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# Near metric properties of function spaces

by

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**Abstract.** "Near metric" properties of the space of continuous real-valued functions on a space $X$ with the compact-open topology or with the topology of pointwise convergence are examined. In particular, it is investigated when these spaces are stratifiable or cometrisable.

**1. Introduction.** For any topological space $X$ it is customary to denote by $C(X)$ the set of all continuous real-valued function on $X$. This structure supports a number of natural topologies, but the most commonly studied are the topology of pointwise convergence ($C(X)$ with this topology is written $C_{\mathrm{p}}(X)$), and the compact-open topology ($C(X)$ with this topology is written $C_{\mathrm{k}}(X)$). A basic open neighbourhood of a point $f$ in $C(X)$ is of the form $B(f,F,\varepsilon)$, where $F$ is a finite subset of $X$, in the case of the pointwise topology, and of the form $B(f,K,\varepsilon)$ here $K$ is a compact subspace of $X$, in the case of the compact-open topology; here $\varepsilon > 0$, and $B(f,S,\varepsilon) = \{g \in C(X) : |f(x) - g(x)| < \varepsilon,\ \forall x \in S\}$ ($S$ is some subset of $X$). Since we are concerned with these function spaces, henceforth we restrict our attention to Tikhonov spaces. A convenient source of information about the pointwise and compact-open topology is McCoy and Ntantu's book [McNt].

Both $C_{\mathrm{p}}(X)$ and $C_{\mathrm{k}}(X)$ are rarely metrisable. The space $X$ has to be countable in the case of the pointwise topology, and $X$ has to be hemicompact (in other words, the family $\mathcal{K}(X)$ of compact subspaces of $X$ has to have countable cofinality) for $C_{\mathrm{k}}(X)$ to be metrisable. Since $C_{\mathrm{p}}(X)$ and $C_{\mathrm{k}}(X)$ are such fundamental objects, it is of some importance to determine when they are "nearly metrisable". There are perhaps two senses in which a space can be "nearly metrisable". First, the topology of the space in question may be "close" in some sense to a metrisable topology, or, second, the space may

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have sufficient structure that it shares many of the desirable properties of metrisable spaces. Gruenhage's articles on generalised metric spaces [Gr1, Gr2] are excellent surveys on near metric properties.

The aim of this paper is to consider two such "near metrisable" properties, one of each type, in the context of the function spaces $C_{\mathrm{p}}(X)$ and $C_{\mathrm{k}}(X)$. It turns out that the situation with respect to $C_{\mathrm{p}}(X)$ is fairly straightforward, but $C_{\mathrm{k}}(X)$ presents a harder challenge, and interesting problems remain.

**2. Cometrisability and stratifiability.** Recall that a space is *submetrisable* if it has a coarser metrisable topology. Submetrisability is a weak condition to place on a space because the original and metrisable topologies are only loosely related. A more stringent condition is that of cometrisability [Gr2]. A space $X$ is *cometrisable* if there is a metric $d$ on $X$ so that the metric topology is coarser than the original topology, and if $U$ is open, $x$ is in $U$, then there is an open $V$ containing $x$ so that the closure of $V$ in the metric topology is contained in $U$.

Observe that both $C_{\mathrm{p}}(X)$ and $C_{\mathrm{k}}(X)$ are locally convex topological vector spaces (and so, *a fortiori*, topological groups). As is very well known, any first countable topological group is metrisable. Consequently, many of the "near metric" structures considered (for example, Moore spaces [Gr1]) are not relevant to the study of function spaces. However, the intensively studied class of stratifiable spaces (see [Gr1]) shares very many properties with the class of metrisable spaces, but does not (necessarily) imply first countability. A space $X$ is *stratifiable* if it is "monotonically perfectly normal", that is to say, for each closed set $A$ and $n$ in $\omega$, there is an open set $U(n,A)$ so that $\bigcap_{n\in\omega} U(n,A) = A = \bigcap_{n\in\omega} \overline{U(n,A)}$, and if $A \subseteq A'$ and $n \le n'$, then $U(n,A) \subseteq U(n',A')$.

It is known that the class of stratifiable spaces is closed under taking countable products, closed images and arbitrary subspaces. Every stratifiable space is monotonically normal and hereditarily paracompact. Another useful feature of stratifiable spaces, especially in the context of locally convex topological vector spaces, are their powerful extension properties. Here, and at a couple of subsequent points, it is convenient to consider $C(X,Y)$, the set of all continuous maps from a space $X$ into another space $Y$, either with the pointwise topology, or the compact-open topology.

THEOREM (Borges–Dugundji, [Bo, Du]). *Let $X$ be stratifiable, $A$ a closed subspace of $X$ and let $L$ be a locally convex topological vector space. Then there is a map $e : C(A,L) \to C(X,L)$ such that*

(1) $e(f)\restriction A = f$;
(2) $e(f)(X)$ *is contained in the closed convex hull of* $f(A)$;

(3) *$e$ is linear*;

(4) *$e$ is continuous when the function spaces are both given either the pointwise or compact-open topologies.*

We will shortly see that every stratifiable space is cometrisable, but we first need an alternative characterisation of stratifiable spaces. (See Theorems 5.25 and 5.27 of [Gr1].)

THEOREM. *A space $(X,\tau)$ is stratifiable if and only if there is a function $g:\omega\times X\to\tau$ such that*

(i) $\{x\}=\bigcap_n g(n,x)$;

(ii) $x\in g(n,x_n)\Rightarrow x_n\to x$;

(iii) *if $H$ is closed and $y\notin H$, then $y\notin\overline{\bigcup\{g(n,x):x\in H\}}$ for some $n$*;

(iv) $y\in g(n,x)\Rightarrow g(n,y)\subseteq g(n,x)$.

Consider a stratifiable space $(X,\tau)$ with function $g$ as above. We may suppose that $g(n+1,x)\subseteq g(n,x)$ for every $x$ in $X$ and $n\in\omega$. Let $\sigma$ be the topology on $X$ generated by all the $g(n,x)$'s. By property (iv), for each $x$ in $X$, the collection $\{g(n,x)\}_{n\in\omega}$ is a local base in $\sigma$ at $x$. Also by (iv), it is clear that these local bases satisfy the conditions of the Collins–Roscoe metrisation theorem [CR]. Thus we derive the following proposition.

PROPOSITION 1. *A space $(X,\tau)$ is stratifiable if and only if there is a function $g:\omega\times X\to\tau$ and a continuous metric $d$ on $X$ such that*

(1) $\{x\}=\bigcap_n g(n,x)$;

(2) *if $H$ is closed and $y\notin H$, then $y\notin\overline{\bigcup\{g(n,x):x\in H\}}$ for some $n\in\omega$*;

(3) *$g(n,x)$ is open in $(X,d)$.*

*Thus, every stratifiable space is cometrisable.*

Cometrisable spaces certainly need not be stratifiable.

