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A note on a question of Abe

by

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Abstract. Assuming large cardinals, we show that every κ-complete filter can be generically extended to a V-ultrafilter with well-founded ultrapower. We then apply this to answer a question of Abe.

1. Weakly precipitous filters. A set $\mathcal{F}$ is a *filter* if it is closed under intersections, $\emptyset \notin \mathcal{F}$, and whenever $A \subseteq B \subseteq \bigcup \mathcal{F}$ with $A \in \mathcal{F}$, then $B \in \mathcal{F}$. In what follows κ is always a regular cardinal $> \omega$. A filter $\mathcal{F}$ is κ-*complete* iff it is closed under intersections of size $< \kappa$.

Definition 1.1. Let $\mathcal{F}$ be a κ-complete filter. We say $\mathcal{F}$ is *weakly precipitous* if there is a partial order $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{G}$ such that it is forced that $\dot{G}$ is a V-κ-complete ultrafilter extending $\mathcal{F}$ with well-founded ultrapower. We say $\mathcal{F}$ is α-*weakly precipitous* if there is a partial order $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{G}$ such that it is forced that $\dot{G}$ is a V-κ-complete ultrafilter extending $\mathcal{F}$ with $j_{\dot{G}}(\alpha)$ in the well-founded part of the ultrapower.

If κ is strongly compact then every κ-complete filter can be extended to a κ-complete ultrafilter. If we use a generic embedding instead of a strongly compact embedding, then (large cardinals imply that) for every κ, every κ-complete filter is weakly precipitous.

Recall that S is *stationary* if for every $f : (\bigcup S)^{<\omega} \to \bigcup S$ there is an $a \in S$ that is closed under f. We have $\mathbb{P}_{<\delta} = \{S \in V_\delta \mid S \text{ is stationary}\}$, ordered by $S \le T$ iff $\bigcup S \supseteq \bigcup T$ and for all $a \in S$, $a \cap (\bigcup T) \in T$. This generalization of stationary and the following theorem appear in [W1] and [W2].

Theorem 1.2 (Woodin). *Assume δ is a Woodin cardinal, $G \subseteq \mathbb{P}_{<\delta}$ is generic, and $j_G : V \to M$ is the generic embedding. Then $M^{<\delta} \subseteq M$ in $V[G]$.*

Lemma 1.3. *Assume $\mathcal{F}$ is a κ-complete filter and there is a Woodin cardinal $> |\bigcup \mathcal{F}|$. Then $\mathcal{F}$ is weakly precipitous.*

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Proof. We may assume $\bigcup\mathcal{F}$ is a cardinal λ and $\lambda\geq\kappa$. Let $\delta>\lambda$ be a Woodin cardinal. The forcing $\mathbb{P}$ that witnesses that $\mathcal{F}$ is weakly precipitous is

$$\mathbb{P}_{<\delta}\restriction\{a\subseteq V_{\lambda+1}\mid |a|<\kappa\;\&\;a\cap\kappa\in\kappa\}.$$

Let $H\subseteq\mathbb{P}$ be generic and $j:V\to M$ the generic embedding (so M is well-founded). It is easy to see (using techniques from [W1]; also see [M2], Chapter 9) that $\mathrm{cp}(j)=\kappa$, $j''\mathcal{F}\in M$, and $j(\kappa)>|j''\mathcal{F}|$ (this last inequality holds since $\mathcal{F}\subseteq V_{\lambda+1}$). Since $j(\mathcal{F})$ is a $j(\kappa)$-complete filter and $j''\mathcal{F}\subseteq j(\mathcal{F})$, there is a $c\in\bigcap j''\mathcal{F}$.

Now in $V[H]$ define a V-ultrafilter $\mathcal{G}$ on λ by $A\in\mathcal{G}$ iff $c\in j(A)$. Clearly, $\mathcal{G}$ is a V-κ-complete ultrafilter extending $\mathcal{F}$. Since $\mathrm{Ult}(V,\mathcal{G})$ can be embedded into M (by the map $k([f])=j(f)(c)$), $\mathrm{Ult}(V,\mathcal{G})$ is well-founded. Finally, standard forcing facts give a name $\dot{G}$ for $\mathcal{G}$. ∎

We can get by with much smaller large cardinals if all we want is α-weakly precipitous.

