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Dirichlet problem for parabolic equations on Hilbert spaces

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Abstract. We study a linear second order parabolic equation in an open subset of a separable Hilbert space, with the Dirichlet boundary condition. We prove that a probabilistic formula, analogous to one obtained in the finite-dimensional case, gives a solution to this equation. We also give a uniqueness result.

1. Introduction. There is a well known correspondence between Markov processes and partial differential equations in finite dimensions, on the whole space, as well as in open subsets. Consider an equation of the form

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\sum_{i,j=1}^{n} q_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(t,x) + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i}u(t,x)\\
&= \frac{1}{2}\operatorname{Tr} Q(x)D^2u(t,x) + \langle A(x), Du(t,x)\rangle, \qquad t>0,\ x\in\mathbb{R}^n,\\
u(0,x) &= \varphi(x).
\end{aligned}
\tag{1.1}
$$

Here $Q(x) = [q_{i,j}(x)]_{i,j=1}^n$ is a symmetric, positive definite matrix and $A(x) = (a_1(x),\ldots,a_n(x))$; $A(x)$ and $\sqrt{Q(x)}$ are Lipschitz continuous and bounded. Denote by X_t^x a diffusion process satisfying the stochastic equation

$$
dX_t = A(X_t)dt + \sqrt{Q(X_t)}dW_t, \qquad X_0 = x,
\tag{1.2}
$$

where W_t is a standard Wiener process. Then the formula $u(t,x) = E\varphi(X_t^x)$ defines the unique solution to equation (1.1). This probabilistic approach to parabolic equations was presented for example in [13] and [12]. In [17] the problem of existence and uniqueness of solution to (1.1) with unbounded coefficients was studied. There are also results concerning parabolic equations in an open set $\mathcal{O}\subset\mathbb{R}^n$, with Dirichlet boundary condition (see [13],

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[12], [21]):

$$\begin{aligned}(1.3)\quad &\frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\operatorname{Tr}Q(x)D^2u(t,x)+\langle A(x),Du(t,x)\rangle,\quad t>0,\ x\in\mathcal{O},\\ &u(0,x)=\varphi(x),\quad x\in\mathcal{O},\quad u(t,x)=0,\quad t>0,\ x\in\partial\mathcal{O}.\end{aligned}$$

Let $\tau_{\mathcal{O}}^x$ denote the time of the first exit of the process X_t^x from $\mathcal{O}$ after $0+$. It is known that if the boundary of $\mathcal{O}$ is smooth enough, $A(x)$ and $\sqrt{Q(x)}$ are Lipschitz continuous and bounded and there exists $\gamma>0$ such that $\langle Q(x)v,v\rangle\ge\gamma|v|$, then for φ bounded and continuous, the function

$$u(t,x)=E\{\varphi(X_t^x):\tau_{\mathcal{O}}^x>t\}$$

satisfies equation (1.3).

The probabilistic approach to parabolic equations in infinite dimensions was presented in [4], [14], [15], [16], [18], [2], [3], [6], [23] and [22]. For more references see e.g. [23]. But most of the results concern equations on the whole space.

First results have been obtained by Gross [14] for the heat equation on a Banach space. In the special case of Hilbert space the equation considered in [14] is of the form

$$(1.4)\qquad \frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\operatorname{Tr}QD^2u(t,x),\qquad u(0,x)=\varphi(x),$$

with a positive, symmetric nuclear operator Q. If φ is bounded, twice continuously differentiable, with bounded derivatives, and W_t is a Wiener process with covariance operator Q, then $u(t,x)=E\varphi(x+W_t)$ satisfies (1.4). It was shown by Gross that the regularity assumption on φ can be weakened when $\operatorname{Tr}QD^2u(t,x)$ is replaced by the so-called Gross Laplacian. This replacement is necessary because of the lack of the smoothing property of the heat semigroup in the infinite-dimensional case. This lack causes problems when considering the heat equation in an open set. Gross' results were generalized by Priola [19] to parabolic and elliptic equations on a halfspace.

A parabolic equation with a term involving the first derivative with respect to the space variable was studied by Piech [18], Cannarsa and Da Prato [2], [3] and Zabczyk [23] (see [23] for more complete references). On a separable Hilbert space H the following equation is considered:

$$\begin{aligned}(1.5)\quad &\frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\operatorname{Tr}QD^2u(t,x)+\langle x,A^*Du(t,x)\rangle,\quad t>0,x\in H,\\ &u(0,x)=\varphi(x),\end{aligned}$$

where Q is a symmetric, nonnegative bounded operator, not necessarily nuclear, A is the generator of a C_0-semigroup on H and φ is a uniformly continuous bounded function on H. Equation (1.5) is related to the Ornstein–

Uhlenbeck process satisfying

$$(1.6)\qquad dX=AX\,dt+dW,\qquad X(0)=x,$$

where W is a Q-Wiener process. As shown in [9], the presence of the unbounded operator A may have a regularizing effect, in the sense that under certain conditions the transition semigroup of X has the smoothing property. It was shown that in this case (1.5) has a unique classical solution which has the form $u(t,x)=E\varphi(X_t^x)$ (see e.g. [23]).

In the present paper we consider the following Dirichlet problem on an open subset $\mathcal{O}$ of a separable Hilbert space H:

$$(1.7)\quad \frac{\partial u}{\partial t}(t,x)=\frac{1}{2}\operatorname{Tr}Q^{1/2}D^2u(t,x)Q^{1/2}+\langle x,A^*Du(t,x)\rangle\quad\text{if }x\in\mathcal{O},\ t>0,$$

$$(1.8)\quad u(0,x)=\varphi(x)\quad\text{if }x\in\mathcal{O},\qquad u(t,x)=0\quad\text{if }x\in\partial\mathcal{O},\ t>0.$$

In (1.8), φ is a continuous bounded function on $\mathcal{O}$. Note that the traces of the operators $QD^2u(t,x)$ and $Q^{1/2}D^2u(t,x)Q^{1/2}$ are equal if both operators are nuclear. We denote by $\tau_{\mathcal{O}}^x$ the first exit time of the process X_t^x from $\mathcal{O}$. In view of finite-dimensional results we define a generalized solution to (1.7), (1.8) by $u(t,x)=P_t^{\mathcal{O}}\varphi(x)=E\{\varphi(X_t^x):\tau_{\mathcal{O}}^x>t\}$ for X_t^x satisfying (1.6). A natural question is whether the generalized solution satisfies (1.7), (1.8) in the classical sense. As in the problem (1.5), the smoothing property of the transition semigroup of the process X plays an important role in our approach. The restricted semigroup $P_t^{\mathcal{O}}$ was studied by Da Prato, Gołdys and Zabczyk in [8] but they only showed that under certain conditions for each bounded Borel φ and $t>0$ the function $P_t^{\mathcal{O}}\varphi(x)$ is Fréchet differentiable.