EXAMPLE 2. *There is a countable cometrisable space which is not stratifiable.*

Proof. Let $(X,\tau)$ be the first example from [NP]. It is a countable (regular) non-stratifiable space. The underlying set of $X$ is $(\omega+1)^2\setminus\omega\times\{\omega\}$. Let $\sigma$ be the natural metrisable topology on $X$ inherited as a subspace of the product $(\omega+1)^2$. The topology $\tau$ refines $\sigma$, and differs from it only at $(\omega,\omega)$. Plainly, $\sigma$ "cometrises" $\tau$. ∎

EXAMPLE 3. *There is a cometrisable space with countable netweight and weight equal to $\omega_1$, which is not stratifiable.*

Proof. Let $\{x_\alpha\}_{\alpha\in\mathfrak{c}\setminus\{0\}}$ be an enumeration of the irrational numbers, and let $x_0=0$. Set $A=\{x_\alpha\}_{\alpha\in\omega_1}$. For each $c\in\mathbb{R}$, set $\partial\bowtie_c=\{(x,y):|y|=|x-c|\}$ and $\bowtie_c=\{(x,y):|y|\le|x-c|\}$. Set $X=(A\times\{0\})\cup(\mathbb{R}^2\setminus\bigcup_{a\in A}\partial\bowtie_a)$.

Refine the usual topology on $X$ (as a subspace of the plane) by declaring all $\bowtie_a$, for $a$ in $A$, to be open. This topology is clearly "cometrised" by the original topology (and hence is $T_3$). As $X$ is the union of two separable metrisable spaces, it clearly has a countable net; and, by construction, has weight precisely $\omega_1$.

In the case when $\omega_1 = \mathfrak{c}$ (and so $A$ consists of all the irrationals), it is well known that $X$ is not stratifiable [H]. A similar argument (but using the fact that every uncountable closed subspace of the reals has size $\mathfrak{c}$, rather than a category argument) will show that $X$ is not stratifiable in the case when $\omega_1 < \mathfrak{c}$. ∎

However, under tight restrictions the implication can be reversed. Below $\mathfrak{b}$ is the minimal cardinality of an unbounded family in $\omega^\omega$.

LEMMA 4. *Let $X = \{x_n : n \in \omega\}$ be a countable metrisable space, $\tau$ be the topology of $X$, $\mathcal{B}$ be a family of open subsets of $X$, and $|\mathcal{B}| < \mathfrak{b}$. Then there exists $g : X \to \tau$ such that*

(1) *$x \in g(x)$ for $x \in X$;*

(2) *for any $U \in \mathcal{B}$, there is a finite $M \subseteq X$ such that $\bigcup\{g(x) : x \in U \setminus M\} \subseteq U$.*

Proof. Let $d$ be a metric on $X$, $B_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}$ for $\varepsilon > 0$, and $x \in X$. Define a function $f_U : \omega \to \omega$ for $U \in \mathcal{B}$. If $x_n \notin U$, then define $f_U(n) = 0$, otherwise choose $f_U(n) > 0$ such that $B_{1/f_U(n)}(x) \subseteq U$. Since $|\mathcal{B}| < \mathfrak{b}$, there exists $f : \omega \to \omega$ such that $f$ is larger than $f_U$ almost everywhere for any $U \in \mathcal{B}$. Put $g(x_n) = B_{1/f(n)}(x)$ for $n \in \omega$. ∎

PROPOSITION 5. *Let $X$ be a cometrisable countable space such that $w(X) < \mathfrak{b}$. Then $X$ is stratifiable.*

Proof. Enumerate $X = \{x_n : n \in \omega\}$. Let $\mathcal{B}$ be a base of $X$, $|\mathcal{B}| < \mathfrak{b}$, $\tau$ be a "cometric" topology of $X$. By the preceding lemma, there exists $h : X \to \tau$ such that

(1) $x \in h(x)$ for $x \in X$;

(2) for any $U \in \mathcal{B}$, there is a finite $M \subseteq X$ such that $\bigcup\{h(x) : x \in V \setminus M\} \subseteq V$, where $V = X \setminus \overline{U}^\tau$.

For $n \in \omega$ and $x \in X$, there is an open $f(n, x)$ such that $x \in f(n, x)$ and $x_i \notin \overline{f(n, x)}$ for $i < n$, $x_i \neq x$. Put $g(n, x) = h(x) \cap f(n, x)$. We then have

(i) $\{x\} = \bigcap_n g(n, x)$;

(ii) if $H$ is closed and $y \notin H$, then $y \notin \overline{\bigcup\{g(n, x) : x \in H\}}$ for some $n \in \omega$.

Hence, $X$ is stratifiable. ∎

**2. The space $C_{\mathrm{p}}(X)$.** It turns out that $C_{\mathrm{p}}(X)$ is stratifiable or cometrisable only in rather uninteresting circumstances. We observe that it is now well known that $C_{\mathrm{p}}(X)$ is stratifiable only if $X$ is countable (see [Ga], for example), but the proof below is new, and the result is entirely new for the cometrisable case. We start with some definitions and easy lemmas.

A continuous map $f : X \to Y$ is a *g-map* if and only if for any $x \in X$ and any neighbourhood $U$ of $x$ there is a neighbourhood $V$ of $x$ such that $\overline{f(V)} \subseteq f(U)$. A continuous map $f : X \to Y$ is said to be *almost open* (or *d-open*) if and only if $f(U) \subseteq \operatorname{int} \overline{f(U)}$ for every $U$ open in $X$.

Lemma 6. *Let $f : X \to Y$ be a continuous map. If $f$ is an almost open $g$-map, then $f$ is open.*

For $M \subseteq A$ let $\pi_M : \prod\{X_\alpha : \alpha \in A\} \to \prod\{X_\alpha : \alpha \in M\}$ be the natural projection.

Lemma 7. *The restriction of an open map to a dense subspace is almost open. In particular, the restriction of a projection map $\pi_M$ to a dense subspace of the product $\prod\{X_\alpha : \alpha \in A\}$ is almost open.*

Lemma 8. *Let $X$ be a space. $X$ is cometrisable if and only if there is a 1-to-1 $g$-map from $X$ onto a metrisable space.*

Lemma 9. *Let $X$, $Y$, $Z$ be spaces, $f : X \to Y$, $k : Y \to Z$ be continuous maps. If $k \circ f$ is a $g$-map and $k$ is 1-to-1, then $f$ is a $g$-map.*

Proposition 10. *Let $\{X_\alpha : \alpha \in A\}$ be a family of separable metrisable spaces and $X$ be a dense subspace of $\prod\{X_\alpha : \alpha \in A\}$. If $X$ is cometrisable, then $A$ is countable.*

Proof. Lemma 8 implies that there is a 1-to-1 $g$-map $h : X \to Z$, where $Z$ is metrisable. There is a countable $B \subseteq A$ such that $h = k \circ f$, where $Y = \pi_B(X)$, $f = \pi_B \restriction X : X \to Y$, and $k : Y \to Z$ is continuous. Lemma 9 implies that $f$ is a $g$-map. Lemma 7 implies that $f$ is almost open. Lemma 6 implies that $f$ is open. Hence, $f$ is a homeomorphism and $B = A$. ∎

Corollary 11. *Let $X$ be a space. Then the following are equivalent*:

(i) $C_{\mathrm{p}}(X)$ *has a dense cometrisable subspace*;
(ii) $C_{\mathrm{p}}(X)$ *has a dense stratifiable subspace*;
(iii) $X$ *is countable*;
(iv) $C_{\mathrm{p}}(X)$ *is metrisable.*

To see this it suffices to observe that $C_{\mathrm{p}}(X)$ is dense in $\mathbb{R}^X$.

