$ a $\mathbb{Z}$-cocycle. For such a cocycle φ define $\tau_\varphi : X \times G \to X \times G$ by setting

$$\tau_\varphi(x,g) = (\tau x, \varphi(x)g).$$

The (not necessarily compact) flow $(X \times G, \tau_\varphi)$ is called a *cocycle extension* of (X,τ). Since X is compact, $X \times G$ is compact (locally compact) iff so is G. Clearly

$$\tau_\varphi^n(x,g) = (\tau^n x, \varphi^{(n)}(x)g).$$

There is a natural action of G on $X \times G$ defined by $(X \times G) \times G \ni ((x,g),h) \mapsto (x, gh) \in X \times G$.

Schmidt's theory of essential values for measurable cocycles ([5]) is easily adapted to the case of a topological flow (see e.g. [4]).

DEFINITION 3.1. Let (X,τ) be a $\mathbb{Z}$-flow, G a topological group and $\varphi : X \to G$ a (continuous) cocycle. A $g_0 \in G$ is an *essential value* of φ if for each nonempty open $U \subset X$, $g_0 \in V \subset G$ there exists an integer N such that

$$U \cap \tau^{-N}U \cap (\varphi^{(N)})^{-1}(V) \neq \emptyset.$$

The set of all essential values of φ will be denoted by $E(\varphi)$.

LEMMA 3.1 ([4]). *The set $E(\varphi)$ is a closed subgroup of G.*

PROPOSITION 3.1 ([4]). *Assume that (X,τ) is topologically ergodic. Then $(X\times G,\tau_\varphi)$ is topologically ergodic if and only if $E(\varphi)=G$.*

Let (X,τ) be a compact $\mathbb{Z}$-flow and Y a compact Hausdorff space. Assume that G is a topological group and $\mathcal{G}\subset \mathrm{Aut}(Y)$ a representation of G. We will denote the action of G on Y by $(g,y)\mapsto S_g(y)$. Additionally assume that $S_g x=\mathrm{id}\, x=x$ for some x implies $g=e$, the unit element of G. Thus we have got a flow (Y,G). Let $\varphi: X\to G$ be a $\mathbb{Z}$-cocycle. Define $\widetilde{\tau}_\varphi : X\times Y\to X\times Y$ by

$$\widetilde{\tau}_\varphi(x,y)=(\tau x, S_{\varphi(x)}(y)).$$

This class of topological flows has been introduced by Glasner and Weiss [3]. We now show a basic property of such kind of extensions.

THEOREM 3. *Assume that the flow (Y,G) is minimal and τ_φ is point transitive. Then the extension $(X\times Y,\widetilde{\tau}_\varphi)\to(X,\tau)$ is relatively minimal. In particular $\widetilde{\tau}_\varphi$ is point transitive.*

Proof. By Proposition 2.1(ii) it suffices to show that there is an $x_0\in X$ such that for each $y\in Y$ the point $(x_0,y)\in X\times Y$ has dense orbit via $\widetilde{\tau}_\varphi$. First observe that if $(x_0,g_0)\in X\times G$ has dense orbit, then for each p, $(\tau^p x_0,e)$ has dense orbit as well. Choose $x_0\in X$ such that (x_0,e) has dense orbit in $X\times G$. Fix $y_0\in Y$. Let $U\subset X$ and $V\subset Y$ be two nonempty open sets. Find $p\in\mathbb{Z}$ such that $\tau^p x_0\in U$. Define $\widetilde{\tau}^p_\varphi(x_0,y_0)=(x_1,y_1)$, $W=\{g\in G:\ S_g(y_1)\in V\}$. Then W is a nonempty ((Y,G) is minimal) open set. There is $n\in\mathbb{Z}$ with $\tau^n_\varphi(T^p x_0,e)\in U\times W$. Then $\widetilde{\tau}^{n+p}_\varphi(x_0,y_0)=\widetilde{\tau}^n_\varphi(x_1,y_1)=(\tau^n x_1, S_{\varphi^{(n)}(x_1)}(y_1))\in U\times V$ because $\varphi^{(n)}(x_1)=\varphi^{(n)}(\tau^p x_0)\in W$. ∎