We show that under conditions similar to [8] the generalized solution satisfies (1.7) and (1.8) in the strong sense. That is, we prove that for $t>0$ and $x\in\mathcal{O}$, $P_t^{\mathcal{O}}\varphi(x)$ is twice continuously differentiable with respect to x and once differentiable with respect to t, and we show that the operator $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ is nuclear and $DP_t^{\mathcal{O}}\varphi(x)$ belongs to the domain of A^*. This means that both sides of (1.7) are well defined. We prove that the equality in (1.7) holds. Moreover, we show that the derivatives of $P_t^{\mathcal{O}}\varphi(x)$ are locally bounded and the nuclear norm of $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ and the norm of $A^*DP_t^{\mathcal{O}}\varphi(x)$ are locally bounded. The estimates obtained depend on the distance of x from the boundary of $\mathcal{O}$ and also on the speed of convergence to zero of $|S(h)x-x|$ when h goes to zero. Finally we prove that $P_t^{\mathcal{O}}\varphi(x)$ converges to φ as t goes to 0, and for $t>0$, $P_t^{\mathcal{O}}\varphi(x)\to 0$ as x goes to a regular point of the boundary. In general, irregular points do not have this property. It was shown in [8] that even for sets so simple as a ball or certain halfspaces the set of irregular points may be dense in the boundary. However there also exist sets whose boundaries consist of regular points only.

We also prove that a strong solution is unique if the process X_t^x exits from the set $\mathcal{O}$ only through regular points of $\partial\mathcal{O}$.

Thus this paper not only answers the question stated in [8] by showing that the generalized solution is a strong solution to (1.7), (1.8), but also provides a uniqueness result. We hope that our result can be useful in studying the more general equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\operatorname{Tr} Q^{1/2}D^2u(t,x)Q^{1/2} + \langle x, A^*Du(t,x)\rangle + \langle F(x), Du(t,x)\rangle \quad \text{if } x\in\mathcal{O},\ t>0,$$

$$u(0,x)=\varphi(x) \quad \text{if } x\in\mathcal{O}, \qquad u(t,x)=0 \quad \text{if } x\in\partial\mathcal{O},\ t>0,$$

where F is Lipschitz.

Let us briefly describe the methods used in the proofs. As we mentioned, the existence of the first order derivative of $P_t^{\mathcal{O}}\varphi(x)$ with respect to x was already shown in [8]. We extend this method to show that there exist derivatives of higher order. The key idea is to write $P_t^{\mathcal{O}}\varphi(x)$ as a series of functions involving the global semigroup P_t. A similar method was used in [21] for the restricted semigroup of a finite-dimensional Wiener process. To show that $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ is of trace class and that $DP_t^{\mathcal{O}}\varphi(x)$ belongs to the domain of A^* we work with the same expansion but now we have to treat it more carefully. The elements of the sum are of the form $P_t\psi_k$ for some bounded Borel ψ_k. The idea is to write them as $P_{t/2}(P_{t/2}\psi_k)$. Since $P_{t/2}\psi_k$ is a smooth function, we can use formulas for the derivatives of the global semigroup acting on a smooth function. Later we use exponential estimates of the probabilities involving exit times of the Ornstein–Uhlenbeck process X_t^x. We use similar methods to show that the derivative of $P_t^{\mathcal{O}}\varphi(x)$ with respect to t exists and (1.7) is satisfied.

We prove that $P_t^{\mathcal{O}}\varphi(x)$ satisfies the boundary condition for regular points via excessive functions and their lower semicontinuity. The proof for the initial condition is straightforward.

Next we show the uniqueness of a solution of (1.7) and (1.8) when the process X_t^x exits $\mathcal{O}$ through regular points only. We show that any solution must be of the form $u(t,x)=P_t^{\mathcal{O}}\varphi(x)$. We prove this using Itô's formula and several levels of approximation of the process X^x as well as of the set $\mathcal{O}$.

2. Notation and results. Let H be a separable Hilbert space with norm $|\cdot|$, and let $\mathcal{O}$ be an open subset of H. We denote by $B_b(\mathcal{O})$ and $C_b(\mathcal{O})$ the sets of bounded Borel functions on $\mathcal{O}$ and bounded continuous functions on $\mathcal{O}$, respectively. Let

$$\|\varphi\|_{\mathcal{O}} = \sup_{x\in\mathcal{O}} |\varphi(x)|;$$

we denote by $\|\cdot\|_{\mathrm{HS}}$ the Hilbert–Schmidt norm and by $\|\cdot\|_1$ the nuclear norm of a linear operator on H.

We consider the following equation:

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\operatorname{Tr} Q^{1/2}D^2u(t,x)Q^{1/2} + \langle x, A^*Du(t,x)\rangle \quad \text{if } x\in\mathcal{O},\ t>0, \tag{2.1}$$

$$u(0,x)=\varphi(x) \quad \text{if } x\in\mathcal{O}, \qquad u(t,x)=0 \quad \text{if } x\in\partial\mathcal{O},\ t>0. \tag{2.2}$$

Here A is the generator of a C_0-semigroup $S(t)$ on H, Q is a symmetric, nonnegative bounded operator, and $\varphi\in C_b(\mathcal{O})$. D and D^2 denote Fréchet derivatives with respect to x.

Problem (2.1)–(2.2) is closely related to the Ornstein–Uhlenbeck process satisfying the equation

$$dX = AX\,dt + dW, \qquad X(0)=x, \tag{2.3}$$

where W is a Q-Wiener process.

One of our assumptions is the following:

$$\exists_{0<\alpha<1}\ \forall_{t\ge 0}\quad \int_0^t \sigma^{-\alpha}\|S(\sigma)Q^{1/2}\|_{\mathrm{HS}}^2\,d\sigma<\infty. \tag{2.4}$$

It is known (see [10]) that under condition (2.4), there exists a unique solution X^x to (2.3) given by

$$X_t^x = S(t)x + \int_0^t S(t-s)\,dW_s. \tag{2.5}$$

Moreover, the process X^x has a continuous version. The transition semigroup P_t of X will be called the *global semigroup*,

$$P_t\varphi(x) = E\varphi(X_t^x) = \int_H \varphi(S(t)x+y)\,N_{Q_t}(dy), \qquad \varphi\in B_b(H), \tag{2.6}$$

where

$$Q_t = \int_0^t S(\sigma)QS^*(\sigma)\,d\sigma, \tag{2.7}$$

and N_{Q_t} denotes the centered Gaussian measure on H with covariance operator Q_t. By (2.4) the operator Q_t is nuclear.

The second important assumption is

$$\forall_{t>0}\quad \operatorname{Im} S(t)\subset \operatorname{Im} Q_t^{1/2}. \tag{2.8}$$

By (2.8) and the closed graph theorem it follows that the operator Λ_t defined by

$$\Lambda_t = Q_t^{-1/2}S(t), \quad t>0,$$

is bounded. Moreover, again by the closed graph theorem, it follows that for each $t > 0$, $S(t)$ is Hilbert–Schmidt, and consequently, by the semigroup property $S(t)$ is nuclear.

Condition (2.8) holds if and only if the global semigroup has the smoothing property (see [9]). Both (2.4) and (2.8) were also assumed in [23] when solving equation (2.1) on the whole H.

For $x \in H$ denote by $\tau_{\mathcal{O}}^x$ the first exit time of the process X_t^x from $\mathcal{O}$,

$$\tau_{\mathcal{O}}^x = \inf\{t > 0 : X_t^x \in \mathcal{O}^c\}.$$

By the Blumenthal 0-1 Law the probability $P(\tau_{\mathcal{O}}^x = 0)$ is either 0 or 1. We say that a point x is *regular* if $P(\tau_{\mathcal{O}}^x = 0) = 1$, otherwise we call it *irregular*.