**3. Completions of cometrisable groups.** The completion of a topological group $G$ is denoted by $\widehat{G}$. Shkarin has an example demonstrating that the completion of a separable stratifiable locally convex topological vector space need not be stratifiable; however, he was able to prove the following.

PROPOSITION (Shkarin, [Shk]). *Let $G$ be a separable stratifiable topological group. Then $\widehat{G}$ is submetrisable.*

On learning Shkarin's proof, we were able to improve both his preconditions and conclusion. We apply this result in the following section. (Note that a space $X$ is a Lindelöf $\Sigma$-space if and only if it is the perfect preimage of a separable metrisable space.)

PROPOSITION 12. *Let $G$ be a dense subgroup of a topological group $H$. Then $H$ is cometrisable provided $G$ is cometrisable and either $G$ is Lindelöf or $H$ is a Lindelöf $\Sigma$-space.*

We say that a topological group $G$ is *invariantly cometrisable* if and only if there is a continuous left-invariant metric $d$ on $G$ such that $d$ "cometrises" $G$. (Left-invariance of $d$ means $d(a.x, a.y) = d(x, y)$ for all $a$, $x$ and $y$ in $G$. Every metrisable topological group has a compatible left-invariant metric—for this, and other basic results on topological groups, see Comfort's article in [KV].) Recall that a topological group $G$ is $\mathbb{R}$-*factorizable* if and only if for any continuous function $f : G \to \mathbb{R}$, there exists a second countable topological group $H$, a continuous homomorphism $\varphi : G \to H$, and a continuous map $h : H \to \mathbb{R}$ such that $f = h \circ \varphi$. Evidently, we may replace the real line here with any second countable space. Tkachenko [Tk] has shown that every Lindelöf topological group is $\mathbb{R}$-factorisable, as is any subgroup of a Lindelöf $\Sigma$-group.

LEMMA 13. *If $G$ is a $\mathbb{R}$-factorisable topological group with a second countable cometric topology, then $G$ is invariantly cometrisable.*

Proof. Let $f : G \to X$ be a continuous 1-to-1 $g$-map of $G$ onto a separable metrisable space $X$. Since $G$ is $\mathbb{R}$-factorisable, there is a second countable topological group $H$, a continuous homomorphism $\varphi : G \to H$, and a continuous map $h : H \to X$ such that $f = h \circ \varphi$. By Lemma 9, $\varphi$ is a 1-to-1 $g$-map. Let $d'$ is a left-invariant metric on $H$ and let $d(x, y) = d'(\varphi(x), \varphi(y))$ for $x, y \in G$. Then $d$ is a left-invariant continuous metric on $G$ and $d$ cometrises $G$. ∎

LEMMA 14. *A topological group $G$ is invariantly cometrisable if and only if there exists a sequence $\{U_n : n \in \omega\}$ of neighbourhoods of the identity such that the following condition holds: For any neighbourhood $W$ of the identity there is a neighbourhood $V$ of the identity such that $\bigcap_{n\in\omega} VU_n \subseteq W$.*

Proof. Observe first, that if $x \in G$, $V$ is an open neighbourhood of the identity, and $\{U_n : n \in \omega\}$ is a sequence of symmetric open neighbourhoods of the identity, then

$$(*)\qquad \forall n \in \omega (xU_n \cap V \neq \emptyset) \Leftrightarrow \Big(x \in \bigcap_{n\in\omega} VU_n\Big).$$

So let $d$ be an invariant cometric. Define $U_n = B(e, 1/n)$ (where $e$ is the identity). Given open $W$ containing $e$, choose, by cometrisability, open $V$ containing the identity so that $\overline{V}^d \subseteq W$. Since $x \in \overline{V}^d$ if and only if $xU_n \cap V \neq \emptyset$ for every $n \in \omega$, the claim for the sequence $\{U_n : n \in \omega\}$ follows from $(*)$.

Now suppose we are given a sequence $\{U_n : n \in \omega\}$ as in the statement of the lemma. We may suppose that the $U_n$'s are symmetric and $U_{n+1}^3 \subseteq U_n$. Hence we may find a left-invariant metric $d$ so that the $U_n$'s form a neighbourhood base at the identity. For a given open $W$ containing $e$, there is an open $V$ containing the identity so that $\bigcap_{n\in\omega} VU_n \subseteq W$. Since we again have $x \in \overline{V}^d$ if and only if $xU_n \cap V \neq \emptyset$ for every $n \in \omega$, the claim follows from $(*)$. ■

Let us agree to call a sequence of open neighbourhoods of the identity as in the statement of Lemma 14 an *invariantly cometrising sequence*. Observe that we may assume the sequence to consist of basic open neighbourhoods, and may add any countable family of open neighbourhoods of the identity to a cometrising sequence.

PROPOSITION 15. *Let $G$ be an invariantly cometrisable dense subgroup of a topological group $H$. Then $H$ is invariantly cometrisable.*

Proof. Denote the neighbourhoods of unity in $G$ by $\mathcal{O}$ and the neighbourhoods of unity in $H$ by $\widehat{\mathcal{O}}$. Lemma 14 implies that there is a sequence $\{U_n : n \in \omega\} \subset \mathcal{O}$ such that the following condition holds:

(1) For any $W \in \mathcal{O}$ there is a $V \in \mathcal{O}$ such that $\bigcap_{n\in\omega} VU_n \subseteq W$.

Put $P_n = H \setminus \overline{G \setminus U_n}^H$ for $n \in \omega$. We show that statement (2) below holds, and so may deduce from Lemma 14 that $H$ is cometrisable.

(2) For any $W \in \widehat{\mathcal{O}}$ there is a $V \in \widehat{\mathcal{O}}$ such that $\bigcap_{n\in\omega} VP_n \subseteq W$.

Let $W \in \widehat{\mathcal{O}}$. There is a $W' \in \widehat{\mathcal{O}}$ such that $\overline{W'W'}^H \subseteq W$. Condition (1) implies that there exists $V' \in \mathcal{O}$ such that $\bigcap_{n\in\omega} V'U_n \subseteq W' \cap G$. Put $V = H \setminus \overline{G \setminus V'}^H$. One can verify that $\bigcap_{n\in\omega} VP_n \subseteq W$. ■

*Proof of Proposition 12.* From Lemma 13 we see that $G$ is invariantly cometrisable. So from Proposition 15 it follows that $H$ is invariantly cometrisable. Thus $H$ is cometrisable. ■

**4. Cometrisability of $C_{\mathrm{k}}(X)$.** McCoy & Ntantu have shown that $C_{\mathrm{k}}(X)$ is submetrisable if and only if $X$ contains a dense $\sigma$-compact subspace. Let $Y$ be an uncountable separable space, all of whose compact subsets are finite. Then $C_{\mathrm{k}}(Y)$ is submetrisable, but by Corollary 11 is not cometrisable. We now give a characterisation of spaces $X$ such that $C_{\mathrm{k}}(X)$ is cometrisable. (Denote the zero function by $\mathbf{0}$.)

