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DISJOINTNESS OF THE CONVOLUTIONS FOR CHACON'S AUTOMORPHISM

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Abstract. The purpose of this paper is to show that if σ is the maximal spectral type of Chacon's transformation, then for any $d \neq d'$ we have $\sigma^{*d} \perp \sigma^{*d'}$. First, we establish the disjointness of convolutions of the maximal spectral type for the class of dynamical systems that satisfy a certain algebraic condition. Then we show that Chacon's automorphism belongs to this class.

Let us consider a measure preserving invertible transformation T of the Lebesgue space (X,μ). We associate with T the unitary operator $\widehat{T} : f(x) \mapsto f(Tx)$ on $L^2(X,\mu)$. Let σ be the maximal spectral type of $\widehat{T}$ restricted to the subspace H of functions with zero mean.

It is an important problem of spectral theory of dynamical systems to investigate properties of convolutions of the maximal spectral type σ (see [2], [3] and [6]–[8]). This question originates from Kolmogorov's well-known problem concerning the *group property* of the spectrum. It was discovered that for some automorphisms the spectral type σ and the convolution $\sigma * \sigma$ are mutually singular (see [5]–[8]). An example is the so-called κ-*mixing* automorphism, i.e. a transformation T with the following property: there exists a subsequence k_j such that $\widehat{T}^{k_j}$ converges weakly to the operator $\kappa\Theta + (1-\kappa)\mathbb{I}$, where Θ is the orthoprojection onto the subspace of constants and $\mathbb{I}$ is the identity operator. This property is known to be generic for measure preserving transformations (see [8]).

Another generic property of automorphisms is the existence of a subsequence k_j such that $\widehat{T}^{k_j} \to \frac{1}{2}\mathbb{I} + \frac{1}{2}\widehat{T}$. This property implies $\sigma \perp \sigma * \sigma$ as well. (This fact was established first by Lemańczyk. Parreau extended this observation by showing that $\sigma \perp \sigma^{*d}$ for all d. Ryzhikov also obtained the same result and used it for solving Rokhlin's problem on homogeneous spectrum (see [2]). Ageev deduced this statement as a consequence of his results concerning spectral multiplicity of $T \times T$.)

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It is known that Chacon's well-known automorphism has the property mentioned above. The following question (raised by del Junco and Lemańczyk [3]) has remained open: are all the d-fold convolutions σ^{*d} of the maximal spectral type σ pairwise singular for Chacon's map? In this paper we show that the answer is affirmative. Namely, we establish (Section 2) that the closure of the powers of Chacon's automorphism contains a sequence of symmetric square polynomials which tends to the operator $\left(\frac{1}{2}\mathbb{I}+\frac{1}{2}\widehat{T}\right)^2$, and we show that this condition implies the disjointness of the convolutions.

1. Disjointness of convolutions. Let $\mathrm{Cl}(T)$ be the set of all operators cK, where c is a positive number and K belongs to the weak closure of the powers of the operator $\widehat{T}$.

THEOREM 1.1. *Let σ be the maximal spectral type of a weakly mixing automorphism T. Suppose that for some sequence a_n of distinct positive numbers the set $\mathrm{Cl}(T)$ contains the polynomials*

$$Q_n(\widehat{T})=\mathbb{I}+a_n\widehat{T}+\widehat{T}^2, \qquad \textit{where } \mathbb{I} \textit{ is the identity operator.}$$

Then all the convolutions σ^{*d} are mutually singular.

