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Gaussian automorphisms whose ergodic self-joinings are Gaussian

by

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Abstract. We study ergodic properties of the class of Gaussian automorphisms whose ergodic self-joinings remain Gaussian. For such automorphisms we describe the structure of their factors and of their centralizer. We show that Gaussian automorphisms with simple spectrum belong to this class.

We prove a new sufficient condition for non-disjointness of automorphisms giving rise to a better understanding of Furstenberg's problem relating disjointness to the lack of common factors. This and an elaborate study of isomorphisms between classical factors of Gaussian automorphisms allow us to give a complete solution of the disjointness problem between a Gaussian automorphism whose ergodic self-joinings remain Gaussian and an arbitrary Gaussian automorphism.

INTRODUCTION

Although the theory of Gaussian dynamical systems is a classical part of modern ergodic theory, little is known about factors of Gaussian systems. Since each Gaussian system with positive entropy is a direct product of a zero entropy Gaussian automorphism and a Bernoulli automorphism with infinite entropy (see e.g. [26]), it follows from [31] that the factor problem is only the problem for zero entropy Gaussian systems, or equivalently, for those automorphisms whose underlying stationary process has singular spectral measure (see [26]).

It was already noticed by the third author in [32] that the Gaussian–Kronecker automorphisms have only those factors which are directly inherited from the Gaussian structure, and which we call here classical factors.

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In this paper we point out that this remains true for all Gaussian automorphisms whose ergodic self-joinings remain Gaussian, in the sense that the Gaussian spaces of the marginal factors span a Gaussian space. We will call such Gaussian automorphisms GAG (*). The GAG property reduces the set of self-joinings to a minimum and therefore the GAG's have many strong ergodic properties.

Recall that any continuous symmetric finite measure σ on $\mathbb{T}$ determines a stationary centered Gaussian process (X_n) whose spectral measure is σ. The corresponding measure-preserving automorphism T_σ will be called a standard Gaussian automorphism. Here we have to consider generalized Gaussian automorphisms, where the Gaussian space is not necessarily the space of a process, but it turns out that the GAG property depends only on the spectral type of the associated unitary operator restricted to the Gaussian space. Thus a continuous symmetric measure σ on $\mathbb{T}$ will be called a GAG measure if T_σ is GAG.

We show that the class of GAG measures is stable for some basic operations, in particular translations. Moreover, a GAG measure is singular with respect to each of its translates.

The set of self-joinings of a given measure-preserving automorphism has a natural structure of a semigroup (see [29] or Section 1.3 below). We show that in the case of standard Gaussian automorphisms the GAG property is characterized by the fact that the semigroup of self-joinings is Abelian. In particular, it follows that all Gaussian automorphisms with simple spectrum are GAG.

We describe the structure of factors of a GAG. These are only classical factors. The only possible isomorphisms between two factors of a GAG are restrictions of Gaussian isomorphisms. By some elementary facts from the representation theory of compact groups, this will allow us to give new examples of non-disjoint automorphisms which have no common non-trivial factors, generalizing an earlier unpublished joint result of del Junco and the third author. We show that the only factors of a GAG which are Gaussian are those determined by subspaces of the Gaussian space, and it follows that the only possible isomorphisms between a GAG and a Gaussian automorphism are also Gaussian. In the standard case we show that all factors of a GAG automorphism are semisimple (in the sense of [14]).

We prove that if two ergodic automorphisms T, S are not disjoint then an ergodic infinite self-joining of T has a common non-trivial factor with S. Building on that we completely solve the problem of disjointness between a GAG and an arbitrary Gaussian automorphism. It turns out that if they are not disjoint then they have a common factor, and even much more, the

(*) GAG comes from the French abbreviation of *gaussiens à autocouplages gaussiens.*

spectral types of the GAG and of the other Gaussian on their Gaussian spaces must have some translations which are not mutually singular. This is an essential improvement of the main disjointness result of T_σ and T_τ in the case of σ and τ concentrated on sets without rational relations from [17].

For other recent results on Gaussian automorphisms see [12], [13], [18], [26], [27], [28].

1. BASIC DEFINITIONS, NOTATION AND INTRODUCTORY RESULTS

1.1. Basic facts from spectral theory of unitary operators. Let H be a Hilbert space. We denote by $\mathcal{L}(H)$ the algebra of all bounded linear operators on H and by $\mathcal{U}(H)$ the Polish group of all unitary operators on H equipped with the strong operator topology.

Given $U \in \mathcal{U}(H)$ and $h \in H$ the spectral measure of h under U is denoted by σ_h (or $\sigma_{h,U}$ if needed). It is the positive finite Borel measure on $\mathbb{T}$ (we shall simply say *measure* on $\mathbb{T}$; throughout, $\mathbb{T}$ is taken as the circle group) given by

$$\widehat{\sigma}_h(n) = \int_{\mathbb{T}} z^n \, d\sigma_h(z) = (U^n h \mid h), \qquad n \in \mathbb{Z}.$$

Define $Z(h) = \overline{\operatorname{span}}\{U^n h : n \in \mathbb{Z}\}$, the cyclic subspace generated by h. The restriction of U to $Z(h)$ is unitarily equivalent, by the correspondence $U^n h \mapsto z^n$, to the operator V of multiplication by the identity function z on $L^2(\mathbb{T}, \sigma_h)$.

The operator U has *simple spectrum* if there exists $h \in H$ such that $H = Z(h)$. Then the spectral representation of U as V yields an isomorphism between the von Neumann algebra $W^*(U)$ generated by U (i.e. the smallest weakly closed subalgebra of $\mathcal{L}(H)$ containing $\{U^n : n \in \mathbb{Z}\}$) and $L^\infty(\mathbb{T}, \sigma_h)$. Moreover, as any bounded operator on $L^2(\mathbb{T}, \sigma_h)$ which commutes with V is the multiplication operator by a bounded Borel function, any operator in $\mathcal{L}(H)$ which commutes with U belongs to $W^*(U)$.

In the general case, up to unitary isomorphism, U is determined by its maximal spectral type σ_U (which is, up to equivalence of measures, some σ_{h_0} with the property that $\sigma_h \ll \sigma_{h_0}$ for each $h \in H$), and its multiplicity function $M_U : \mathbb{T} \to \{1, \ldots, \infty\}$ defined σ_U-a.e. Then $W^*(U)$ is still isomorphic to $L^\infty(\mathbb{T}, \sigma_U)$ in such a way that U corresponds to the identity function z. The spectral projector corresponding to the indicator function χ_A of a Borel subset of $\mathbb{T}$ will be denoted by π_A and the spectral subspace $\pi_A H$ by H_A. Since $\pi_A \in W^*(U)$, H_A is invariant under any operator $W \in \mathcal{L}(H)$ which commutes with U. When the multiplicity function takes a constant value n on A, the restriction of U to H_A is unitarily isomorphic to the multiplication by z on $L^2(A, \sigma_U|_A, H')$, where H' is an n-dimensional Hilbert space.

We refer to [24] or [23] for other definitions and results in spectral theory.

1.2. Joinings of automorphisms of standard Borel spaces. Let $(X,\mathcal{B},\mu)$ be a standard Borel probability space. To any automorphism $T : (X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$, i.e. $T\in \operatorname{Aut}(X,\mathcal{B},\mu)$, corresponds the unitary operator U_T on $L^2(X,\mathcal{B},\mu)$ given by

$$U_T(f)=f\circ T.$$

The spectral analysis of T is meant as the spectral analysis of U_T, but it is often restricted to the subspace $L^2_0(X,\mathcal{B},\mu)$ of functions with zero mean. Since U_T preserves the subspace of real functions, σ_U must be the type of a symmetric measure (σ is said to be *symmetric* if $\sigma(A)=\sigma(\overline{A})$ for any Borel subset $A\subset\mathbb{T}$). Given a positive Borel measure σ on $\mathbb{T}$ we define $\widetilde{\sigma}$ by $\widetilde{\sigma}(A):=\sigma(\overline{A})$. For spectral measures of U_T we then have $\sigma_{\bar f}=\widetilde{\sigma}_f$.

The mapping $T\mapsto U_T$ allows us to embed $\operatorname{Aut}(X,\mathcal{B},\mu)$ as a closed subgroup of $\mathcal{U}(L^2(X,\mathcal{B},\mu))$. The strong topology restricted to $\operatorname{Aut}(X,\mathcal{B},\mu)$ is then the usual weak topology on $\operatorname{Aut}(X,\mathcal{B},\mu)$.

The *centralizer* $C(T)$ of T is the closed subgroup $\{S\in\operatorname{Aut}(X,\mathcal{B},\mu): ST=TS\}$ of $\operatorname{Aut}(X,\mathcal{B},\mu)$. By a *factor* of T we mean any T-invariant sub-σ-algebra $\mathcal{A}$ of $\mathcal{B}$ (more precisely, the corresponding factor is the quotient action of T on $(X/\mathcal{A},\mathcal{A},\mu)$). If there is no ambiguity on T, we shall also say that $\mathcal{B}$ *has* $\mathcal{A}$ *as its factor* or that $\mathcal{B}$ *is an extension of* $\mathcal{A}$, which we denote by $\mathcal{B}\to\mathcal{A}$.

Given $F\subset L^2(X,\mathcal{B},\mu)$, let $\mathcal{B}(F)$ denote the smallest sub-σ-algebra of $\mathcal{B}$ which makes all the elements of F measurable (all σ-algebras are considered modulo null sets). It is a factor of T if F is T-invariant. Any compact subgroup $\mathcal{K}$ of $C(T)$ determines the *compact* factor

$$\mathcal{B}/\mathcal{K}=\{B\in\mathcal{B}: SB=B \text{ for every } S\in\mathcal{K}\}.$$

If T is weakly mixing then $\mathcal{B}/\mathcal{K}$ cannot be trivial (cf. e.g. [3]). Moreover, we have the *saturation condition* (see [15]):

$$\mathcal{K}=\{S\in C(T): SA=A \text{ for each } A\in\mathcal{B}/\mathcal{K}\}.$$

Let $T_i:(X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2,\ldots,$ be a finite or infinite sequence of automorphisms. We denote by $J(T_1,T_2,\ldots)$ the set of all *joinings* of them, identified with the set of all $T_1\times T_2\times\ldots$-invariant probability measures λ on $(X_1\times X_2\times\ldots,\mathcal{B}_1\otimes\mathcal{B}_2\otimes\ldots)$ whose marginals are equal to μ_i (more precisely, the joining is the corresponding automorphism $T_1\times T_2\times\ldots$ on $(X_1\times X_2\times\ldots,\mathcal{B}_1\otimes\mathcal{B}_2\otimes\ldots,\lambda)$, which we also denote $(T_1\times T_2\times\ldots,\lambda)$). The subset $J^{\mathrm{e}}(T_1,T_2,\ldots)$ of *ergodic joinings* consists of those λ for which this action of $T_1\times T_2\times\ldots$ is ergodic. If each T_i is ergodic then $J^{\mathrm{e}}(T_1,T_2,\ldots)$ is the set of extremal points of $J(T_1,T_2,\ldots)$ and the ergodic decomposition of each joining consists of joinings.

In the case of $T_1=T_2=\ldots=T$ we speak about *self-joinings* and use the notation $J_n(T)$, $J_n^{\mathrm{e}}(T)$ in the case of n copies of T, and $J_\infty(T)$, $J_\infty^{\mathrm{e}}(T)$ in the case of infinitely many copies.

If $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is an automorphism, then with each $S\in C(T)$ we associate the *graph self-joining* λ_S given by

$$\lambda_S(A\times B)=\mu(A\cap S^{-1}B). \tag{1}$$

If T is ergodic then λ_S is ergodic. If $\mathcal{A}$ is a factor of T then the *relative product over* $\mathcal{A}$ is the self-joining $\mu\otimes_{\mathcal{A}}\mu$ in $J_2(T)$ given by

$$\mu\otimes_{\mathcal{A}}\mu(A\times B)=\int\limits_{X/\mathcal{A}}E(A\,|\,\mathcal{A})E(B\,|\,\mathcal{A})\,d\mu. \tag{2}$$

Assume now that T is ergodic. If $\mathcal{A}=\mathcal{B}/\mathcal{K}$ for a compact subgroup $\mathcal{K}\subset C(T)$ then the ergodic decomposition of the relative product $T\times T$ over $\mathcal{A}$ is given by

$$\mu\otimes_{\mathcal{A}}\mu=\int\limits_{\mathcal{K}}\lambda_S\,dS,$$

where dS stands for the normalized Haar measure on $\mathcal{K}$. A converse also holds—Veech in [36] shows that a factor $\mathcal{A}$ is of the form $\mathcal{B}/\mathcal{K}$ whenever its ergodic decomposition contains solely graph joinings.

In the case of two automorphisms $T_i:(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$, easy extensions of formulas (1) and (2) allow us to define joinings between T_1 and T_2 when an isomorphism $S:(X_1,\mathcal{B}_1,\mu_1,T_1)\to(X_2,\mathcal{B}_2,\mu_2,T_2)$ is given, or when there is an isomorphism between a non-trivial factor of T_1 and a factor of T_2 (in the latter case we say that T_1 and T_2 *have a common factor*).

An automorphism T is said to be *relatively weakly mixing over* $\mathcal{A}$, or we say that $\mathcal{B}\to\mathcal{A}$ *is relatively weakly mixing*, if $\mu\otimes_{\mathcal{A}}\mu\in J_2^{\mathrm{e}}(T)$. A notion complementary to weak mixing is distality (see [39] for the definition). If $\mathcal{K}\subset C(T)$ is a compact subgroup then $\mathcal{B}\to\mathcal{B}/\mathcal{K}$ is an example of a distal extension. Moreover, if $\mathcal{A}\subset\mathcal{A}'\subset\mathcal{B}$ are factors and $\mathcal{B}\to\mathcal{A}$ is distal then so is $\mathcal{A}'\to A$. Given a factor $\mathcal{A}$ there exists exactly one factor $\widehat{\mathcal{A}}$ such that $\mathcal{A}\subset\widehat{\mathcal{A}}\subset\mathcal{B}$, $\mathcal{B}\to\widehat{\mathcal{A}}$ is relatively weakly mixing and $\widehat{\mathcal{A}}\to\mathcal{A}$ is distal (see [4], Th. 6.17 and the final remark on page 139). The decomposition $\mathcal{B}\to\widehat{\mathcal{A}}\to\mathcal{A}$ is called the *Furstenberg decomposition* of $\mathcal{B}\to\mathcal{A}$. It follows that, given a factor $\mathcal{A}$, there exists a smallest factor $\widehat{\mathcal{A}}\supset\mathcal{A}$ such that T is relatively weakly mixing over $\widehat{\mathcal{A}}$. In particular, if $\mathcal{K}\subset C(T)$ is a compact subgroup then $\widehat{\mathcal{B}/\mathcal{K}}=\mathcal{B}$, that is, there is no proper factor $\mathcal{A}\supset\mathcal{B}/\mathcal{K}$ with $\mathcal{B}\to\mathcal{A}$ relatively weakly mixing. If T is weakly mixing then the only factor independent of $\mathcal{B}/\mathcal{K}$ is the trivial (one-point) factor.

Two automorphisms $T_i:(X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$, are said to be *disjoint* if $J(T_1,T_2)=\{\mu_1\otimes\mu_2\}$ (cf. [3]). We will then write $T_1\perp T_2$. If T_1 and T_2 are disjoint then they cannot have a common factor, but the converse does not hold. On the other hand, a sufficient but not necessary condition for $T_1\perp T_2$ is that their maximal spectral types on the spaces of functions of zero mean be mutually singular (see [7]); one then says that T_1 and T_2 are *spectrally disjoint*.

1.3. Intertwining Markov operators. We now discuss basic properties of operators associated with joinings of two automorphisms. These operators were introduced by Vershik. For more details, we refer to [29].

Let $(X_i, \mathcal{B}_i, \mu_i)$ be probability spaces, $i = 1, 2$. To each probability measure λ on $\mathcal{B}_1 \otimes \mathcal{B}_2$ whose marginals are μ_i ($i = 1, 2$) corresponds the operator $\Phi_\lambda : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$, given by

$$(\Phi_\lambda f \mid g) = \int_{X_1 \times X_2} f(x_1)\overline{g(x_2)}\, d\lambda(x_1, x_2) \tag{3}$$

for every $f \in L^2(X_1, \mathcal{B}_1, \mu_1)$, $g \in L^2(X_2, \mathcal{B}_2, \mu_2)$. It may be considered as the conditional expectation operator with respect to the σ-algebra $X_1 \times \mathcal{B}_2$ (the σ-algebra generated by the second coordinate), restricted to $L^2(\mathcal{B}_1 \times X_2, \lambda)$ (more precisely, $E(f \otimes 1 \mid X \times \mathcal{B}_2) = 1 \otimes \Phi_\lambda f$).

Assume now that $T_i : (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i)$ are automorphisms, $i = 1, 2$. Since $(\Phi_\lambda(f \circ T_1) \mid g \circ T_2) = (\Phi_\lambda U_{T_1} f \mid U_{T_2} g) = (U_{T_2}^{-1} \Phi_\lambda U_{T_1} f \mid g)$, λ is $T_1 \times T_2$-invariant iff $U_{T_2}^{-1} \Phi_\lambda U_{T_1} = \Phi_\lambda$ and thus

$$\lambda \in J(T_1, T_2) \quad \text{iff} \quad \Phi_\lambda U_{T_1} = U_{T_2} \Phi_\lambda.$$

Such operators $\Phi = \Phi_\lambda$ have the Markov properties:

$$\Phi f \geq 0 \quad \text{if } f \geq 0, \qquad \Phi 1 = 1, \qquad \Phi^* 1 = 1. \tag{4}$$

Conversely, each operator $\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ satisfying (4) defines a measure λ on $\mathcal{B}_1 \otimes \mathcal{B}_2$ with marginals μ_i ($i = 1, 2$) and such that $\Phi = \Phi_\lambda$, by

$$\lambda(A \times B) = \int_B \Phi(\chi_A)\, d\mu_2, \qquad A \in \mathcal{B}_1,\ B \in \mathcal{B}_2.$$

Therefore, we have a one-to-one correspondence between $J(T_1, T_2)$ and the set of all Markov operators $\Phi : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ satisfying (4) and the intertwining relation $\Phi U_{T_1} = U_{T_2} \Phi$.

