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Nonreflecting stationary subsets of $P_\kappa\lambda$

by

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Abstract. We explore the possibility of forcing nonreflecting stationary sets of $P_\kappa\lambda$. We also present a $P_\kappa\lambda$ generalization of Kanamori's weakly normal filters, which induces stationary reflection.

1. Introduction. Throughout this paper κ denotes a regular uncountable cardinal and λ a cardinal $\geq \kappa$. For any such pair (κ, λ), $P_\kappa\lambda$ denotes the set $\{x \subset \lambda : |x| < \kappa\}$. For $x \in P_\kappa\lambda$, let $\kappa_x = |x \cap \kappa|$, $P_{\kappa_x} x = \{s \subset x : |s| < \kappa_x\}$, and $\widehat{x} = \{y \in P_\kappa\lambda : x \subset y\}$.

We say $X \subset P_\kappa\lambda$ is *unbounded* if $X \cap \widehat{x} \neq \emptyset$ for any $x \in P_\kappa\lambda$. Let $FSF_{\kappa,\lambda}$ be the filter generated by $\{\widehat{x} : x \in P_\kappa\lambda\}$. Every filter on $P_\kappa\lambda$ is assumed to be fine, that is, extending $FSF_{\kappa,\lambda}$. If F is a filter, F^+ denotes the set $\{X \subset P_\kappa\lambda : P_\kappa\lambda - X \notin F\}$.

We say $X \subset P_\kappa\lambda$ is *closed* if $\bigcup_{\alpha<\delta} x_\alpha \in X$ for any $\subset$-increasing chain $\langle x_\alpha \mid \alpha < \delta\rangle$ in X with $\delta < \kappa$; X is a *club* if it is closed and unbounded. We say $S \subset P_\kappa\lambda$ is *stationary* if $S \cap X \neq \emptyset$ for any club X. Let $CF_{\kappa,\lambda}$ denote the club filter on $P_\kappa\lambda$ generated by the club subsets of $P_\kappa\lambda$.

All the notions defined above for $P_\kappa\lambda$ can be naturally translated into $P_{\kappa_x} x$ if κ_x is regular uncountable. For instance, $X \subset P_{\kappa_x} x$ is unbounded if for any $y \in P_{\kappa_x} x$ there is $z \in X$ such that $y \subset z$, and $FSF_{\kappa_x,x}$ denotes the filter on $P_{\kappa_x} x$ generated by $\{\widehat{s} \cap P_{\kappa_x} x : s \in P_{\kappa_x} x\}$ which is a κ_x-complete filter on $P_{\kappa_x} x$.

In the next section for certain large $T \subset P_\kappa\kappa^+$ we force a stationary set $S \subset P_\kappa\kappa^+$ such that $S \cap P_{\kappa_x} x$ is nonstationary for any $x \in T$.

As the counterpart, in the third section, we present a generalization of weakly normal filters on regular cardinals due to Kanamori and show that the existence of such filters gives the reflection of stationary sets of $P_\kappa\lambda$.

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The last section is devoted to the forcing giving different type of nonreflection from that of the earlier sections.

2. Forcing a nonreflecting stationary set. Adding nonreflecting stationary sets to regular uncountable cardinals is an elementary technique and has important applications such as destroying supercompactness [3]. The forcing notion $\mathbb{Q}$ ordered by end extension is as follows:

$$p \in \mathbb{Q} \quad \text{if} \quad p \subset \kappa,\ |p| < \kappa,\ \text{and } p \cap \alpha \text{ is nonstationary for any } \alpha < \kappa.$$

Assuming $\kappa^{<\kappa} = \kappa$, $\mathbb{Q}$ is κ^+-c.c. and $<\kappa$-distributive, hence all cardinals are preserved. We try to provide a generalization to $P_\kappa\lambda$.

Definition 2.1. Let $X \subset P_\kappa\lambda$ be stationary. For $x \in P_\kappa\lambda$ we say X *reflects at* x if $X \cap P_{\kappa_x}x$ is stationary in $P_{\kappa_x}x$; X is *nonreflecting* if it does not reflect at any $x \in P_\kappa\lambda$.

A simple way is the forcing below:

$$\begin{aligned} p \in \mathbb{P}_0 \quad &\text{if} \quad p \subset P_\kappa\kappa^+,\ |p| < \kappa,\ \text{and } p \cap P_{\kappa_x}x \text{ is nonstationary} \\ &\qquad \text{for any } x \in P_\kappa\lambda, \\ p \le_{\mathbb{P}_0} q \quad &\text{if} \quad p \supset q \text{ and there is no pair } x \in p - q \text{ and } y \in q \\ &\qquad \text{such that } x \subset y. \end{aligned}$$

Although $\mathbb{P}_0$ is also κ^+-c.c., $<\kappa$-distributivity is not clear. We take a union at limit stages as in the proof of distributivity of $\mathbb{Q}$ in order to get an extension of the conditions defined at earlier stages. Since $P_\kappa\lambda$ is not linearly ordered, $\bigcup_{\alpha<\beta} p_\alpha \cap P_{\kappa_x}x$ may be stationary for some x even if $p_\alpha \cap P_{\kappa_x}x$ is nonstationary for any $\alpha < \beta$.

We handle this problem at the expense of narrowing the set where the generic stationary set does not reflect. Gitik's idea [6] for shooting a club subset of $P_\kappa\lambda$ is used.

Theorem 2.2. *Let $V \subset W$ be two models of ZFC with the same ordinals, $(\kappa^+)^V = (\kappa^+)^W$, C a club subset of κ of V-inaccessibles, κ an inaccsessible cardinal in W, and $T = \{x \in P_\kappa\kappa^+ : V \models$ "$|x|$ is not inaccessible"$\}$. Then there is a cardinal preserving forcing notion $\mathbb{P} \in W$ such that $\Vdash_{\mathbb{P}}$ "there is a stationary $S \subset P_\kappa\kappa^+$ such that $S \cap P_{\kappa_x}x$ is nonstationary for any $x \in T$".*

Proof. The forcing notion $\mathbb{P}$ is defined in W by

$$\begin{aligned} p \in \mathbb{P} \quad &\text{if} \quad p \subset P_\kappa\kappa^+,\ |p| < \kappa,\ \text{and } p \cap P_{\kappa_x}x \text{ is nonstationary} \\ &\qquad \text{for any } x \in T. \end{aligned}$$

We define

$$\begin{aligned} p \le_{\mathbb{P}} q \quad &\text{if} \quad p \supset q \text{ and there is no pair } x \in p - q \text{ and } y \in q \\ &\qquad \text{such that } x \subset y. \end{aligned}$$

We show $\mathbb{P}$ is κ^+-c.c. and $<\kappa$-distributive, hence all cardinals are preserved.