Proof. Let us fix integers $d'>d>1$ and show that $\sigma^{*d}\perp\sigma^{*d'}$. Suppose that an operator $J: H^{\otimes d}\to H^{\otimes d'}$ satisfies

$$J\,\underbrace{\widehat{T}\otimes\ldots\otimes\widehat{T}}_{d}=\underbrace{\widehat{T}\otimes\ldots\otimes\widehat{T}}_{d'}\,J,$$

where H is the subspace in $L^2(X,\mu)$ of functions with zero mean. It is enough to prove that $J=0$. Indeed, it is evident that σ^{*d} is the spectral type of the operator $\widehat{T}^{\otimes d}$ restricted to the subspace $H^{\otimes d}$. Suppose that $\sigma^{*d}\not\perp\sigma^{*d'}$. Then there are two cyclic subspaces $C_1\subset H^{\otimes d}$ and $C_2\subset H^{\otimes d'}$ with the same spectral measure. Let J be an operator establishing a unitary equivalence between the restriction of $\widehat{T}^{\otimes d}$ to C_1 and the restriction of $\widehat{T}^{\otimes d'}$ to C_2 which is zero on $C_1^{\perp}$. Then, evidently, $JT^{\otimes d}=T^{\otimes d'}J$ and $J\neq 0$.

For any $K\in\mathrm{Cl}(T)$ we have $JK^{\otimes d}=\gamma(K)K^{\otimes d'}J$, where $\gamma(K)$ is a positive constant that depends on K. In particular, for $K=Q_n(\widehat{T})$,

$$J(\mathbb{I}+a_n\widehat{T}+\widehat{T}^2)^{\otimes d}=\gamma_n\,(\mathbb{I}+a_n\widehat{T}+\widehat{T}^2)^{\otimes d'}J,\qquad \gamma_n=\frac{1}{(2+a_n)^{d'-d}}.$$

The left part of this equation can be represented in the form $J\sum_{i=0}^{d}a_n^iW_i^{(d)}$, where

$$W_i^{(d)}=\sum_{\substack{(r_1,\ldots,r_d)\\ r_k\in\{-1,0,1\},\,|r_1|+\ldots+|r_d|=d-i}}\widehat{T}^{1+r_1}\otimes\ldots\otimes\widehat{T}^{1+r_d}.$$

Since the dimension of the space spaned by $W_k^{(d)}$ is not greater than $d+1$, there exists a non-trivial sequence of reals c_i such that

$$J\sum_{n=1}^{d+2} c_n Q_n(\widehat{T})^{\otimes d} = 0.$$

This implies that

$$\sum_{n=1}^{d+2} \gamma_n c_n Q_n(\widehat{T})^{\otimes d'} J = 0.$$

We will show that the operators $W_i^{(d')}J$ are linearly independent. It will follow that the operators $Q_n(\widehat{T})^{\otimes d'}J$, $1 \le n \le k$, are linearly independent if and only if $k \le d'+1$. (This follows directly from the representation $Q_n(\widehat{T})^{\otimes d'} = \sum_{i=0}^{d'} a_n^i W_i^{(d')}$ and the fact that the a_n are distinct.) Thus, the linear combination above cannot be zero because $d+2 = (d+1)+1 \le d'+1$ (recall that $d < d'$). This contradiction completes the proof.

The only thing we must show is that the $W_i^{(d')}J$ are linearly independent. Indeed, any non-trivial linear combination $\sum_i c_i W_i^{(d')}J$ has the form $V(\widehat{T},\dots,\widehat{T})J = 0$, where V is some non-trivial polynomial of d' variables. If $J \ne 0$, then there exists a function f such that $Jf \ne 0$. Let us pass to the spectral representation of $\widehat{T}$. Namely, set

$$U : L^2(\mathbb{T},\sigma) \to L^2(\mathbb{T},\sigma) : \phi(z) \mapsto z\phi(z)$$

and let $\Phi : L^2(X,\mu) \to L^2(\mathbb{T},\sigma)$ be the unitary operator that conjugates $\widehat{T}$ and U: $\Phi\widehat{T} = U\Phi$.