THEOREM 16. *The function space $C_{\mathrm{k}}(X)$ is invariantly cometrisable if and only if*

(CK) *there is a $\sigma$-compact subset $Y$ of $X$ such that for every compact subset $K$ of $X$, there is a compact subset $L$ satisfying $K \subseteq \overline{L \cap Y}$.*

Proof. Let us start by supposing that $X$ satisfies (CK), so that there is a $Y = \bigcup_{n\in\omega} K_n$, the $K_n$'s compact, such that for every compact subspace $K$ of $X$, there is a compact $L$ with $K \subseteq \overline{L \cap Y}$. We aim to show that $\{B(\mathbf{0}, K_n, 1/m) : m, n \geq 1\}$ is an invariantly cometrising sequence as in Lemma 14.

CLAIM. *For each compact subspace $K$ and $\varepsilon > 0$,*

$$\bigcap_{n,m\geq 1} (B(\mathbf{0}, L, \varepsilon/3) + B(\mathbf{0}, K_n, 1/m)) \subseteq B(\mathbf{0}, K, \varepsilon).$$

Proof. Take any continuous map $h$ in the left hand side. We need to show that $|h(x)| < \varepsilon$ for all $x \in K$. By continuity of $h$ and the fact that $\overline{L \cap Y}$ contains $K$, it suffices to show that $|h(x)| \leq 2\varepsilon/3$ for all $x \in L \cap K_n$ and all $n \geq 1$.

Fix $n$. Choosing $m$ large enough, we get $h \in B(\mathbf{0}, L, \varepsilon/3) + B(\mathbf{0}, K_n, \varepsilon/3)$. Then $h = f + g$, where $|f(x)| < \varepsilon/3$ for all $x \in L$ and $|g(x)| < \varepsilon/3$ for all $x \in K_n$. Thus, for every $x$ in $L \cap K_n$, $|h(x)| = |f(x)+g(x)| \leq |f(x)|+|g(x)| \leq 2\varepsilon/3$. ∎

Now suppose that $C_{\mathrm{k}}(X)$ is invariantly cometrisable. Then $C_{\mathrm{k}}(X)$ has an invariantly cometrising sequence as in Lemma 14, which we may assume to be of the form $\{B(\mathbf{0}, K_n, 1/m) : m, n \geq 1\}$. Let $Y = \bigcup_{n\geq 1} K_n$. Take any compact $K \subseteq X$. By the definition of an invariantly cometrising sequence, there is a compact $L$ (which we may assume to contain $K$) and a $\delta > 0$ so that

$$\bigcap_{n,m\geq 1} (B(\mathbf{0}, L, \delta) + B(\mathbf{0}, K_n, 1/m)) \subseteq B(\mathbf{0}, K, 1).$$

CLAIM. *The closure of $L \cap Y$ contains $K$.*

Proof. Suppose not. So there is an $x_0$ in $K$, and an open $U$ containing $x_0$ with $U \cap (L \cap Y) = \emptyset$. Pick a continuous $h : X \to [0, 1]$ such that $h(x_0) = 1$ and $h(y) = 0$ for $y \notin U$. So $h \notin B(\mathbf{0}, K, 1)$. We show that $h \in \bigcap_{n,m\geq 1}(B(\mathbf{0}, L, \delta) + B(\mathbf{0}, K_n, 1/m))$.

Fix $m$ and $n$. Note that $h(y)=0$ for all $y\in K_n\cap L$. By continuity of $h$, pick open $V$ containing $K_n\cap L$ such that $|h(y)|<\delta$ for all $y\in V$. Also, pick continuous $\phi_n : X\to[0,1]$ so that $\phi_n(y)=1$ for $y\in L\setminus V$ and $\phi_n(y)=0$ for $y\in K_n$.

Set $g_n=\phi_n.h$ and $f_n=h-g_n$. Obviously, $h=f_n+g_n$. Since $\phi_n(y)=0$ for all $y\in K_n$, we have $g_n(y)=0$ for all $y\in K_n$, and $g_n\in B(\mathbf{0},K_n,1/m)$. We show that $|f_n(y)|<\delta$ for every $y$ in $L$, and so $f_n\in B(\mathbf{0},L,\delta)$.

Take any $y$ in $V$. Then $|f_n(y)|=|h(y)|\cdot|1-\phi_n(y)|\le|h(y)|$, and $|h(y)|<\delta$ by choice of $V$. So now suppose, $y$ is a point of $L\setminus V$. Then $f_n(y)=h(y)-\phi_n(y)h(y)=0$, since $\phi_n(y)=1$ by choice of $\phi_n$. ■

Evidently, $\sigma$-compact (and, *a fortiori*, countable) spaces have property (CK). To aid identification of other spaces with property (CK) we have the following tests.

Let $\mathcal{F}$ be a family of subsets of $\omega$. Recall that $\mathcal{F}$ has the *strong intersection property* if the intersection of any finite subfamily of $\mathcal{F}$ is infinite, and that a subset $A$ of $\omega$ is a *pseudo-intersection* of $\mathcal{F}$ if $F\setminus A$ is finite for every $F$ in $\mathcal{F}$. The cardinal $\mathfrak{p}$ is defined to be

$$\mathfrak{p}=\min\{|\mathcal{F}| : \mathcal{F}\text{ is a subfamily of }[\omega]^\omega\text{ with strong finite intersection property which has no infinite pseudo-intersections}\}.$$

A convenient source of information about $\mathfrak{p}$ is van Douwen's survey article in [KV].

LEMMA 17. *Let $X$ be a space, $\chi(X)<\mathfrak{p}$, $C$ be a countable dense subset of $X$, and suppose $K$ is a compact subspace of $X$, with $\chi(K,X)<\mathfrak{p}$, and $d(K)<\mathfrak{p}$. Then there exists $D\subseteq C$ such that $K$ is the set of cluster points of $D$.*

Proof. Let $E\subseteq K$ be a dense subset of $K$ such that $|E|<\mathfrak{p}$, and let $\mathcal{U}$ be an outer base of $K$ such that $|\mathcal{U}|<\mathfrak{p}$. Since $\mathfrak{p}=\mathfrak{p}_\chi$ (see [KV], p. 129, Theorem 6.2) and $\chi(X)<\mathfrak{p}$, there is a sequence $\xi_x\subseteq C$ converging to $x$ for any $x\in E$. Put $\mathcal{A}=\{C\cap U : U\in\mathcal{U}\}$ and $\mathcal{B}=\{\xi_x : x\in E\}$. Since $\mathfrak{p}=\mathfrak{p}_2$ (see [KV], p. 119, Theorem 3.8), there is $D\subseteq C$ such that $D\cap B$ is infinite for $B\in\mathcal{B}$ and $D$ is a pseudo-intersection of $\mathcal{A}$. ■

PROPOSITION 18. *Let $X$ be a separable space. If $\chi(K,X)<\mathfrak{p}$ and $d(K)<\mathfrak{p}$ for every compact $K\subseteq X$, then $X$ has property* (CK).

Proof. This follows from Lemma 17 and the definition of property (CK). ■

Let us observe that a space $X$ satisfies the preconditions of Proposition 18 if and only if $\chi(\mathcal{Z}(X))<\mathfrak{p}$, where $\mathcal{Z}(X)$ is the space of compact subsets of $X$ with the Vietoris topology.