Assume first $\mathcal{A} \subset \mathbb{P}$ is an antichain of size κ^+ and work for a contradiction. Since $|\bigcup p| < \kappa$ for every $p \in \mathcal{A}$ and κ is inaccessible, we may assume $\{\bigcup p : p \in \mathcal{A}\}$ forms a Δ-system with a root r. Since $|P(r)| < \kappa$, we can find $p, q \in \mathcal{A}$ such that $p \cap P(r) = q \cap P(r)$. Let $s = p \cup q$. Suppose that $t \in s - p$, $u \in p$ and $t \subsetneq u$. Then $t \in q$ and $t \subset u \subset \bigcup p$. Thus $t \subset \bigcup p \cap \bigcup q = r$ contradicting $t \notin p$. So there are no such t and u. Pick $x \in T$. Both $p \cap P_{\kappa_x} x$ and $q \cap P_{\kappa_x} x$ are nonstationary. Hence $s \cap P_{\kappa_x} x = (p \cap P_{\kappa_x} x) \cup (q \cap P_{\kappa_x} x)$ is nonstationary. Now $s \in \mathbb{P}$ with $s \leq p, q$. This contradicts $\mathcal{A}$ being an antichain. So $\mathbb{P}$ is κ^+-c.c.

Second let $\delta < \kappa$, D_α be an open dense subset of $\mathbb{P}$ for each $\alpha < \delta$, and p any condition. We will find $q \in \bigcap_{\alpha<\delta} D_\alpha$ with $q \leq p$.

Choose sufficiently large λ and let

$$\mathcal{B} = \langle H(\lambda), \in, \kappa, \kappa^+, P_\kappa\kappa^+, C, \delta, \mathbb{P}, \Vdash, \langle D_\alpha \mid \alpha < \delta\rangle, p\rangle.$$

We can find $\mathcal{M} \prec \mathcal{B}$ such that $|\mathcal{M}| = \kappa$, $\mathcal{M} \cap \kappa^+ \in \kappa^+$, and ${}^{<\kappa}(\mathcal{M} \cap \kappa^+) \subset \mathcal{M}$. Fix a bijection $g \in V$ from κ to $\mathcal{M} \cap \kappa^+$. We can build an increasing continuous chain $\langle \mathcal{M}_\xi \mid \xi < \kappa\rangle$ of elementary submodels of $\mathcal{M}$ such that $\delta + 1 \subset \mathcal{M}_0$ and for every $\xi < \kappa$, $|\mathcal{M}_\xi| < \kappa$, ${}^\xi\mathcal{M}_\xi \subset \mathcal{M}_{\xi+1}$, and $g''\xi \subset \mathcal{M}_\xi$.

There is a club $E \subset C$ such that for every $\xi \in E$, $g''\xi = \mathcal{M}_\xi \cap \kappa^+$, $|\xi| = \xi$, and $g''\xi \cap \kappa = \xi = \mathcal{M}_\xi \cap \kappa$. Note that $\mathcal{M}_\xi \cap \kappa^+ \in P_\kappa\kappa^+ \cap \mathcal{M}_{\xi+1}$ for every $\xi \in E$. If $q \in \mathbb{P} \cap \mathcal{M}_\xi$ with $\xi \in E$, then $\mathcal{M}_\xi \models$ "$|q| < \kappa$". Hence $q = f''\eta$ for some $\eta < \xi$ and $f \in \mathcal{M}_\xi$. Thus $q \subset \mathcal{M}_\xi$. By the same argument we have $q \subset P(\mathcal{M}_\xi \cap \kappa^+)$. In fact, $x \subsetneq \mathcal{M}_\xi \cap \kappa^+$ for every $x \in q$. Let $\langle \xi_\alpha \mid \alpha < \kappa\rangle$ be an increasing enumeration of E.

With the above remark in mind we inductively define a decreasing sequence of conditions $\langle p_{\xi_\alpha} \mid \alpha \leq \delta\rangle$ such that for every $\alpha < \delta$, $p_\alpha \in \mathcal{M}_{\xi_\alpha+1} \cap D_\beta$ for every $\beta < \alpha$.

Since $D_0 \in \mathcal{M}_{\xi_0}$ because $\delta+1 \subset \mathcal{M}_0 \subset \mathcal{M}_{\xi_0}$, there is $q_0 \in \mathcal{M}_{\xi_0} \cap D_0$ with $q_0 \leq p$. Set $p_0 = q_0 \cup \{\mathcal{M}_{\xi_0} \cap \kappa^+\}$. Since $V \models$ "$|\mathcal{M}_{\xi_0}| = \xi_0$ is inaccessible", p_0 is a condition. By the former remark $p_0 \in \mathcal{M}_{\xi_0+1}$.

Suppose that $\alpha = \beta + 1$ and p_β is defined. Then $p_\beta \in \mathcal{M}_{\xi_\beta+1} \subset \mathcal{M}_{\xi_\alpha}$, and $D_\beta \in \mathcal{M}_{\xi_\alpha}$ since $\beta \in \xi_\alpha = \mathcal{M}_{\xi_\alpha} \cap \kappa$. So we can find $q_\alpha \in \mathcal{M}_{\xi_\alpha} \cap D_\beta$ such that $q_\alpha \leq p_\beta$. Set $p_\alpha = q_\alpha \cup \{\mathcal{M}_{\xi_\alpha} \cap \kappa^+\}$. This is well defined as before.

Next suppose that α is a limit ordinal $\leq \delta$ and p_β is defined for all $\beta < \alpha$. Set $p_\alpha = \bigcup_{\beta<\alpha} p_\beta$.

Since $p_\beta \in \mathcal{M}_{\xi_\beta+1}$ and $\xi_\beta + 1 \leq \xi_{\beta+1} < \xi_\alpha$ for all $\beta < \alpha$, $\{p_\beta : \beta < \alpha\} \in {}^\alpha\mathcal{M}_{\xi_\alpha} \subset \mathcal{M}_{\xi_\alpha+1}$. Hence $p_\alpha \in \mathcal{M}_{\xi_\alpha+1}$. Since $\mathcal{M}_{\xi_\beta+1} \cap \kappa^+$ is the greatest element of $p_{\beta+1}$, we have $\bigcup p_\alpha = \bigcup_{\beta<\alpha}(\mathcal{M}_{\xi_{\beta+1}} \cap \kappa^+) = \mathcal{M}_{\xi_\alpha} \cap \kappa^+$.