Then for the function $F = \Phi^{\otimes d'}Jf$ on $\mathbb{T}^{d'}$ we have

$$0 = \Phi^{\otimes d'} V(\widehat{T},\dots,\widehat{T})(Jf) = V(z_1,\dots,z_{d'})F.$$

Thus, F is supported on the manifold $\mathcal{N} = \{V(z_1,\dots,z_{d'}) = 0\}$. It is not hard to prove that, since V is a polynomial, we have $\sigma^{\otimes d'}(\mathcal{N}) = 0$. Indeed, suppose, for simplicity, that $d' = 2$. Then there are finitely many points $z_1^{(j)}$ such that $\mathcal{N} \cap (\{z_1^{(j)}\} \times \mathbb{T})$ is not finite. It is known that a transformation is weakly mixing iff it has continuous spectrum. Hence, $(\sigma \times \sigma)(\mathcal{N}) = 0$, because $\widehat{T}$ is weakly mixing. Thus, $Jf = 0$ and J must be zero; but $J \ne 0$, and we have proved that the $W_i^{(d')}$ are linearly independent.

2. Chacon's automorphism. Let $h_1 = 1$ and $h_{j+1} = 3h_j + 1$ be the sequence of heights. Note that $h_j = (3^j - 1)/2$. *Chacon's automorphism* T is the rank-1 transformation that is built via a cutting-and-stacking construction described below (see [4] and [1]). At the jth stage we cut a tower of height h_j into 3 equal subtowers, add one spacer to the top of the middle subtower and stack these towers together.

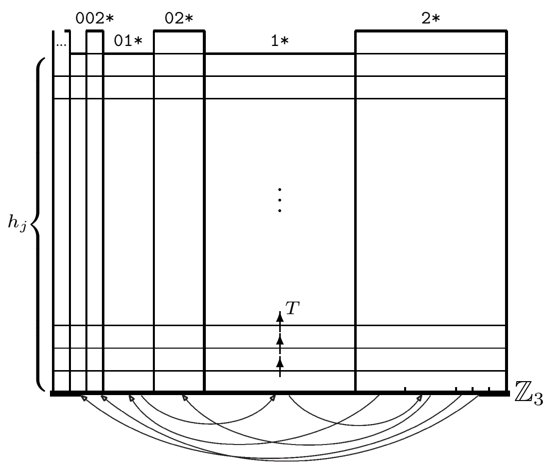

Fig. 1. Chacon's automorphism

Our purpose is to prove the following

THEOREM 2.1. *Let σ be the maximal spectral type of Chacon's automorphism. Then for any $d \neq d'$ we have $\sigma^{*d} \perp \sigma^{*d'}$.*

This theorem is a direct corollary of Theorem 1.1 and Lemma 2.3.

We begin with a definition of Chacon's map which will be more convenient in what follows. Namely, for each $j \geq 1$, we may consider T as an integral automorphism over the 3-adic rotation, by identifying the base B_j of the jth tower with the group $\mathbb{Z}_3$ of 3-adic integers in the following way. $\mathbb{Z}_3$ may be considered as the set of all sequences $a_1a_2\ldots$, where $a_k \in \{\mathtt{0},\mathtt{1},\mathtt{2}\}$. Consider a point $x \in B_j$. When cutting the jth tower into 3 subtowers we get a partition $B_j = B_{j,0} \sqcup B_{j,1} \sqcup B_{j,2}$ such that

$$B_{j,0} \xrightarrow{T^{h_j}} B_{j,1} \xrightarrow{T^{h_j+1}} B_{j,2} \xrightarrow{\cdots} B_{j,0}.$$

Suppose that $x \in B_j \simeq [0,1]$. We associate with x its ternary decomposition $a_1a_2a_3\ldots$ (A more geometric way is to put $a_1 = a$ if $x \in B_{j,a}$, and to define $a_2a_3\ldots$ similarly considering $x - a/3 \in B_{j,0} = B_{j+1}$ instead of x.) Then T can be viewed as the integral automorphism over the map

$$R : \mathbb{Z}_3 \to \mathbb{Z}_3 : a_1a_2a_3\ldots \mapsto a_1a_2a_3 + \mathtt{100}\ldots$$

with the ceiling function $h_j + \phi$, where

$$\phi(a) = \begin{cases} 0 & \text{if } a = \mathtt{22\ldots20*\ldots}, \\ 1 & \text{if } a = \mathtt{22\ldots21*\ldots}, \end{cases}$$

where $*$ designates an arbitrary element of $\{\mathtt{0},\mathtt{1},\mathtt{2}\}$. (Note that the conditional measure $\mu(\cdot|B_j)$ coincides after identification with the Haar measure λ on $\mathbb{Z}_3$.)