Note that if $\lambda = \mu_1 \otimes \mu_2$ then Φ_λ is the projector onto the subspace of constant functions. If $\lambda = \lambda_S$ is the graph joining corresponding to an isomorphism S between T_1 and T_2, then $\Phi_\lambda = U_S$. Finally, if λ is the relative product over a factor $\mathcal{A}$ of $(X, \mathcal{B}, \mu, T)$, then Φ_λ is the conditional expectation projector $\pi_{\mathcal{A}} = \pi_{L^2(\mathcal{A})} = E(\cdot \mid \mathcal{A})$.

It is also clear that the class of Markov intertwining operators is closed under composition. If $T_i : (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2, 3$, are automorphisms, $\lambda \in J(T_1, T_2)$ and $\lambda' \in J(T_2, T_3)$, then $\Phi_{\lambda'} \Phi_\lambda$ corresponds to a joining $\varrho \in J(T_1, T_3)$ which we call the *composition* of λ and λ'. It can be described as the factor $\mathcal{B}_1 \otimes \mathcal{B}_3$ in the relative product $\lambda \otimes_{\mathcal{B}_2} \lambda'$ of λ and

λ' over $\mathcal{B}_2$ (after obvious identifications of $X_1 \times \mathcal{B}_2$, $\mathcal{B}_2 \times X_3$ with $\mathcal{B}_2$ and $\mathcal{B}_1 \times X_2 \times \mathcal{B}_3$ with $\mathcal{B}_1 \otimes \mathcal{B}_3$).

If $\lambda \in J(T_1, T_2)$ then Φ_λ^* is the Markov operator corresponding to the joining $\lambda^* \in J(T_2, T_1)$ obtained from λ by the exchange of coordinates. Since it is clear that $\Phi_\lambda|_{L_0^2(X_1,\mu_1)} = 0$ iff $\lambda = \mu_1 \otimes \mu_2$, we have

$$\lambda = \mu_1 \otimes \mu_2 \quad \text{iff} \quad \Phi_\lambda^* \circ \Phi_\lambda|_{L_0^2(X_1,\mu_1)} = 0. \tag{5}$$

So, for any automorphism T, $J_2(T)$ has a semigroup structure. By a slight abuse of notation, we let $J_2(T)$ mean also the corresponding operator semigroup; it contains every U_S for $S \in C(T)$ and every $\pi_{\mathcal{A}}$ for $\mathcal{A}$ a factor of T.

If we assume that T (i.e. U_T) has simple spectrum then $J_2(T) \subset W^*(U_T)$ since every bounded operator which commutes with U_T belongs to $W^*(U_T)$. Therefore in this case $J_2(T)$ is commutative. More generally, we shall say that T has *commuting self-joinings* if $J_2(T)$ is Abelian.

Directly from this discussion, we have the following:

PROPOSITION 1. *Let T be an automorphism of $(X, \mathcal{B}, \mu)$ and $S \in C(T)$. If T has commuting self-joinings or if $U_S \in W^*(U_T)$ then*

$$J_2(T) \subset J_2(S), \qquad C(T) \subset C(S),$$

and any T-invariant σ-algebra $\mathcal{A} \subset \mathcal{B}$ is also S-invariant. ∎

COROLLARY 1. *Let S and T be two commuting ergodic automorphisms. If both S and T have commuting self-joinings then $J_2^{\mathrm{e}}(T) = J_2^{\mathrm{e}}(S)$.*

Proof. As we have already noticed, $J_2^{\mathrm{e}}(S)$ and $J_2^{\mathrm{e}}(T)$ are the sets of extremal points of $J_2(S)$ and $J_2(T)$. From Proposition 1, $J_2(S) = J_2(T)$ and therefore $J_2^{\mathrm{e}}(S) = J_2^{\mathrm{e}}(T)$. ∎

LEMMA 1. *Let T be an automorphism of $(X, \mathcal{B}, \mu)$ and assume that T has commuting self-joinings. Then for every factor $\mathcal{A}$ of T and every $\lambda \in J_2(T)$,*

$$\Phi_\lambda(L^2(\mathcal{A}, \mu)) \subset L^2(\mathcal{A}, \mu)$$

and

$$(\mathcal{A} \otimes \mathcal{A}) \cap (X \times \mathcal{B}) = X \times \mathcal{A} \bmod \lambda.$$

In particular, these assertions hold if T has simple spectrum.

Proof. The first assertion follows from $\Phi_\lambda \pi_{\mathcal{A}} = \pi_{\mathcal{A}} \Phi_\lambda$. Now, the second assertion follows from the first one. Indeed, if $f, g \in L^2(\mathcal{A}, \mu)$ and also $f \otimes g \in L^2(\mathcal{A} \otimes \mathcal{A}, \lambda)$ then

$$E(f \otimes g \mid X \times \mathcal{B}) = E(f \otimes 1 \mid X \times \mathcal{B})(1 \otimes g) = 1 \otimes (\Phi_\lambda(f) g)$$

and since $\Phi_\lambda(f) \in L^2(\mathcal{A}, \mu)$, we have $E(f \otimes g \mid X \times \mathcal{B}) \in L^2(X \times \mathcal{A}, \lambda)$. Since the products of functions as above span $L^2(\mathcal{A} \otimes \mathcal{A}, \lambda)$, $E(\cdot \mid X \times \mathcal{B})$ sends $L^2(\mathcal{A} \otimes \mathcal{A}, \lambda)$ to $L^2(X \times \mathcal{A}, \lambda)$ and the result follows. ∎

Remark 1. Without the assumption that T has commuting self-joinings, the assertions of Lemma 1 remain true for any self-joining λ and any factor $\mathcal{A}$ such that $\pi_{\mathcal{A}}$ commutes with Φ_λ. That is in particular the case when $\pi_{\mathcal{A}}$ is a spectral projector, i.e. if the spectral types of U_T on $L^2(\mathcal{A},\mu)$ and $L^2(\mathcal{A},\mu)^\perp$ are mutually singular. This can also be seen from the following general observation.

If T_i are automorphisms of $(X_i,\mathcal{B}_i,\mu_i)$ $(i=1,2)$ and $\lambda\in J(T_1,T_2)$ then

$$\sigma_{\Phi_\lambda f,U_{T_2}} \ll \sigma_{f,U_{T_1}} \quad \text{for every } f\in L^2(X_1,\mathcal{B}_1,\mu_1). \tag{6}$$

Indeed,

$$\widehat{\sigma}_{\Phi_\lambda f,U_{T_2}}(n) = (U_{T_2}^n\Phi_\lambda f \mid \Phi_\lambda f) = (\Phi_\lambda U_{T_1}^n f\mid \Phi_\lambda f) = (U_{T_1}^n f\mid \Phi_\lambda^*\Phi_\lambda f).$$

Recall that given a Hilbert space H, $U\in\mathcal{U}(H)$ and $f,g\in H$, the correlation measure $\sigma_{f,g,U}$ is defined by $\widehat{\sigma}_{f,g,U}(n)=(U^nf\mid g)$, $n\in\mathbb{Z}$, and it satisfies $\sigma_{f,g,U}\ll\sigma_{f,U}$ (see e.g. [24], p. 18). Here we find $\sigma_{\Phi_\lambda f,U_{T_2}}=\sigma_{f,\Phi_\lambda^*\Phi_\lambda f,U_{T_1}}$ and $\sigma_{\Phi_\lambda f,U_{T_2}}\ll\sigma_{f,U_{T_1}}$ follows.

2. GAUSSIAN AUTOMORPHISMS AND JOININGS

2.1. Gaussian spaces and generalized Gaussian automorphisms. For general properties of Gaussian spaces and Gaussian dynamical systems, we refer to [22] and [1], Chapter 14. By a *Gaussian probability space* we mean a standard probability space $(X,\mathcal{B},\mu)$ together with an infinite-dimensional closed real subspace H^{r} of $L_0^2(X,\mathcal{B},\mu)$ such that

(7) $\mathcal{B}(H^{\mathrm{r}})=\mathcal{B}$ and each non-zero function of H^{r} has a Gaussian distribution.

We refer to the subspace $H=H^{\mathrm{r}}+iH^{\mathrm{r}}$ as the *complex Gaussian space* of X (or simply as the *Gaussian space* of X). We define a *generalized Gaussian automorphism*, or simply a *Gaussian automorphism*, as an ergodic automorphism of $(X,\mathcal{B},\mu)$ such that H is invariant under U_T. In the classical case where H^{r} is the space of a real stationary centered process $(f\circ T^n)$, i.e. where $U_T|_H$ has simple spectrum, we call T a *standard Gaussian automorphism*. We want first to point out that important features of the Gaussian structure do not depend on the automorphism T. For instance, this is the case of the *decomposition into Wiener chaos*:

$$L^2(X,\mathcal{B},\mu)=\bigoplus_{n=0}^{\infty}H^{(n)}$$

where $H^{(0)}$ is the subspace of constant functions, $H^{(1)}=H$ and, for $n>1$, $H^{(n)}$ is defined inductively as the orthocomplement of $\bigoplus_{k<n}H^{(k)}$ in the span of all products $f_1\ldots f_m$ of functions in H, $m\le n$. The nth chaos $H^{(n)}$ is isometric to the symmetric tensor power $H^{\odot n}$ in such a way that

$f_1 \odot \ldots \odot f_n$ corresponds to the projection of the product $f_1 \ldots f_n$, and we shall constantly identify $H^{(n)}$ with $H^{\odot n}$. In fact, when $f_1, \ldots, f_n \in H$ are pairwise orthogonal, their product is orthogonal to the chaos $H^{(k)}$, $k < n$ (see [1], Corollary on p. 358), so it belongs to $H^{(n)}$ and we can identify it with $f_1 \odot \ldots \odot f_n$. Moreover, $H^{\odot n}$ is spanned by such tensor products of pairwise orthogonal functions (see Lemma 6.13 in [22]).

The following lemma is well known but since no explicit reference seems to exist, we give a proof for the reader's convenience. Given two Gaussian spaces H_i, we call operators $U : H_1 \to H_2$ which map H_1^{r} into H_2^{r} *real operators*.

LEMMA 2. *For two given Gaussian probability spaces* $(X_i, \mathcal{B}_i, \mu_i)$, *and their Gaussian spaces* H_i $(i = 1, 2)$, *any real isometry* U *from* H_1 *onto* H_2 *extends to a unique measure-theoretic isomorphism* $T_U : (X_2, \mathcal{B}_2, \mu_2) \to (X_1, \mathcal{B}_1, \mu_1)$ *such that* $f \circ T_U = Uf$ *for every* $f \in H_1$. *Moreover*, *the operator* $U_{T_U} : L^2(X_1, \mathcal{B}_1, \mu_1) \to L^2(X_2, \mathcal{B}_2, \mu_2)$ *maps each chaos* $H_1^{(n)}$ *onto* $H_2^{(n)}$, *and its restriction to* $H_1^{(n)}$ *is given by* $U^{\odot n}$.

Proof. Let $(Y, \mathcal{C}, \nu)$ be the product space $\mathbb{R}^{\mathbb{N}}$ equipped with its Borel σ-algebra and ν the infinite product of normalized centered Gaussian distributions. Since orthogonal Gaussian random variables in a Gaussian space are independent (e.g. Proposition 2.4 of [22]), any orthonormal basis $(f_i)_{i \geq 0}$ of H_1^{r} yields a metric isomorphism $S_1 : (X_1, \mathcal{B}_1, \mu_1) \to (Y, \mathcal{C}, \nu)$, $S_1(x) = (f_i(x))_{i \geq 0}$. Since $(Uf_i)_{i \geq 0}$ is an orthonormal basis of H_2^{r}, we also have an isomorphism $S_2 : (X_2, \mathcal{B}_2, \mu_2) \to (Y, \mathcal{C}, \nu)$, $S_2(x) = (Uf_i(x))_{i \geq 0}$. Now, $Uf_i(x) = f_i \circ T_U(x)$ μ_2-a.e., $i \geq 0$, implies $S_2(x) = S_1(T_U(x))$ μ_2-a.e. and thus we define $T_U = S_1^{-1} \circ S_2$.

The second part of the lemma now follows directly from the fact that the nth chaos is spanned by the products $f_1 \ldots f_n$, where $f_j \in H_1$, $j = 1, \ldots, n$, are pairwise orthogonal. ∎

Such an isomorphism T_U will be called a *Gaussian isomorphism*. By the uniqueness property, $T_{UV} = T_V \circ T_U$ whenever the product is defined.

Assume that $(X, \mathcal{B}, \mu)$ is a Gaussian probability space, and H its Gaussian space. We denote by $\mathcal{L}^{\mathrm{r}}(H)$ the algebra of bounded real operators on H and by $\mathcal{U}^{\mathrm{r}}(H)$ the group of real unitary operators on H. Note that, since any two real infinite-dimensional separable Hilbert spaces are isometric, it follows from Lemma 2 that the set $\{T_U : U \in \mathcal{U}^{\mathrm{r}}(H)\}$ is up to a Gaussian isomorphism the same for all Gaussian probability spaces.

Of course, any automorphism T of $(X, \mathcal{B}, \mu)$ which preserves H is equal to T_U where U is the restriction of U_T to H. In particular, we have the following.

PROPOSITION 2. *Let T be a Gaussian automorphism, H its Gaussian space and U the restriction of U_T to H. If V is a real unitary operator on H commuting with U, then $T_V \in C(T)$. Conversely, any $S \in C(T)$ such that $U_S H = H$ is equal to T_V for some $V \in \mathcal{U}^{\mathrm{r}}(H)$ commuting with U.* ■

However, we emphasize that such an automorphism S need not be ergodic. So, it is a Gaussian automorphism iff it is ergodic.

Let $C^{\mathrm{r}}(U)$ denote the subgroup of those operators in $\mathcal{U}^{\mathrm{r}}(H)$ which commute with U. The subgroup of all S in $C(T)$ which preserve the Gaussian space will be called the *Gaussian centralizer* of T and denoted by $C^{\mathrm{g}}(T)$. The map $V \mapsto T_V$ is a topological isomorphism from $C^{\mathrm{r}}(U)$ onto $C^{\mathrm{g}}(T)$. Indeed, for each $n \geq 1$ the map $V \mapsto V^{\odot n}$ from $C^{\mathrm{r}}(U)$ to $\mathcal{U}(H^{\odot n})$ is continuous in the strong topology, hence by Lemma 2, $V \mapsto T_V$ is continuous, and conversely, $S \mapsto V = U_S|_H$ is clearly continuous.

It is well known that up to isomorphism there exists exactly one standard Gaussian automorphism whose Gaussian process has a given continuous symmetric spectral type σ. We will denote it by T_σ (the unicity of T_σ also follows from Lemma 2 since up to unitary isomorphism a unitary operator U with simple spectrum is determined by its maximal spectral type). In this case, if $H = Z(h)$ with $h \in H^{\mathrm{r}}$ (and $\sigma_h = \sigma$), the space H^{r} is generated by finite sums $\sum a_n U^n h$ with $a_n \in \mathbb{R}$. Hence, in the representation sending h to 1, H^{r} corresponds to the space of Hermitian functions in $L^2(\mathbb{T}, \sigma)$ (we say that f is *Hermitian* if $f(\overline{z}) = \overline{f(z)}$ σ-a.e.). It follows that the real operators in $W^*(U)$ correspond to the Hermitian functions in $L^\infty(\mathbb{T}, \sigma)$, and $C^{\mathrm{r}}(U)$ corresponds to the group of Hermitian functions of modulus 1 in $L^\infty(\mathbb{T}, \sigma)$. This latter group is denoted by $\mathcal{F}_\sigma$, the strong operator topology corresponding to the L^2-topology on $\mathcal{F}_\sigma$.

In the general case of an automorphism $T = T_U$ where U is a real unitary operator on H, the spectral type σ of U is a symmetric measure on the circle, and the spectral type of $U^{\odot n}$ is

$$\sigma^{(n)} := \underbrace{\sigma * \ldots * \sigma}_{n}.$$

Thus, as in the standard case, the maximal spectral type of T_U is $\exp \sigma$; moreover, T is ergodic iff σ is continuous and then it is weakly mixing. In this case, we shall say that T is a *Gaussian automorphism of type* σ. Then the Gaussian space H can be spectrally identified with a closed subspace of $L^2(\mathbb{T}, \sigma, H')$, where H' is a separable Hilbert space, the action of U on H being identified with the multiplication by z. The real operators in $W^*(U)$ still correspond to the symmetric functions in $L^\infty(\mathbb{T}, \sigma)$. Note also that $H^{\otimes n}$ and thus $H^{\odot n}$ can be identified with closed subspaces of $L^2(\mathbb{T}^n, \sigma^n, H'^{\otimes n})$, with $U^{\odot n}$ corresponding to the multiplication by $z_1 \ldots z_n$ (in the standard case, $H^{\odot n}$ corresponds to $L^2_{\mathrm{sym}}(\mathbb{T}^n,\sigma^n)$, the subspace of functions invariant under permutations of coordinates).