Suppose that $x \in T$ and $p_\alpha \cap P_{\kappa_x} x$ is stationary. Clearly $x \subset \bigcup p_\alpha$, hence $V \models$ "$|x| \leq |\mathcal{M}_{\xi_\alpha} \cap \kappa^+|$". Since $|\mathcal{M}_{\xi_\alpha} \cap \kappa^+| = \xi_\alpha$ is inaccessible in V, it follows that $(|x|)^V < \xi_\alpha$. Now $x \subset \mathcal{M}_\alpha \cap \kappa^+ = g''\xi_\alpha = \bigcup_{\zeta<\xi_\alpha} g''\zeta$. Since $V \models$ "ξ_α is regular", there is $\zeta < \xi_\alpha$ such that $x \subset g''\zeta$. Then $x \subsetneq g''\xi_\gamma = \mathcal{M}_{\xi_\gamma} \cap \kappa^+ \in p_\gamma$ for some $\gamma < \alpha$.

Let $y \in p_\alpha \cap P_{\kappa_x} x$. We know $y \subset x \subsetneq \mathcal{M}_{\xi_\gamma} \cap \kappa^+ \in p_\gamma$ and $y \in p_\mu$ for some $\mu \in (\gamma, \alpha)$. Since $p_\mu \le p_\gamma$, we have $y \in p_\gamma$. Thus $p_\alpha \cap P_{\kappa_x} x = p_\gamma \cap P_{\kappa_x} x$, which is nonstationary as $p_\gamma \in \mathbb{P}$. Hence p_α is a condition and belongs to $\bigcap_{\beta<\alpha} D_\beta$.

Thus we can define p_δ as a desired condition showing $\mathbb{P}$ is $<\kappa$-distributive.

Let G be W-generic for $\mathbb{P}$ and $S = \bigcup G$. By an easy density argument S is unbounded in $P_\kappa \kappa^+$. We show S is stationary in $P_\kappa \kappa^+$ in $W[G]$ and does not reflect at x if $x \in T$. Note that $P_\kappa \kappa^+ \cap W = P_\kappa \kappa^+ \cap W[G]$.

Let $D \subset P_\kappa \kappa^+$ be a club in $W[G]$ and $\dot{D}$ its name. Assume $p \Vdash$ "$\dot{D}$ is a club of $P_\kappa \kappa^+$".

Take a sufficiently large λ and choose $\mathcal{N} \prec \langle H(\lambda), \in, \kappa, \kappa^+, P_\kappa \kappa^+, C, \mathbb{P}, \Vdash, \dot{D}, p\rangle$ such that $|\mathcal{N}| = \kappa$, $\mathcal{N} \cap \kappa^+ \in \kappa^+$, and ${}^{<\kappa}(\mathcal{N} \cap \kappa^+) \subset \mathcal{N}$. Fix a bijection $h \in V$ from κ to $\mathcal{N} \cap \kappa^+$ and an increasing continuous chain $\langle \mathcal{N}_{\nu_\alpha} \mid \alpha < \kappa\rangle$ of elementary submodels of $\mathcal{N}$ such that for every $\alpha < \kappa$, $|\mathcal{N}_{\nu_\alpha}| < \kappa$, ${}^{\nu_\alpha}\mathcal{N}_{\nu_\alpha} \subset \mathcal{N}_{\nu_\alpha+1}$, $h''\nu_\alpha \cap \kappa = \nu_\alpha \in C$ is a cardinal, and $h''\nu_\alpha = \mathcal{N}_{\nu_\alpha} \cap \kappa^+$. (Hence $h''\nu_\alpha = \mathcal{N}_{\nu_\alpha} \cap \kappa$ and $\langle \nu_\alpha \mid \alpha < \kappa\rangle$ is increasing continuous.)

We inductively define a descending sequence of conditions $\langle p_n \mid n \in \omega\rangle$ as follows.

By elementarity there are $p' \in \mathcal{N}_{\nu_0}$ and $x_0 \in \mathcal{N}_{\nu_0}$ such that $p' \le p$ and $p' \Vdash$ "$x_0 \in \dot{D}$". Let $p_0 = p' \cup (\mathcal{N}_{\nu_0} \cap \kappa^+)$. As before $p_0 \in \mathbb{P} \cap \mathcal{N}_{\nu_0+1} \subset \mathcal{N}_{\nu_1}$.

Suppose $p_n \in \mathcal{N}_{\nu_n+1}$ is defined. We know $\bigcup p_n = \mathcal{N}_{\nu_n} \cap \kappa^+ \in \mathcal{N}_{\nu_n+1}$ and $p_n \Vdash$ "$\dot{D}$ is a club". Since $\mathcal{N}_{\nu_n} \cap \kappa^+ \in P_\kappa \kappa^+ \cap \mathcal{N}_{\nu_n+1}$, there are $p'_n \in \mathcal{N}_{\nu_n+1}$ and $x_{n+1} \in \mathcal{N}_{\nu_n+1}$ such that $p'_n \le p_n$, $\mathcal{N}_{\nu_n} \cap \kappa^+ \subset x_{n+1}$ and $p'_n \Vdash$ "$x_{n+1} \in \dot{D}$". Then $x_{n+1} \subset \mathcal{N}_{\nu_n+1} \cap \kappa^+ \subset \mathcal{N}_{\nu_{n+1}}$. Set $p_{n+1} = p'_n \cup \{\mathcal{N}_{\nu_{n+1}} \cap \kappa^+\}$.

Let $q = (\bigcup_{n\in\omega} p_n) \cup \{\mathcal{N}_{\nu_\omega} \cap \kappa^+\}$. As before $q \in \mathbb{P}$ and extends all p_n. Since $\mathcal{N}_{\nu_n} \cap \kappa^+ \subset x_{n+1} \subset \mathcal{N}_{\nu_n+1} \cap \kappa^+ \subset \mathcal{N}_{\nu_{n+1}} \cap \kappa^+$, we have $\mathcal{N}_{\nu_\omega} \cap \kappa^+ = \bigcup_{n\in\omega} x_n$. For every $n \in \omega$, $p_n \Vdash$ "$x_n \in \dot{D}$". Hence $q \Vdash$ "$x_n \in \dot{D}$ for every $n \in \omega$ and $\dot{D}$ is closed". Hence $q \le p$ and $q \Vdash$ "$\mathcal{N}_{\nu_\omega} \cap \kappa^+ \in \dot{D} \cap \bigcup G$". We have shown that S is stationary.