It is convenient to redefine the function ϕ so that $\phi(a) = 0$ if $a = \texttt{00...01*...}$ The new system is conjugate to Chacon's automorphism. Let us describe precisely the sets where ϕ is constant:

$$\phi(a) = \begin{cases} 0 & \text{if } a \in \texttt{(0)1*...}, \\ 1 & \text{if } a \in \texttt{(0)2*...}, \end{cases}$$

where (0)1*... and (0)2*... abbreviate the following two sets:

(0)1*... :	1*	(0)2*... :	2*
	01*		02*
	001*		002*
	0001*		0002*
	...		...

Each of these two *tables* should be meant as a code of a partition of some set in $\mathbb{Z}_3$. A row of a table designates an element of a partition, for example, 01* is the set of sequences $a_1 a_2 \ldots$ such that $a_1 = \texttt{0}$ and $a_2 = \texttt{1}$. Here * means an arbitrary element of $\{\texttt{0}, \texttt{1}, \texttt{2}\}$ (more exactly, we assume that any symbol can appear at this position), and a * at the end of a line abbreviates **...

It is a simple corollary from the definition of Chacon's transformation that

$$\widehat{T}^{-h_j} \xrightarrow{\text{w}} \lambda(\texttt{(0)1*})\mathbb{I} + \lambda(\texttt{(0)2*})\widehat{T} = \tfrac{1}{2}\mathbb{I} + \tfrac{1}{2}\widehat{T},$$

where λ is the Haar measure on $\mathbb{Z}_3$, and $\mathbb{I}$ is the identity operator. Indeed, fix measurable sets A and C. Since Chacon's map is a rank-1 transformation, for any $\varepsilon > 0$ there exists j_0 such that for all $j \geq j_0$ we have $\mu(A \triangle A_j) < \varepsilon$ and $\mu(C \triangle C_j) < \varepsilon$, where A_j and C_j are the unions of levels of the jth tower. Then the base B_j can be uniquely divided into sets $B_j^{(0)}$ and $B_j^{(1)}$ so that for any level $L = T^k B_j$ except one, the set $T^{h_j} L$ has the form $L^{(0)} \sqcup T^{-1} L^{(1)}$, where $L = L^{(0)} \sqcup L^{(1)}$ and $L^{(\alpha)} = T^k B_j^{(\alpha)}$. Moreover, $\mu(B_j^{(0)} | B_j) = \lambda(\texttt{(0)1*}) = \mu(B_j^{(1)} | B_j) = \lambda(\texttt{(0)2*}) = 1/2$. It follows directly from this picture that

$$\mu(T^{h_j} A_j \cap C_j) \approx \tfrac{1}{2}\mu(A_j \cap C_j) + \tfrac{1}{2}\mu(T^{-1} A_j \cap C_j)$$

with precision $1/h_j$. Taking into account the fact that A_j and C_j approximate A and C respectively we get the desired convergence

$$\widehat{T}^{-h_j} = (\widehat{T}^{h_j})^* \xrightarrow{\text{w}} \tfrac{1}{2}\mathbb{I} + \tfrac{1}{2}(\widehat{T}^{-1})^* = \tfrac{1}{2}\mathbb{I} + \tfrac{1}{2}\widehat{T}.$$