LEMMA 1.4. *Assume $\mathcal{F}$ is a κ-complete filter and there is a measurable cardinal $\delta>|\bigcup\mathcal{F}|$. Then $\mathcal{F}$ is δ-weakly precipitous.*

Proof. We may assume $\bigcup\mathcal{F}=\lambda$ a cardinal and $\lambda\geq\kappa$. Since $\delta>\lambda$ is measurable,

$$S=\{a\subseteq V_\delta\mid a\cap\kappa\in\kappa\;\&\;|a\cap V_{\lambda+1}|<\kappa\;\&\;|a|=\delta\}$$

is stationary ([W1]). Let $\mathbb{P}$ be all stationary subsets of S ordered by inclusion, and $H\subseteq\mathbb{P}$ generic. Then we have an embedding $j:V\to(M,E)$ with $\mathrm{cp}(j)=\kappa$, δ in the well-founded part of (M,E), $j(\delta)=\delta$, $j''\mathcal{F}\in M$, and $|j''\mathcal{F}|<j(\kappa)$ (this is all standard—see [W1] or [M2]). Now we argue as above to get a V-κ-complete ultrafilter $\mathcal{G}$ extending $\mathcal{F}$. Let $j_{\mathcal{G}}:V\to\mathrm{Ult}(V,\mathcal{G})$ and $k:\mathrm{Ult}(V,G)\to(M,E)$ be the canonical maps. Then $j_{\mathcal{G}}(\delta)$ is in the well-founded part of $\mathrm{Ult}(V,G)$ since $k(j_{\mathcal{G}}(\delta))=j(\delta)=\delta$. ∎

2. A question of Abe. It is possible that one can use large cardinals (and weakly precipitous filters) instead of precipitous filters. For example, in [M1] Magidor proves that if there is a precipitous ideal on ω_1 and a measurable cardinal then all Σ^1_3 sets are Lebesgue measurable. If we use Theorem 1.2 instead of a precipitous ideal on ω_1, Magidor's proof gives that all Σ^1_3 sets are Lebesgue measurable from a measurable cardinal above a Woodin cardinal. Magidor goes on to show that all Σ^1_4 sets are Lebesgue measurable from other precipitous ideals. Using Magidor's ideas from this proof and Theorem 1.2, one sees that a measurable cardinal above n Woodin cardinals implies that all Σ^1_{n+2} sets are Lebesgue measurable.

In this section we give another example of this in answering a question of Abe from [A]. The following definition and two theorems are due to Abe and appear in [A].

Definition 2.1 (Abe). Assume $\mathcal{F}$ is a filter on $\mathcal{P}_\kappa\lambda$ (all filters on $\mathcal{P}_\kappa\lambda$ are κ-complete and fine). $\mathcal{F}$ is *weakly normal* iff $\forall f$ if $\{a \in \mathcal{P}_\kappa\lambda \mid f(a) \in a\} \in \mathcal{F}$ then $\exists\beta < \lambda$ such that $\{a \in \mathcal{P}_\kappa\lambda \mid f(a) < \beta\} \in \mathcal{F}$. Further, $\mathcal{F}$ is *semi-weakly normal* iff $\forall f$ if $\{a \in \mathcal{P}_\kappa\lambda \mid f(a) \in a\} \in \mathcal{F}^+$ then $\exists\beta < \lambda$ such that $\{a \in \mathcal{P}_\kappa\lambda \mid f(a) < \beta\} \in \mathcal{F}^+$.

Theorem 2.2 (Abe). *Assume $\mathcal{F}$ is a filter on $\mathcal{P}_\kappa\lambda$. Then $\mathcal{F}$ is weakly normal iff $\mathcal{F}$ is semi-weakly normal and there is no sequence of* $\mathrm{cof}(\lambda)$ *many disjoint $\mathcal{F}$-positive sets.*

Theorem 2.3 (Abe). *If λ is regular and there is a weakly normal filter on $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.*

This last result generalizes the well-known result of Solovay [S].

Also in [A], Abe proved a similar result when $\mathrm{cof}(\lambda) \le \kappa$ and asked if one can compute $\lambda^{<\kappa}$ when $\kappa < \mathrm{cof}(\lambda) < \lambda$. Abe could answer this question assuming that a certain filter was precipitous—we show that λ-weak precipitousness suffices.

Theorem 2.4. *Assume β is regular and there is a filter $\mathcal{F}$ on $\mathcal{P}_\kappa\beta$ that has no β sequence of disjoint sets from $\mathcal{F}^+$ (and there is a measurable cardinal $>\beta$). Then there is a weakly normal filter on $\mathcal{P}_\kappa\beta$.*

Remark. Matsubara has proved that if there is a β saturated precipitous ideal on $\mathcal{P}_\kappa\beta$ then $\beta^{<\kappa} = 2^{<\kappa}\cdot\beta$ ([M3], [M4]). Our result (combined with 2.3) eliminates the precipitous assumption (at the expense of a large cardinal). We also seem to have a weaker saturation hypothesis.