Let us finally notice that if we decompose the Gaussian space H of T into $H = Z(f_1)\oplus Z(f_2)\oplus\ldots$ with f_i real and $\sigma=\sigma_{f_1}\gg\sigma_{f_2}\gg\ldots$, then the factors $\mathcal{B}(Z(f_i))$ are independent and T is isomorphic to the direct product $T_{\sigma_{f_1}}\times T_{\sigma_{f_2}}\times\ldots$ Conversely, any direct product of Gaussian automorphisms is a Gaussian automorphism since a sum of independent Gaussian variables remains Gaussian.

2.2. Classical factors of a Gaussian automorphism. Assume that T is a Gaussian automorphism with the Gaussian space $H = H^{\mathrm{r}} + iH^{\mathrm{r}}$. We define a *Gaussian factor* of T as a factor $\mathcal{B}(H_1)$, where H_1 is the subspace of H spanned by a non-trivial U_T-invariant subspace H_1^{r} of H^{r}. In the standard case $T=T_\sigma$, this factor is isomorphic to T_τ where τ is the maximal spectral type of U_T on H_1. Conversely, every T_τ with $\tau\ll\sigma$ appears as a Gaussian factor of T_σ, or of any Gaussian automorphism of type σ. Also, by the remarks at the end of the previous section, a Gaussian automorphism of type σ appears as a factor of the infinite direct product $T_\sigma\times T_\sigma\times\ldots$, which we shall denote by T_σ^∞.

In general, $\mathcal{B}(H)\to\mathcal{B}(H_1)$ is always relatively weakly mixing. Indeed, if we let $H_2^{\mathrm{r}} = H^{\mathrm{r}}\ominus H_1^{\mathrm{r}}$, then H_2^{r} is U_T-invariant, $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$ are independent and $\mathcal{B}(H)$ is the smallest factor containing both of them. So, T is represented as a direct product $T_1\times T_2$ and the relative product of $T\times T$ over $\mathcal{B}(H_1)$ is isomorphic to $T_1\times T_2\times T_2$, which is ergodic by the weak mixing property of T.

Consider the decomposition into Wiener chaos

$$L^2_0(\mathcal{B}(H_1)) = H_1^{(1)}\oplus H_1^{(2)}\oplus\ldots$$

with $H_1^{(1)} = H_1$. As $H_1^{(n)}$ is spanned by the products $f_1\ldots f_n$ where $f_i\in H_1$ $(1\le i\le n)$ and $f_1,\ldots,f_n$ are pairwise orthogonal, $H_1^{(n)}$ is contained in $H^{(n)}$ and the inclusion map $H_1^{(n)}\to H^{(n)}$ corresponds to the natural embedding of $H_1^{\odot n}$ in $H^{\odot n}$. It follows (see also [22], Cor. 2.6, Prop. 7.7) that

$$H_1^{(n)} = L^2(\mathcal{B}(H_1))\cap H^{(n)}, \tag{8}$$

and, if we denote by $\pi : H\to H_1$ and $\pi^{(n)} : H^{(n)}\to H_1^{(n)}$ the orthogonal projectors, then

$$\pi^{(n)} = \pi^{\odot n}. \tag{9}$$

Assume that $(H_\alpha^{\mathrm{r}})_{\alpha\in\Lambda}$ is a family of closed invariant real subspaces of H^{r}. Then we have

$$\mathcal{B}\Big(\bigcap_{\alpha\in\Lambda} H_\alpha\Big) = \bigcap_{\alpha\in\Lambda}\mathcal{B}(H_\alpha). \tag{10}$$

Indeed, when Λ is finite, (10) was proved in [34] (see also Remark 5, in Section 3.2 below). Thus, since H is separable, it suffices to consider the case when (H_α^{r}) is a decreasing sequence of subspaces. Let then $H_0^{\mathrm{r}} = \bigcap H_\alpha^{\mathrm{r}}$, and $\pi_\alpha : H \to H_\alpha$, $\pi_0 : H \to H_0$ be the corresponding projectors. By (9), for each $n \geq 1$ the sequence $(\pi_\alpha^{(n)}) = (\pi_\alpha^{\odot n})$ converges weakly to $\pi_0^{(n)}$ and thus $\pi_{\mathcal{B}(H_\alpha)}$ converges weakly to $\pi_{\mathcal{B}(H_0)}$. Therefore $\bigcap \mathcal{B}(H_\alpha) = \mathcal{B}(H_0)$.

In particular, we have

Proposition 3. *If $\mathcal{A}$ is an arbitrary factor of a Gaussian automorphism T then there exists a smallest Gaussian factor of T containing $\mathcal{A}$.* ■

This smallest Gaussian factor will be called the *Gaussian cover* of $\mathcal{A}$ and will be denoted by $\widehat{\mathcal{A}}^{\mathrm{g}}$.

Now, we have a larger class of factors of a Gaussian automorphisms which arise directly from the Gaussian structure.

Definition 1. If $\mathcal{B}(H_1)$ is a Gaussian factor of a Gaussian automorphism T and $\mathcal{K}_1$ is a compact subgroup of $C^{\mathrm{g}}(T|_{\mathcal{B}(H_1)})$ then the factor $\mathcal{A} = \mathcal{B}(H_1)/\mathcal{K}_1$ is called a *classical factor* of T.

In order to study most properties of classical factors, with no loss of generality we can restrict ourselves to the case $\mathcal{A} = \mathcal{B}(H)/\mathcal{K}$ where $\mathcal{K}$ is a compact subgroup of $C^{\mathrm{g}}(T)$. Then the ergodic decomposition of the relative product, $\mu \otimes_{\mathcal{A}} \mu = \int_{\mathcal{K}} \lambda_S \, dS$, where dS stands for the normalized Haar measure on $\mathcal{K}$, corresponds to the decomposition of the orthogonal projector $\pi_{\mathcal{A}} = \pi_{L^2(\mathcal{A})}$,

$$\pi_{\mathcal{A}} = \int_{\mathcal{K}} U_S \, dS.$$

In particular, since each U_S preserves the chaos, $\pi_{\mathcal{A}}$ preserves the chaos and thus

$$L^2(\mathcal{A}) = \bigoplus_{n=0}^{\infty} L^2(\mathcal{A}) \cap H^{(n)}. \tag{11}$$

This fact has already been observed in the standard case in [18].

The analysis of compact factors in the general case relies upon analysis of compact subgroups of $C^{\mathrm{g}}(T) = C^{\mathrm{r}}(U)$. We shall discuss it later on (Section 3.4). Let us finish this section by some description in the standard case. Then $C^{\mathrm{r}}(U)$ is spectrally identified with the group $\mathcal{F}_\sigma$ defined in the previous section.

Proposition 4. *Let $\mathcal{K}$ be a compact subgroup of $\mathcal{F}_\sigma$. Then there exists a countable measurable partition $\mathcal{P}$ of the circle such that every function in $\mathcal{K}$ is σ-a.e. constant on each element of $\mathcal{P}$.*

Proof. Let $m_{\mathcal{K}}$ stand for the normalized Haar measure of $\mathcal{K}$. By a standard result (see e.g. [35], p. 65), since the embedding $\mathcal{K} \subset \mathcal{F}_\sigma$ may be seen as a Borel map from $\mathcal{K}$ to $L^2(\mathbb{T},\sigma)$, there is a Borel function F on $\mathcal{K}\times\mathbb{T}$ such that, for $m_{\mathcal{K}}$-almost every g,

$$F(g,z) = g(z) \quad \text{for } \sigma\text{-almost every } z.$$

Then, for σ-almost every z, the map $F_z : g \mapsto F(g,z) = g(z)$ is a measurable group homomorphism from $\mathcal{K}$ to $\mathbb{T}$, and thus it is a continuous character of $\mathcal{K}$ (e.g. see [20]). Since the group of characters of $\mathcal{K}$ is countable, the result follows. ∎

The partition $\mathcal{P}$ corresponding to $\mathcal{K}$ is moreover symmetric, in the sense that $A \in \mathcal{P}$ iff $\overline{A} \in \mathcal{P}$. When $A = \overline{A}$, the corresponding constants must be real, i.e. they must be equal to ± 1.

Example 1. The most classical example is the *even factor*, where $\mathcal{K} = \{1,-1\}$. Moreover, it is clear that, for any symmetric countable measurable partition $\mathcal{P}$ of the circle, the subgroup of all $g \in \mathcal{F}_\sigma$ which are constant on each element of $\mathcal{P}$ is compact and thus yields a classical factor.

Let A be a subset of the circle such that $\sigma(A) > 0$ and $\sigma(A\cap\overline{A}) = 0$. We first consider the Gaussian factor $\mathcal{B}_A = \mathcal{B}(H_{A\cup\bar{A}})$ of T_σ, where $H_{A\cup\bar{A}}$ is the spectral subspace of H associated with $A\cup\overline{A}$. We then have a classical factor $\mathcal{A}_A = \mathcal{B}_A/\mathcal{K}_A$, where $\mathcal{K}_A$ is the group of all $g \in \mathcal{F}_{\sigma|_{A\cup\bar{A}}}$ which are constant on A (and on $\overline{A}$). Now, given any classical factor $\mathcal{A}_1 = \mathcal{B}(H_1)/\mathcal{K}_1$, if we take for A a subset of some element of the partition associated with $\mathcal{K}_1$ such that $H_{A\cup\bar{A}} \subset H_1$, then $\mathcal{K}_A$ contains the restrictions to $A\cup\overline{A}$ of all functions in $\mathcal{K}_1$. It follows that $\mathcal{A}_A \subset \mathcal{A}_1$. So, every classical factor contains some factor $\mathcal{A}_A$ (see Section 3.5 for the case of generalized Gaussian automorphisms).

2.3. Gaussian joinings. Let $T_j : (X_j,\mathcal{B}_j,\mu_j) \to (X_j,\mathcal{B}_j,\mu_j)$, $j = 1,2$, be Gaussian automorphisms, with T_j of type σ_j and H_j its Gaussian space. We say that a joining λ of T_1 and T_2 is a *Gaussian joining* if, given any $f_1 \in H_1^{\mathrm{r}}$ and $f_2 \in H_2^{\mathrm{r}}$, the function $(x_1,x_2) \mapsto f_1(x_1) + f_2(x_2)$ on the probability space $(X_1\times X_2, \mathcal{B}_1\otimes\mathcal{B}_2, \lambda)$ has a Gaussian distribution whenever it is not identically 0. We shall naturally identify H_1 and H_2 with subspaces of $L^2(X_1\times X_2,\lambda)$. So, $\lambda \in J(T_1,T_2)$ is a Gaussian joining iff

$$H^{\mathrm{r}} = \overline{H_1^{\mathrm{r}} + H_2^{\mathrm{r}}} \text{ is a Gaussian space.}$$

We denote by $J^{\mathrm{g}}(T_1,T_2)$ the set of Gaussian joinings of T_1 and T_2.

First notice that every Gaussian joining is ergodic. Indeed, since $\sigma_{f_1+f_2} \ll \sigma_{f_1} + \sigma_{f_2}$, the spectral type of $T_1\times T_2$ on H is absolutely continuous with respect to $\sigma_1+\sigma_2$, hence continuous.

LEMMA 3. *Assume that λ is a Gaussian joining between T_1 and T_2. Then Φ_λ maps each chaos $H_1^{(n)}$ into $H_2^{(n)}$ and moreover*

$$\Phi_\lambda|_{H_1^{(n)}} = (\Phi_\lambda|_{H_1})^{\odot n}.$$

Proof. We can write $\Phi_\lambda = \pi_2 \circ i_1$, where $i_1 : L^2(\mathcal{B}(H_1)) \to L^2(\mathcal{B}(H))$ is the natural embedding, and $\pi_2 : L^2(\mathcal{B}(H)) \to L^2(\mathcal{B}(H_2))$ is the orthogonal projector. Since i_1 and π_2 have a decomposition along the chaos into the corresponding symmetric tensor products, the result follows. ■

This proves that λ is uniquely determined by the operator $\Phi = \Phi_\lambda|_{H_1}$. It is a real operator from H_1 to H_2, satisfying

$$\|\Phi\| \le 1 \quad \text{and} \quad \Phi U_{T_1}|_{H_1} = U_{T_2}|_{H_2}\Phi. \tag{12}$$

Note that, in particular, for every $f_1 \in H_1$ and $f_2 \in H_2$, since $\int f_1 \bar{f}_2 \, d\lambda = \int \Phi f_1 \bar{f}_2 \, d\mu_2$, we have

$$\|f_1 + f_2\|_H^2 = \|f_1\|_{H_1}^2 + 2\mathrm{Re}(\Phi f_1 \mid f_2)_{H_2} + \|f_2\|_{H_2}^2. \tag{13}$$

We will now show that any real operator $\Phi : H_1 \to H_2$ satisfying (12) corresponds to a Gaussian joining of T_1 and T_2. Let such a Φ be given. The first step is to construct a "joining" of the unitary operators $U_j = U_{T_j}|_{H_j}$. The formula (13) defines a Hermitian form $\|\cdot\|^2$ on $H_1 \oplus H_2$, which is non-negative since $\|\Phi\| \le 1$, but in general not definite (when $\|\Phi f_1\|_{H_2} = \|f_1\|_{H_1}$, we have $\|f_1 - \Phi f_1\|^2 = 0$). By passing to the corresponding quotient we obtain a pre-Hilbert space which after completion yields a Hilbert space H, containing isometric embeddings of H_1 and H_2. Then, for $f_1 \in H_1$ and $f_2 \in H_2$, we have

$$(f_1 \mid f_2)_H = (\Phi f_1 \mid f_2)_{H_2}.$$

Since $\Phi U_1 = U_2 \Phi$, we have $(\Phi U_1 f_1 \mid U_2 f_2)_{H_2} = (\Phi f_1 \mid f_2)_{H_2}$ whence $\|U_1 f_1 + U_2 f_2\|_H = \|f_1 + f_2\|_H$. Therefore the map $f_1 + f_2 \mapsto U_1 f_1 + U_2 f_2$ has a unique extension to an isometry $U : H \to H$. The same argument for the map $f_1 + f_2 \mapsto U_1^* f_1 + U_2^* f_2$ shows that $U \in \mathcal{U}(H)$. By construction, for $j = 1, 2$, H_j is U-invariant and $U|_{H_j} = U_j$.

Moreover, if we let $H^{\mathrm{r}} = \overline{H_1^{\mathrm{r}} + H_2^{\mathrm{r}}}$, then clearly $H = H^{\mathrm{r}} + iH^{\mathrm{r}}$ and U preserves H^{r}.

We can then identify H^{r} with the real Gaussian space of a Gaussian probability space $(X, \mathcal{B}, \mu)$ and consider the automorphism T_U given by Lemma 2 (and the property $\sigma_{f_1+f_2} \ll \sigma_{f_1} + \sigma_{f_2}$ shows that T_U is ergodic, hence a Gaussian automorphism). Therefore $\mathcal{B}(H_j)$ is a Gaussian factor of T_U and by the uniqueness assertion in Lemma 2, $T_U|_{\mathcal{B}(H_j)}$ is isomorphic to T_j. We also have $\mathcal{B} = \mathcal{B}(H) = \mathcal{B}(H_1) \vee \mathcal{B}(H_2)$.

Finally, T_U is isomorphic to a Gaussian joining between T_1 and T_2, and the corresponding Markov operator Φ_λ satisfies

$$(\Phi_\lambda f_1 \mid f_2)_{H_2} = (f_1 \mid f_2)_H = (\Phi f_1 \mid f_2)_{H_2},$$

for every $f_1 \in H_1$ and $f_2 \in H_2$, whence $\Phi_\lambda|_{H_1} = \Phi$.

We have proved the following:

THEOREM 1. *For two given Gaussian automorphisms T_1 and T_2 with Gaussian spaces H_1 and H_2, the mapping $\lambda \mapsto \Phi_\lambda|_{H_1}$ establishes a* 1-1 *correspondence between Gaussian joinings of T_1 and T_2 and real operators from H_1 to H_2 of norm at most* 1 *intertwining $U_{T_1}|_{H_1}$ and $U_{T_2}|_{H_2}$.* ■

In the case of a single Gaussian automorphism $T = T_1 = T_2$ we speak about *Gaussian self-joinings* and we use the notation $J_2^{\mathrm{g}}(T)$. By Theorem 1 it is a semigroup isomorphic to the semigroup

$$\mathcal{S}_T = \{\Phi \in \mathcal{L}^{\mathrm{r}}(H) : \|\Phi\| \le 1,\ \Phi U_T = U_T \Phi\}.$$

It is compact in the weak topology of $J_2(T)$ which corresponds to the weak operator topology (in this topology, the multiplication is only separately continuous). Also, up to an obvious identification, $C^{\mathrm{g}}(T) \subset J_2^{\mathrm{g}}(T)$.

If $T = T_\sigma$ then $\mathcal{S}_{T_\sigma}$ can naturally be identified with the semigroup of Hermitian functions in the unit ball of $L^\infty(\mathbb{T},\sigma)$. If $\lambda \in J_2^{\mathrm{g}}(T_\sigma)$, the action of Φ_λ on $L^2(\mathbb{T},\sigma)$ is the multiplication by a Hermitian function ϕ_λ of modulus ≤ 1 and, by Lemma 3, the action of Φ_λ on $H^{(n)} \simeq L^2_{\mathrm{sym}}(\mathbb{T}^n,\sigma^n)$ is given by

$$\Phi_\lambda^{(n)} g(z_1,\ldots,z_n) = \phi_\lambda(z_1)\cdot\ldots\cdot\phi_\lambda(z_n) g(z_1,\ldots,z_n). \tag{14}$$

In particular, $J_2^{\mathrm{g}}(T_\sigma)$ is commutative.