Suppose that S reflects at some $x \in T$. By our definition of the ordering some condition $p \in G$ must reflect at x, contrary to $p \in \mathbb{P}$. ∎

Proposition 2.3. *Let V and W be as in* 2.2. *Then*:

(1) *No κ^+-supercompact embedding $j : W \to M$ lifts to $k : W[G] \to M[k(G)]$.*

(2) *Suppose κ is supercompact in both V and W, and W is a generic extension of V by Radin forcing. Then κ is not κ^+-Shelah in $W[G]$.*

Proof. (1) We know κ^+ and $<\kappa$-sequences are the same in all of W, M, $W[G]$, and $M[k(G)]$. Suppose j lifts to k. Then $k(S)$ is a nonreflecting stationary subset of $P_{j(\kappa)}j(\kappa^+)$ in $M[k(G)]$, $j''\kappa^+ \in P_{j(\kappa)}j(\kappa^+)$, and $j''\kappa^+ \cap j(\kappa) = \kappa$. So $k(S) \cap P_\kappa j''\kappa^+ = j''S$ is nonstationary in $P_\kappa j''\kappa^+$. This contradicts the stationarity of S in $W[G]$.

(2) Let V be a model of ZFC + GCH + κ is supercompact and W a generic extension by a Radin forcing using a measure sequence long enough for κ to remain supercompact. The conditions in 2.2 are satisfied and V and W have the same cardinals. We know T belongs to the κ^+-Shelah filter. This is the same in $W[G]$ if κ remains κ^+-Shelah. However every stationary subset of $P_\kappa\kappa^+$ reflects on a set in the κ^+-Shelah filter. ∎

3. A generalization of Kanamori's weak normality. Reflection of stationary subsets of $P_\kappa\lambda$ is a large cardinal property. It is easily derived from supercompactness of κ. In fact κ being λ-Shelah, a weakening of supercompactness from the point of view of combinatorial property is sufficient [4], [11].

Solovay's theorem says that for λ regular the sup-function is one-to-one on a set in a supercompact ultrafilter on $P_\kappa\lambda$. We have the same result for not only the filter canonically defined by the λ-Shelah property [8], [2], but also for strongly normal λ-saturated filters on $P_\kappa\lambda$ (see [1]).

The motivation of this section is what strength of saturation of ideals on $P_\kappa\lambda$ provides stationary reflection.

Kanamori [9] defined weakly normal filters for regular uncountable cardinals as follows:

Definition 3.1. Let F be a filter on κ. We say F is *weakly normal* if any regressive function on κ is bounded on a set in F.

On the other hand strongly normal filters on $P_\kappa\lambda$ were investigated [5] as a weakening of supercompact ultrafilters.

Definition 3.2. A function $f : P_\kappa\lambda \to P_\kappa\lambda$ is called *set regressive* if $f(x) \in P_{\kappa_x}x$ for every $x \in P_\kappa\lambda$.

A filter F on $P_\kappa\lambda$ is *strongly normal* if any set regressive function defined on $X \in F^+$ is constant on some $Y \in P(X) \cap F^+$. This is equivalent to the following: for any $\{X_s : s \in P_\kappa\lambda\} \subset F$, $\varDelta_\prec X_s := \{x : x \in X_s \text{ for every } s \in P_{\kappa_x}x\} \in F$. Clearly every strongly normal filter is normal.

Kanamori's idea gives us the following notion of WN-filter.

Definition 3.3. Let F be a filter on $P_\kappa\lambda$. We write $WN(F)$ and call F a *WN-filter* (1) if for any set regressive function f on $P_\kappa\lambda$ there is $a \in P_\kappa\lambda$ such that $\{x \in P_\kappa\lambda : f(x) \subset a\} \in F$.

(1) The basic properties of κ-complete *WN*-filters are studied by S. Kawano in his master thesis at University of Osaka Prefecture, March 1999, where they are called strongly weakly normal filters.

REMARK 3.4. Note that we do not assume any completeness for WN-filters.

It turns out that the existence of a WN-filter provides the stationary reflection of sets in the range corresponding to its completeness. Kanamori's weak normality provides the reflection of stationary subsets of a weakly inaccessible cardinal. So WN-filter may be a natural generalization of Kanamori's notion.

The following is straightforward.

LEMMA 3.5. (1) *Every filter extension of a WN-filter is also WN.*

(2) *If F is a WN-filter on $P_\kappa\lambda$, then $F\restriction\delta$ is a WN-filter on $P_\kappa\delta$ for any $\delta \in [\kappa, \lambda)$. ($F\restriction\delta := \{X \subset P_\kappa\delta : \{x \in P_\kappa\lambda : x \cap \delta \in X\} \in F\}$.)*

(3) *If there is a WN-filter on $P_\kappa\lambda$ for some $\lambda \geq \kappa$, then κ is weakly inaccessible.*

LEMMA 3.6. *Let* $\mathrm{Reg} = \{x \in P_\kappa\lambda : x \cap \kappa$ *is regular*$\}$. *Then every WN-filter on $P_\kappa\lambda$ extends $CF_{\kappa,\lambda}\restriction\mathrm{Reg}$.*

Proof. Suppose $X = \{x \in P_\kappa\lambda : x \cap \kappa \notin \kappa\} \in F^+$. For each $x \in X$ let $\alpha_x \in x \cap \kappa$ with $\alpha_x \not\subset x$. There are $a \in P_\kappa\lambda$ and $X' \in P(X) \cap F^+$ so that $\alpha_x \in a$ for every $x \in X'$. Then $\bigcup(a \cap \kappa) < \kappa$ whereas $\bigcup(a \cap \kappa) \not\subset x$ for any $x \in X'$. Contradiction. Hence $\{x : x \cap \kappa \in \kappa\} \in F$.

Let $f : \lambda \times \lambda \to P_\kappa\lambda$ and assume $Y = \{x : f''(x \times x) \not\subset P(x)\} \in F^+$. For $x \in Y$ define $g(x) \in x \times x$ such that $f(g(x)) \not\subset x$. For some $b \in P_\kappa\lambda$, $Y' = \{x \in Y : g(x) \subset b\} \in F^+$. Then $f''(b \times b) \not\subset x$ for any $x \in Y'$. This is absurd since $f''(b \times b) \in P_\kappa\lambda$. Hence $CF_{\kappa,\lambda} \subset F$.