EXAMPLE 3.1. Let $X=\{-1,1\}^{\mathbb{Z}}$, $\tau: X\to X$ be the shift, $G=\mathbb{Z}$. Define $\varphi: X\to\mathbb{Z}$ by $\varphi(x)=x[0]$. We will prove that τ_φ is topologically ergodic. By Proposition 3.1 it suffices to show that $E(\varphi)=\mathbb{Z}$. Fix $n\in\mathbb{Z}$. Let $B=[a_{-k},a_{-k+1},\dots,a_0,a_1,\dots,a_k]$ be a block of -1's and 1's and put $U=\{x\in X:\ x[-k,k]=B\}$. We now show that there is an integer N such that $U\cap\tau^{-N}U\cap(\varphi^{(N)})^{-1}(\{n\})\neq\emptyset$. To do this define a sequence $x_0\in X$ in the following way. Set $x_0[-k,k]=B$. Define $s=-(a_0+a_1+\ldots+a_k+a_{-k}+a_{-k+1}+\ldots+a_{-1})+n$. If $s<0$ then add $|s|$ times the symbol -1 to the right of $x_0[-k,k]$. If $s>0$ then add s times the symbol 1 to the right of $x_0[-k,k]$. Next add the block B. Put 1's at the other positions of x_0. Now $x_0,\tau^{k+|s|+k}(x)\in U$ and $\varphi^{(k+|s|+k)}(x_0)=n$. Thus τ_φ is topologically ergodic, thus point transitive.

Let $Y=\{y\in\mathbb{C}:\ |y|=1\}$ and let $z_0\in Y$ be such that $\{z_0^n:\ n\in\mathbb{Z}\}$ is dense in Y. Define a minimal action of $G=\mathbb{Z}$ on Y by $S_n(y)=z_0^n y$.

By Theorem 3, the homeomorphism $\widetilde{\tau}_\varphi : X \times Y \to X \times Y$, $\widetilde{\tau}_\varphi(x,y) = (\tau x, z_0^{\varphi(x)} y)$, is point transitive and the extension $\widetilde{\tau}_\varphi \to \tau$ is rel. min.

In [1] there is given a construction of so called cocycles of product type. Example 1.7 in [1] presents such a cocycle which is continuous and a coboundary in measure-theoretic sense, so nonergodic. We will show that this cocycle is point transitive.

Example 3.2. Let $(a_n)_{n\geq 1}$ be a sequence of positive integers. Set

$$\Omega = \prod_{n=1}^{\infty} \{0,1,\ldots,a_n - 1\}.$$

Equip Ω with the product discrete topology to get a compact Abelian group with addition defined by

$$(x+y)[n] = x[n] + y[n] + \varepsilon_n \bmod a_n,$$

where $\varepsilon_1 = 0$ and

$$\varepsilon_n = \begin{cases} 0, & x[n] + y[n] + \varepsilon_n < a_n, \\ 1, & x[n] + y[n] + \varepsilon_n \geq a_n. \end{cases}$$

Let $x_0 = (1,0,0,0,\ldots)$ be the unit element of the group Ω. Define $\tau : \Omega \to \Omega$ by $\tau x = x_0 + x$. Then $\overline{\mathrm{O}}(x_0) = \overline{\{kx_0 : k \in \mathbb{Z}\}} = \Omega$. Actually the flow (Ω, τ) is minimal. Let $\varphi : \Omega \setminus \{-x_0\} \to \mathbb{R}$ be of the form

$$\varphi(x) = \sum_{n=1}^{\infty} (b_n(\tau x) - b_n(x)),$$

where $b_n(x) = \beta_n(x[n])$, $\beta_n : \{0,1,\ldots,a_n-1\} \to \mathbb{R}$. Since for $x \neq -x_0 = (a_1 - 1, a_2 - 1, a_3 - 1, \ldots)$, τx differs from x only in finitely many places, the summation above is in fact finite. In measure-theoretic ergodic theory the cocycle φ is called a cocycle of product type. Actually the definition of such cocycles works if we replace $\mathbb{R}$ by any Abelian topological group ([1]). Now assume additionally that

1. $a_n \geq 3$, $n \geq 1$,
2. $\sum_n 1/a_n < \infty$,
3. $\beta_{2k+1} = 0$, $k \geq 1$,
4. $\beta_{2n}(k) = \begin{cases} 1/n, & k = 1, \\ 0, & k \neq 1. \end{cases}$

Then φ is continuous on $\Omega \setminus \{-x_0\}$ and $|\varphi(x)| \leq 2/\ell(x)$, where $\ell(x) = \min\{n \geq 1 : x[n] < a_n - 1\} < \infty$ for $x \neq -x_0$ (see [1]). Now, if $x_n \to -x_0$ then $\ell(x_n) \to \infty$, hence $\varphi(x_n) \to 0$. Setting $\varphi(-x_0) = 0$ we get a continuous cocycle $\varphi : \Omega \to \mathbb{R}$.