It is also not hard to check using the same technique that

$$\widehat{T}^{-kh_j} \xrightarrow{\text{w}} P_k(\widehat{T}) = \int\limits_{\mathbb{Z}_3} \widehat{T}^{\phi^{(k)}(a)}\, d\lambda(a) = \sum_{t=0}^{k} c_{k,t} \widehat{T}^t,$$

where

$$\phi^{(k)}(a) = \sum_{t=0}^{k-1} \phi(R^{-t}a).$$

(Here we have used the fact that λ is invariant under R.) Note that $P_k(\widehat{T})$ is a polynomial in $\widehat{T}$. Let $\widetilde{P}_k(\widehat{T}) = \widehat{T}^{-r_k} P_k(\widehat{T})$, where r_k is the smallest power of $\widehat{T}$ in $P_k(\widehat{T})$. Evidently, $\widetilde{P}_k(\widehat{T}) \in \mathrm{Cl}(\widehat{T})$ as well. Below several polynomials $\widetilde{P}_k(\widehat{T})$ are given ([1]):

$$\begin{aligned}
\widetilde{P}_1(\widehat{T}) &= \tfrac{1}{2}\mathbb{I} + \tfrac{1}{2}\widehat{T}, & \widetilde{P}_4(\widehat{T}) &= \tfrac{2}{9}\mathbb{I} + \tfrac{5}{9}\widehat{T} + \tfrac{2}{9}\widehat{T}^2,\\
\widetilde{P}_2(\widehat{T}) &= \tfrac{1}{6}\mathbb{I} + \tfrac{4}{6}\widehat{T} + \tfrac{1}{6}\widehat{T}^2, & \widetilde{P}_5(\widehat{T}) &= \tfrac{1}{18}\mathbb{I} + \tfrac{8}{18}\widehat{T} + \tfrac{8}{18}\widehat{T}^2 + \tfrac{1}{18}\widehat{T}^3,\\
\widetilde{P}_3(\widehat{T}) &= \tfrac{1}{2}\mathbb{I} + \tfrac{1}{2}\widehat{T}, & \widetilde{P}_6(\widehat{T}) &= \tfrac{1}{6}\mathbb{I} + \tfrac{4}{6}\widehat{T} + \tfrac{1}{6}\widehat{T}^2.
\end{aligned}$$

One can notice that all the polynomials $\widetilde{P}_{3^n}$ coincide with P_1 (Lemma 2.2). The deeper Lemma 2.3 proves the following observation: the polynomials $\widetilde{P}_{3^n+1}$ are symmetric square polynomials that tend to P_1^2 as $n \to \infty$.

LEMMA 2.2. *Let* $l_n = (3^n - 1)/2$. *Then*

$$\phi^{(3^n)}(a) = \begin{cases} l_n & \text{if } a \in *^n(\mathtt{0})\mathtt{1}*, \\ l_n + 1 & \text{if } a \in *^n(\mathtt{0})\mathtt{2}*, \end{cases} \qquad \text{where} \quad *^n = \underbrace{*\ldots*}_{n},$$

and $P_{3^n}(\widehat{T}) = \frac{1}{2}\widehat{T}^{l_n} + \frac{1}{2}\widehat{T}^{l_n+1}$.

Proof. This lemma is proved by induction on n. The case $n = 0$ is trivial. We will establish the lemma for $n = 1$. The proof for arbitrary n is completely analogous. Consider three translations of the function ϕ:

		$t=0$	$t=1$	$t=2$
		`1*`	`2*`	`0*`
$\phi(R^{-t}a) = 0$	on	`01*`	`11*`	`21*`
		`001*`	`101*`	`201*`
		`2*`	`0*`	`1*`
$\phi(R^{-t}a) = 1$	on	`02*`	`12*`	`22*`
		`002*`	`102*`	`202*`

Let A_v^t be the set on which $\phi(R^{-t}a) = v$. Fixing v_0, v_1, v_2 we calculate $A_{v_0}^0 \cap A_{v_1}^1 \cap A_{v_2}^2$. It can be easily checked that it is non-empty only when $v_1 + v_2 + v_3$ is either 1 or 2. Suppose that $v_0 = v_1 = 0$ and $v_2 = 1$. Then the only non-trivial intersection is $\mathtt{1*} \cap \mathtt{1(0)1*} \cap \mathtt{1*} = \mathtt{1(0)1*}$. Moreover, in all similar chains sets are ordered. In the intersection considered we have $\mathtt{1*} \subset \mathtt{11*}, \mathtt{101*}, \ldots$ So, any intersection is uniquely described by the longer code, e.g., `1(0)1*`. All intersections in our case are represented in the