We define the *restricted semigroup* by the formula

$$P_t^{\mathcal{O}}\varphi(x) = E\{\varphi(X_t^x) : \tau_{\mathcal{O}}^x > t\}, \quad x \in \mathcal{O},\ \varphi \in B_{\mathrm{b}}(\mathcal{O}).$$

The function

$$u(t,x) = P_t^{\mathcal{O}}\varphi(x) = E\{\varphi(X_t^x) : \tau_{\mathcal{O}}^x > t\} \tag{2.9}$$

is called the *generalized solution* to (2.1), (2.2). In view of finite-dimensional results this is a natural candidate for a strong solution of (2.1), (2.2).

DEFINITION 2.1. We say that a continuous bounded function $u(t,x)$ on $[0,\infty) \times \mathcal{O}$ is a *strong solution* of (2.1) and (2.2) if it satisfies

(i) for each $t > 0$ the Fréchet derivatives $Du(t,x)$ and $D^2u(t,x)$ exist and they are continuous functions of $x \in \mathcal{O}$, locally bounded as functions of two variables, i.e.

$$\forall_{x\in\mathcal{O}}\ \forall_{t>0}\ \exists_{0<t_1<t<T}\ \exists_{r>0} \sup_{(s,y)\in[t_1,T]\times B(x,r)} \{|Du(s,y)| + \|D^2u(s,y)\|\} < \infty, \tag{2.10}$$

(ii) for $t > 0$, $Du(t,x) \in D(A^*)$, $|A^*Du(t,x)|$ and $\|Q^{1/2}D^2u(t,x)Q^{1/2}\|_1$ are locally bounded, and for fixed $t > 0$, $Q^{1/2}D^2u(t,x)Q^{1/2}$ is continuous as a function of $x \in \mathcal{O}$ into the space of linear trace class operators on H,

(iii) for each $x \in \mathcal{O}$, $\frac{\partial}{\partial t}u(t,x)$ exists for $t > 0$ and is a continuous function of t,

(iv) equation (2.1) is satisfied in the classical sense,

(v) $u(0,x) = \varphi(x)$ for $x \in \mathcal{O}$,

(vi) for any sequences $\{x_n\}$ of points in $\mathcal{O}$ converging to a regular point x of the boundary, and $\{t_n\}$, $t_n > 0$, converging to $t > 0$, $u(t_n,x_n)$ converges to zero.

REMARK 2.2. By the compactness of $\{x\} \times [\gamma,T]$ for $0 < \gamma < T < \infty$, it follows that (2.10) is equivalent to

$$\forall_{x\in\mathcal{O}}\ \forall_{0<\gamma<T<\infty}\ \exists_{r>0} \sup_{(s,y)\in[\gamma,T]\times B(x,r)} \{|Du(s,y)| + \|D^2u(s,y)\|\} < \infty.$$

THEOREM 2.3. *Assume that* (2.4) *and* (2.8) *are satisfied. If* $\operatorname{Im} S(t) \subset D(A)$ *for all* $t > 0$ *and there exist* $t_0, C > 0$ *and* $\delta \geq 0$ *such that for all* $0 < t \leq t_0$,

$$\|AS(t)\| \leq C/t^\delta, \tag{2.11}$$

$$\|\Lambda_t\| \leq C/t^\delta, \tag{2.12}$$

then for $\varphi \in C_{\mathrm{b}}(\mathcal{O})$, $u(t,x) = P_t^{\mathcal{O}}\varphi(x)$ *is a strong solution of* (2.1), (2.2).

REMARK 2.4. If $\operatorname{Im} S(t) \subset D(A)$ then, since A is closed, $AS(t)$ is a bounded operator. Consequently, $S(t)A$ can be extended to a bounded operator; this extension is equal to $AS(t)$. Moreover, $\operatorname{Im} S^*(t) \subset D(A^*)$ and $\|A^*S^*(t)\| = \|AS(t)\|$.

Assumption (2.11) is satisfied for analytic semigroups with $\delta = 1$ (see [11]).

EXAMPLE 2.5. Let

$$H = L^2(0,1), \quad A = \frac{d^2}{d\xi^2}, \quad D(A) = H^2(0,1) \cap H_0^1(0,1), \quad Q = \mathrm{Id}$$

and let $\mathcal{O}$ be an open set in H. The operator A has eigenvalues $-n^2\pi^2$, $n = 1, 2, \ldots$ The semigroup $S(t)$ generated by A, and the operator Q_t, have the same eigenvectors as A. The corresponding eigenvalues are $e^{-n^2\pi^2 t}$ and $\frac{1-e^{-2n^2\pi^2 t}}{2n^2\pi^2}$, respectively. All the assumptions of Theorem 2.3 are satisfied: (2.4) holds for all $0 < \alpha < 1/2$, (2.11) is satisfied for $\delta = 1$ and (2.12) with $\delta = 1/2$. Thus for each $\varphi \in C_{\mathrm{b}}(\mathcal{O})$ the function $P_t^{\mathcal{O}}\varphi(x)$ is a strong solution to (2.1), (2.2).

If $\mathcal{O} = B(0,r)$ is a ball in H then, as shown in [8], every $z \in \partial\mathcal{O}$ such that $\langle Qz, z\rangle > 0$, $z \in D(A)$ or $z \in D((-A)^{1/2})$ is regular, and consequently, for $t > 0$,

$$\lim_{\substack{x\in\mathcal{O}\\ x\to z}} P_t^{\mathcal{O}}\varphi(x) = 0.$$

But in [8] it was also shown that there exists a dense set of irregular points.

Under a weaker assumption on the initial function we obtain the following theorem that deals only with equation (2.1):

THEOREM 2.6. *Let the assumptions of Theorem* 2.3 *be satisfied. Then for each* $\varphi \in B_{\mathrm{b}}(\mathcal{O})$ *the function* $u(t,x) = P_t^{\mathcal{O}}\varphi(x)$ *satisfies* (i)–(iv) *of the Definition* 2.1. *Moreover,* $\operatorname{Im} D^2P_t^{\mathcal{O}}\varphi(x) \subset D(A^*)$ *and* $A^*D^2P_t^{\mathcal{O}}\varphi(x)$ *is a bounded operator.*

The next theorem provides the following result about the uniqueness of a solution.

THEOREM 2.7. *Assume that* (2.4) *is satisfied, the solution* X^x *of* (2.3) *exits from the open set* $\mathcal{O}$ *only through regular points of the boundary and* *for all $x \in \mathcal{O}$ and $t > 0$, $P(\tau_{\mathcal{O}}^x = t) = 0$. If $u(t,x)$ is a strong solution of* (2.1) *and* (2.2), *then $u(t,x) = E\{\varphi(X_t^x) : \tau_{\mathcal{O}}^x > t\}$.*

REMARK 2.8. There exist sets $\mathcal{O}$ for which all points of $\partial\mathcal{O}$ are regular. The method used in the proof of Theorem 3.1 of [8] shows that for each $\gamma \in \mathbb{R}$ and $a \in D(A^*)$ such that $\langle Qa, a\rangle > 0$, the sets $\{x \in H : \langle x, a\rangle > \gamma\}$ and $\{x \in H : \langle x, a\rangle < \gamma\}$ have regular boundaries. A finite intersection of sets of this form also has a boundary consisting of regular points only.