COROLLARY 19. *Let $X$ be a separable space. If either $X$ has a perfectly normal hereditarily separable compactification, or $w(X) < \mathfrak{p}$, then $X$ has property* (CK).

EXAMPLE 20. *The Sorgenfrey line, $S$, has property* (CK), *and so $C_{\mathrm{k}}(S)$ is* (*invariantly*) *cometrisable.*

A space $X$ is said to be an $\aleph_0$ *space* if there is a countable family $\mathcal{N}$ of subsets of $X$ such that if $K$ is a compact subspace of $X$, and $U$ is an open set containing $K$, then there is an $N \in \mathcal{N}$ such that $K \subseteq N \subseteq U$. A map $f$ from a topological space $X$ onto a space $Y$ is said to be *compact-covering* if it is continuous and for every compact subset $L$ of $Y$ there is a compact subset $K$ of $X$ such that $f(K) = L$. Compact-covering maps are useful in the study of $C_{\mathrm{k}}(X)$ because if $f : X \to Y$ is compact-covering then the natural map of $C_{\mathrm{k}}(Y)$ into $C_{\mathrm{k}}(X)$ is a linear topological embedding. It is known that a space is $\aleph_0$ if and only if it is the compact-covering image of a separable metrisable space (see [Gr1]).

PROPOSITION 21. *Every $\aleph_0$ space, $X$, has property* (CK), *and so $C_{\mathrm{k}}(X)$ is* (*invariantly*) *cometrisable.*

Proof. Every $\aleph_0$ space is the compact-covering image of a separable metrisable space. So if $X$ is $\aleph_0$, then $C_{\mathrm{k}}(X)$ embeds as a linear subspace of $C_{\mathrm{k}}(Y)$ for some separable metrisable space $Y$. Thus, it suffices to show that every separable metrisable space $Y$ has property (CK). This can easily be done directly, but since every separable metrisable space has a metrisable compactification, this also follows from Corollary 19. ∎

The situation with dense cometrisable subspaces of $C_{\mathrm{k}}(X)$ is less clear. It is quite possible for a nonsubmetrisable $C_{\mathrm{k}}(X)$ to have a dense subgroup which is countable (and hence submetrisable). For example, we may take $X$ to be the uncountable discrete space of size $\omega_1$. For dense cometrisable *subgroups* of $C_{\mathrm{k}}(X)$ we have a positive result, albeit only in certain cases.

PROPOSITION 22. *If $C_{\mathrm{k}}(X)$ contains a dense cometrisable subgroup, $G$, and either $G$ is Lindelöf or $C_{\mathrm{k}}(X)$ is a Lindelöf $\Sigma$-space, then $C_{\mathrm{k}}(X)$ is invariantly cometrisable.*

Proof. Apply Proposition 12, with $H = C_{\mathrm{k}}(X)$. ∎

QUESTION A. *If $C_{\mathrm{k}}(X)$ contains a dense cometrisable subspace, is it true that the entirety of $C_{\mathrm{k}}(X)$ is cometrisable?*

QUESTION B. *For which spaces $X$ is $C_{\mathrm{k}}(X)$ a Lindelöf $\Sigma$-space?*

**5. Local conditions for stratifiability of topological groups.** A topological group is metrisable if and only if it is first countable. In this section we give a similar local condition for topological groups to be stratifiable. Since the topology of $C_{\mathrm{k}}(X)$ is given in terms of a local basis this is of practical importance—as will be demonstrated in the following section.

Let $X$ be a space. A collection $\mathcal{P}$ of pairs of subsets of $X$ is said to be a *local pairbase* at a point $x$ in $X$ if whenever $U$ is an open neighbourhood of $x$ there is a $P = (P_1, P_2) \in \mathcal{P}$ such that $P_1$ is open and $x \in P_1 \subseteq P_2 \subseteq U$; and $\mathcal{P}$ is a *pairbase* (for the whole of the space $X$) if it is a local pairbase for every point of $X$. A collection $\mathcal{P}$ of pairs of subsets of $X$ is said to be *cushioned* if for every $\mathcal{P}' \subseteq \mathcal{P}$, we have $\overline{\bigcup\{P_1 : (P_1, P_2) \in \mathcal{P}'\}} \subseteq \bigcup\{P_2 : (P_1, P_2) \in \mathcal{P}'\}$, and *$\sigma$-cushioned* if it can be written as a countable union of cushioned subcollections. A point in a space which has a $\sigma$-cushioned local pairbase is called a *$\sigma$-$m_3$ point*; and a point which has a cushioned local pairbase is called an *$m_3$ point*. Finally, a space each of whose points is a ($\sigma$-) $m_3$ point, is said to be ($\sigma$-) $m_3$. It can easily be shown that every monotonically normal space is an $m_3$ space. Also, it is well known that a space is stratifiable if and only if it has a $\sigma$-cushioned pairbase. (See [Bu] for further information about $m_3$ spaces.)

A useful method of checking for cushioning is contained in the following lemma.

LEMMA 23. *Let $X$ be a space, and let $\mathcal{P}$ be a collection of pairs of subsets of $X$. Then $\mathcal{P}$ is cushioned if, and only if, for each $x$ in $X$, there is an open neighbourhood $U_x$ of $x$ such that $U_x \cap P_1 \neq \emptyset$ implies $x \in P_2$, for all $(P_1, P_2) \in \mathcal{P}$.*

Proof. Suppose first that $\mathcal{P}$ is cushioned. Take any $x$ in $X$, and define $U_x = X \setminus \overline{\bigcup\{P_1 : x \notin P_2, (P_1, P_2) \in \mathcal{P}\}}$. As $\mathcal{P}$ is cushioned, $U_x$ is an open neighbourhood of $x$. Clearly, $U_x$ satisfies the other condition.

Now suppose, for each $x$ in $X$, there is an open $U_x$, as in the statement of the lemma. Take any $\mathcal{P}' \subseteq \mathcal{P}$ and consider an arbitrary $x \in \overline{\bigcup\{P_1 : (P_1, P_2) \in \mathcal{P}'\}}$. Then $U_x \cap P_1' \neq \emptyset$ for some $(P_1', P_2') \in \mathcal{P}'$, and, by hypothesis, $x \in P_2' \subseteq \bigcup\{P_2 : (P_1, P_2) \in \mathcal{P}'\}$. ∎

Now we can give a local condition for separable topological groups to be stratifiable. It is known that there are nonseparable topological groups with an $m_3$ point which fail to be stratifiable [Ga].

THEOREM 24. *Let $G$ be a separable topological group. Then $G$ is stratifiable if and only if $G$ has a $\sigma$-$m_3$ point.*

Proof. We only need to show that if the separable topological group $G$ has a $\sigma$-$m_3$ point, then it has a $\sigma$-cushioned pairbase. Taking translations we may suppose that the identity has a local pairbase $\mathcal{P}$ which can be written $\mathcal{P} = \bigcup_{n\in\omega} \mathcal{P}_n$, where each $\mathcal{P}_n$ is cushioned. Let $D = \{g_m : m \in \omega\}$ be a dense subset of $G$. Define $\mathbb{P} = \bigcup_{m,n\in\omega}\{\langle g_m.P_1^n, g_m.P_2^n\rangle : \langle P_1^n, P_2^n\rangle \in \mathcal{P}_n\}$. It is immediate that $\mathbb{P}$ is $\sigma$-cushioned. We show that $\mathbb{P}$ is a pairbase for $G$.