We say that a self-joining λ in $J_2(T)$ has a *Gaussian disintegration* if its ergodic decomposition consists a.e. of Gaussian joinings, i.e. if there is a probability Borel measure P on $J_2^{\mathrm{g}}(T)$ such that

$$\lambda = \int\limits_{J_2^{\mathrm{g}}(T)} \varrho\, dP(\varrho).$$

Then Φ_λ still preserves the chaos and, for each $n \ge 0$,

$$\Phi_\lambda|_{H^{(n)}} = \int\limits_{J_2^{\mathrm{g}}(T)} (\Phi_\varrho|_H)^{\odot n}\, dP(\varrho). \tag{15}$$

Finally, the set of joinings with Gaussian disintegration is a semigroup, and in the standard case it is commutative.

3. GAUSSIAN AUTOMORPHISMS WITH GAUSSIAN SELF-JOININGS

3.1. Definition, basic properties and examples of GAG

DEFINITION 2. A Gaussian automorphism T is called a *GAG* if $J_2^{\mathrm{e}}(T) = J_2^{\mathrm{g}}(T)$.

The following lemma characterizes the GAG property in terms of stationary processes.

LEMMA 4. *A Gaussian automorphism T of type σ is a GAG if and only if, for every ergodic automorphism $S : (Y,\mathcal{C},\nu) \to (Y,\mathcal{C},\nu)$, for every real-valued $f_i \in L_0^2(Y,\nu)$ for which the processes $(f_i \circ S^n)_{n\in\mathbb{Z}}$ are Gaussian, $\sigma_{f_i} \ll \sigma$ $(i = 1,2)$ and $f_1 + f_2 \neq 0$, the function $f_1 + f_2$ has a Gaussian distribution.*

Proof. Only the necessity requires a proof. Let $\mathcal{B}_i = \mathcal{B}(Z(f_i))$ and $\mathcal{A} = \mathcal{B}_1 \vee \mathcal{B}_2$. Then $S|_{\mathcal{B}_1}$ and $S|_{\mathcal{B}_2}$ are isomorphic to Gaussian factors of T, and $\mathcal{A}$ can be identified with an ergodic joining λ of them. Then λ can be extended to an ergodic joining of T. Indeed, taking $\varrho \in J_2(T)$ which corresponds to the Markov operator $\Phi \circ \pi|_{\mathcal{A}}$ we obtain a self-joining whose restriction to $\mathcal{A} \otimes \mathcal{A}$ is λ, and almost every ergodic component $\widetilde{\lambda}$ of ϱ has the same restriction, since λ is ergodic. Now $\widetilde{\lambda} \in J_2^{\mathrm{g}}(T)$ and $f_1 + f_2$ belongs to its Gaussian space. ■

A few observations directly follow from Lemma 4:

PROPOSITION 5. *Let T be a Gaussian automorphism of type σ.*

(i) *T is a GAG iff T_σ is a GAG.*

(ii) *If T is a GAG and S is a Gaussian automorphism of type $\eta \ll \sigma$, then S is also a GAG.*

(iii) *If T is a GAG then every finite or infinite ergodic self-joining λ of T is Gaussian. More precisely, in $L^2(X \times X \times \ldots, \lambda)$, the sum of the coordinate Gaussian spaces spans a Gaussian space.* ■

For (iii), which might be stated *"GAG of order two implies GAG of all orders"*, it is sufficient to remark that, in the setting of Lemma 4, we have $\sigma_{f_1+f_2} \ll \sigma_{f_1} + \sigma_{f_2} \ll \sigma$ and then proceed inductively for a sum $f_1(x_1) + \ldots + f_n(x_n)$ where each f_i is a real function in the Gaussian space of T.

In connection with the classical problem whether 2-mixing implies 3-mixing, in [15] one proposes to consider a stronger property: the only ergodic self-joining $\lambda \in J_\infty^{\mathrm{e}}(T)$ whose restriction to any two coordinates is the product measure (one says that such a joining is *pairwise independent*) is the product measure. A remarkable result of B. Host in [9] shows that this property holds for automorphisms with singular spectra. For other results of this type see [6], [15], [29]. The following result has already been shown in [33] in the case of Gaussian–Kronecker automorphisms.

PROPOSITION 6. *If T_σ is a GAG then any ergodic self-joining which is pairwise independent is globally independent.*

Proof. Take $\lambda \in J^{\mathrm{e}}_{\infty}(T_\sigma)$. In view of Proposition 5(iii), λ is Gaussian. Now in the Gaussian space of $(T^{\infty}_{\sigma}, \lambda)$ the coordinate Gaussian spaces are pairwise orthogonal since λ is pairwise independent, hence they are globally independent. ■

It follows from Proposition 5 that the GAG property is a property of the measure σ itself. We will say that a symmetric measure σ on $\mathbb{T}$ is a *GAG measure* if the corresponding standard Gaussian automorphism T_σ is a GAG. We now consider the problem of existence of GAG measures.

We shall say that a symmetric measure σ on the circle has the *Foiaş–Strătilă* (*FS*) *property* if, for each ergodic automorphism $S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$, any real-valued $f \in L^2_0(Y, \nu)$ with $\sigma_f = \sigma$ is a Gaussian variable. Recall that the Foiaş–Strătilă theorem (see [2], or [1], p. 375) asserts that this property holds for continuous symmetric measures concentrated on $K \cup \overline{K}$, where K is a Kronecker set (T_σ is then called a *Gaussian–Kronecker* automorphism). By Lemma 4, again since $\sigma_{f_1+f_2} \ll \sigma_{f_1} + \sigma_{f_2}$, we have

$$\mathrm{FS} \Rightarrow \mathrm{GAG}.$$

Some easy extensions of the Foiaş–Strătilă theorem are given in [17]: a continuous symmetric measure σ such that the group $\{z^n : n \in \mathbb{Z}\}$ is dense in $\mathcal{F}_\sigma$ in the L^2-topology (a *symmetric Kronecker measure*) satisfies the FS property. If σ has the FS property and $\eta \ll \sigma$ then η also has the FS property. Some measures with the FS property which are not Kronecker are also constructed in [17].

The next theorem characterizes the GAG property and exhibits a much larger class of GAG automorphisms.

THEOREM 2. *T_σ is a GAG if and only if it has commuting self-joinings. In particular, if T_σ has simple spectrum then it is a GAG.*

In order to prove Theorem 2, we will need the following well-known lemma.

LEMMA 5. *Let σ, τ be continuous symmetric measures on the circle. Then there exists an $S \in C^{\mathrm{g}}(T_\sigma)$ which is isomorphic to T_τ.*

Proof. Let H denote the Gaussian space of T_σ. Recall that $C^{\mathrm{g}}(T_\sigma)$ and $\mathcal{F}_\sigma$ are isomorphic in such a way that, if S corresponds to the function g, then $U_S|_H$ is spectrally conjugate to the operator V_g of multiplication by g on $L^2(\mathbb{T}, \sigma)$. Since both measures σ and τ are continuous, the standard Borel spaces $(\mathbb{T}, \sigma)$ and $(\mathbb{T}, \tau)$ are Borel isomorphic. Since both measures are symmetric, we can choose a $g : (\mathbb{T}, \sigma) \to (\mathbb{T}, \tau)$ which establishes an isomorphism and satisfies $g(\overline{z}) = \overline{g(z)}$ σ-a.e., that is, $g \in \mathcal{F}_\sigma$. Now, the spectral type of V_g is the image of σ by g and, since g is one-to-one (σ-a.e.),

V_g has simple spectrum. Hence the corresponding $S \in C^{\mathrm{g}}(T_\sigma)$ is isomorphic to T_τ. ∎

Remark 2. Assume that in the above lemma we consider instead of τ a sequence (τ_j), $j \geq 1$, of continuous symmetric measures on the circle. By considering a measurable partition consisting of symmetric sets A_j with $\sigma(A_j) > 0$, we can find $g \in \mathcal{F}_\sigma$ such that $g|_{A_j}$ establishes an isomorphism $(\mathbb{T}, \sigma|_{A_j}) \to (\mathbb{T}, \tau_j)$. Then S_g is isomorphic to $T_{\tau_1} \times T_{\tau_2} \times \ldots$ Thus, in $C^{\mathrm{g}}(T_\sigma)$ we can find an S isomorphic to any given Gaussian automorphism. Conversely, given any Gaussian automorphism T, $C^{\mathrm{g}}(T)$ contains an automorphism isomorphic to T_σ.

Proof of Theorem 2. If T_σ is a GAG then every self-joining of it has a Gaussian disintegration, and we have already noticed that in the case of a standard Gaussian automorphism this implies the commutativity property. Conversely, from Lemma 5 we can find in $C^{\mathrm{g}}(T_\sigma)$ some GAG automorphism $S \simeq T_\tau$ (e.g. a Gaussian–Kronecker automorphism) and then S has commuting self-joinings. If T_σ also has commuting self-joinings, then $J_2^{\mathrm{e}}(T_\sigma) = J_2^{\mathrm{e}}(S) = J_2^{\mathrm{g}}(S)$ by Corollary 1. Now, S and T_σ have the same Gaussian space, and the result follows. ∎

Remark 3. The class of standard Gaussian automorphisms which are GAG is larger than the class of simple spectrum Gaussian automorphisms. The second author has recently proved that if σ is a GAG then so is $\sigma * \sigma$ (this result will be published elsewhere). No Gaussian automorphism of the form $T_{\sigma*\sigma}$ has simple spectrum.

Directly from Theorem 2 and the existence of a mixing simple spectrum Gaussian automorphism (see [21]) we obtain

Corollary 2. *There exists a mixing GAG.* ∎

Remark 4. According to Proposition 6, we hence obtain some mixing Gaussian automorphisms for which the product measure is the only pairwise independent self-joining. It follows that such automorphisms are mixing of all orders (see [15]). The fact that a mixing Gaussian automorphism is mixing of all orders is classical ([19]). We do not know, however, whether the assertion of Proposition 6 is satisfied for all Gaussian automorphisms with zero entropy.

3.2. The centralizer and the structure of factors for a GAG. Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a Gaussian automorphism, and H be its Gaussian space. If T is a GAG and $S \in C(T)$ then the graph self-joining λ_S is Gaussian and, by Lemma 3, the Markov operator $\Phi_{\lambda_S} = U_S$ preserves the Gaussian space. Hence

Proposition 7. *If T is a GAG then $C(T) = C^{\mathrm{g}}(T)$.* ∎

We have already seen that if $\mathcal{A}$ is a Gaussian factor of T then T is relatively weakly mixing over $\mathcal{A}$. In order to show that, for a GAG, the converse holds we first recall that any joining λ of two automorphisms $T_i : (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, determines a factor of T_1 and a factor of T_2. Indeed, the intersection $(\mathcal{B}_1 \times X_2) \cap (X_1 \times \mathcal{B}_2) \pmod{\lambda}$ can be naturally identified with a factor $\mathcal{B}_1(\lambda)$ of T_1 on the one hand, and with a factor $\mathcal{B}_2(\lambda)$ of T_2 on the other hand. The restriction of λ to $\mathcal{B}_1(\lambda) \otimes \mathcal{B}_2(\lambda)$ is then a graph joining. More precisely, $L^2(\mathcal{B}_1(\lambda))$ is the subspace of all $f \in L^2(\mathcal{B}_1)$ such that $\|\Phi_\lambda f\| = \|f\|$, or equivalently, $\Phi_\lambda^* \Phi_\lambda f = f$. Note in passing that, as $\Phi_\lambda^* \Phi_\lambda$ is a non-negative self-adjoint operator of norm ≤ 1, the orthogonal projector $\pi_{\mathcal{B}_1(\lambda)}$ is the weak limit

$$\pi_{\mathcal{B}_1(\lambda)} = \lim_{k \to \infty} (\Phi_\lambda^* \Phi_\lambda)^k. \tag{16}$$

Let us now come back to the case of a Gaussian self-joining λ of the Gaussian automorphism T. Let H_1 and H_2 be the subspaces of $L^2(X \times X, \lambda)$ corresponding to the coordinate Gaussian spaces, both isomorphic to H. Then $\mathcal{B}(H_1) = \mathcal{B} \times X$ and $\mathcal{B}(H_2) = X \times \mathcal{B}$ and, since H_1 and H_2 are contained in the Gaussian space of $(T \times T, \lambda)$, by (10), we have $\mathcal{B}(H_1) \cap \mathcal{B}(H_2) = \mathcal{B}(H_1 \cap H_2)$ and thus both $\mathcal{B}_i(\lambda)$ are Gaussian factors. Furthermore, the natural isomorphism between $\mathcal{B}_1(\lambda)$ and $\mathcal{B}_2(\lambda)$, given by the restriction of Φ_λ to $L^2(\mathcal{B}_1(\lambda))$, is a Gaussian isomorphism.

Remark 5. As λ is a Gaussian joining, it follows from (16) and Lemma 3 that $\pi_{\mathcal{B}_1(\lambda)}$ is decomposed along the chaos into the tensor products of its restriction to H. This yields a direct proof that $\mathcal{B}_1(\lambda)$ is a Gaussian factor. Since for two Gaussian factors $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$ of a Gaussian automorphism, $\mathcal{B}(H_1) \vee \mathcal{B}(H_2)$ can be seen as their Gaussian joining, this also proves (10) for a finite family of Gaussian factors.

Lemma 6. *Let T be a GAG. Every factor $\mathcal{A}$ of T such that T is relatively weakly mixing over $\mathcal{A}$ is a Gaussian factor.*

Proof. Let $\lambda := \mu \otimes_{\mathcal{A}} \mu \in J_2^{\mathrm{e}}(T) = J_2^{\mathrm{g}}(T)$. Then both factors $\mathcal{B}_1(\lambda)$ and $\mathcal{B}_2(\lambda)$ are equal to $\mathcal{A}$. Thus $\mathcal{A}$ is a Gaussian factor. ∎

In particular, we have proved that if T is a GAG then the Gaussian cover $\widehat{\mathcal{A}}^{\mathrm{g}}$ of any factor $\mathcal{A}$ of T equals $\widehat{\mathcal{A}}$. We will now show how to make use of Veech's theorem (see [36]) to obtain the following.

Theorem 3. *Let T be a GAG. For every factor $\mathcal{A}$ of T there exists a compact subgroup $\mathcal{K} \subset C^{\mathrm{g}}(T|_{\widehat{\mathcal{A}}})$ such that $\mathcal{A} = \widehat{\mathcal{A}}/\mathcal{K}$. In other words, each factor of a GAG is classical.*

Proof. Since the restriction of T to $\widehat{\mathcal{A}} = \widehat{\mathcal{A}}^{\mathrm{g}}$ is still a GAG, there is no harm to assume that $\widehat{\mathcal{A}}^{\mathrm{g}} = \mathcal{B}$. Let then

$$\mu \otimes_{\mathcal{A}} \mu = \int \varrho \, dP(\varrho)$$

denote the ergodic decomposition of $\mu \otimes_{\mathcal{A}} \mu$. Since the restriction of $\mu \otimes_{\mathcal{A}} \mu$ to $\mathcal{A} \otimes \mathcal{A}$ is the diagonal joining (the graph self-joining corresponding to the identity), the restriction of this decomposition to $\mathcal{A} \otimes \mathcal{A}$ is trivial and

$$\mathcal{A} \subset \mathcal{B}_1(\varrho) \quad \text{and} \quad \mathcal{A} \subset \mathcal{B}_2(\varrho)$$

for P-a.a. ϱ. Now, since the $\mathcal{B}_i(\lambda)$ are Gaussian factors, $\mathcal{B} = \widehat{\mathcal{A}}^{\mathrm{g}} = \mathcal{B}_1(\varrho) = \mathcal{B}_2(\varrho)$ and ϱ is P-a.e. a graph measure. By Veech's theorem, $\mathcal{A} = \mathcal{B}/\mathcal{K}$ for some compact group $\mathcal{K} \subset C(T)$ and the result follows from Proposition 7. ∎

Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a Gaussian automorphism and let $\mathcal{A}$ be a compact factor of some Gaussian factor $\mathcal{B}(H_1)$ of T. Then

$$\widehat{\mathcal{A}}^{\mathrm{g}} = \mathcal{B}(H_1). \tag{17}$$

Indeed, let $\widehat{\mathcal{A}}^{\mathrm{g}} = \mathcal{B}(H_1')$. Clearly, $H_1' \subset H_1$. On the other hand, $\mathcal{B}(H_1)$ cannot contain a non-trivial factor $\mathcal{B}(H_2)$ independent of its compact factor $\mathcal{A}$, and thus H_1 cannot contain a non-trivial Gaussian subspace H_2 orthogonal to H_1'.

Assume now that $\mathcal{A}_i = \mathcal{B}(H_i)/\mathcal{K}_i$, $i = 1, 2$, are classical factors of a Gaussian automorphism T. We then have

(18) *if* $\mathcal{A}_1 \subset \mathcal{A}_2$ *then* $H_1 \subset H_2$, $\mathcal{K}_2 H_1 \subset H_1$ *and* $\mathcal{K}_1 \supset \mathcal{K}_2|_{H_1}$.