3. Proof of the existence theorem. First we will show Theorem 2.6. Let us gather here a few results that will be frequently used in the proof. The first one is an extension of the interpolation lemma given in [8].

LEMMA 3.1. *Let U be an open subset of H. For $x \in U$ define $d_U(x) = \operatorname{dist}(x, U^c) \wedge 2$. Then*

$$\|Dg(x)\| \le \frac{4}{d_U(x)}\sqrt{\|g\|_U}\sqrt{\|g\|_U + \|D^2 g\|_U} \quad \text{if } g \in C^2(U),$$

$$\|D^2 g(x)\| \le \left(\frac{8}{d_U(x)}\right)^{3/2} (\|g\|_U)^{1/4}(\|g\|_U + \|D^2 g\|_U)^{1/4} \times (\|Dg\|_U + \|D^3 g\|_U)^{1/2} \quad \text{if } g \in C^3(U),$$

$$\|D^3 g(x)\| \le \left(\frac{16}{d_U(x)}\right)^{7/4} (\|g\|_U)^{1/8}(\|g\|_U + \|D^2 g\|_U)^{1/8} \times (\|Dg\|_U + \|D^3 g\|_U)^{1/4}(\|D^2 g\|_U + \|D^4 g\|_U)^{1/2} \quad \text{if } g \in C^4(U).$$

Proof. The first inequality was shown in [8]. Now let V be an open ball with center at x and radius $d_U(x)/2$. We apply the first inequality to V and functions of the form $f_v(x) = \langle Dg(x), v\rangle$, $v \in H$, to obtain

$$\|D^2 g(x)\| \le \frac{4}{d_V(x)}\sqrt{\|Dg\|_V}\sqrt{\|Dg\|_V + \|D^3 g\|_V} \le \frac{8}{d_U(x)}\sqrt{\|Dg\|_V}\sqrt{\|Dg\|_U + \|D^3 g\|_U}. \tag{3.1}$$

For $y \in V$ we have

$$\|Dg(y)\| \le \frac{4}{d_U(y)}\sqrt{\|g\|_U}\sqrt{\|g\|_U + \|D^2 g\|_U}.$$

Now, since $d_U(y) \ge d_U(x)/2$, we obtain

$$\|Dg\|_V \le \frac{8}{d_U(x)}\sqrt{\|g\|_U}\sqrt{\|g\|_U + \|D^2 g\|_U}.$$

Combining this with (3.1) we get the second inequality of the lemma. The proof of the last one is similar. ∎

Let $W_h(y) = \langle h, Q_t^{-1/2} y\rangle$. On the probability space $(H, \mathcal{B}(H), N_{Q_t})$, W_h is a real-valued centered Gaussian random variable with variance $|h|^2$. Let us introduce the Wick products $G_{n_1,\dots,n_k}(W_{h_1},\dots,W_{h_k})$, defined recursively in the following way (cf. [20]):

$$G_{0,\dots,0}(x_1,\dots,x_k) = 1,$$
$$\frac{\partial}{\partial x_j} G_{n_1,\dots,n_k}(x_1,\dots,x_k) = n_j G_{n_1,\dots,n_j - 1,\dots,n_k}(x_1,\dots,x_k),$$
$$E G_{n_1,\dots,n_k}(W_{h_1},\dots W_{h_k}) = 0 \quad \text{for } (n_1,\dots,n_k) \ne (0,\dots,0).$$

It is easy to see that $G_{n_1,\dots,n_k}(W_{h_1},\dots,W_{h_k})$ is the orthogonal projection of the product $W_{h_1}^{n_1}\dots W_{h_k}^{n_k}$ onto the Wiener chaos of order $n_1 + \dots + n_k$ in the space $L^2(H, N_{Q_t})$ and therefore $G_{n_1}(W_{h_1})$ is a Hermite polynomial of order n_1.

LEMMA 3.2. *Assume that* (2.4) *holds for $\alpha = 0$ and* (2.8) *is satisfied. Then for all $\varphi \in B_b(H)$ and $h_1,\dots,h_k \in H$,*

$$D^{n_1+\dots+n_k} P_t\varphi(x)(h_1^{\otimes n_1} \otimes \dots \otimes h_k^{n_k}) = \int_H G_{n_1,\dots,n_k}(W_{\Lambda_t h_1}(y),\dots,W_{\Lambda_t h_k}(y))\varphi(S(t)x + y)\, N_{Q_t}(dy). \tag{3.2}$$

Moreover, putting $n = n_1 + \dots + n_k$,

$$\forall_{t>0} \quad \|D^n P_t\varphi\|_{H^{\otimes n}} \le C_n \|\varphi\|_H \|\Lambda_t\|^n. \tag{3.3}$$

Proof. By the Cameron–Martin formula we can write

$$P_t\varphi(x+h) = \int_H \varphi(S(t)x + y)\exp\{\langle \Lambda_t h, Q_t^{-1/2} y\rangle - \tfrac{1}{2}|\Lambda_t h|^2\}\, N_{Q_t}(dy).$$

And since

$$\exp\{\langle \Lambda_t h, Q_t^{-1/2} y\rangle - \tfrac{1}{2}|\Lambda_t h|^2\} = \sum_{n_1=0}^{\infty} G_{n_1}(W_h),$$

it is easy to check that (3.2) holds for $D^{n_1} P_t\varphi(h^{\otimes n_1})$. The general case follows by polarization and linearity.

Since the vector $(W_{h_1},\dots,W_{h_n})$ is Gaussian, we have an explicit formula:

$$G_{1,\dots,1}(W_{h_1},\dots,W_{h_n}) = \sum_{P \in \mathcal{P}_n} \Big[\prod_{\{i\}\in P} W_{h_i} \prod_{\{k,j\}\in P} (-E W_{h_k} W_{h_j})\Big],$$

where $\mathcal{P}_n$ denotes the class of all distinct partitions P of the set $\{1,\dots,n\}$ into disjoint subsets which contain either one or two elements, i.e. every $P \in \mathcal{P}_n$ has the form $P = \{\{i_1, i_2\},\dots,\{i_{2m-1}, i_{2m}\},\{i_{2m+1}\},\dots,\{i_n\}\}$, P is unordered. We also have for $p \in \mathbb{N}$,

$$(3.4)\qquad \int_H \langle \Lambda_t g, Q_t^{-1/2}y\rangle^{2p} N_{Q_t}(dy) \le \widetilde{C}_p\Big[\int_H \langle \Lambda_t g, Q_t^{-1/2}y\rangle^{2} N_{Q_t}(dy)\Big]^p = \widetilde{C}_p |\Lambda_t g|^{2p}.$$