[1] See www.geocities.com/apri7 for the first 122 polynomials $P_k(z)$.

following table:

$$
\begin{array}{llll}
0,0,1: & \texttt{1(0)1*}, & 1,1,0: & \texttt{0(0)2*}, \\
0,1,0: & \texttt{0(0)1*}, & 1,0,1: & \texttt{2(0)2*}, \\
\underline{1,0,0}: & \underline{\texttt{2(0)1*}}, & \underline{0,1,1}: & \underline{\texttt{1(0)2*}}, \\
\cup: & \texttt{*(0)1*}, & \cup: & \texttt{*(0)2*}.
\end{array}
$$

It is evident that $\phi^{(3)}(a) = 1$ iff $a \in \texttt{*(0)1*}$.

Lemma 2.3. *$\widehat{T}^{-l_n} P_{3^n+1}(\widehat{T})$ are square polynomials,*

$$
\begin{aligned}
\widehat{T}^{-l_n} P_{3^n+1}(\widehat{T}) &= \frac{(3^{n+1}-1) + 2(3^{n+1}+1)\widehat{T} + (3^{n+1}-1)\widehat{T}^2}{4 \cdot 3^{n+1}} \\
&\to \left(\frac{1}{2} + \frac{1}{2}\widehat{T}\right)^2, \qquad n \to \infty.
\end{aligned}
$$

Proof. First, note that

$$
\phi^{(3^n+1)}(a) = \phi^{(3^n)}(a) + \phi(R^{-3^n} a).
$$

Since both $\phi^{(3^n)}$ and $\phi \circ R^{-3^n}$ take two values, these functions are uniquely described by the two corresponding partitions (see the discussion above). Let us see how these partitions look (Figs. 2 and 3).

`*...*1*`	`*...*2*`
`*...*01*`	`*...*02*`
`*...*001*`	`*...*002*`
`*...*0001*`	`*...*0002*`

Fig. 2. Partitions for $\phi^{(3^n)}$

`1*`	`2*`
`01*`	`02*`
`001*`	`002*`
`.....`	`.....`
`0...1*`	`0...2*`
`0...02*`	`0...00*`
`0...011*`	`0...012*`
`0...0101*`	`0...0102*`
`0...01001*`	`0...01002*`

Fig. 3. Partitions for $\phi \circ R^{-3^n}$

Suppose that $\phi^{(3^n)} - l_n$ and $\phi \circ R^{-3^n}$ equal v on the sets C_v and A_v respectively. It can be easily seen from Figures 2 and 3 that

$$C_0 \cap A_0 = \bigcup_{p=0}^{n-1} \mathtt{0}^p\mathtt{1*}^{n-1-p}\mathtt{(0)1*} \cup \mathtt{0}^n\mathtt{1(0)1*},$$
$$C_1 \cap A_1 = \bigcup_{p=0}^{n-1} \mathtt{0}^p\mathtt{2*}^{n-1-p}\mathtt{(0)2*} \cup \mathtt{0}^n\mathtt{0(0)2*},$$

and that

$$\lambda(C_0 \cap A_0) = \lambda(C_1 \cap A_1) = \frac{1}{2}\sum_{p=0}^{n-1}\frac{1}{3^{p+1}} + \frac{1}{2\cdot 3^{n+1}} = \frac{3^{n+1}-1}{4\cdot 3^{n+1}} \to \frac{1}{4}$$

as $n \to \infty$. To complete the proof we only have to recall that if $P_k(z) = \sum_{t=0}^{k} c_{k,t}z^t$, then $\sum_{t=0}^{k} c_{k,t} = 1$.

Proof of Theorem 2.1. It is shown in Lemma 2.3 that $1 + (2+\varepsilon_n)\widehat{T} + \widehat{T}^2 \in \mathrm{Cl}(T)$ with distinct ε_n. Thus, Theorem 2.1 follows immediately from Theorem 1.1.

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