Take any $g \in G$, and $U$ an open neighbourhood of the identity. Note that $g.U$ is a basic neighbourhood of $g$. By continuity of multiplication, there is a $\langle P_1, P_2\rangle \in \mathcal{P}$ such that $P_1$ is an open neighbourhood of the identity, and $P_2.P_2 \subseteq U$. By continuity of inversion, there is an open neighbourhood $V$ of the identity such that $V = V^{-1}$ and $V \subseteq P_1$.

The set $g.V$ is an open neighbourhood of $g$, so we may pick $g_m \in g.V \cap D$. Note that $\langle g_m.P_1, g_m.P_2\rangle \in \mathbb{P}$. From $g_m \in g.V$, we have $g^{-1}g_m \in V$, so ${g_m}^{-1} \in V^{-1} = V$. Now we see that:

(1) $g \in g_m.V \subseteq g_m.P_1$, open, and
(2) $g_m.P_1 \subseteq g_m.P_2 \subseteq g.V.P_2 \subseteq g.P_2.P_2 \subseteq g.U$.

This demonstrates that $\mathbb{P}$ is, indeed, a pairbase. ∎

**6. Stratifiability of $C_{\mathrm{k}}(X)$.** In this section sufficient conditions for $C_{\mathrm{k}}(X)$ to be stratifiable are given. A simple necessary condition follows from Theorem 16. To see this observe that any space with a dense $\sigma$-(compact metrisable) subspace is separable.

LEMMA 25. *Suppose $X$ has a $G_\delta$ diagonal*; *or has a point countable base*; *or has hereditarily normal cube*; *or has*, *hereditarily*, *any other property making compact spaces metrisable. Then $X$ is separable whenever $C_{\mathrm{k}}(X)$ is stratifiable.*

We concentrate on the case when $X$ is (separable) metrisable. Two simple facts about separable metrisable spaces are used repeatedly: every separable metrisable space has a metrisable compactification, and every separable metrisable space is the perfect preimage of a zero-dimensional separable metrisable space. The first few lemmas give methods of identifying new spaces, $X$, such that $C_{\mathrm{k}}(X)$ is stratifiable, from old.

LEMMA 26. *Let $X$ be a separable metrisable space and $K$ be a metrisable compact space. If $C_k(X)$ is stratifiable then $C_{\mathrm{k}}(X \times K)$ is stratifiable.*

Proof. If $K$ is finite then the lemma is clear. Suppose that $K$ is not finite. Since

$$C_k(X \times K) \hookrightarrow C_k(X, C_k(K)) \hookrightarrow C_k(X, \mathbb{R}^\omega) \hookrightarrow C_k(X)^\omega,$$

we see that $C_k(X \times K)$ is indeed stratifiable. ∎

PROPOSITION 27. *Suppose $C_k(Y)$ to be stratifiable. If one of the following conditions holds*, *then $C_{\mathrm{k}}(X)$ is stratifiable.*

(1) *$X$ is a closed subspace of $Y$, and $Y$ is stratifiable*;
(2) *$X$ is a compact-covering image of $Y$*;

(3) *$X$ is a perfect preimage of $Y$, and both $X$ and $Y$ are separable metrisable.*

Proof. The first part of the claim follows from the Borges–Dugundji Extension Theorem. The extender given by the theorem embeds $C_{\mathrm{k}}(X)$ in $C_{\mathrm{k}}(Y)$.

If $X$ is the compact-covering image of $Y$, then $C_{\mathrm{k}}(X)$ embeds in $C_{\mathrm{k}}(Y)$.

For the third part let $X$ and $Y$ be separable metrisable. It is known that $X$, as a separable metrisable space which is the perfect preimage of a separable metrisable space $Y$, can be embedded as a closed subspace of a product $Y \times K$, where $K$ is a compact metrisable space (in fact a compactification of $X$). Thus the claim follows from the preceding lemma. ∎

By standard techniques, from part (2) of Proposition 27, one may derive a locally finite sum theorem.

LEMMA 28. *Let $X$ be a separable metrisable space and $\mathcal{F}$ be a locally finite family of closed subsets of $X$ such that $C_{\mathrm{k}}(F)$ is stratifiable for $F \in \mathcal{F}$. Then $C_{\mathrm{k}}(X)$ is stratifiable.*

Let $X$ be a space. Denote by $\mathcal{O}(X)$ the clopen subsets of $X$, set $\mathcal{O}_*(X) = \mathcal{O}(X) \setminus \{\emptyset\}$, write $\mathcal{K}(X)$ for the compact subsets of $X$, and set $\mathcal{K}_*(X) = \mathcal{K}(X) \setminus \{\emptyset\}$. The following lemma yields an internal characterisation of separable zero-dimensional spaces, $X$, such that $C_{\mathrm{k}}(X)$ is stratifiable. Although highly useful as a technical result, it is not very informative about such spaces $X$. It is convenient to introduce some additional notation. Write $\overline{B}(\mathbf{0}, K, \varepsilon)$ for $\{g \in C(X) : |g(x)| \le \varepsilon\}$, and $B(f, K)$ for $\{g \in C(X, \{0,1\}) : g(x) = f(x)$ for all $x \in K\}$. The $B(\mathbf{0}, K)$ form a local basis at $\mathbf{0}$ in $C(X, \{0,1\})$ as $K$ runs over compact subsets of $X$.

LEMMA 29. *Let $X$ be a zero-dimensional separable metrisable space. Then the following conditions are equivalent*:

(1) *$C_{\mathrm{k}}(X)$ is stratifiable*;

(2) *$C_{\mathrm{k}}(X, \{0,1\})$ is stratifiable*;

(3) *there exist maps $k : \mathcal{O}_*(X) \to \mathcal{K}_*(X)$ and $E : \mathcal{K}_*(X) \to \mathcal{K}_*(X)$ such that $k(U) \subseteq U$ for $U \in \mathcal{O}_*(X)$ and if $V \cap K \ne \emptyset$ then $k(V) \cap E(K) \ne \emptyset$ for $V \in \mathcal{O}_*(X)$ and $K \in \mathcal{K}_*(X)$.*

Proof. As $C_{\mathrm{k}}(X, \{0,1\})$ embeds in $C_{\mathrm{k}}(X)$, the first implication is immediate. So let us suppose that $C_{\mathrm{k}}(X, \{0,1\})$ is stratifiable. Write $B(f, K)$ for $\{g \in C_{\mathrm{k}}(X, \{0,1\}) : g(x) = f(x)$ for all $x \in K\}$. Then $\mathbf{0}$ has a cushioned local pairbase $\mathcal{P}$. For each $K \in \mathcal{K}_*(X)$ pick $\langle B(\mathbf{0}, E(K)), B(\mathbf{0}, K_2)\rangle$ in $\mathcal{P}$ so that $B(\mathbf{0}, E(K)) \subseteq B(\mathbf{0}, K_2) \subseteq B(\mathbf{0}, K)$. This defines $E : \mathcal{K}_*(X) \to \mathcal{K}_*(X)$. Now we define $k : \mathcal{O}_*(X) \to \mathcal{K}_*(X)$. So take any $U \in \mathcal{O}_*(X)$. Let $f_U = \chi_U$ (the characteristic function of $U$). As $\mathcal{P}$ is cushioned, there is a $B(f_U, k'(U))$

such that $B(f_U, k'(U)) \cap P_1 \neq \emptyset$ implies $f_U \in P_2$, for all $\langle P_1, P_2\rangle \in \mathcal{P}$. Let $k(U) = k'(U) \cap U$. Observe that $U \cap K \neq \emptyset$ if and only if $f_U \notin B(\mathbf{0}, K)$, and $k(U) \cap E(K) = \emptyset$; and $B(f_U, k'(U)) \cap B(\mathbf{0}, E(K)) = \emptyset$ if and only if $k(U) \cap E(K) \neq \emptyset$. So we see that the maps $k$ and $E$ have the properties required.