Applying Hölder's inequality and (3.4) (alternatively, Schwarz' inequality and the formula for mixed moments of jointly Gaussian random variables) to

$$\int_H |G_{1,\ldots,1}(W_{\Lambda_t h_1}(y),\ldots,W_{\Lambda_t h_n}(y))|\, N_{Q_t}(dy)$$

we get estimate (3.3). ∎

Let $\varphi\in B_{\rm b}(\mathcal{O})$. Define

$$\widehat{P}_t^{\mathcal{O}}\varphi(x) = \begin{cases} P_t^{\mathcal{O}}\varphi(x) & \text{if } x\in\mathcal{O},\\ 0 & \text{if } x\in\mathcal{O}^{\rm c}.\end{cases}$$

LEMMA 3.3. *Let $t\ge s>0$, $x\in\mathcal{O}$ and $\varphi\in B_{\rm b}(\mathcal{O})$. Then*

$$(3.5)\qquad |P_t^{\mathcal{O}}\varphi(x) - P_s(\widehat{P}_{t-s}^{\mathcal{O}}\varphi)(x)| \le \|\varphi\|_{\mathcal{O}} P(\tau_{\mathcal{O}}^x \le s).$$

The above lemma is proved in [8] and [23]. The next lemma, which gives estimates for the exit probabilities, is taken from [23]; another version of it can be found in [8].

LEMMA 3.4. *Assume that condition* (2.4) *holds. Then*

(i) *the process $Z(t)=\int_0^t S(t-s)\,dW_s$ has a continuous version and for each $T>0$ there exist positive constants C_1, C_2 such that for all $t\in(0,T]$ and $r>0$,*

$$(3.6)\qquad P(\sup_{s\le t}\|Z(s)\|\ge r)\le C_1 e^{-C_2r^2/t^\alpha},$$

(ii) *for all $x\in\mathcal{O}$ and $T>0$ there exist positive constants r_0, t_0, C_3, C_4 (depending on x, T and $\mathcal{O}$) such that for all $t\in(0,t_0]$,*

$$(3.7)\qquad \sup_{y\in B(x,r_0)} P(\tau_{\mathcal{O}}^y\le t)\le C_3 e^{-C_4/t^\alpha}.$$

The previous results are used to show the smoothness of $P_t^{\mathcal{O}}\varphi$ as a function of $x\in\mathcal{O}$. We need one more lemma to prove that for arbitrary $\varphi\in B_{\rm b}(\mathcal{O})$, $x\in\mathcal{O}$ and $t>0$ the operator $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ is of trace class and that $DP_t^{\mathcal{O}}\varphi(x)$ is in the domain of A^*. This result is due to Gołdys (see also [23]).

LEMMA 3.5. *Assume that* (2.4) *and* (2.8) *hold. Then*

(i) *we have*

$$(3.8)\qquad DP_t\phi(x) = \int_H S^*(t)D\phi(S(t)x+y)\,N_{Q_t}(dy)\qquad \textit{if } \phi\in C_{\rm b}^1(H),$$

$$(3.9)\qquad D^2P_t\phi(x) = \int_H S^*(t)D^2\phi(S(t)x+y)S(t)\,N_{Q_t}(dy)\qquad \textit{if } \phi\in C_{\rm b}^2(H).$$

(ii) *$DP_t\phi(x)\in D(A^*)$ for all $t>0$, $x\in H$ and $\phi\in B_{\rm b}(H)$ if and only if $\operatorname{Im} S(t)\subset D(A)$.*

(iii) *The operator $D^2P_t\phi$ is of trace class for all $t>0$, $x\in H$ and $\phi\in B_{\rm b}(H)$.*

Proof of Theorem 2.6. STEP 1. First we need to show that for all $t>0$, $P_t^{\mathcal{O}}\varphi(x)$ is of class C^3 with respect to the space variable $x\in\mathcal{O}$, with locally bounded derivatives. Since we have Lemma 3.1 this fact follows by a simple extension of the proof for the first derivative given in [8]. For completeness we recall here the method of [8] and show for example the existence of the second derivative of $P_t^{\mathcal{O}}\varphi$, and its continuity and boundedness on small balls in $\mathcal{O}$. The proof for the third derivative is analogous. In fact extending Lemma 3.1 in a natural way we can conclude that $P_t^{\mathcal{O}}\varphi$ is infinitely differentiable.

Fix $x\in\mathcal{O}$ and $0<t<T$. Let r_0, t_0 be as in Lemma 3.4(ii). Taking t_0 smaller if necessary we can assume that for all $s\le t_0$ assumptions (2.11) and (2.12) are satisfied. In what follows C will always denote a positive constant depending only on x, T and the set $\mathcal{O}$. This constant can be different in different expressions below.

For $k\in\mathbb{N}$, $k\ge1$ we define functions ψ_k as follows:

$$(3.10)\qquad \psi_k(y)=P_{t/k}\widehat{P}_{(k-1)t/k}^{\mathcal{O}}\varphi(y)\qquad \text{for } y\in H.$$

By Lemmas 3.3 and 3.4 the sum $\psi_1(y)+\sum_{k=2}^\infty(\psi_k(y)-\psi_{k-1}(y))$ converges uniformly to $P_t^{\mathcal{O}}\varphi(y)$ for $y\in B(x,r_0)$. Each ψ_k is in $C_{\rm b}^\infty(H)$, hence to prove that $P_t^{\mathcal{O}}\varphi$ is twice differentiable at x it suffices to show that

$$(3.11)\qquad \|D^2\psi_1\|_{B(x,r_0/2)}+\sum_{k=2}^\infty\|D^2(\psi_k-\psi_{k-1})\|_{B(x,r_0/2)}<\infty.$$

For each $k\ge1$ the norm $\|D^2\psi_k\|_{B(x,r_0/2)}$ is finite. For $k>2$ such that $t/k<t_0$ we will use Lemma 3.1 with $g_k=\psi_k-\psi_{k-1}$, $U=B(x,r_0)$. By Lemma 3.2 and assumption (2.12) we have for $y\in B(x,r_0/2)$,

$$\|Dg_k(y)\|\le\|\varphi\|_{\mathcal{O}}(\|\Lambda_{t/k}\|+\|\Lambda_{t/(k-1)}\|)\le C\|\varphi\|_{\mathcal{O}}(k/t)^\delta,$$
$$\|D^2g_k(y)\|\le C\|\varphi\|_{\mathcal{O}}(k/t)^{2\delta},$$
$$\|D^3g_k(y)\|\le C\|\varphi\|_{\mathcal{O}}(k/t)^{3\delta}.$$

As a consequence of Lemma 3.1 we get

$$\|D^2(\psi_k-\psi_{k-1})(y)\|\le C(\|\psi_k-\psi_{k-1}\|_{B(x,r_0)})^{1/4}\|\varphi\|_{\mathcal{O}}^{3/4}(k/t)^{2\delta},$$

which, by Lemmas 3.3 and 3.4(ii), can be estimated by

$$C\|\varphi\|_{\mathcal{O}} e^{-((k-1)/t)^{\alpha}/C}(k/t)^{2\delta}.$$

Thus the series in (3.11) is convergent and for $y \in B(x, r_0/2)$,

$$D^2 P_t^{\mathcal{O}}\varphi(y) = D^2\psi_1(y) + \sum_{k=2}^{\infty} D^2(\psi_k - \psi_{k-1})(y) \tag{3.12}$$

and we get (i) in the definition of the strong solution.