Indeed, $\mathcal{B}(H_1) = \widehat{\mathcal{A}}_1^{\mathrm{g}} \subset \widehat{\mathcal{A}}_2^{\mathrm{g}} = \mathcal{B}(H_2)$ whence $H_1 \subset H_2$. Moreover, $\mathcal{K}_2$ preserves $\widehat{\mathcal{A}}_1^{\mathrm{g}}$, whence it preserves H_1. We have

$$\mathcal{A}_1 \subset \mathcal{A}_2 \cap \mathcal{B}(H_1) = \mathcal{B}(H_1)/(\mathcal{K}_2|_{H_1})$$

and (18) follows from the saturation property.

Suppose moreover that T is a GAG and assume that R establishes an isomorphism of $\mathcal{A}_1$ and $\mathcal{A}_2$. Take a $\varrho \in J_2^{\mathrm{e}}(T) = J_2^{\mathrm{g}}(T)$ such that $\varrho|_{\mathcal{A}_1 \otimes \mathcal{A}_2}$ is the graph of R. Then $\mathcal{A}_i \subset \mathcal{B}_i(\varrho)$ and it follows that $\mathcal{B}(H_i) = \widehat{\mathcal{A}}_i^{\mathrm{g}} \subset \mathcal{B}_i(\varrho)$, $i = 1, 2$. Thus $\varrho|_{\mathcal{B}(H_1) \otimes \mathcal{B}(H_2)}$ is the graph of a Gaussian isomorphism S between $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$, extending R. We now have two compact subgroups $\mathcal{K}_1$ and $S^{-1}\mathcal{K}_2 S$ that fix all elements of $\mathcal{A}_1$. By the saturation property, $\mathcal{K}_1 = S^{-1}\mathcal{K}_2 S$ and we have proved the following:

Proposition 8. *Let T be a GAG. If $\mathcal{B}(H_1)/\mathcal{K}_1$ and $\mathcal{B}(H_2)/\mathcal{K}_2$ are isomorphic then there exists a Gaussian isomorphism S of the factor $\mathcal{B}(H_1)$ onto the factor $\mathcal{B}(H_2)$ such that $\mathcal{K}_1 = S^{-1}\mathcal{K}_2 S$.* ∎

3.3. Semisimplicity and the GAG property. Generalizing the earlier notions of minimal self-joinings ([25]) and 2-fold simplicity ([36], [15]), in [14] one introduces the concept of semisimplicity. An automorphism $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is called *semisimple* if for each $\lambda\in J_2^{\mathrm{e}}(T)$ the extension $(\mathcal{B}\otimes\mathcal{B},\lambda)\to(\mathcal{B}\times X,\lambda)$ is relatively weakly mixing. If T is a Gaussian automorphism and $\lambda\in J_2^{\mathrm{g}}(T)$ then $\mathcal{B}\times X$ is a Gaussian factor of $(T\times T,\lambda)$, so the extension above is relatively weakly mixing. Hence

each GAG is semisimple.

An open question remains whether semisimplicity of a Gaussian automorphism implies the GAG property.

The results of the previous section can also be obtained from the results of [14] since we have shown that the family of Gaussian factors coincides with the family of factors relative to which a GAG is weakly mixing, so the Gaussian factors define the smallest *natural family* for a GAG (in the sense of [14]). We should also notice that an earlier analysis of the structure of factors in the Gaussian–Kronecker case has been done by the third author in [32].

We will now show that factors of a standard GAG are also semisimple. To this end we first need the following.

PROPOSITION 9. *Assume that $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is semisimple and has commuting self-joinings. Then each of its factors is also semisimple.*

Proof. Let $\mathcal{A}$ be a factor of T and assume that $\lambda\in J_2^{\mathrm{e}}(T|_{\mathcal{A}})$. Extend λ to a $\widetilde{\lambda}\in J_2^{\mathrm{e}}(T)$. Consider now the Furstenberg decomposition $(\mathcal{A}\otimes\mathcal{A},\lambda)\to\widehat{\mathcal{A}}\to\mathcal{A}\times X$ of $(\mathcal{A}\otimes\mathcal{A},\lambda)\to\mathcal{A}\times X$. In particular, $\widehat{\mathcal{A}}\to\mathcal{A}\times X$ is distal. Since $(\mathcal{B}\otimes\mathcal{B},\widetilde{\lambda})\to\mathcal{B}\times X$ is relatively weakly mixing, $\widehat{\mathcal{A}}\subset\mathcal{B}\times X$. It follows from Lemma 1 that

$$(\mathcal{A}\otimes\mathcal{A})\cap(\mathcal{B}\times X)=\mathcal{A}\times X,$$

whence $\widehat{\mathcal{A}}=\mathcal{A}\times X$ and thus $(\mathcal{A}\otimes\mathcal{A},\lambda)\to\mathcal{A}\times X$ is relatively weakly mixing, which shows that $\mathcal{A}$ is semisimple. ∎

COROLLARY 3. *If T_σ is a GAG then each of its factors is semisimple.* ∎

3.4. GAG measures. Mutual singularity and translations. We now pass to some facts relating the GAG property and measure-theoretical properties.

PROPOSITION 10. *Let σ be a GAG measure. If $\sigma_1,\sigma_2\ll\sigma$ are symmetric measures and $\sigma_1\perp\sigma_2$, then T_{σ_1} and T_{σ_2} are disjoint.*

Proof. It is enough to show that a self-joining λ of T_{σ_1} and T_{σ_2} is the product measure in the case when λ is ergodic. Then it follows directly from Lemma 4 that, in $L^2(\lambda)$, the sum of the Gaussian spaces H_1, H_2 of the factors spans a Gaussian space (i.e. $\lambda\in J^{\mathrm{g}}(T_{\sigma_1},T_{\sigma_2})$). Since the spectral types σ_i on H_i are mutually singular, these subspaces are moreover orthogonal. Therefore $\mathcal{B}(H_1)$ and $\mathcal{B}(H_2)$ must be independent and λ is the product measure. ∎

COROLLARY 4. *If σ is a GAG measure then for each $z \in \mathbb{T} \setminus \{1\}$,*

$$\sigma \perp \sigma * \delta_z.$$

Proof. Assume that $\sigma \not\perp \sigma * \delta_{z_0}$ for some $z_0 \in \mathbb{T} \setminus \{1\}$. We can then find symmetric measures $\eta_1, \eta_2 \ll \sigma$ such that $\eta_1 \perp \eta_2$ but η_1 and a certain translation of η_2 are not mutually singular. It follows from Proposition 3 of [17] that T_{η_1} and T_{η_2} have a non-trivial common factor, and in particular, they are not disjoint. This contradicts Proposition 10. ∎

REMARK 6. We state as a question whether the condition formulated in Proposition 10 is in fact equivalent to the GAG property.

We will see that the sum of two mutually singular measures which are GAG need not be GAG (Remark 7). We have, however,

LEMMA 7. *If σ_1, σ_2 are GAG measures and $T_{\sigma_1} \perp T_{\sigma_2}$, then $\sigma_1 + \sigma_2$ is a GAG measure.*

Proof. Put $\sigma = \sigma_1 + \sigma_2$ and let H denote the Gaussian space of T_σ. Then $H = H_1 \oplus H_2$ where the spectral type of H_i is σ_i, $i = 1, 2$. Take $\lambda \in J_2^{\mathrm{e}}(T_\sigma)$. We have to show that, whenever $f, g \in H^{\mathrm{r}}$, the function $h : (x, y) \mapsto f(x) + g(y)$ is a Gaussian variable on the corresponding probability space. We write $f = f_1 + f_2$, $g = g_1 + g_2$, where $f_1, g_1 \in H_1^{\mathrm{r}}$ and $f_2, g_2 \in H_2^{\mathrm{r}}$. Let $h_i(x, y) = f_i(x) + g_i(y)$, $i = 1, 2$. Since σ_i is a GAG measure, $(h_i \circ (T_\sigma^n \times T_\sigma^n))$ is a Gaussian process, and its spectral measure is absolutely continuous with respect to σ_i. So, the factor of $(T_\sigma \times T_\sigma, \lambda)$ generated by h_i is isomorphic to a factor of T_{σ_i}. By the disjointness condition the variables h_1 and h_2 are independent, whence their sum h remains Gaussian. ∎

Given a subset $\mathcal{L} \subset \mathrm{Aut}(X, \mathcal{B}, \mu)$ we denote by $\mathrm{gp}(\mathcal{L})$ the subgroup of $\mathrm{Aut}(X, \mathcal{B}, \mu)$ generated by $\mathcal{L}$. In order to state some other properties of GAG measures we need the following lemma.

LEMMA 8. *Let S and T be two commuting ergodic automorphisms of $(X, \mathcal{B}, \mu)$. Let $F = \overline{\mathrm{gp}(S)}$ and $G = \overline{\mathrm{gp}(S, T)}$. If G/F is compact and $J_2^{\mathrm{e}}(T) \subset J_2^{\mathrm{e}}(S)$ then $J_2^{\mathrm{e}}(T) = J_2^{\mathrm{e}}(S)$.*

Proof. Take $\lambda \in J_2^{\mathrm{e}}(S)$. For $R \in G$, the image of λ under $R \times R$ depends only on the coset $\widetilde{R} = RF$ of R in G/F. We denote it by $\lambda_{\widetilde{R}}$. Let

$$\widetilde{\lambda} = \int_{G/F} \lambda_{\widetilde{R}} \, d\widetilde{R} \tag{19}$$

where $d\widetilde{R}$ is the normalized Haar measure on G/F. Each $\lambda_{\widetilde{R}}$ is still an ergodic self-joining of S and thus (19) appears as the ergodic decomposition of $\widetilde{\lambda}$. By construction, $\widetilde{\lambda}$ is also $T \times T$-invariant. Let

$$\widetilde{\lambda} = \int_{J_2^{\mathrm{e}}(T)} \varrho \, dP(\varrho) \tag{20}$$

be its ergodic decomposition as a $T \times T$-invariant measure.

Then, by assumption, $\varrho \in J_2^{\mathrm{e}}(T) \subset J_2^{\mathrm{e}}(S)$ P-a.e., and the two decompositions (19) and (20) must coincide. Hence almost every $\lambda_{\widetilde{R}}$ is $T\times T$-invariant, which readily implies that λ itself is $T\times T$-invariant. Since G/F is generated by the coset of T, it follows that the decompositions (19) and (20) are trivial and hence $\lambda \in J_2^{\mathrm{e}}(T)$. ∎

Now, if T is a standard Gaussian automorphism and $S \in C^{\mathrm{g}}(T)$, we have $J_2^{\mathrm{g}}(T) \subset J_2^{\mathrm{g}}(S)$. So, if T is a standard GAG and $S \in C(T)$, then $J_2^{\mathrm{e}}(T) = J_2^{\mathrm{g}}(T) \subset J_2^{\mathrm{g}}(S) \subset J_2^{\mathrm{e}}(S)$. Therefore, if moreover $\overline{\mathrm{gp}(S,T)}/\overline{\mathrm{gp}(S)}$ is compact, we see by Lemma 8 that S is also a GAG.

Proposition 11. *Assume that σ is a GAG measure. Then*

(i) *the image of σ under the map $z \mapsto z^N$ is a GAG measure* ($N \in \mathbb{Z} \setminus \{0\}$);

(ii) *for each decomposition $\sigma = \sigma_0 + \widetilde{\sigma}_0$ with $\sigma_0 \perp \widetilde{\sigma}_0$ and each $z_0 \in \mathbb{T}$ the measure $\tau = \sigma_0 * \delta_{z_0} + \widetilde{\sigma}_0 * \delta_{\bar{z}_0}$ is a GAG measure.*

Proof. Let $T = T_\sigma$ and let H be its Gaussian space. For each case, it is enough to find $S \in C(T)$ such that $\overline{\mathrm{gp}(S,T)}/\overline{\mathrm{gp}(S)}$ is compact and the spectral type of $U_S|_H$ is the given measure (note that in both cases, the measure is continuous and thus S will be ergodic).

For (i), $S = T^N$ has the required properties. To see (ii) let us define $g \in \mathcal{F}_\sigma$ by $g(z) = z_0$ σ-a.e. and $g(z) = \bar{z}_0$ $\widetilde{\sigma}_0$-a.e. Then g spans a compact subgroup of $\mathcal{F}_\sigma$ and if $R \in C^{\mathrm{g}}(T)$ corresponds to g, it follows that $S = TR$ is a GAG, while its spectral type on H is the image of σ under $z \mapsto zg(z)$, which is equal to τ. ∎

Remark 7. If T_σ is a GAG, then T_σ^N is a *standard GAG*. Indeed, since $\sigma \perp \sigma * \delta_u$ for each $u \in \mathbb{T} \setminus \{1\}$, the map $z \mapsto z^N$ is one-to-one σ-a.e. and it follows that the multiplicity function of T_σ^N on H remains 1.

Note also that if σ is a GAG measure and τ is defined as in (ii), with $z_0 \neq 1$, then, by Corollary 4, $\sigma \perp \tau$ but $\sigma + \tau$ cannot be a GAG measure, although σ and τ are GAG.

Example 2. Let σ be a GAG measure. By the remark above, $\sigma * \delta_{-1}$ is GAG but $\sigma + \sigma * \delta_{-1}$ is not. Let $(X, \mathcal{B}, \mu)$ be the Gaussian probability space and H be the Gaussian space of T_σ. Consider the direct product $T = T_\sigma \times T_\sigma$. It is a Gaussian automorphism with Gaussian space $\widetilde{H} = H_1 \oplus H_2$, where H_1 and H_2 are the two natural copies of H in $L^2(X \times X, \mu \otimes \mu)$.

Put $F : X \times X \to X \times X$, $F(x,y) = (x,-y)$. We have $F = T_V$, where $V : \widetilde{H} \to \widetilde{H}$ is given by $V(f_1 + f_2) = f_1 - f_2$ for $f_1 \in H_1$, $f_2 \in H_2$, and thus $F \in C^{\mathrm{g}}(T)$. Now, let $S = F \circ T$. Then U_S preserves H_1 and H_2, and its restrictions to H_1 and H_2 are, up to a natural identification, $U_{T_\sigma}|_H$ and $-U_{T_\sigma}|_H$, which have spectral types σ and $\sigma * \delta_{-1}$ respectively. Also, both have simple spectrum and, since $\sigma \perp \sigma * \delta_{-1}$, it follows that $U_S|_{\widetilde{H}}$ has simple spectrum and its spectral type is $\tau = \sigma + \sigma * \delta_{-1}$. Thus S is isomorphic to T_τ and, in particular, S is not a GAG.

Therefore, it may happen that the composition of a generalized GAG with a compact element of its Gaussian centralizer is not a GAG (cf. remarks before Proposition 11).

Note that S and T have the common (classical for both) factor $\mathcal{B}_F$ of those sets which are fixed under F. However, if R denotes the switching of coordinates $(x,y) \mapsto (y,x)$ and if we let $\mathcal{K}$ be the finite group generated by F and R, then $\mathcal{K}$ is contained in $C^{\mathrm{g}}(T)$ and the factor $\mathcal{A} = (\mathcal{B} \otimes \mathcal{B})/\mathcal{K}$ of $\mathcal{B}_F$ is still a common factor of T and S, but for the latter automorphism it is not a classical factor. Indeed, suppose that $\mathcal{A}$ may be written in the form $\mathcal{A} = \mathcal{B}(H')/\mathcal{K}'$ as a classical factor of S. Then, since $T \in C^{\mathrm{g}}(S)$ and S is standard, H' is also U_T-invariant and $\mathcal{K}' \subset C^{\mathrm{g}}(T|_{\mathcal{B}(H')})$. By (17) and (18) (applied for T), $H' = \widetilde{H}$ and $\mathcal{K}' = \mathcal{K}$. This is a contradiction since S and R do not commute.

Assume moreover that T_σ has simple spectrum, i.e. each chaos $H^{(n)}$ is cyclic under U_{T_σ} and $\sigma^{(m)} \perp \sigma^{(n)}$ for every $m \neq n$. It easily follows that, under U_S, each $\widetilde{H}^{(n)} = (H_1 \oplus H_2)^{\odot n}$ is the orthogonal sum of finitely many cyclic subspaces and that the convolution powers of τ are still mutually singular. Therefore $+\infty$ is not an essential value of the multiplicity function of T_τ. It was already noticed by Vershik in [38] that Gaussian automorphisms T_τ for which $+\infty$ is not an essential value of the multiplicity function have strong Gaussian behavior, for example all isomorphisms between such automorphisms must be Gaussian. The fact that the spectral types on the chaos are mutually orthogonal also implies that, for each $\lambda \in J_2(T_\tau)$, Φ_λ preserves the chaos (see Remark 1). Despite all this, we deduce that such automorphisms need not be GAG.

3.5. More about classical factors. We give here a more precise description of classical factors of generalized Gaussian automorphisms and typical examples. We shall make use of some basic facts of representation theory for which we refer to [8].

Consider a separable Hilbert space H, a unitary operator $U \in \mathcal{U}(H)$, and a compact subgroup $\mathcal{K} \subset \mathcal{U}(H)$ each of whose elements commutes with U. By a direct consequence of the Peter–Weyl theorem (see [8], Theorem 27.44), H is a direct sum of finite-dimensional $\mathcal{K}$-invariant subspaces on which the action of $\mathcal{K}$ is irreducible. Given such a subspace F_0, of dimension m, let $(f_1, \ldots, f_m)$ be an orthonormal basis of F_0 and let F be the closed U-invariant subspace generated by F_0. Then $U|_F$ has spectral multiplicity $\leq m$ and thus admits a spectral representation in some closed subspace $\widetilde{F}$ of $L^2(\mathbb{T}, \tau, \mathbb{C}^m)$, where τ is its maximal spectral type. Let us denote by $\widetilde{f}$ the vector-valued function corresponding to $f \in F$ and by $\widetilde{V}$ the operator on $\widetilde{F}$ corresponding to $V \in \mathcal{L}(F)$.