Suppose $Z = \{x : x \cap \kappa \text{ is singular}\} \in F^+$. Since κ is weakly inaccessible and Z is stationary, we may assume $x \cap \kappa$ is a cardinal for all $x \in Z$. Let $c_x \subset x$ be cofinal with order type $< x \cap \kappa$. We have $c \in P_\kappa\lambda$ and $Z' \in F^+$ such that $c_x \subset c$ for all $x \in Z'$. Then $x \cap \kappa \subset \bigcup(c \cap \kappa) < \kappa$ for all $x \in Z'$. Contradiction. ∎

COROLLARY 3.7. *If there is a WN-filter, then κ is weakly Mahlo.*

Let $WCF_{\kappa,\lambda}$ denote the minimal strongly normal filter on $P_\kappa\lambda$. It is known that $WCF_{\kappa,\lambda}$ is proper if and only if κ is Mahlo or $\kappa = \nu^+$ with $\nu^{<\nu} = \nu$. In addition, $X \in WCF_{\kappa,\lambda}$ if and only if there is a function $f : P_\kappa\lambda \to P_\kappa\lambda$ such that $\{x \in P_\kappa\lambda : f''P_{\kappa_x}x \subset P(x)\} \subset X$. If κ is Mahlo, then $\{x \in P_\kappa\lambda : x \cap \kappa \text{ is inaccessible}\} \in WCF_{\kappa,\lambda}$.

We observe a relationship between strong normality and the WN property.

PROPOSITION 3.8. (1) *If $WCF_{\kappa,\lambda}$ is proper, then every WN-filter extends $WCF_{\kappa,\lambda}$.*

(2) *Every strongly normal κ-saturated filter on $P_\kappa\lambda$ is WN.*

(3) *If a WN-filter F is κ-complete, then F is normal κ-saturated. Hence neither $WCF_{\kappa,\lambda}$ nor any restriction of $CF_{\kappa,\lambda}$ is WN.*

Proof. (1) Suppose $X \in F^+$ and $f(x) \in P_{\kappa_x}x$ for all $x \in X$. For some $a \in P_\kappa\lambda$, $Y = \{x \in X : x \cap \kappa \in \kappa$ and $f(x) \subset a\}$ is stationary. For every $x \in Y$, $f(x) \subset a$ and $|P(a)| < \kappa$. Hence f is constant on some unbounded subset of Y. Thus $X \in WCF^+_{\kappa,\lambda}$.

(2) Let $g(x) \in P_{\kappa_x}x$ for all $x \in P_\kappa\lambda$ and $\mathcal{A} = \{g^{-1}(\{y\}) : y \in P_\kappa\lambda\} \cap F^+$. Since F is κ-saturated, we have $|\mathcal{A}| < \kappa$. Set $b = \bigcup\{y : g^{-1}(\{y\}) \in \mathcal{A}\}$. Then $b \in P_\kappa\lambda$, $\bigcup\mathcal{A} \in F$, and $g(x) \subset b$ for every $x \in \bigcup\mathcal{A}$.

(3) Assume h is regressive on $Z \in F^+$. There is $c \in P_\kappa\lambda$ such that $Z' = \{x \in Z : h(x) \in c\} \in F^+$. By κ-completeness, h is constant on some set in F^+. Hence F is normal.

Next assume that $\{W_\xi : \xi < \kappa\}$ is a disjoint partition of $P_\kappa\lambda$ into F-positive sets. Let $W'_\xi = W_\xi \cap \widehat{\{\xi\}}$ and $k(x) = \xi$ if $x \in W'_\xi$. Then we have $d \in P_\kappa\lambda$ such that $W = \{x \in P_\kappa\lambda : k(x) \subset d\} \in F$. Choose any $\xi \notin d$. Then $W'_\xi \cap W = \emptyset$. Contradiction. ∎

Note that every normal κ-saturated filter is strongly normal if κ is inaccessible.

COROLLARY 3.9. *Let κ be Mahlo and F a filter on $P_\kappa\lambda$. Then F is κ-complete and WN if and only if F is normal κ-saturated.*

A standard forcing argument shows that a noninaccessible cardinal can carry WN-filters.

THEOREM 3.10. *Suppose F is a WN-filter on $P_\kappa\lambda$ and $\mathbb{P}$ is μ-c.c. with $\mu < \kappa$ in V. Then F generates a WN-filter in $V^{\mathbb{P}}$. Hence it is consistent that κ is not inaccessible and there exists a WN-filter on $P_\kappa\lambda$. In fact $P_\kappa\lambda$ can carry a normal, non-strongly normal, κ-saturated filter.*

Now we observe the stationary reflection under the existence of a WN-filter with some degree of completeness.

DEFINITION 3.11. For $\omega < \mu \le \kappa$, $X \subset P_\kappa\lambda$ is said to be a *$<\mu$-club* if X is unbounded and closed under $\subset$-increasing chains of length $< \mu$. Let $CF^\mu_{\kappa,\lambda}$ denote the filter generated by the $<\mu$-clubs. So, $CF^\kappa_{\kappa,\lambda} = CF_{\kappa,\lambda}$.

We say $S \subset P_\kappa\lambda$ is *$CF^\mu_{\kappa,\lambda}$-stationary* if $S \in (CF^\mu_{\kappa,\lambda})^+$, that is, $S \cap X \ne \emptyset$ for any $X \in CF^\mu_{\kappa,\lambda}$.

PROPOSITION 3.12. (1) *For μ a cardinal, $CF^\mu_{\kappa,\lambda}$ is a κ-complete normal filter on $P_\kappa\lambda$.*

(2) *$CF^\mu_{\kappa,\lambda} \supseteq CF^{\mu'}_{\kappa,\lambda}$ for $\mu < \mu'$.*

(3) *If μ is singular, then $CF^\mu_{\kappa,\lambda} = CF^{\mu^+}_{\kappa,\lambda}$.*

(4) *For μ a limit cardinal, $CF^\mu_{\kappa,\lambda} = \bigcap_{\delta<\mu} CF^\delta_{\kappa,\lambda}$.*

THEOREM 3.13. *Let $\omega \le \mu < \kappa$ and F be a μ^+-complete WN-filter on $P_\kappa\lambda$. If S is $CF^{\mu^+}_{\kappa,\lambda}$-stationary, then $\{x \in P_\kappa\lambda : S \cap P_{\kappa_x}x$ is stationary in $P_{\kappa_x}x\} \in F$.*

Proof. Suppose otherwise that S is $CF^{\mu^+}_{\kappa,\lambda}$-stationary and $X = \{x \in P_\kappa\lambda : S \cap P_{\kappa_x}x$ is nonstationary in $P_{\kappa_x}x\} \in F^+$. We assume that for every $x \in X$, $x \cap \kappa$ is a regular cardinal $> \mu$, and $C_x \subset P_{\kappa_x}x$ is a club with $C_x \cap S = \emptyset$. Let $F' = F{\restriction}X := \{Y \subset P_\kappa\lambda : Y \cup (P_\kappa\lambda - X) \in F\}$.