STEP 2. We now prove that if (2.4) is satisfied then

$$\sum_{k=1}^{\infty}(t/k)^2\|S(t/k)Q^{1/2}\|_{\mathrm{HS}}^2 < \infty. \tag{3.13}$$

Since $\{S(t)\}_{t\ge 0}$ forms a C_0-semigroup, there exist constants $N > 0$ and $a \in \mathbb{R}$ such that $\|S(\sigma)\| \le Ne^{a\sigma}$ (see [11]). Define $M = \sup_{\sigma\le T}\|S(\sigma)\| \vee 1$. From (2.4) we have

$$\begin{aligned}\infty > \int_0^t \sigma^{-\alpha}\|S(\sigma)Q^{1/2}\|_{\mathrm{HS}}^2\,d\sigma &\ge t^{-\alpha}\sum_{k=1}^{\infty}\int_{t/(k+1)}^{t/k}\|S(\sigma)Q^{1/2}\|_{\mathrm{HS}}^2\,d\sigma\\ &\ge t^{-\alpha}\sum_{k=1}^{\infty}\int_0^{t/k-t/(k+1)}\left\|S\left(\frac{t}{k+1}+\sigma\right)Q^{1/2}\right\|_{\mathrm{HS}}^2 d\sigma.\end{aligned}$$

By the semigroup property,

$$\begin{aligned}\left\|S\left(\frac{t}{k}\right)Q^{1/2}\right\|_{\mathrm{HS}} &\le \left\|S\left(\frac{t}{k}-\frac{t}{k+1}-\sigma\right)\right\|\left\|S\left(\frac{t}{k+1}+\sigma\right)Q^{1/2}\right\|_{\mathrm{HS}}\\ &\le M\left\|S\left(\frac{t}{k+1}+\sigma\right)Q^{1/2}\right\|_{\mathrm{HS}}\end{aligned}$$

for $0 \le \sigma \le t/(k(k+1))$. Thus we get

$$\infty > \frac{1}{M^2}\sum_{k=1}^{\infty}\frac{t}{k(k+1)}\left\|S\left(\frac{t}{k}\right)Q^{1/2}\right\|_{\mathrm{HS}}^2$$

and (3.13) follows.

STEP 3. Now we show that for arbitrary $t > 0$ and $x \in \mathcal{O}$ the operator $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ is of trace class and $DP_t^{\mathcal{O}}\varphi(x) \in D(A^*)$, which means that the right hand side of (2.1) is well defined for $u(t,x) = P_t^{\mathcal{O}}\varphi(x)$. We also prove local boundedness of $A^*DP_t^{\mathcal{O}}\varphi(x)$ and $\|Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}\|_1$.

Fix $x \in \mathcal{O}$, $0 < t < T$ and $k_0 \ge 2$ such that for all $k > k_0$, $|S(t/k)x - x| < r_0/4$ and $t/k < t_0$. To prove that $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ is nuclear it suffices to show that

$$\|Q^{1/2}D^2\psi_{k_0}(x)Q^{1/2}\|_1 + \sum_{k=k_0+1}^{\infty}\|Q^{1/2}D^2(\psi_k - \psi_{k-1})(x)Q^{1/2}\|_1 < \infty, \tag{3.14}$$

where ψ_k are defined in (3.10). Then we will also have

$$\begin{aligned}\operatorname{Tr} Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2} &= \operatorname{Tr} Q^{1/2}D^2\psi_{k_0}(x)Q^{1/2}\\ &\quad + \sum_{k=k_0+1}^{\infty}\operatorname{Tr} Q^{1/2}D^2(\psi_k-\psi_{k-1})(x)Q^{1/2}.\end{aligned}$$

Set

$$\Psi_k(x) = P_{t/(2k)}\widehat{P}^{\mathcal{O}}_{(k-1)t/k}\varphi(x) - P_{t/(k-1)-t/(2k)}\widehat{P}^{\mathcal{O}}_{(k-2)t/(k-1)}\varphi(x). \tag{3.15}$$

By the smoothing property of the global semigroup P_t we see that $\Psi_k \in C_{\mathrm{b}}^2(H)$. Applying Lemma 3.5 to

$$D^2(\psi_k - \psi_{k-1})(x) = D^2(P_{t/(2k)}\Psi_k)(x)$$

we get

$$\begin{aligned}&D^2(\psi_k - \psi_{k-1})(x)\\ &\quad = \int_H S^*\left(\frac{t}{2k}\right)D^2\Psi_k\left(S\left(\frac{t}{2k}\right)x + y\right)S\left(\frac{t}{2k}\right)N_{Q_{t/(2k)}}(dy),\end{aligned} \tag{3.16}$$

and consequently we have

$$\begin{aligned}&\|Q^{1/2}D^2(\psi_k - \psi_{k-1})(x)Q^{1/2}\|_1\\ &\quad \le \left\|S\left(\frac{t}{2k}\right)Q^{1/2}\right\|_{\mathrm{HS}}^2\int_H\left\|D^2\Psi_k\left(S\left(\frac{t}{2k}\right)x+y\right)\right\|N_{Q_{t/(2k)}}(dy).\end{aligned} \tag{3.17}$$

First we estimate the integral in (3.17). From the definition of Ψ_k and Lemma 3.2 we see that

$$\begin{aligned}\|D^2\Psi_k\|_H &\le C_2\|\varphi\|_{\mathcal{O}}(\|\Lambda_{t/(2k)}\|^2 + \|\Lambda_{t/(k-1)-t/(2k)}\|^2)\\ &\le C\|\varphi\|_{\mathcal{O}}(k/t)^{2\delta}.\end{aligned} \tag{3.18}$$

Thus by (3.18) and Lemma 3.4(i),

$$\begin{aligned}&\int_{|y|\ge r_0/4}\left\|D^2\Psi_k\left(S\left(\frac{t}{2k}\right)x+y\right)\right\|N_{Q_{t/(2k)}}(dy)\\ &\qquad \le C\|\varphi\|_{\mathcal{O}}\left(\frac{k}{t}\right)^{2\delta}P\left(\left|\int_0^{t/(2k)}S\left(\frac{t}{2k}-u\right)dW_u\right| \ge \frac{r_0}{4}\right)\\ &\qquad \le C\|\varphi\|_{\mathcal{O}}\left(\frac{k}{t}\right)^{2\delta}e^{-(k/t)^{\alpha}/C}.\end{aligned} \tag{3.19}$$

If $|y| < r_0/4$ then $S(t/(2k))x+y \in B(x,r_0/2)$, since $|S(t/(2k))x - x| < r_0/4$ and by Lemmas 3.3 and 3.4,

$$\sup_{z\in B(x,r_0)} |\Psi_k(z)| \le 2\|\varphi\|_{\mathcal{O}} \sup_{z\in B(x,r_0)} P\left(\tau_{\mathcal{O}}^z \le \frac{t}{k-1} - \frac{t}{2k}\right) \le C\|\varphi\|_{\mathcal{O}} e^{-(k/t)^\alpha/C}. \tag{3.20}$$