Any operator $W \in W^*(U)$ commutes with each element of $\mathcal{K}$ and thus yields an intertwining between the actions of $\mathcal{K}$ on F_0 and on WF_0. By [8], Theorem 27.13, its restriction to F_0 must be of the form a constant times an isometry. Passing to the spectral representation, for every $g \in L^\infty(\mathbb{T}, \tau)$ there hence exists a constant $c(g)$ such that for $j, k = 1, \ldots, m$,

$$(g \cdot \widetilde{f}_j \mid g \cdot \widetilde{f}_k) = c(g)(\widetilde{f}_j \mid \widetilde{f}_k).$$

If $j \neq k$ the inner product is zero and it follows that, for any two Borel subsets A' and A'' of the circle,

$$(\chi_{A'} \widetilde{f}_j \mid \chi_{A''} \widetilde{f}_k) = (\chi_{A' \cap A''} \widetilde{f}_j \mid \chi_{A' \cap A''} \widetilde{f}_k) = 0.$$

Hence the cyclic subspaces $Z(f_j)$ are pairwise orthogonal. For $j = k$, we find that for every Borel subset A of the circle,

$$\sigma_{f_j}(A) = (\chi_A \widetilde{f}_j \mid \widetilde{f}_j) = \|\chi_A \widetilde{f}_j\|^2$$

does not depend on j. Thus the f_j's have the same spectral measure, equivalent to τ, and $U|_F$ has uniform multiplicity m.

We may therefore suppose that τ is the common spectral measure of the f_j's and that the spectral isomorphism maps f_j to the constant function $\widetilde{f}_j : z \mapsto e_j$, where $(e_1, \ldots, e_m)$ is the canonical basis of $\mathbb{C}^m$ (and $\widetilde{F} = L^2(\mathbb{T}, \tau, \mathbb{C}^m)$). To the action of $\mathcal{K}$ on F_0 corresponds an irreducible representation ξ of $\mathcal{K}$ in the unitary group $U(m)$ such that, for $V \in \mathcal{K}$ and $j = 1, \ldots, m$, we have $\widetilde{Vf_j}(z) = \xi(V)e_j$. Then, for every $g \in L^\infty(\mathbb{T}, \tau)$, since $\widetilde{V}$ commutes with the multiplication by g, we get $\widetilde{V}(g \cdot \widetilde{f}_j)(z) = g(z)\,\xi(V)e_j$, and it follows that

$$\widetilde{V} f(z) = \xi(V) \cdot f(z) \quad \text{for every } f \in L^2(\mathbb{T}, \tau, \mathbb{C}^m). \tag{21}$$

By separability of H, we get:

Proposition 12. *H is the orthogonal sum of U-invariant and $\mathcal{K}$-invariant subspaces F such that $U|_F$ has a uniform finite multiplicity m and,* *in some spectral representation of F as $L^2(\mathbb{T},\sigma_{U|_F},\mathbb{C}^m)$, $\mathcal{K}$ acts by some irreducible representation in $U(m)$ according to formula* (21). ∎

Now, let T be a Gaussian automorphism of type σ and let $\mathcal{A}=\mathcal{B}(H)/\mathcal{K}$ be a classical factor of T, where H is a U_T-invariant Gaussian subspace and $\mathcal{K}$ is a compact subgroup of $C^{\mathrm{g}}(T|_{\mathcal{B}(H)})$. Consider a decomposition $\sigma=\sigma_0+\widetilde{\sigma}_0$, where σ_0 is the restriction of σ to a Borel subset A_0 of $\mathbb{T}$ such that $A_0\cup\overline{A}_0=\mathbb{T}$ and $\sigma(A_0\cap\overline{A}_0)=0$, and let $H_0=H_{A_0}=\{f\in H:\sigma_f\ll\sigma_0\}$. Since H_0 is a spectral subspace of $U=U_T|_H$, it must be $\mathcal{K}$-invariant. Moreover we have $H=H_0\oplus\overline{H}_0$. Any operator $V_0\in\mathcal{L}(H_0)$ commuting with $U|_{H_0}$ has a unique extension to a real operator $V\in\mathcal{L}(H)$ commuting with U, by letting $Vf=\overline{V_0\overline{f}}$ on $\overline{H}_0$. Therefore $\mathcal{K}$ is determined by its action on H_0, and conversely, any compact subgroup of $\mathcal{U}(H_0)$ commuting with U yields a compact subgroup of $C^{\mathrm{g}}(T|_{\mathcal{B}(H)})$.

This allows us to extend the construction in Example 1 to generalized Gaussian automorphisms.

Example 3. Consider $T=T_\sigma^{(m)}$, the m-fold direct product $T_\sigma\times\ldots\times T_\sigma$, and its Gaussian space $H=H_\sigma^m\approx L^2(\mathbb{T},\sigma,\mathbb{C}^m)$. Then H_0 corresponds to the subspace $L^2(\mathbb{T},\sigma_0,\mathbb{C}^m)$. Every operator $R\in U(m)$ yields a unitary operator V_R on H_0, by letting $\widetilde{V_R}f(z)=R\cdot f(z)$ in the spectral representation, and V_R extends to a real unitary operator on H commuting with U. We thus get a subgroup $\mathcal{U}_{\sigma_0}(m)$ of $C^{\mathrm{g}}(T)$ and a compact factor $\mathcal{B}(H_\sigma^m)/\mathcal{U}_{\sigma_0}(m)$ which we shall denote by $\mathcal{A}_{\sigma_0}(m)$.

If we come back to the general case of a Gaussian automorphism T of type σ and a classical factor $\mathcal{A}=\mathcal{B}(H)/\mathcal{K}$ of it, since H_0 is $\mathcal{K}$-invariant, by Proposition 12 we find a Gaussian subspace $F_0\subset H_0$ such that $U|_{F_0}$ has a uniform multiplicity m and in a spectral representation $\mathcal{K}$ acts by (21). Let τ_0 be the spectral type of $U|_{F_0}$, $F=F_0\oplus\overline{F}_0$ and $\tau=\tau_0+\widetilde{\tau}_0$. Then $U|_F$ still has uniform multiplicity m, so we can identify $T|_{\mathcal{B}(F)}$ with $T_\tau^{(m)}$. The factor $\mathcal{A}\cap\mathcal{B}(F)$ is a compact factor of $\mathcal{B}(F)$ determined by some subgroup of $\mathcal{U}_{\tau_0}(m)$, hence it contains $\mathcal{A}_{\tau_0}(m)$. This proves:

Proposition 13. *Every classical factor of a Gaussian automorphism of type $\sigma=\sigma_0+\widetilde{\sigma}_0$ contains a factor $\mathcal{A}_{\tau_0}(m)$ for some measure $\tau_0\ll\sigma_0$ and some positive integer m.* ∎

In particular, if an automorphism has a common factor with a GAG T of type σ, we can assert that it has a factor isomorphic to some $\mathcal{A}_{\tau_0}(m)$ with $\tau_0\ll\sigma_0$.

The next proposition gives a new example of non-disjoint automorphisms without common factors.

PROPOSITION 14. *Let $\sigma = \sigma_0 + \widetilde{\sigma}_0$ with $\sigma_0 \perp \widetilde{\sigma}_0$, and let $m_1 \neq m_2$ be two positive integers. Then $\mathcal{A}_{\sigma_0}(m_1)$ and $\mathcal{A}_{\sigma_0}(m_2)$ are not disjoint, but if moreover σ is a GAG measure, they have no common factor.*

Proof. Assume $m_1 < m_2$. Then $\mathcal{B}(H_\sigma^{m_1})$ and thus $\mathcal{A}_{\sigma_0}(m_1)$ appear naturally as factors of $\mathcal{B}(H_\sigma^{m_2})$. Since $\mathcal{A}_{\sigma_0}(m_2)$ is a compact factor, it cannot be independent of any other non-trivial factor of $\mathcal{B}(H_\sigma^{m_2})$. This yields a joining of $\mathcal{A}_{\sigma_0}(m_1)$ and $\mathcal{A}_{\sigma_0}(m_2)$ which is different from the product measure.

Suppose now that σ is a GAG measure and $\mathcal{A}_i$ are non-trivial factors of $\mathcal{A}_{\sigma_0}(m_i)$, $i = 1, 2$. As $\mathcal{A}_i$ is also a factor of $\mathcal{B}(H_\sigma^{m_i})$, it is classical and can be written as $\mathcal{A}_i = \mathcal{B}(H_i)/\mathcal{K}_i$. By (18), since $\mathcal{A}_i \subset \mathcal{A}_{\sigma_0}(m_i)$, H_i is invariant under $\mathcal{U}_{\sigma_0}(m_i)$. It follows that the multiplicity of $U_{T_\sigma^{(m_i)}}$ on H_i must be equal to m_i. Indeed, passing to the spectral representation in $L^2(\mathbb{T}, \sigma, \mathbb{C}^{m_i})$, if H_i is spanned by m cyclic subspaces $Z(f_1), \ldots, Z(f_m)$, then every $\widetilde{f} \in \widetilde{H}_i$ takes σ-a.e.(z) its values in the subspace spanned by the $\widetilde{f}_j(z)$. By separability, this subspace is σ-a.e. invariant under $U(m_i)$, hence equal either to $\{0\}$ or to $\mathbb{C}^{m_i}$, and it follows that $m \geq m_i$.

On the other hand, if $\mathcal{A}_1$ and $\mathcal{A}_2$ were isomorphic, by Proposition 8, the restrictions of $U_{T_\sigma^{(m_i)}}$ to H_i ($i = 1, 2$) would be unitarily equivalent and therefore they would have the same multiplicity. ■

3.6. Factors and Gaussian isomorphisms. We will now decide which factors of a GAG can be isomorphic to a Gaussian automorphism. Let $(X, \mathcal{B}, \mu)$ be a Gaussian probability space with Gaussian space H. Recall (see [16]) that for each non-zero $f \in H^{(n)}$ ($n \geq 1$) there exists $\alpha_* \in \mathbb{R}_+$ such that

$$\int_X e^{\alpha|f|^{2/n}}\, d\mu \begin{cases} < \infty & \text{for each } \alpha < \alpha_*, \\ = \infty & \text{for each } \alpha > \alpha_*. \end{cases} \tag{22}$$

Assume now that T and S are Gaussian automorphisms with Gaussian spaces H and J respectively. It follows that the distributions of a real variable from $H^{(n)}$ and of a real variable from $J^{(m)}$ are different whenever $n \neq m$. Moreover, (22) easily extends to every $f \in \bigoplus_{k=1}^{n} H^{(k)} \setminus \bigoplus_{k=1}^{n-1} H^{(k)}$ and therefore, if I is an isomorphism between a factor $\mathcal{A}_1$ of S and a factor $\mathcal{A}_2$ of T and if $f \in L^2(\mathcal{A}_1) \cap J^{(n)}$, then either $U_I(f) \in H^{(1)} \oplus \ldots \oplus H^{(n)}$, or the orthogonal projection of $U_I f$ on $H^{(k)}$ is not zero for infinitely many k's.

To each $a \in (0, 1)$ we assign the Gaussian self-joining λ_a of S corresponding, by Theorem 1, to the operator $f \mapsto af$ on J (which clearly belongs to $\mathcal{S}_S$). Let then $\Phi_a = \Phi_{\lambda_a}$. In view of Lemma 3, for each $n \geq 1$,

$$\Phi_a(f) = a^n f \quad \text{for each } f \in J^{(n)}.$$

So, each chaos $J^{(n)}$ is the eigenspace of Φ_a corresponding to the eigenvalue a^n. Note also that Φ_a commutes with every Markov operator of a self-joining of S with a Gaussian disintegration and, in particular, if $\mathcal{A}$ is a classical factor of S, then $\pi_{\mathcal{A}}\Phi_a = \Phi_a\pi_{\mathcal{A}}$.

PROPOSITION 15. *Let S and T be Gaussian automorphisms, let J and H be the Gaussian spaces of S and T respectively, and assume that T is a GAG. Let $\mathcal{A}$ be a factor of T and $\mathcal{A}'$ a* classical *factor of S. Suppose that I establishes an isomorphism of the factors $\mathcal{A}$ and $\mathcal{A}'$. Then for each $m \geq 1$,*

$$U_I(H^{(m)} \cap L^2(\mathcal{A})) = J^{(m)} \cap L^2(\mathcal{A}').$$

Proof. Fix $a \in (0,1)$ and let λ_a and Φ_a be defined as above. Then we consider $\widetilde{\lambda}_a$,the restriction of λ_a to $\mathcal{A}' \otimes \mathcal{A}'$. The corresponding Markov operator $\Phi_{\widetilde{\lambda}_a}$ equals $\pi_{\mathcal{A}'}\Phi_a|_{L^2(\mathcal{A}')}$ but, since $\mathcal{A}'$ is a classical factor, $\pi_{\mathcal{A}'}\Phi_a = \Phi_a\pi_{\mathcal{A}'}$ and thus $\Phi_{\widetilde{\lambda}_a} = \Phi_a|_{L^2(\mathcal{A}')}$. So, for each $n \geq 0$, $J^{(n)} \cap L^2(\mathcal{A}')$ is an eigenspace of $\Phi_{\widetilde{\lambda}_a}$ and $L^2(\mathcal{A}')$ is the orthogonal sum of these eigenspaces. Consider now the ergodic self-joining λ of $\mathcal{A}$ corresponding to $\widetilde{\lambda}_a$ by I. Then $\Phi_\lambda\pi_{\mathcal{A}}$ is the Markov operator of some self-joining of T and, since T is a GAG, it preserves the chaos. Therefore, for all $m \geq 0$, Φ_λ preserves the subspace $H^{(m)} \cap L^2(\mathcal{A})$ and it follows that this subspace can be decomposed according to the eigenspaces of Φ_λ:

$$H^{(m)} \cap L^2(\mathcal{A}) = \bigoplus_{n=0}^{\infty} H^{(m)} \cap U_I^*(J^{(n)} \cap L^2(\mathcal{A}')).$$

However, $H^{(m)} \cap U_I^*(J^{(n)} \cap L^2(\mathcal{A}')) = \{0\}$ whenever $n \neq m$ since in view of (22) a non-zero variable from $H^{(m)}$ has a different distribution from a variable from $J^{(n)}$. Since

$$L^2(\mathcal{A}) = \bigoplus_{m=0}^{\infty} H^{(m)} \cap L^2(\mathcal{A}),$$

the result follows. ∎

The corollary below generalizes a fact which was well known for Gaussian automorphisms with simple spectrum (cf. [38], Th. 5).

COROLLARY 5. *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a GAG with Gaussian space H. Any isomorphism between T and another Gaussian automorphism is Gaussian. Moreover, any U_T-invariant subspace J of $L_0^2(X, \mu)$ which is spanned by a real subspace of Gaussian variables is contained in H.* ∎

The second assertion states that the only factors of a GAG which are isomorphic to some Gaussian automorphism are its (natural) Gaussian factors; in other words, factors determined by non-trivial compact subgroups are not isomorphic to any Gaussian automorphism.

EXAMPLE 4. As an application we give a construction of weakly isomorphic but not isomorphic automorphims using a GAG. Let $T = T_\sigma^\infty$, where

T_σ is a GAG. Consider T and $T_1 = S \times T$, where S is a non-trivial compact factor of T_σ. Clearly, T and T_1 are weakly isomorphic. They cannot, however, be isomorphic, since T_1 is a non-trivial compact factor of T.

This generalizes a result from [32], where the same construction of two weakly isomorphic but not isomorphic Gaussian systems was carried out using T_σ with simple spectrum and some spectral arguments.

COROLLARY 6. *Let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a Gaussian automorphism with Gaussian space H. Assume that $J \subset L_0^2(X, \mu)$ is a T-invariant Gaussian subspace and that the spectral type η of U_T on J is a GAG measure. Then $J \subset H$.*

Proof. Consider the family Θ of all closed T-invariant Gaussian subspaces $J' \subset L_0^2(X, \mu)$ whose spectral type is η. By the GAG property, if J_1, $J_2 \in \Theta$, then $\overline{J_1 + J_2} \in \Theta$, and it follows that there exists a biggest element, say F, in Θ. Now, clearly, if $S \in C(T)$ and $J' \in \Theta$ then $U_S(J') \in \Theta$. According to Remark 2, we can take $S \in C^{\mathrm{g}}(T)$ which is a GAG. Then we must have $SF = F$ and the Gaussian space of S is still H. So, by Corollary 5, $F \subset H$ and the result follows. ∎

REMARK 8. The proof of the above corollary allows us to define, similarly to the notion of the Kronecker factor in ergodic theory, a biggest GAG factor of type absolutely continuous with respect to a given GAG measure: whenever $S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ is an ergodic automorphism of a standard probability space and σ is a GAG measure, there exists a biggest factor $\mathcal{A} \subset \mathcal{C}$ such that $S|_{\mathcal{A}}$ is a GAG of type $\ll \sigma$.

REMARK 9. Let us see the particular meaning of Corollary 6 in terms of stationary processes. Suppose that $(X_n), (Y_n)$ are two stationary centered Gaussian processes defined on $(\Omega, \mathcal{F}, P)$ such that (X_n, Y_n) is also stationary. Assume that $\mathcal{B}((X_n)) = \mathcal{F}$ and that the spectral measure of (Y_n) is σ with T_σ a GAG. Then $(X_n + Y_n)$ is a stationary centered Gaussian process. Indeed, $\mathcal{B}((Y_n))$ is a Gaussian factor which is GAG and whose Gaussian space is given by the span of $\{Y_n : n \in \mathbb{Z}\}$. By Corollary 6 we have $Y_n \in \operatorname{span}\{X_k : k \in \mathbb{Z}\}$.