It remains to show that if $X$ has maps $E$ and $k$ as in (3), then $C_{\mathrm{k}}(X)$ is stratifiable. By Theorem 24 it is sufficient to show that the zero function, $\mathbf{0}$, of $C_{\mathrm{k}}(X)$ is a $\sigma$-$m_3$ point. Define $\mathcal{P}_n = \{\langle B(\mathbf{0}, E(K), 1/(4n)), \overline{B}(\mathbf{0}, K, 1/n)\rangle : K \in \mathcal{K}_*(X)\}$, and $\mathcal{P} = \bigcup_{n\geq 1} \mathcal{P}_n$. Clearly, $\mathcal{P}$ is a local pairbase at $\mathbf{0}$.

We show that $\mathcal{P}_n$ is cushioned for each $n \geq 1$. We will apply Lemma 23. To this end, take $\mathbf{0} \neq f \in C_{\mathrm{k}}(X)$. If $\|f(x)\| \leq 1/n$ for all points $x$, then $f \in \overline{B}(\mathbf{0}, K, 1/n)$ for all $K \in \mathcal{K}_*(X)$, and there is nothing to do. Otherwise, we may pick a closed and open set $U'$ such that $\|f\|^{-1}[1/n, \infty) \subseteq U' \subseteq \|f\|^{-1}(1/(2n), \infty)$. Set $U_f = B(f, k(U'), 1/(4n))$.

Suppose $g \in U_f \cap B(\mathbf{0}, E(K), 1/(4n))$. Then for all $x$ in $k(U') \cap E(K)$, $\|f(x)\| < 1/(2n)$. Since $k(U') \subseteq U'$, we must have $k(U') \cap E(K) = \emptyset$. Hence, by hypothesis, $U' \cap K = \emptyset$, and thus, for all $x$ in $K$, $\|f(x)\| < 1/n$, or in other words, $f \in B(\mathbf{0}, K, 1/n)$, as required. ■

We are going to show that for every Polish space (in other words, a separable completely metrisable space), $X$ say, $C_{\mathrm{k}}(X)$ is stratifiable. Since every Polish space is the perfect image of the irrationals $\mathbb{P}$, Proposition 27(2) allows us to concentrate on showing that $C_{\mathrm{k}}(\mathbb{P})$ is stratifiable. This, in turn, is facilitated by Lemma 29.

Let $\mathcal{T}$ be the set of nondecreasing functions from $\mathbb{P}$, and let $\mathcal{T}_{\mathrm{b}}$ be the set of nondecreasing bounded functions from $\mathbb{P}$. Clearly, $\mathcal{T}$ is a closed subset of $\mathbb{P}$ and $\mathcal{T}$ is homeomorphic to $\mathbb{P}$. Partially order $\mathbb{P}$ by $g \leq f$ if and only if $g(n) \leq f(n)$ for all $n$. For $M \subseteq \mathbb{P}$, define $m(M) = \{f \in M : g \not\leq f$ for any $g \in M\}$.

Lemma 30. *For each closed subset $M$ of $\mathbb{P}$, and any $g \in M$, there exists $f \in m(M)$ such that $f \leq g$.*

Note that if an infinite $M \subseteq \mathbb{P}$ is closed and discrete, then any infinite subset of $M$ is unbounded.

Lemma 31. *Let $M \subseteq \mathcal{T}_{\mathrm{b}}$ be a discrete closed subset of $\mathbb{P}$. Then $m(M)$ is finite.*

Proof. Suppose, for a contradiction, that $L = m(M)$ is infinite. Put $n_* = \min\{n \in \omega : |\{f(n) : f \in L\}| = \omega\}$. There exists an infinite $N \subseteq L$ such that for different $f, g \in N$, we have $f(n_*) \neq g(n_*)$ and $f(i) = g(i)$ for $i < n_*$. Take $f \in N$. There exists $g \in N$ such that $g(n_*) > \max f$. One can see that $f \leq g$, hence $g \notin m(M)$, a contradiction. ■

Lemma 32. *Let $U$ be a clopen subset of $\mathcal{T}$. Then $m(U)$ is finite.*

Proof. Let $\mathcal{F}$ be the set of all nondecreasing finite sequences of nonnegative integers. For $b = (b_0, b_1, \ldots, b_n) \in \mathcal{F}$, put $U(b) = \{f \in \mathcal{T} : f(i) = b_i$ for $i \leq n\}$. Then $\mathcal{B} = \{U(b) : b \in \mathcal{F}\}$ is a base for $\mathcal{T}$ such that if any two members of the base meet, then one is contained in the other. Note that for $V \in \mathcal{B}$, $|m(V)| = 1$ and $m(V) \subseteq \mathcal{T}_{\mathrm{b}}$. There exists $\mathcal{B}' \subseteq \mathcal{B}$ such that $\mathcal{B}'$ is a partition of $U$. Then $m(U) \subseteq \bigcup\{m(V) : V \in \mathcal{B}'\}$ and $|m(U) \cap V| \leq 1$ for $V \in \mathcal{B}'$. Hence, $m(U) \subseteq \mathcal{T}_{\mathrm{b}}$ and $m(U)$ is a closed discrete subspace of $\mathbb{P}$. Since $m(U) = m(m(U))$, Lemma 31 implies that $m(U)$ is finite. ∎

For $f \in \mathcal{T}$, write $K(f)$ for $\{g \in \mathcal{T} : g \leq f\}$.

Lemma 33. *$\mathcal{B}_0 = \{B(\mathbf{0}, K(f)) : f \in \mathcal{T}\}$ forms a closure-preserving clopen neighbourhood base at $\mathbf{0}$ for $C_{\mathrm{k}}(\mathcal{T}, \{0,1\})$.*

Proof. Clearly, $\mathcal{B}_0$ is a clopen base at $\mathbf{0}$. Let $\mathcal{B}' \subseteq \mathcal{B}_0$ and $\chi \in C_{\mathrm{k}}(\mathcal{T}, \{0,1\}) \setminus \bigcup\mathcal{B}'$. We show that $\chi \notin \overline{\bigcup\mathcal{B}'}$. There exist $R \subseteq \mathcal{T}$ and a clopen $U \subseteq \mathcal{T}$ such that $\mathcal{B}' = \{B(\mathbf{0}, K(f)) : f \in R\}$ and $\chi = \chi(U)$. By Lemma 32, $m(U)$ is finite and therefore, compact. Take any $f \in R$. It is sufficient to show that $B(\chi(U), m(U)) \cap K(f) \neq \emptyset$. Since $\chi(U) \notin B(\mathbf{0}, K(f))$, we have $U \cap K(f) \neq \emptyset$; in other words, there exists $g \in U$ such that $g \leq f$. Lemma 30 implies that there exists $h \in m(U)$ such that $h \leq g \leq f$, that is to say, $m(U) \cap K(f) \neq \emptyset$. ∎

Theorem 34. *If the space $X$ is the compact-covering image of a Polish space, then $C_{\mathrm{k}}(X)$ is stratifiable.*

We observe the following corollary of Theorem 34 and the Borges–Dugundji Theorem.