Using Lemma 3.1 and next Lemma 3.2, estimates (2.12) for Λ_t and (3.20) we obtain

$$\begin{aligned}\int_{|y|<r_0/4} \left\| D^2\Psi_k\left(S\left(\frac{t}{2k}\right)x+y\right)\right\| N_{Q_{t/(2k)}}(dy) &\le C \int_{|y|<r_0/4} \|\Psi_k\|_{B(x,r_0)}^{1/4} (\|\Psi_k\|_H + \|D^2\Psi_k\|_H)^{1/4} \\ &\quad \times (\|D\Psi_k\|_H + \|D^3\Psi_k\|_H)^{1/2} N_{Q_{t/(2k)}}(dy) \\ &\le C\|\varphi\|_{\mathcal{O}} (k/t)^{2\delta} e^{-(k/t)^\alpha/C}.\end{aligned} \tag{3.21}$$

Applying (3.19) and (3.21) to (3.17) we get

$$\|Q^{1/2}D^2(\psi_k - \psi_{k-1})(x)Q^{1/2}\|_1 \le C\|\varphi\|_{\mathcal{O}} \left\|S\left(\frac{t}{2k}\right)Q^{1/2}\right\|_{\mathrm{HS}}^2 \left(\frac{k}{t}\right)^{2\delta} e^{-(k/t)^\alpha/C}. \tag{3.22}$$

In view of (3.13) the sum of these terms over k is finite, since $k^\gamma e^{-k^\alpha/C} \to 0$ as $k\to\infty$ for each $\gamma\in\mathbb{R}$. This completes the proof of (3.14).

The proof that $DP_t^{\mathcal{O}}\varphi(x)$ belongs to the domain of A^* is similar. It suffices to show that the series

$$A^*D\psi_{k_0}(x) + \sum_{k=k_0+1}^{\infty} A^*D(\psi_k - \psi_{k-1})(x) \tag{3.23}$$

is convergent in H; then, as A^* is a closed operator, we get the desired result.

By (3.8) we have

$$D(\psi_k - \psi_{k-1})(x) = \int_H S^*\left(\frac{t}{2k}\right) D\Psi_k\left(S\left(\frac{t}{2k}\right)x+y\right) N_{Q_{t/(2k)}}(dy).$$

Moreover,

$$\int_H \left\|A^*S^*\left(\frac{t}{2k}\right)D\Psi_k\left(S\left(\frac{t}{2k}\right)x+y\right)\right\| N_{Q_{t/(2k)}}(dy) \le \left\|S\left(\frac{t}{2k}\right)A\right\| \int_H \left\|D\Psi_k\left(S\left(\frac{t}{2k}\right)x+y\right)\right\| N_{Q_{t/(2k)}}(dy).$$

We can estimate the integral in the last expression in a similar way to (3.17) and use assumption (2.11) to see that

$$\|A^*D(\psi_k - \psi_{k-1})(x)\| \le C(k/t)^\gamma e^{-(k/t)^\alpha/C}$$

for some $\gamma > 0$. Thus

$$|A^*D\psi_{k_0}(x)| + \sum_{k=k_0+1}^{\infty} |A^*D(\psi_k - \psi_{k-1})(x)| < \infty$$

and the series in (3.23) is convergent.

By (3.9) we also have $D^2P_t^{\mathcal{O}}\varphi(x)v \in D(A^*)$ for each $v\in H$. Since A^* is closed it follows that $A^*D^2P_t^{\mathcal{O}}\varphi(x)$ is a bounded operator.

Notice that from the above discussion we can infer that for all $x\in\mathcal{O}$ and $0<t_1<T<\infty$ there exists $r>0$ such that both $\|Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(y)Q^{1/2}\|_1$ and $|A^*DP_t^{\mathcal{O}}\varphi(y)|$ are uniformly bounded for $(t,y)\in[t_1,T]\times B(x,r)$. Moreover, $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ is continuous as a function of $x\in\mathcal{O}$ into the space of nuclear operators on H.

If we take $r = r_0/(16M)$, where $M = \sup_{t\in[0,T]} \|S(t)\| \vee 1$ and $k_0 > 2\vee T/t_0$ is such that $|S(h)x - x| < r_0/8$ for any $h \le T/k_0$, then for $y\in B(x,r)$ and $t_1\le t\le T$ we get $|S(t/k)y - y| \le r_0/4$. Now we can follow Step 3 of the proof and use the estimates

$$\begin{aligned} \left\|S\left(\frac{t}{2k}\right)Q^{1/2}\right\|_{\mathrm{HS}} &\le C\left\|S\left(\frac{t_1}{2k}\right)Q^{1/2}\right\|_{\mathrm{HS}}, \\ \left\|S\left(\frac{t}{2k}\right)A\right\| &\le C\left\|S\left(\frac{t_1}{2k}\right)A\right\|, \\ \|\Lambda_{t/(2k)}\| &\le C\left(\frac{k}{t_1}\right)^\delta, \\ Ce^{-(k/t)^\alpha/C} &\le Ce^{-(t_1/T)^\alpha (k/t_1)^\alpha/C} \end{aligned} \tag{3.24}$$

to obtain

$$\sup_{(t,y)\in[t_1,T]\times B(x,r)} \|Q^{1/2}D^2\psi_{k_0}(y)Q^{1/2}\|_1 + \sum_{k=k_0+1}^{\infty} \sup_{(t,y)\in[t_1,T]\times B(x,r)} \|Q^{1/2}D^2(\psi_k - \psi_{k-1})(y)Q^{1/2}\|_1 < \infty. \tag{3.25}$$

Thus

$$\sup_{(t,y)\in[t_1,T]\times B(x,r)} \|Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(y)Q^{1/2}\|_1 < \infty.$$

The argument for $|A^*DP_t^{\mathcal{O}}\varphi(y)|$ is similar.

The continuity of $Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}$ follows from (3.25) and the continuity of $Q^{1/2}D^2\psi_k(y)Q^{1/2}$. Thus we get (ii) of Definition 2.1.

STEP 4. In this part we show that for fixed $x \in \mathcal{O}$ the right derivative of $P_t^{\mathcal{O}}\varphi(x)$ with respect to t exists for each $t > 0$ and satisfies

$$\frac{\partial^+}{\partial t} P_t^{\mathcal{O}}\varphi(x) = \frac{1}{2} \operatorname{Tr} Q^{1/2} D^2 P_t^{\mathcal{O}}\varphi(x) Q^{1/2} + \langle x, A^* D P_t^{\mathcal{O}}\varphi(x)\rangle. \tag{3.26}$$