We show that $C = \{y \in P_\kappa\lambda : \{x \in X : y \in C_x\} \in F'\}$ is a $<\mu^+$-club.

Pick any $z \in P_\kappa\lambda$. Note that $\{x \in X : z \in P_{\kappa_x}x\} \in F'$. Let $f_0(x) \in C_x$ so that $z \subset f_0(x)$ if exists, $f_0(x) = \emptyset$ otherwise. We have $a_0 \in P_\kappa\lambda$ and $X_0 \in F'$ such that $z \subset f_0(x) \subset a_0$ for every $x \in X_0$.

Suppose that $a_n \in P_\kappa\lambda$, $X_n \in F'$, and f_n are defined such that $f_n(x) \subset a_n$ for every $x \in X_n$. Since $\{x \in X : a_n \in P_{\kappa_x}x\} \in F'$, we can define a set regressive f_{n+1} so that $\{x : a_n \subset f_{n+1}(x) \in C_x\} \in F'$, and find $a_{n+1} \in P_\kappa\lambda$ and $X_{n+1} \in F'$ such that $a_n \subset f_{n+1}(x) \subset a_{n+1}$ for any $x \in X_{n+1}$.

Set $y = \bigcup_{n\in\omega} a_n$ and $Y = \bigcap_{n\in\omega} X_n$. Then $a \in P_\kappa\lambda$ and $Y \in F'$. For every $x \in Y$ we have

$$z \subset f_0(x) \subset a_0 \subset \ldots \subset a_n \subset f_{n+1}(x) \subset a_{n+1} \subset \ldots \subset y$$

and $f_n(x) \in C_x$ for every $n \in \omega$. Since C_x is closed and $x \cap \kappa$ is regular $> \mu \ge \omega$, $y = \bigcup_{n\in\omega} f_n(x) \in C_x$. Hence $z \subset y \in C$, which is unbounded.

To show C is closed let $\langle y_\alpha \mid \alpha < \mu\rangle$ be an increasing chain in C and $w = \bigcup_{\alpha<\mu} y_\alpha$. Clearly $w \in P_\kappa\lambda$. Since $\{x \in X : y_\alpha \in C_x\} \in F'$ for every $\alpha < \mu$, there is $Z \in F'$ such that $y_\alpha \in C_x$ for every $x \in Z$ and $\alpha < \mu$. Since C_x is a club of $P_{\kappa_x}x$ and $x \cap \kappa$ is a regular cardinal $> \mu$, it follows that $w \in C_x$ for every $x \in Z$. Hence $w \in C$ and C is $<\mu^+$-closed.

Now we know $C \subset P_\kappa\lambda$ is a $<\mu^+$-club and $S\cap C \ne \emptyset$. Let $y \in S\cap C$. Then $\{x \in X : y \in C_x\} \in F'$, hence $\{x \in X : S \cap C_x \ne \emptyset\} \ne \emptyset$. Contradiction. ∎

COROLLARY 3.14. *If there is a strongly normal κ-saturated filter F on $P_\kappa\lambda$, then every stationary subset of $P_\kappa\lambda$ reflects on a set in F.*

Relating to 2.2 and the remark at the beginning of this section we ask the following:

QUESTION. *Is the existence of a normal κ^+-saturated filter on $P_\kappa\kappa^+$ sufficient for reflection of stationary subsets of $P_\kappa\kappa^+$?*

4. Another type of reflection. It is well known that every stationary subset of a weakly compact cardinal reflects. We mentioned a $P_\kappa\lambda$ analogue in the last section and present another one here. Let CF_λ denote the filter generated by the closed unbounded subsets of λ.

Proposition 4.1. *Let λ be weakly compact and $X \subset P_\kappa\lambda$. Then $X \in CF_{\kappa,\lambda}$ if and only if $\{\alpha < \lambda : X \cap P_\kappa\alpha \in CF_{\kappa,\alpha}\} \in CF_\lambda$.*

Proof. Since the forward direction is true whenever $\mathrm{cf}(\lambda) \geq \kappa$, we only have to show the converse.

Let $E \subset \{\alpha < \lambda : X \cap P_\kappa\alpha \in CF_{\kappa,\alpha}\}$ be a club. For $\alpha \in E$ there is a function $f_\alpha : \alpha \times \alpha \to \alpha$ such that $C_{f_\alpha} \cap \{x \in P_\kappa\alpha : x \cap \kappa \in \kappa\} \subset X \cap P_\kappa\alpha$ ($C_{f_\alpha} := \{x \in P_\kappa\alpha : f''_\alpha(x \times x) \subset x\}$). Since κ is weakly compact, there is $f : \lambda \times \lambda \to \lambda$ such that for every $\alpha \in E$ there is $\beta \in E$ so that $\alpha < \beta$ and $f{\restriction}(\alpha \times \alpha) = f_\beta{\restriction}(\alpha \times \alpha)$. Suppose that $x \in C_f$ and $x \cap \kappa \in \kappa$. There are α and β as above with $x \in P_\kappa\alpha$. Then $x \in C_{f_\beta} \subset X$. Hence $C_f \cap \{x \in P_\kappa\lambda : x \cap \kappa \in \kappa\} \subset X \in CF_{\kappa,\lambda}$. ∎

Corollary 4.2. *If λ is weakly compact, then every stationary subset of $P_\kappa\lambda$ reflects at $P_\kappa\alpha$ for stationary many $\alpha < \lambda$.*

Remark 4.3. There is an easy limitation as follows. Let

$$c(\kappa,\lambda) = \min\{|C| : C \subset P_\kappa\lambda \text{ is a club}\}.$$

If the conclusion of the corollary holds, then $c(\kappa,\lambda) = \lambda \cdot \sup_{\alpha<\lambda} c(\kappa,\alpha)$. So it fails in L for λ singular with countable cofinality.