K. Itô in [11], §29, raised the problem of whether each stationary process is a function of a stationary Gaussian process. The Itô problem is equivalent to the question of whether each dynamical system is a factor of a Gaussian automorphism. Vershik in [38] answered this question in the negative by showing that $T_\sigma \times T_\sigma$ cannot be a factor of a standard Gaussian automorphism whenever T_σ has simple spectrum. This sort of examples can also be constructed using the GAG's as follows from Corollary 6. All theorems giving rise to classes of automorphisms disjoint from the Gaussian automorphisms yield counterexamples to the Itô question, historically the first one being proved by the third author in [33] (see also [13]). We will give, however, another class of transformations that are not factors of any Gaussian system.

PROPOSITION 16. *Assume that T is a GAG. If an automorphism R is a non-trivial distal extension of T then R cannot be a factor of any Gaussian automorphism.*

Proof. If T is a factor of a Gaussian automorphism S then Corollary 6 says in particular that S is a relatively weakly mixing extension of T; if R is a factor of S then R has to be relatively weakly mixing over T, a contradiction. ∎

4. ON THE DISJOINTNESS PROBLEM

4.1. Disjointness and common factors. Let $(X,\mathcal{B},\mu)$, $(Y,\mathcal{C},\nu)$ be two probability spaces. Let λ be a probability measure on $X\times Y$ with marginals μ and ν respectively. Denote by λ^∞ the measure on $(X^\infty\times Y, \mathcal{B}^\infty\otimes\mathcal{C})$ which is the infinite relative product of $(X\times Y,\mathcal{B}\otimes\mathcal{C},\lambda)$ over Y, i.e.

$$\int\limits_{X^\infty\times Y}\prod_{i=1}^n f_i(x_i)\, g(y)\, d\lambda^\infty(x,y)=\int\limits_Y\prod_{i=1}^n(\varPhi_\lambda f_i)g\, d\nu \tag{23}$$

for each $f_i\in L^\infty(X,\mu)$, $g\in L^\infty(Y,\nu)$ $(x=(x_1,x_2,\ldots)\in X^\infty)$. The following lemma is a relative version of the 0-1 law.

LEMMA 9. *Under the above assumptions, each function $g\in L^2(X^\infty\times Y, \lambda^\infty)$ invariant under all permutations of coordinates in X is $(X^\infty\times\mathcal{C})$-measurable.*

Proof. Let $\mathcal{B}_n$ and $\mathcal{B}_n'$ denote the sub-σ-algebras on X^∞ generated by the coordinates $x_1,\ldots,x_n$ and $x_{n+1},\ldots$ respectively. The assumption that g is invariant under permutations of X-coordinates implies that it is $\bigcap_{n\ge1}(\mathcal{B}_n'\otimes\mathcal{C})$-measurable. Then it follows from (23) that, for every $n\ge1$ and every $f\in L^\infty(\mathcal{B}_n\otimes\mathcal{C},\lambda^\infty)$,

$$\int fg\, d\lambda^\infty=\int E(f\,|\,X^\infty\otimes\mathcal{C})E(g\,|\,X^\infty\otimes\mathcal{C})\, d\lambda^\infty=\int fE(g\,|\,X^\infty\otimes\mathcal{C})\, d\lambda^\infty,$$

whence $g=E(g\,|\,X^\infty\otimes\mathcal{C})$ a.e. ∎

The following theorem contributes to a better understanding of Furstenberg's problem on the relation between disjointness and the lack of common factors (see [3]).

THEOREM 4. *Let $S:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$, $T:(Y,\mathcal{C},\nu)\to(Y,\mathcal{C},\nu)$ be two automorphisms. If S and T are not disjoint then T has a common* *factor with an infinite self-joining of S. If T and S are additionally ergodic then T has a common factor with an ergodic infinite self-joining of S.*

P r o o f. Let λ be a joining of S and T, $\lambda \neq \mu \otimes \nu$. The measure λ^∞ defined in Lemma 9 is a joining between an infinite self-joining of S and T. Take a function $g \in L^2(Y,\nu)$ such that $E^\lambda(g(y) \mid \mathcal{B} \times Y)$ is not constant and consider the function $G = E^{\lambda^\infty}(g(y) \mid \mathcal{B}^\infty \times Y)$. Then G is invariant under all permutations of coordinates in X and hence by Lemma 9 it is $(\mathcal{B}^\infty \times Y) \cap (X^\infty \times \mathcal{C})$-measurable. As the projection of λ^∞ on each factor $X \times Y$ is λ and $E^\lambda(g(y) \mid \mathcal{B} \times Y)$ is not constant, G is not constant. Therefore $(\mathcal{B}^\infty \times Y) \cap (X^\infty \times \mathcal{C})$ is not trivial, and it is isomorphic both to a factor of an infinite self-joining of S and to a factor of T.

Assume now that T and S are ergodic. It follows that there exists $\varrho \in J_\infty(S)$ such that T and (S^∞, ϱ) have a common factor. This common factor is necessarily ergodic, so it is a common factor of T and almost every ergodic component of ϱ. ■

Remark 10. Theorem 4 simplifies (and unifies) the proofs of a number of classical theorems on disjointness. Let us mention three of them:

(i) *zero entropy automorphisms* $\perp$ *K-automorphisms* ([30], also [3]);
(ii) *distal automorphisms* $\perp$ *weakly mixing automorphisms* ([3]);
(iii) *rigid automorphisms* $\perp$ *mildly mixing automorphisms* ([5]).

Indeed, it is enough to show that the classes under consideration have no common factors, that they are closed under taking factors and that the left-hand class is closed under taking joinings. The latter property follows from the fact that each automorphism has a biggest factor with zero entropy (its Pinsker factor) for (i), a biggest distal factor for (ii), and a biggest factor with a given rigidity sequence for (iii).

Observe also that the classical fact: Id is disjoint from any ergodic T, follows from the above scheme.

Remark 11. We notice, however, that the necessary condition for non-disjointness in Theorem 4 is not sufficient, since an ergodic automorphism T can be disjoint from some non-trivial factor of an ergodic self-joining of itself. For example, it is known that an ergodic self-joining of a weakly mixing automorphism can have a rotation as a factor, as the following construction shows (details are left to the reader).

Let T be a weakly mixing automorphism. Choose a measurable $\psi : X \to \mathbb{T}$ so that the extension $T_\psi : (x,z) \mapsto (Tx, \psi(x)z)$ on $X \times \mathbb{T}$ ($\mathbb{T}$ is equipped with the Lebesgue measure) remains weakly mixing. Then, on $X \times \mathbb{T}^2$,

$$\widetilde{T}_\psi : (x,z,z') \mapsto (x, \psi(x)z, zz')$$

is still weakly mixing.

Now, take a $z_0 \in \mathbb{T}$ which is not a root of 1 and let

$$S(x, z, z', z'') = (x, \psi(x)z, zz', z_0 zz'')$$

on $X \times \mathbb{T}^3$. Then S can be identified with an ergodic self-joining of $\widetilde{T}_\psi$ but it has z_0 as an eigenvalue and thus has a rotation as a factor.

For a GAG we can improve the assertion of Theorem 4.

THEOREM 5. *Let T be a GAG of type σ and S be an ergodic automorphism. Assume that $T \not\perp S$. Then S and T_σ^∞ have a common factor. In particular, if $T_\sigma^\infty \not\perp S$ then S and T_σ^∞ itself have a common factor.*

Proof. By Theorem 4 the only thing we have to note is that each ergodic infinite self-joining of T is a factor of T_σ^∞. ∎

REMARK 12. It follows from Proposition 13 that, if an ergodic automorphism S is not disjoint from a GAG of type σ, it has a factor isomorphic to some $\mathcal{A}_\tau(m)$ where $\tau \ll \sigma$ and $\tau \perp \widetilde{\tau}$. Moreover, since a compact factor cannot be independent of any other non-trivial factor, a classical factor of T_σ^∞ cannot be disjoint from any Gaussian automorphism of type σ, and therefore we have here a necessary and sufficient condition for non-disjointness.

Moreover, we recall that quite a similar condition of non-disjointness for an ergodic automorphism S and a simple automorphism T was given by del Junco and Rudolph ([15], Th. 4.1): S must then have a factor isomorphic to some compact factor of a finite direct product $T \times \ldots \times T$. Again, Theorem 4 allows us to simplify the proof in [15]. Indeed, as in the case of a GAG, any infinite ergodic self-joinings of T is isomorphic to a factor of T^∞, and any non-trivial factor of T^∞ contains a compact factor of a finite direct product of copies of T (for an analysis of the factors of T^∞, see [13]).

We will also need a result which was already used in [13] to prove that Gaussian automorphisms are disjoint from simple automorphisms.

PROPOSITION 17. *Assume that $S : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is ergodic and T is a Gaussian automorphism. Assume moreover that $S \not\perp T$. Then there exists an ergodic infinite self-joining of S which has a factor isomorphic to a* classical *factor of T.*

Proof. By Theorem 4 we can find a factor $\mathcal{A}$ of T, an ergodic infinite self-joining λ of S and a factor of (S^∞, λ) which is isomorphic to $\mathcal{A}$. In view of Remark 2, we can find in $C^{\mathrm{g}}(T)$ a standard GAG T'. Then every factor of T' is classical and is also a classical factor of T. Thus $\widetilde{\mathcal{A}} := \bigvee_{n\in\mathbb{Z}} {T'}^n \mathcal{A}$ is a classical factor of T. Clearly, $\widetilde{\mathcal{A}}$ can be viewed as an infinite ergodic self-joining of $\mathcal{A}$, hence it is isomorphic to a factor of an ergodic infinite self-joining of (S^∞, λ), whence to a factor of an ergodic infinite self-joining of S. ∎

REMARK 13. It follows from the proof that each factor $\mathcal{A}$ of a Gaussian automorphism is contained in a classical factor $\mathcal{A}'$ isomorphic to an infinite self-joining of $\mathcal{A}$.

4.2. Disjointness from a GAG—spectral point of view. Assume that $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is a Gaussian automorphism of type σ, with Gaussian space H. Let $\lambda \in J_2(T)$ be a self-joining with a Gaussian disintegration. As we already noticed in Section 2.3, Φ_λ then preserves the chaos. In contrast to the ergodic case, $\Phi_\lambda|_H = 0$ does not imply $\Phi_\lambda = 0$ on $L_0^2(X, \mu)$, as the well-known example of the even factor shows: in this case the corresponding Markov operator is the orthogonal projector on the sum of the even chaos. However, such a behavior is more general. We write $\sigma = \sigma_0 + \widetilde{\sigma}_0$ with $\sigma_0 \perp \widetilde{\sigma}_0$. Put $H_0 = \{f \in H : \sigma_f \ll \sigma_0\}$. Hence $H = H_0 \oplus \overline{H}_0$, where $\overline{H}_0 = \{\overline{f} : f \in H_0\} = \{g \in H : \sigma_g \ll \widetilde{\sigma}_0\}$.

PROPOSITION 18. *Let T be a Gaussian automorphism with Gaussian space H. If $\lambda \in J_2(T)$ has a Gaussian disintegration and $\lambda \neq \mu \otimes \mu$ then $\Phi_\lambda|_{H^{(2n)}} \neq 0$ for each $n \geq 1$.*

Proof. Let

$$\Phi_\lambda = \int_{J_2^{\mathrm{g}}(T)} \Phi_\varrho \, dP(\varrho)$$

correspond to the ergodic decomposition of λ. Let us remark first that for some $f \in H_0$, there is a set of positive P-measure on which $\Phi_\varrho f \neq 0$. Indeed, if not, by separability, we get $\Phi_\varrho|_{H_0} = 0$ a.e., whence also $\Phi_\varrho|_{\overline{H}_0} = 0$ a.e., and thus $\Phi_\varrho|_H = 0$ a.e. Since $\varrho \in J_2^{\mathrm{g}}(T)$ a.s., it follows that $\Phi_\varrho = 0$ on $L^2(X, \mu)$ a.s. and therefore the same holds for Φ_λ, a contradiction.

Now take a "good" $f \in H_0$. We have $\Phi_\varrho f \in H_0$ since H_0 is a spectral subspace and thus $\Phi_\varrho f \perp \overline{\Phi_\varrho f} = \Phi_\varrho \overline{f}$. Therefore in the natural identification of $H^{(2)}$ with $H^{\odot 2}$, $|f|^2 = f \odot \overline{f}$ and $|\Phi_\varrho f|^2 = \Phi_\varrho(f) \odot \overline{\Phi_\varrho(f)}$. It follows that

$$\Phi_\lambda(|f|^2) = \Phi_\lambda(f \odot \overline{f}) = \int |\Phi_\varrho f|^2 \, dP(\varrho) > 0.$$

This proves the case $n = 1$. For the general case, it is enough to replace f by $f^{\odot n}$. We still have $H_0^{\odot n} \perp \overline{H}_0^{\odot n}$ (in $H^{\odot n}$) and this still implies

$$\Phi_\varrho(f^{\odot n} \odot \overline{f}^{\odot n}) = (\Phi_\varrho f)^{\odot n} \odot (\Phi_\varrho \overline{f})^{\odot n} = |(\Phi_\varrho f)^{\odot n}|^2 \neq 0$$

for ϱ in a set of positive P-measure. ∎

Since the relative product over a classical factor has a Gaussian disintegration, we have the following.

COROLLARY 7. *Assume that $\mathcal{A}$ is a non-trivial classical factor of a Gaussian automorphism T with Gaussian space H. Then, for each $n \geq 1$,*

$$L^2(\mathcal{A}) \cap H^{(2n)} \neq \{0\}. \ ∎$$

REMARK 14. As $H_0 \perp \overline{H}_0$, $H_0 \otimes \overline{H}_0$ may be viewed as a subspace of $H^{\odot 2}$ (the embedding being given by $f \otimes g \mapsto f \otimes g + g \otimes f$). The proof of Proposition 18 yields more precisely $\Phi_\lambda|_{H_0\otimes \overline{H}_0} \neq 0$, and thus if $\mathcal{A}$ is a non-trivial classical factor then $L^2(\mathcal{A}) \cap (H_0 \otimes \overline{H}_0) \neq \{0\}$.

EXAMPLE 5. Let $\mathcal{K} = \{S_z : z \in \mathbb{T}\}$ be the compact subgroup of $C^{\mathrm{g}}(T)$ where $U_{S_z} = V_z$ is defined on H by $V_z f = zf$ if $f \in H_0$ and $V_z f = \overline{z} f$ if $f \in \overline{H}_0$, for $z \in \mathbb{T}$, and let $\mathcal{A} = \mathcal{B}/\mathcal{K}$ (if T is standard, then $\mathcal{A}$ is of the type given in Example 1). We have

$$\pi_{\mathcal{A}} = \int_{\mathbb{T}} V_z \, dz.$$

Clearly, $\pi_{\mathcal{A}}|_H = 0$. On the second chaos we have

$$V_z(f \odot g) = V_z f \odot V_z g = \begin{cases} f \odot g & \text{if } f \in H_0 \text{ and } g \in \overline{H}_0, \\ z^2 f \odot g & \text{if } f, g \in H_0, \\ \overline{z}^2 f \odot g & \text{if } f, g \in \overline{H}_0. \end{cases}$$

Hence $\pi_{\mathcal{A}}(f \odot g) = f \odot g$ in the first case, and 0 in the other two cases. It follows that the restriction of $\pi_{\mathcal{A}}$ to $H^{\odot 2}$ is the orthogonal projector on $H_0 \otimes \overline{H}_0$.

For T a GAG, Proposition 18 gives rise to a necessary condition of non-disjointness of T with another automorphism.

COROLLARY 8. *Let T be a GAG and $S : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ an automorphism. Assume that S and T are not disjoint. Then the maximal spectral type of S and $\sigma * \sigma$ are not mutually singular* (*or more generally*, *the same holds for $\sigma^{(2n)}$ for all $n \geq 1$*).

Proof. Let $\lambda \in J(T, S)$ be different from the product measure and put $\Psi := \Phi_\lambda^* \circ \Phi_\lambda$. It follows from (5) that $\Psi \neq 0$ on $L_0^2(X, \mu)$. Thus Proposition 18 implies that Ψ does not vanish identically on the second chaos of T. Consequently, $\Phi_\lambda|_{H^{(2)}} : H^{(2)} \to L^2(Y, \mathcal{C}, \nu)$ is non-zero and the assertion follows from (6). ∎

4.3. Disjointness of a GAG from a Gaussian automorphism. Assume that σ is a GAG measure and let τ be a continuous symmetric measure. Our first aim is to prove that if $T_\sigma \not\perp T_\tau$ then σ and a translation of τ are not mutually singular. In fact, the same conclusion holds and the proof will be the same under the weaker hypothesis that T_τ and some Gaussian automorphism of type σ are not disjoint.

Then, by Proposition 17 and the fact that an ergodic infinite self-joining of a GAG of type σ is still a Gaussian automorphism of type σ, there exist a (classical) factor $\mathcal{A}$ of a GAG T of type σ, a classical factor $\mathcal{A}'$ of T_τ and an isomorphism I from $\mathcal{A}'$ to $\mathcal{A}$. The factor $\mathcal{A}$ may be written as $\mathcal{A} = \mathcal{B}(H)/\mathcal{K}$, where H is an invariant Gaussian subspace of T and $\mathcal{K}$ is a compact subgroup of $C^{\mathrm{g}}(T|_{\mathcal{B}(H)})$. Similarly we can find an invariant Gaussian subspace F of T_τ and a compact subgroup $\mathcal{L} \subset C^{\mathrm{g}}(T|_{\mathcal{B}(F)})$ such that $\mathcal{A}' = \mathcal{B}(F)/\mathcal{L}$. We will show that the existence of such an I implies that the spectral type of $U|_T$ on H and some translation of the spectral type of $U|_{T_\tau}$ on F are not mutually singular. Therefore, with no loss of generality we shall assume that H and F are the whole Gaussian spaces of T and T_τ respectively.