Corollary 35. *Let $X$ be the compact-covering image of a Polish space. Then any closed convex subspace of $C_{\mathrm{k}}(X)$ is a retract. Indeed, every closed and convex subspace of $C_{\mathrm{k}}(X)$ is an absolute retract in the class of all stratifiable spaces.*

Remark. Suppose $X$ is such that $C_{\mathrm{k}}(X)$ is stratifiable, and $Y$ is any separable metrisable space. Then $C_{\mathrm{k}}(X)^\omega = C_{\mathrm{k}}(X, \mathbb{R}^\omega)$ is stratifiable, and, since $Y$ is (homeomorphic to) a subspace of $\mathbb{R}^\omega$, $C_{\mathrm{k}}(X, Y)$ is stratifiable. Thus, for every Polish space $X$, and separable metrisable $Y$, $C(X, Y)$, the space of all self maps of $X$, and the space of autohomeomorphisms of $X$ (all with the compact-open topology) are stratifiable.

The authors do not know how to characterise the compact-covering images of Polish spaces. Of course, in one direction, it is well known that a space is analytic if and only if it is the continuous image of a Polish space. In the other direction, it is equally well known that a space is $\aleph_0$ if and only if it is the compact-covering image of a separable metrisable space.

**7. Stratifiable subspaces of $C_{\mathrm{k}}(X)$.** Next we give criteria for subspaces of $C_{\mathrm{k}}(X)$ to be stratifiable. Let $X$ be a space, $Y \subseteq X$. Set $C_{\mathrm{k}}(X|Y) = \pi_Y(C_{\mathrm{k}}(X))$, where $\pi_Y$ is the natural projection $C_{\mathrm{k}}(X) \to C_{\mathrm{k}}(Y)$.

THEOREM 36. *Let $X$ be a metrisable compactum, $Y \subseteq X$. Then $C_{\mathrm{k}}(X|Y)$ is stratifiable.*

Proof. We may assume that $\overline{Y} = X$. Let $d$ be a continuous metric on $X$. There is a base $\mathcal{B}$ of $X$ such that $\{U \in \mathcal{B} : \operatorname{diam}_d U > \varepsilon\}$ is finite for any $\varepsilon > 0$. Fix $x(U) \in U \cap Y$ for $U \in \mathcal{B}$. For a compact $K \subseteq Y$ put $E(K) = K \cup \{x(U) : U \in \mathcal{B}$ and $U \cap K \neq \emptyset\}$. Observe that $E(K)$ is compact. Put $\mathcal{P}_n = \{\langle B(\mathbf{0}, K, 1/n), B(\mathbf{0}, E(K), 1/(2n))\rangle : K \subseteq Y$ is compact$\}$ for $n \in \omega \setminus \{0\}$.

CLAIM. *$\mathcal{P}_n$ is cushioned.*

Proof. For every $f \in C(X)$ there exists a finite $\mathcal{B}_f \subseteq \mathcal{B}$ such that $f^{-1}([1/n, \infty)) \subseteq \bigcup \mathcal{B}_f \subseteq f^{-1}(3/(4n), \infty)$. Put $M_f = \{x(U) : U \in \mathcal{B}_f\}$ and $V_f = B(f, M_f, 1/(4n))$.

Let $\langle P_1, P_2\rangle = \langle B(\mathbf{0}, K, 1/n), B(\mathbf{0}, E(K), 1/(2n))\rangle \in \mathcal{P}_n$. We show that if $f \notin P_1$ then $V_f \cap P_2 = \emptyset$. Since $f \notin P_1$ there is $U \in \mathcal{B}_f$ such that $U \cap f^{-1}([1/n, \infty)) \cap K \neq \emptyset$ and $f(U) \subseteq (3/(4n), \infty)$. Hence $x(U) \in M_f$ and $x(U) \in E(K)$. If $g \in V_f$ then $g(x(U)) > 3/(4n) - 1/(4n) = 1/(2n)$. Therefore $g \notin P_2$. Hence $V_f \cap P_2 = \emptyset$.

Lemma 23 implies that $\mathcal{P}_n$ is cushioned. ∎

*Continuation of the proof of Theorem 36.* Clearly, $\mathcal{P} = \bigcup \mathcal{P}_n$ is a pair-base at $\mathbf{0}$. The Claim implies that $\mathcal{P}$ is $\sigma$-cushioned. Then Theorem 24 implies that $C_{\mathrm{k}}(X|Y)$ is stratifiable. ∎

COROLLARY 37. *Let $X$ be a separable metrisable space. If $Y \subseteq C_{\mathrm{k}}(X)$ is countable then $Y$ is stratifiable.*

Proof. Since $C_{\mathrm{k}}(X)$ is homeomorphic to $C_{\mathrm{k}}(X, (0,1))$ we may assume that $Y \subseteq C_{\mathrm{k}}(X, (0,1))$. Embedding $X$ into a Hilbert cube via $Y$, and taking the closure, we see that there exists a metrisable compactification $bX$ of $X$ such that $Y \subseteq C_{\mathrm{k}}(bX|X)$. Theorem 36 implies that $Y$ is stratifiable. ∎

PROBLEM C. *Let $X$ be a separable metrisable space and $Y \subseteq C_{\mathrm{k}}(X)$ be $\sigma$-compact. Is $Y$ stratifiable?*

Naturally, another sufficient condition for all countable subspaces to be stratifiable is given by Proposition 5.

PROPOSITION 38. *Suppose $X$ has property* (CK) *and* $\operatorname{cof} \mathcal{K}(X) < \mathfrak{b}$. *Then every countable subspace of $C_{\mathrm{k}}(X)$ is stratifiable.*

**8. Questions.** Besides the questions already mentioned, there are two key unresolved problems. The first is whether the conditions of Theorem 34 are necessary as well as sufficient.

CONJECTURE D. *If $X$ is metrisable, and $C_k(X)$ is stratifiable, then $X$ is completely metrisable.*

The second concerns the famous "$M_1$-$M_3$" problem. A space is said to be $M_1$ if it has a $\sigma$-closure preserving base. Evidently, an $M_1$ space has a $\sigma$-cushioned pairbase, and so is stratifiable. Whether the converse is true is the content of the $M_3$-$M_1$ problem. Interestingly, we have so far been unable to determine if $C_k(X)$ is $M_1$ when it is stratifiable, not even when $X$ is $\sigma$-compact and completely metrisable.

QUESTION E. *If $X$ is separable and completely metrisable, then is $C_k(X)$ an $M_1$ space?*

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