We use the following poset in order to force a stationary subset S of $P_\kappa\lambda$ such that $S \cap P_\kappa\alpha$ is nonstationary for any $\alpha \in (\kappa,\lambda)$:

$$\begin{aligned} \langle X,\alpha\rangle \in \mathbb{R} \quad &\text{if} \quad \alpha < \lambda,\ X \subset P_\kappa\alpha,\ \text{and } X \cap P_\kappa\beta \text{ is nonstationary} \\ &\qquad \text{for any } \beta \in (\kappa,\alpha]. \\ \langle X,\alpha\rangle \leq \langle Y,\beta\rangle \quad &\text{if} \quad \beta \leq \alpha \text{ and } Y = X \cap P_\kappa\beta. \end{aligned}$$

Lemma 4.4. $\mathbb{R}$ *is* $<\mathrm{cf}(\lambda)$*-distributive.*

Proof. Assume $\delta < \mathrm{cf}(\lambda)$ and $D_\xi \subset \mathbb{R}$ is open dense for $\xi < \delta$. Let $\langle X,\alpha\rangle \in \mathbb{R}$. We inductively define a descending sequence $\langle\langle X_\xi,\alpha_\xi\rangle \mid \xi \leq \delta\rangle$ such that $\langle \alpha_\xi \mid \alpha < \delta\rangle$ is strictly increasing, $\alpha_{\xi+1}$ is a successor ordinal and $\langle X_{\xi+1},\alpha_{\xi+1}\rangle \in D_\xi$ for every $\xi < \delta$.

Let $\langle X_0,\alpha_0\rangle \leq \langle X,\alpha\rangle$ be arbitrary.

Suppose that $\langle X_\xi,\alpha_\xi\rangle$ is defined. There is $\langle X',\beta\rangle \in D_\xi$ such that $\langle X',\beta\rangle \leq \langle X_\xi,\alpha_\xi+2\rangle \leq \langle X_\xi,\alpha_\xi\rangle$. Set $X_{\xi+1} = X'$ and $\alpha_{\xi+1} = \beta+1$. Then $\langle X_{\xi+1},\alpha_{\xi+1}\rangle$ is an element of D_ξ stronger than $\langle X_\xi,\alpha_\xi\rangle$.

Let ζ be a limit ordinal and $\langle X_\xi,\alpha_\xi\rangle$ be defined for every $\xi < \zeta$. Set $X_\zeta = \bigcup_{\xi<\zeta} X_\xi$ and $\alpha_\zeta = \sup_{\xi<\zeta}\alpha_\xi$. Clearly $X_\zeta \subset P_\kappa\alpha_\zeta$ and $\alpha_\zeta < \lambda$. If $\langle X_\zeta,\alpha_\zeta\rangle \in \mathbb{R}$, the induction step can be continued. Otherwise we stop it. If we can define $\langle X_\delta,\alpha_\delta\rangle$, it is an extension of $\langle X,\alpha\rangle$ lying in $\bigcup_{\xi<\delta} D_\xi$.

Suppose our induction stops at some stage, say ζ. Then ζ must be a limit ordinal and $X_\zeta \cap P_\kappa\alpha$ is stationary for some $\alpha \leq \alpha_\zeta$.

If $\alpha < \alpha_\zeta$ and $\alpha < \alpha_\xi$ for some $\xi < \zeta$, then $X_\xi = X_\zeta \cap P_\kappa\alpha_\xi$ and $X_\zeta \cap P_\kappa\alpha = X_\xi \cap P_\kappa\alpha$ is stationary, contrary to $\langle X_\xi, \alpha_\xi\rangle \in \mathbb{R}$. Thus X_ζ must be stationary in $P_\kappa\alpha_\zeta$.

Assume first that $\mathrm{cf}(\alpha_\zeta) < \kappa$. For every $x \in X_\zeta$ there is $\xi < \zeta$ such that $x \in P_\kappa\alpha_\xi$ hence $\bigcup x \le \alpha_\xi < \alpha_\zeta$. Thus X_ζ is not unbounded in $P_\kappa\alpha_\zeta$. Contradiction.

Let $\mathrm{cf}(\alpha_\zeta) \ge \kappa$. Set $\beta_{\xi+1} = \alpha_\xi + 1$ for $\xi < \zeta$ and $\beta_\xi = \alpha_\xi$ if ξ is a limit ordinal. Then $C = \{\beta_\xi : \xi < \zeta\}$ is a club of α_ζ. We show $\{\bigcup x : x \in X_\zeta\} \cap C = \emptyset$, contradicting the stationarity of X_ζ. Let $x \in X_\zeta$ and η be the least ordinal such that $x \in X_\eta$. By our construction η is a successor ordinal, say, $\xi + 1$. At stage $\xi + 1$, $\alpha_{\xi+1} = \beta + 1$ for some β and $\langle X_{\xi+1}, \alpha_{\xi+1}\rangle \le \langle X_{\xi+1}, \beta\rangle \le \langle X_\xi, \alpha_\xi + 2\rangle$. Hence $x \in X_{\xi+1} = X_{\xi+1} \cap P_\kappa\beta$. Thus $x \subset \beta$ and $x \not\subset \alpha_\xi + 2$. Hence $\beta_{\xi+1} = \alpha_\xi + 1 < \bigcup x \le \alpha_{\xi+1} < \alpha_{\xi+1} + 1 = \beta_{\xi+2}$. Now we are done. ∎

Theorem 4.5. *Let $\kappa < \lambda$ be regular cardinals in V, G $\mathbb{R}$-generic over V, and $S = \bigcup\{X : \langle X, \alpha\rangle \in G$ for some $\alpha\}$. Then $V[G] \models$ "S is stationary in $P_\kappa\lambda$ and $S \cap P_\kappa\alpha$ is nonstationary for any $\alpha \in (\kappa, \lambda)$".*

Proof. By the above lemma κ, λ are regular in $V[G]$ and $P_\kappa\lambda \cap V = P_\kappa\lambda \cap V[G]$.

To show S is stationary let $\langle X, \alpha\rangle \in G$ so that $\langle X, \alpha\rangle \Vdash$ "$\dot{C} \subset P_\kappa\lambda$ is a club". Pick any $\langle X_0, \alpha_0\rangle \le \langle X, \alpha\rangle$. There are $\langle X_1, \alpha_1\rangle \le \langle X_0, \alpha_0\rangle$ and $x_1 \in P_\kappa\lambda$ such that $\langle X_1, \alpha_1\rangle \Vdash$ "$x_1 \in \dot{C}$". Choose $y_1 \in P_\kappa\lambda$ such that $x_1 \subset y_1$ and $\sup(y_1) \ge \alpha_1 + \kappa$. Set $X_2 = X_1 \cup \{y_1\}$ and $\alpha_2 = \bigcup y_1$. Then $\langle X_2, \alpha_2\rangle$ is a stronger condition than $\langle X_1, \alpha_1\rangle$.