For any symmetric Borel subsets $A, B \subset \mathbb{T}$ we consider the spectral Gaussian subspaces

$$H_A = \{f \in H : \sigma_f \ll \sigma|_A\}, \qquad F_B = \{g \in F : \sigma_g \ll \tau|_B\}$$

and we define

$$\mathcal{A}_A = \mathcal{B}(H_A) \cap \mathcal{A}, \qquad \mathcal{A}'_B = \mathcal{B}(F_B) \cap \mathcal{A}'.$$

We also have

$$\mathcal{A}_A = \mathcal{B}(H_A)/\mathcal{K}_A, \qquad \mathcal{A}'_B = \mathcal{B}(F_B)/\mathcal{L}_B,$$

where $\mathcal{K}_A = \mathcal{K}|_{\mathcal{B}(H_A)}$ and $\mathcal{L}_B = \mathcal{L}|_{\mathcal{B}(F_B)}$.

By $\mathcal{B}_{\mathrm{sym}}(\mathbb{T})$ we denote the sub-σ-algebra of $\mathcal{B}(\mathbb{T})$ consisting of all sets which are invariant under the map $z \mapsto \overline{z}$. The pointwise isomorphism I induces a Boolean (mod null sets) isomorphism $\mathcal{A} \to \mathcal{A}'$ still denoted by I.

Lemma 10. *Under these assumptions*, *there exists a unique map*

$$I^* : \mathcal{B}(\mathbb{T})_{\mathrm{sym}} \bmod \sigma \to \mathcal{B}(\mathbb{T})_{\mathrm{sym}} \bmod \tau$$

such that $I(\mathcal{A}_A) = \mathcal{A}'_{I^*(A)}$ *for each symmetric Borel subset* $A \subset \mathbb{T}$.

Proof. Fix a symmetric Borel set $A \subset \mathbb{T}$. We have to show that $I(\mathcal{A}_A)$ is of the form $\mathcal{A}'_B$ for some symmetric Borel set $B \subset \mathbb{T}$. Clearly, the set B is then uniquely determined mod τ and it will not change if A is modified by a σ-null set.

Take a standard GAG $S \in C^{\mathrm{g}}(T_\tau)$ and consider the classical factor of T_τ constructed from $I(\mathcal{A}_A)$ as in the proof Proposition 17,

$$\widetilde{\mathcal{A}} = \bigvee_{n\in\mathbb{Z}} S^n I(\mathcal{A}_A).$$

Since T_τ is standard, the only invariant subspaces of F are its spectral subspaces, and $\widetilde{\mathcal{A}} = \mathcal{B}(F_B)/\mathcal{L}'$ for some symmetric Borel set $B \subset \mathbb{T}$ and some compact subgroup $\mathcal{L}' \subset C^{\mathrm{g}}(T_\tau|_{\mathcal{B}(F_B)})$. Again since T_τ is standard, any classical factor of T_τ is S-invariant. It follows that $\widetilde{\mathcal{A}} \subset \mathcal{A}'$, hence $\widetilde{\mathcal{A}} \subset \mathcal{A}' \cap \mathcal{B}(F_B) = \mathcal{A}'_B$, and moreover, $\widetilde{\mathcal{A}}$ is a compact factor of $\mathcal{A}'_B$.

The factor $I^{-1}\widetilde{\mathcal{A}} \subset \mathcal{A}$ of T is generated by a family of factors isomorphic to $\mathcal{A}_A$. Since T is a GAG, it follows from Proposition 8 that the Gaussian cover of each factor in this family is Gaussian isomorphic to $\mathcal{B}(H_A)$. As H_A

is a spectral subspace of H, this actually implies that each of these factors is contained in $\mathcal{B}(H_A)$. Therefore

$$\mathcal{A}_A \subset I^{-1}\widetilde{\mathcal{A}} \subset \mathcal{A} \cap \mathcal{B}(H_A) = \mathcal{A}_A,$$

and $I(\mathcal{A}_A) = \widetilde{\mathcal{A}}$.

Now, $I^{-1}(\mathcal{A}'_B)$ is of the form $\mathcal{B}(H')/\mathcal{K}'$ and, since it contains $\mathcal{A}_A$ as a compact factor, we must have $H' = H_A$. Thus

$$\mathcal{A}_A \subset I^{-1}(\mathcal{A}'_B) \subset \mathcal{A} \cap \mathcal{B}(H_A) = \mathcal{A}_A$$

and finally $I(\mathcal{A}_A) = \mathcal{A}'_B$. ∎

We shall now show that I^* corresponds to some point mapping on the circle. Since it is defined only on symmetric sets, let us fix a Borel subset A_0 (for example, the upper half of the circle) such that

$$A_0 \cup \overline{A}_0 = \mathbb{T}, \quad \sigma(A_0 \cap \overline{A}_0) = 0.$$

Lemma 11. *There exists a Borel map $\psi : \mathbb{T} \to A_0$ such that $I^*(A) = \psi^{-1}(A)$ for every symmetric Borel subset $A \subset \mathbb{T}$ and $\psi_*\tau \ll \sigma$. Moreover, $\psi(\overline{z}) = \psi(z)$ τ-a.e.*

Proof. The first condition is equivalent to saying that $\psi^{-1}(A) = I^*(A \cup \overline{A})$ for every Borel subset A of A_0. When $A = A_0$, since $A_0 \cup \overline{A}_0 = \mathbb{T}$, we have $\mathcal{A}_{A\cup\overline{A}} = \mathcal{A}$ whence $I^*(A \cup \overline{A}) = \mathbb{T}$. If $\sigma(A) = 0$ then $H_A = \{0\}$, which implies that $F_{I^*A} = \{0\}$ and hence $\tau(I^*A) = 0$. For the first assertion of the lemma, it now suffices to show that

$$I^*\Big(\bigcap_{i=1}^{\infty} A_i\Big) = \bigcap_{i=1}^{\infty} I^*A_i \tag{24}$$

and

$$I^*\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \bigcup_{i=1}^{\infty} I^*A_i \tag{25}$$

for every sequence $(A_i)_{i\ge1}$ of symmetric Borel subsets of $\mathbb{T}$.

By (10), if $A = \bigcap_{i=1}^{\infty} A_i$ then $\mathcal{B}(H_A) = \bigcap_{i=1}^{\infty} \mathcal{B}(H_{A_i})$ and thus $\mathcal{A}_A = \bigcap_{i=1}^{\infty} \mathcal{A}_{A_i}$. The same holds for the factors $\mathcal{A}'_{I^*(A_i)}$ and (24) follows.

Now, let $A = \bigcup_{i=1}^{\infty} A_i$. Clearly, $I^*A \supset \bigcup_{i=1}^{\infty} I^*(A_i)$. On the other hand, as H_A is the smallest Gaussian space which contains all H_{A_i} $(i \ge 1)$, $\bigvee_{i=1}^{\infty} \mathcal{A}_{A_i}$ is a compact factor of $\mathcal{B}(H_A)$. Hence $\mathcal{A}_A \subset \mathcal{B}(H_A)$ cannot contain a factor independent of $\bigvee_{i=1}^{\infty} \mathcal{A}_{A_i}$ and $\mathcal{A}'_{I^*A}$ cannot contain a factor independent of $\bigvee_{i=1}^{\infty} \mathcal{A}'_{I^*A_i}$. Therefore $\bigcup_{i=1}^{\infty} I^*A_i = I^*A$.

For the second assertion, let $\widetilde{\psi}(z) = \psi(\overline{z})$. Then $\widetilde{\psi}^{-1}(A) = \overline{\psi^{-1}(A)} = \psi^{-1}(A)$ for every Borel subset $A \subset A_0$, whence $\widetilde{\psi} = \psi$ τ-a.e. ∎

We identify H with a closed subspace of a space $L^2(\mathbb{T},\sigma,\mathcal{H})$ by the spectral representation and we identify $H^{(2)} = H\odot H$ with a closed subspace of $L^2(\mathbb{T}^2,\sigma\otimes\sigma,\mathcal{H}\otimes\mathcal{H})$, in such a way that

$$U_T g(w,z) = wzg(w,z).$$

Let $A\in\mathcal{B}_{\mathrm{sym}}(\mathbb{T})$. Then H_A corresponds to the functions supported by A and $H_A^{(2)}$ to the subspace of all functions supported by $A\times A$. More precisely, the multiplication by χ_A corresponds to the spectral projector $H\to H_A$ and it commutes with the action of $C^{\mathrm{g}}(T)$ on H. It follows that the multiplication by $\chi_{A\times A} = \chi_A\odot\chi_A$ commutes with the action of $C^{\mathrm{g}}(T)$ on $H^{(2)}$ and thus, since $\mathcal{A} = \mathcal{B}(H)/\mathcal{K}$ with $\mathcal{K}\subset C^{\mathrm{g}}(T)$, it maps $L^2(\mathcal{A})\cap H^{(2)}$ onto $L^2(\mathcal{A})\cap H_A^{(2)} = L^2(\mathcal{A}_A)\cap H^{(2)}$. Similarly, we identify spectrally $L^2(\mathcal{A}')\cap F^{(2)}$ with a closed subspace of $L^2_{\mathrm{sym}}(\mathbb{T}^2,\tau\otimes\tau)$ and then, for a given $B\in\mathcal{B}_{\mathrm{sym}}(\mathbb{T})$, the multiplication by $\chi_{B\times B}$ corresponds to the orthogonal projector on $L^2(\mathcal{A}'_B)\cap F^{(2)}$.

By Proposition 15, the isometry $U_I : L^2(\mathcal{A})\to L^2(\mathcal{A}')$ maps $L^2(\mathcal{A})\cap H^{(2)}$ onto $L^2(\mathcal{A}')\cap F^{(2)}$ and $L^2(\mathcal{A}_A)\cap H^{(2)}$ onto $L^2(\mathcal{A}'_{I^*A})\cap F^{(2)}$. Thus, for $g\in L^2(\mathcal{A})\cap H^{(2)}$, if $B = I^*A$, then

$$U_I(\chi_{A\times A}\cdot g) = \chi_{B\times B}\cdot U_I g. \tag{26}$$

Now, by Corollary 7, $L^2(\mathcal{A})\cap H^{(2)}\neq\{0\}$. Put $\sigma_0 = \sigma|_{A_0}$, so that we have a decomposition $\sigma = \sigma_0 + \widetilde{\sigma}_0$ with $\sigma_0\perp\widetilde{\sigma}_0$ as in the previous section. If we let again $H_0 = \{f\in H : \sigma_f \ll \sigma_0\}$, by Remark 14, we still have $L^2(\mathcal{A})\cap(H_0\otimes\overline{H}_0)\neq\{0\}$. Here we consider $H_0\otimes\overline{H}_0$ as a subspace of $H^{(2)}$, which corresponds to a subspace of functions supported by $(A_0\times\overline{A}_0)\cup(\overline{A}_0\times A_0)$, but it is also naturally identified with the subspace of their restrictions to $A_0\times\overline{A}_0$.

Then $L^2(\mathcal{A})\cap(H_0\otimes\overline{H}_0)$ is represented by a closed non-zero subspace of $L^2(A_0\times\overline{A}_0,\sigma\otimes\sigma,\mathcal{H}\otimes\mathcal{H})$ and the restriction of U_I yields an isometry, still denoted by U_I, from this subspace to a subspace of $L^2_{\mathrm{sym}}(\mathbb{T}^2,\tau\otimes\tau)$. On $L^2(\mathcal{A})\cap(H_0\otimes\overline{H}_0)$, (26) now becomes

$$U_I(\chi_{A\times\bar{A}}\cdot g) = \chi_{B\times B}\cdot U_I g \tag{27}$$

whenever A is a Borel subset of A_0 and $B = I^*(A\cup\overline{A})$.

By Lemma 11, $\chi_{B\times B}(w,z) = \chi_{A\times\bar{A}}(\psi(w),\overline{\psi(z)})$ and (27) extends to

$$U_I(f\cdot g) = f\circ(\psi,\overline{\psi})\cdot U_I g \tag{28}$$

for every $f\in L^\infty(A_0\times\overline{A}_0)$ which is measurable with respect to the σ-algebra generated by the sets of the form $A\times\overline{A}$. But this σ-algebra consists of all sets invariant under the map $(w,z)\mapsto(\overline{z},\overline{w})$. Hence (28) holds for every $f\in L^\infty(A_0\times\overline{A}_0)$ invariant under $(w,z)\mapsto(\overline{z},\overline{w})$. On the other hand, from

$U_I U_T = U_{T_\tau} U_I$, we have

$$U_I(f \cdot g) = f \cdot U_I g \tag{29}$$

when $f(w,z) = wz$, and (29) remains valid for every continuous f which may be written as a function of wz.

Now, take $f(w,z) = \|wz\|$ (where $\|u\|$ denotes the distance from $u \in \mathbb{T}$ to 1 on the complex plane). Then both (29) and (28) hold, so

$$f(w,z)U_I g(w,z) = U_I(f \cdot g)(w,z) = f(\psi(w), \overline{\psi(z)})U_I g(w,z)$$

for $g \in L^2(\mathcal{A}) \cap (H_0 \otimes \overline{H}_0)$. Since we can find a non-zero g belonging to this intersection and then $U_I g \neq 0$, there exists a set of positive $\tau \otimes \tau$-measure on which $f(w,z) = f(\psi(w), \overline{\psi(z)})$, i.e.

$$\|wz\| = \|\psi(w)\overline{\psi(z)}\|.$$

Therefore on a set of positive $\tau \otimes \tau$-measure

$$wz = \psi(w)\overline{\psi(z)}.$$

We can then find z and a set of w of positive τ-measure such that

$$\psi(w) = wz\psi(z)$$

and since by Lemma 11, $\psi_*\tau \ll \sigma$ we see that $\sigma \not\perp \tau * \delta_{z\psi(z)}$.

We have proved the following result.

Theorem 6. *Let T be a GAG of type σ and let T_τ be a standard Gaussian automorphism. Suppose that T and T_τ are not disjoint. Then there exists $z_0 \in \mathbb{T}$ such that $\sigma \not\perp \tau * \delta_{z_0}$.* ∎

Corollary 9. *Let T be a GAG of type σ and S be a generalized Gaussian automorphism of type τ and assume $T \not\perp S$. Then there exists $z_0 \in \mathbb{T}$ such that $\sigma \not\perp \tau * \delta_{z_0}$.*

Proof. All we need to show is that $T_\sigma^\infty \not\perp T_\tau$. By Proposition 17 we can find a factor $\mathcal{A}$ of T_σ^∞ isomorphic to a classical factor of S. But a classical factor of S cannot be disjoint from T_τ. ∎

Using Theorem 6 and the fact that T_σ and T_τ have a common factor whenever σ and a certain translation of τ are not mutually singular, we can now give more precise forms of some results of Section 3.4 (see Proposition 10 and Lemma 7) and of [17] (see Th. 4 and Lemma 2).

Corollary 10. *Let σ and τ be two continuous symmetric measures on $\mathbb{T}$ with $\sigma \perp \tau$.*

(i) *If σ, τ are GAG measures then $\sigma + \tau$ is a GAG measure if and only if $\sigma \perp \tau * \delta_z$ for each $z \in \mathbb{T}$.*

(ii) *If σ and τ have the FS property then $\sigma + \tau$ has the FS property if and only if $\sigma \perp \tau * \delta_z$ for each $z \in \mathbb{T}$.* ∎

Finally, Theorem 6 allows us to give examples of zero-entropy Gaussian automorphisms which are disjoint from any GAG.

EXAMPLE 6. Let ϱ be a continuous symmetric probability Borel measure on $\mathbb{T}$ which is quasi-invariant under a countable subgroup D of the circle, i.e.

$$\varrho * \delta_z \ll \varrho \quad \text{ for each } z \in D.$$

Assume that ϱ is ergodic, where by ergodicity we mean that each Borel subset invariant under all translations by D must be of trivial ϱ-measure (for examples of ergodic ϱ see e.g. [10], IV.8.3, [24], III.3). From the ergodicity of ϱ follows the weaker property that if $0 < \tau \ll \varrho$ then there exists $z \in D \setminus \{1\}$ such that $\tau \not\perp \tau * \delta_z$. Indeed, if not, since D is countable, there exists a Borel set A such that $\tau(\mathbb{T} \setminus A) = 0$ and $\tau(zA) = 0$ for each $z \in D \setminus \{1\}$. Then for each Borel subset $B \subset A$,

$$\tau\Big(\bigcup_{z\in D} zB\Big) = \tau(B).$$

If B has positive τ-measure, then $\varrho(\mathbb{T} \setminus \bigcup_{z\in D} zB) = 0$ and it follows that

$$\tau\Big(\mathbb{T} \setminus \bigcup_{z\in D} zB\Big) = 0.$$

Therefore $\tau(B) = \tau(A)$ and since τ is continuous, we obtain a contradiction.

Suppose $T_\varrho \not\perp T$, where T is a GAG of type σ. Then there exists a non-zero measure $\tau \ll \varrho$ such that for some $z \in \mathbb{T}$, $\tau * \delta_z \ll \sigma$. Since τ is not singular with respect to a translation of itself, the same holds for σ and we obtain a contradiction with Corollary 4.

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