We will prove that $E(\varphi) = \mathbb{R}$. It suffices to show that $1/m \in E(\varphi)$ for each negative integer m.

Fix $m < 0$ and $\varepsilon > 0$. Let $V = (1/m - \varepsilon, 1/m + \varepsilon)$. Let $U \subset \Omega$ be an open set. We can assume that $U = [\alpha_1, \ldots, \alpha_t]$. Find $t < n_1 < \ldots < n_s$ such that

$$\left| \frac{1}{n_1} + \ldots + \frac{1}{n_s} + \frac{1}{m} \right| < \varepsilon.$$

Let $x = (\alpha_1, \ldots, \alpha_t, 1, 1, \ldots)$. Put $q_1 = 1$, $q_n = a_1 \ldots a_n$, $n \geq 1$, $N = a_{2n_1} q_{2n_1} + \ldots + a_{2n_s} q_{2n_s}$. By [1],

$$\tau^N(x)[k] = \begin{cases} \alpha_k, & k \leq t, \\ 1, & k \neq 2n_i,\ k \neq 2n_i + 1, \\ 0, & k = 2n_i, \\ 2, & k = 2n_i + 1. \end{cases}$$

Clearly $x \in U \cap \tau^{-N}(U)$. Now

$$\begin{aligned} \varphi^{(N)}(x) &= \sum_{i=0}^{N-1} \varphi(\tau^i x) = \sum_{i=0}^{N-1} \sum_{k=1}^{\infty} (b_k(\tau^{i+1}x) - b_k(\tau^i x)) \\ &= \sum_{k=1}^{\infty} (b_k(\tau^N x) - b_k(x)) = \sum_{k=1}^{\infty} (\beta_{2k}(\tau^N x[2k]) - \beta_{2k}(x[2k])) \\ &= \beta_{2n_1}(\tau^N x[2n_1]) - \beta_{2n_1}(x[2n_1]) + \ldots \\ &\quad + \beta_{2n_s}(\tau^N x[2n_s]) - \beta_{2n_s}(x[2n_s]) \\ &= -\frac{1}{n_1} - \frac{1}{n_2} - \ldots - \frac{1}{n_s} \in V. \end{aligned}$$

Thus $E(\varphi) = \mathbb{R}$ and by Proposition 3.1, τ_φ is point transitive. Now, by Theorem 3, for each compact minimal flow $(Y, \mathbb{R})$ the extension $\widetilde{\tau}_\varphi \to \tau$ is relatively minimal, hence $\widetilde{\tau}_\varphi$ is minimal. It is shown in [1] that from the measure-theoretic point of view, φ is a coboundary, i.e. there is a measurable function $f : \Omega \to \mathbb{R}$ such that $\varphi(x) = f(\tau x) - f(x)$. Therefore the sets $A_y = \{(x, S_{f(x)}(y)) : x \in X\} \subset X \times T$, where $y \in Y$, are $\widetilde{\tau}_\varphi$-invariant and hence $\widetilde{\tau}_\varphi$ is strongly nonergodic. Thus we have obtained a minimal flow for which the product measure decomposes into measures concentrated on graphs of functions $\Omega \ni x \mapsto S_{f(x)}(y) \in Y$, $y \in Y$.

REFERENCES

[1] J. Aaronson, M. Lemańczyk, C. Mauduit and H. Nakada, *Koksma's inequality and group extensions of Kronecker transformations*, in: Algorithms, Fractals, and Dynamics, Plenum Press, New York, 1995, 27–50.

[2] R. Ellis, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.

[3] S. Glasner and B. Weiss, *On the construction of minimal skew products*, Israel J. Math. 34 (1979), 321–336.

[4] M. Lemańczyk and K. K. Mentzen, *Topological ergodicity of real cocycles over minimal rotations*, preprint.
[5] K. Schmidt, *Cocycles of Ergodic Transformation Groups*, Lecture Notes in Math. 1, MacMillan of India, 1977.
[6] J. de Vries, *Elements of Topological Dynamics*, Kluwer Acad. Publ., 1993.

Faculty of Mathematics and Computer Science
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: mentzen@mat.uni.torun.pl

Received 4 May 1999;
revised 20 August 1999 (3748)