Proof (of Theorem 2.4). Let $\mathcal{F}$ be a filter on $\mathcal{P}_\kappa\beta$ with no β sequence of disjoint sets from $\mathcal{F}^+$. Since there is a measurable cardinal $> \beta$ there is a partial order $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{G}$ such that $\mathbb{P}$ forces $\dot{G} \supseteq \mathcal{F}$ is a V-κ-complete ultrafilter on $\mathcal{P}_\kappa\beta$ with $j_{\dot{G}}(\beta)$ in the well-founded part of the ultrapower. Now fix $f : \mathcal{P}_\kappa\beta \to \mathrm{Ord}$ and $p \in \mathbb{P}$ such that $p \Vdash$ "$[f] = \sup(j''_{\dot{G}}\beta)$".

Define a filter $\mathcal{E}$ on $\mathcal{P}_\kappa\beta$ by $A \in \mathcal{E}$ iff $p \Vdash A \in \dot{G}$. It is easy to see that $\mathcal{E}$ is a κ-complete fine filter on $\mathcal{P}_\kappa\beta$, $\mathcal{F} \subseteq \mathcal{E}$, and there is no β sequence of disjoint $\mathcal{E}$ positive sets. Because $p \Vdash$ "$[f] = \sup(j''_{\dot{G}}\beta)$", we have

$$\forall\gamma < \beta\ \{a \in \mathcal{P}_\kappa\beta \mid f(a) \ge \gamma\} \in \mathcal{E}. \tag{1}$$

Note that $T \in \mathcal{E}^+$ iff $\exists q \le p$ such that $q \Vdash T \in \dot{G}$. Using this, and again the fact that $p \Vdash$ "$[f] = \sup(j''_{\dot{G}}\beta)$", we have

(2) $\forall g$ if $\{a \in \mathcal{P}_\kappa\beta \mid g(a) < f(a)\} \in \mathcal{E}^+$ then
$$\exists\gamma < \beta \text{ with } \{a \in \mathcal{P}_\kappa\beta \mid g(a) < \gamma\} \in \mathcal{E}^+.$$

Finally, define a filter $\mathcal{D}$ on $\mathcal{P}_\kappa\beta$ by
$$A \in \mathcal{D} \text{ iff } \{a \in \mathcal{P}_\kappa\beta \mid a \cap f(a) \in A\} \in \mathcal{E}.$$

Clearly, $\mathcal{D}$ is a κ-complete filter on $\mathcal{P}_\kappa\beta$. Using (1) and the fact that $\mathcal{E}$ is fine, we find that $\mathcal{D}$ is fine. Note that $A \in \mathcal{D}^+$ iff $\{a \in \mathcal{P}_\kappa\beta \mid a \cap f(a) \in A\} \in \mathcal{E}^+$. So there is no β sequence of disjoint $\mathcal{D}$ positive sets. Using (2) we see that $\mathcal{D}$ is semi-weakly normal. Therefore $\mathcal{D}$ is a weakly normal filter on $\mathcal{P}_\kappa\beta$. ∎

COROLLARY 2.5. *Assume* $\mathrm{cof}(\lambda) \geq \kappa$ *and there is a filter on* $\mathcal{P}_\kappa\lambda$ *with no* λ *sequence of disjoint* $\mathcal{F}$*-positive sets* (*and there is a measurable cardinal* $> \lambda$). *Then* $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.

Proof. If λ is regular then we use 2.4 and 2.3.

So assume $\kappa \leq \mathrm{cof}(\lambda) < \lambda$. Let $\mathcal{F}$ be a filter on $\mathcal{P}_\kappa\lambda$ with no λ sequence of disjoint, positive sets. It is easy to see that there is a $\gamma < \lambda$ and $S \in \mathcal{F}^+$ such that S cannot be split into γ many disjoint positive sets. Replace $\mathcal{F}$ with $\mathcal{F}\restriction S$ (so there is no γ sequence of disjoint positive sets) and take $\gamma \geq \mathrm{cof}(\lambda)$.

Now given any regular β with $\gamma \leq \beta < \lambda$, let $\mathcal{F}_\beta$ be the projection of $\mathcal{F}$ to $\mathcal{P}_\kappa\beta$ ($\mathcal{F}_\beta = \{\{a \cap \beta \mid a \in S\} \mid S \in \mathcal{F}\}$). So $\mathcal{F}_\beta$ is a κ-complete fine filter on $\mathcal{P}_\kappa\beta$ with no β sequence of disjoint positive sets (no γ sequence in fact). So by 2.4 and 2.3, $\beta^{<\kappa} = 2^{<\kappa} \cdot \beta$.

Finally, since $\mathrm{cof}(\lambda) \geq \kappa$, we have $\lambda^{<\kappa} = \bigcup_{\beta<\lambda} \beta^{<\kappa}$, and therefore $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$. ∎

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