We inductively define a decreasing sequence of conditions $\langle\langle X_n, \alpha_n\rangle \mid n \in \omega\rangle$ and increasing sequences $\langle x_n \mid n \in \omega\rangle$, $\langle y_n \mid n \in \omega\rangle$. Suppose that $\langle X_n, \alpha_n\rangle$, x_n, and y_n are defined so that $x_n \subset y_n$ and $\langle X_n, \alpha_n\rangle \Vdash$ "$x_n \in \dot{C}$". Since $\langle X_n, \alpha_n\rangle \le \langle X, \alpha\rangle$, $\langle X_n, \alpha_n\rangle \Vdash$ "$(\exists x \in \dot{C})(y_n \subset x)$". Hence there are $\langle X'_n, \alpha'_n\rangle \le \langle X_n, \alpha_n\rangle$ and $x_{n+1} \in P_\kappa\lambda$ such that $\langle X'_n, \alpha'_n\rangle \Vdash$ "$y_n \subset x_{n+1} \in \dot{C}$". Pick $y_{n+1} \in P_\kappa\lambda$ such that $x_{n+1} \subset y_{n+1}$ and $\bigcup y_{n+1} \ge \alpha'_n + \kappa \ge \alpha_n + \kappa$. Set $X_{n+1} = X'_n \cup \{y_{n+1}\}$ and $\alpha_{n+1} = \bigcup y_{n+1}$.

It is clear that $X_{n+1} \subset P_\kappa\alpha_{n+1}$, $\alpha_{n+1} < \lambda$, and $X_{n+1} \cap P_\kappa\alpha'_n = X'_n$. For every $\gamma < \alpha_{n+1}$, $X_{n+1} \cap P_\kappa\gamma = X'_n \cap P_\kappa\gamma$ is nonstationary in $P_\kappa\gamma$. Since $\alpha'_n + \kappa \le \alpha_{n+1}$, there is $\eta \in (\alpha'_n, \alpha_{n+1})$ with $\eta \notin y_{n+1}$. Since $X'_n \subset P_\kappa\alpha'_n$, we have $\eta \notin x$ for any $x \in X_{n+1}$. Hence X_{n+1} is nonstationary in $P_\kappa\alpha_{n+1}$ and $\langle X_{n+1}, \alpha_{n+1}\rangle$ is well defined.

Let $x = \bigcup_{n\in\omega} x_n$, $\beta = \sup_{n\in\omega} \alpha_n$, and $Y = \bigcup_{n\in\omega} X_n \cup \{x\}$.

Clearly $x = \bigcup_{n\in\omega} y_n$, $\beta = \bigcup x < \lambda$, $Y \subset P_\kappa\beta$, and $Y \cap P_\kappa\alpha_n = X_n$ for every $n \in \omega$.

Let $\gamma < \beta$ and n be the least integer such that $\gamma \le \alpha_n$. Then $Y \cap P_\kappa\gamma = (Y \cap P_\kappa\alpha_n) \cap P_\kappa\gamma = X_n \cap P_\kappa\gamma$ is nonstationary. If $y \in Y - \{x\}$, then $y \in X_n$

for some $n \in \omega$ hence $\bigcup y \le \alpha_n < \beta$. Since $\kappa \le \beta$, $\mathrm{cf}(\beta) = \omega < \kappa$, and $x \in P_\kappa\beta$, there is $z \in P_\kappa\beta$ such that $\bigcup z = \beta$ and $z \cap x = \emptyset$. Thus z is included in no element in Y. Hence Y is nonstationary in $P_\kappa\beta$. We have shown $\langle Y, \beta\rangle \in \mathbb{R}$ and $\langle Y, \beta\rangle \le \langle X_n, \alpha_n\rangle$ for every $n \in \omega$. Hence $\langle Y, \beta\rangle \Vdash$ "$(\forall n \in \omega)(x_n \in \dot{C})$". Now $\langle Y, \beta\rangle \le \langle X, \alpha\rangle$ and $\langle Y, \beta\rangle \Vdash$ "$x \in \dot{S} \cap \dot{C} \ne \emptyset$". This shows stationarity of S.

Any two compatible conditions are in fact comparable with respect to "$\subset$". Thus $X \subset X'$ or $X \supset X'$ whenever $\langle X, \alpha\rangle$, $\langle X', \alpha'\rangle \in G$. Let $\delta < \lambda$. By an easy density argument there is a least ordinal $\xi > \delta$ such that $\langle X, \xi\rangle \in G$ for some X. Then $S \cap P_\kappa\delta = X \cap P_\kappa\delta$ is nonstationary. ∎

We conclude this section by showing that supercompactness of κ has no influence on this type of reflection.

LEMMA 4.6. *If* $\mathrm{cf}(\lambda) \ge \kappa$, *then* $\mathbb{R}$ *is* $<\kappa$*-directed closed.*

Proof. Suppose $\delta < \kappa$ and $\{\langle X_\xi, \alpha_\xi\rangle : \xi < \delta\} \subset \mathbb{R}$ is directed. Any two members are comparable. Set $X = \bigcup_{\xi<\delta} X_\xi$ and $\alpha = \sup\{\alpha_\xi : \xi < \delta\}$. Clearly $X \subset P_\kappa\alpha$ and $\alpha < \lambda$.

For every $\beta < \alpha$ there is $\xi < \delta$ such that $\beta < \alpha_\xi$. For such ξ, $X \cap P_\kappa\beta = X_\xi \cap P_\kappa\beta$ is nonstationary. We show X is nonstationary in $P_\kappa\alpha$.

First assume $\alpha = \alpha_\zeta$ for some $\zeta < \delta$. Then $X = X_\zeta$ is nonstationary. Otherwise $\mathrm{cf}(\alpha) \le \delta < \kappa$. Since every $x \in X$ is a subset of α_ξ for some $\xi < \delta$, $\bigcup x < \alpha$. Hence X is not unbounded. ∎

Forcing by $\mathbb{R}$ after Laver preparation we get a model in which the following holds:

THEOREM 4.7. *It is consistent that κ is supercompact and there is a stationary set $X \subset P_\kappa\lambda$ such that $X \cap P_\kappa\alpha$ is nonstationary for any $\alpha < \lambda$.*

Koszmider [10], forcing a (κ, λ) semimorass, proved the existence of a stationary subset of $P_\kappa\lambda$ which is nonstationary in $P_\kappa X$ for any $X \subset \lambda$ with $|X| \ge \kappa$ in the generic extension. The forcing is $<\kappa$-closed and κ^+-c.c. There is a κ-Kurepa tree and $2^\kappa \ge \lambda$ in his model.

While in our model in Theorem 4.7 no κ-Kurepa tree exists and it is possible to make $2^\kappa = \kappa^+$.

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