If $0 < h < t$ then

$$\begin{aligned}\frac{P_{t+h}^{\mathcal{O}}\varphi(x) - P_t^{\mathcal{O}}\varphi(x)}{h} \\ = \frac{P_h \widehat{P}_t^{\mathcal{O}}\varphi(x) - P_t^{\mathcal{O}}\varphi(x)}{h} + \frac{P_{t+h}^{\mathcal{O}}\varphi(x) - P_h \widehat{P}_t^{\mathcal{O}}\varphi(x)}{h}\end{aligned} \tag{3.27}$$

and by Lemmas 3.3 and 3.4(ii) the second term on the right hand side of (3.27) converges to zero as $h \to 0$. Let $M = \sup_{s\le 2t} \|S(s)\| \vee 1$. By Lemma 3.4(i),

$$\begin{aligned}\Bigg|\frac{1}{h} \int\limits_{|y|\ge r_0/(8M)} (\widehat{P}_t^{\mathcal{O}}\varphi(S(h)x + y) - P_t^{\mathcal{O}}\varphi(x))\, N_{Q_h}(dy)\Bigg| \\ \le 2\|\varphi\|_{\mathcal{O}} \frac{1}{h} C e^{-1/(Ch^\alpha)} \xrightarrow[h\to 0]{} 0\end{aligned}$$

and (3.27) takes the form

$$\begin{aligned}\frac{P_{t+h}^{\mathcal{O}}\varphi(x) - P_t^{\mathcal{O}}\varphi(x)}{h} \\ = \frac{1}{h} o(h) + \frac{1}{h} \int\limits_{|y|<r_0/(8M)} (\widehat{P}_t^{\mathcal{O}}\varphi(S(h)x + y) - P_t^{\mathcal{O}}\varphi(x))\, N_{Q_h}(dy).\end{aligned} \tag{3.28}$$

Suppose that h is so small that $|S(u)x - x| < r_0/(8M)$ for each $u \le h$. If $|y| < r_0/(8M)$ then $S(h)x + y$ is in the ball $B(x, r_0/(4M))$ and here $P_t^{\mathcal{O}}\varphi$ is three times continuously differentiable, with bounded derivatives. We use Taylor's formula to see that the right hand side of (3.28) is equal to

$$\begin{aligned}&\frac{1}{h} o(h) + \frac{1}{h} \int\limits_{|y|<r_0/(8M)} \langle DP_t^{\mathcal{O}}\varphi(x), S(h)x - x + y\rangle\, N_{Q_h}(dy) \\ &+ \frac{1}{2h} \int\limits_{|y|<r_0/(8M)} \langle (D^2 P_t^{\mathcal{O}}\varphi(x))(S(h)x - x + y), S(h)x - x + y\rangle\, N_{Q_h}(dy) \\ &+ \frac{1}{6h} \int\limits_{|y|<r_0/(8M)} \langle D^3 P_t^{\mathcal{O}}\varphi(x + \theta_{hy})(S(h)x - x + y)(S(h)x - x + y), \\ &\qquad\qquad S(h)x - x + y\rangle\, N_{Q_h}(dy) \\ &= \frac{1}{h} o(h) + I_1(h) + I_2(h) + I_3(h),\end{aligned} \tag{3.29}$$

where $\theta_{hy} = \eta_{hy}(S(h)x - x + y)$ for some $0 \le \eta_{hy} \le 1$. Notice that if $|y| < r_0/(8M)$ then $|\theta_{hy}| < r_0/(4M)$.

Since the measure N_{Q_h} is symmetric we have

$$I_1(h) = \frac{1}{h}\langle DP_t^{\mathcal{O}}\varphi(x), S(h)x - x\rangle P\Big(|Z(h)| \le \frac{r_0}{8M}\Big), \tag{3.30}$$

where $Z(h) = \int_0^h S(h-u)\, dW_u$. Moreover,

$$\begin{aligned}\frac{1}{h}\langle DP_t^{\mathcal{O}}\varphi(x), S(h)x - x\rangle &= \Big\langle DP_t^{\mathcal{O}}\varphi(x), \frac{1}{h} A \int_0^h S(u)x\, du\Big\rangle \\ &= \Big\langle A^* DP_t^{\mathcal{O}}\varphi(x), \frac{1}{h} \int_0^h S(u)x\, du\Big\rangle \\ &\xrightarrow[h\to 0]{} \langle A^* DP_t^{\mathcal{O}}\varphi(x), x\rangle\end{aligned} \tag{3.31}$$

and

$$\lim_{h\to 0} P\Big(|Z(h)| \le \frac{r_0}{8M}\Big) = 1.$$

Thus from (3.30) we get

$$\lim_{h\to 0} I_1(h) = \langle A^* DP_t^{\mathcal{O}}\varphi(x), x\rangle. \tag{3.32}$$

Next, $I_2(h)$ in (3.29) can be written as

$$\begin{aligned}I_2(h) &= \frac{1}{2h}\langle D^2 P_t^{\mathcal{O}}\varphi(x)(S(h)x - x), S(h)x - x\rangle P\Big(|Z(h)| \le \frac{r_0}{8M}\Big) \\ &+ \frac{1}{2h} \int_H \langle D^2 P_t^{\mathcal{O}}\varphi(x)y, y\rangle\, N_{Q_h}(dy) \\ &- \frac{1}{2h} \int\limits_{|y|\ge r_0/M} \langle D^2 P_t^{\mathcal{O}}\varphi(x)y, y\rangle\, N_{Q_h}(dy) \\ &= I_{21}(h) + I_{22}(h) + I_{23}(h).\end{aligned} \tag{3.33}$$

As in (3.31) we get

$$I_{21}(h) = \frac{1}{2}\Big\langle A^* D^2 P_t^{\mathcal{O}}\varphi(x)(S(h)x - x), \frac{1}{h}\int_0^h S(u)x\, du\Big\rangle P\Big(|Z(h)| \le \frac{r_0}{8M}\Big).$$

Recall that by the end of Step 3, $A^* D^2 P_t^{\mathcal{O}}\varphi(x)$ is a bounded operator. Since $S(h)x - x \to 0$ and $h^{-1}\int_0^h S(u)x\, du \to x$ we have

$$\lim_{h\to 0} I_{21}(h) = 0. \tag{3.34}$$

Finally, by Hölder's inequality we get

$$|I_{23}(h)| \le \|D^2 P_t^{\mathcal{O}} \varphi(x)\| \frac{1}{2h} \Big(\int\limits_{|y| \ge r_0/(8M)} N_{Q_h}(dy) \Big)^{1/2} \Big(\int\limits_H |y|^4 N_{Q_h}(dy) \Big)^{1/2}.$$

By Corollary 2.17 of [10], for each $m \in \mathbb{N}$ there exists a constant $C_m > 0$ such that

$$\int\limits_H |y|^{2m} N_{Q_h}(dy) \le C_m (\operatorname{Tr} Q_h)^m. \tag{3.35}$$

Thus by Lemma 3.4 and (3.35),

$$|I_{23}(h)| \le C \|D^2 P_t^{\mathcal{O}} \varphi(x)\| \frac{1}{h} e^{-1/(Ch^\alpha)} \operatorname{Tr} Q_h \xrightarrow[h\to 0]{} 0. \tag{3.36}$$

The term $I_{22}(h)$ in (3.33) can be written as follows:

$$\begin{aligned} I_{22}(h) &= \frac{1}{2} \operatorname{Tr} Q^{1/2} D^2 P_t^{\mathcal{O}} \varphi(x) Q^{1/2} \\ &\quad + \frac{1}{2} \Big(\frac{1}{h} \operatorname{Tr} Q_h D^2 P_t^{\mathcal{O}} \varphi(x) - \operatorname{Tr} Q^{1/2} D^2 P_t^{\mathcal{O}} \varphi(x) Q^{1/2} \Big). \end{aligned} \tag{3.37}$$

We will show that the last term in (3.37) converges to zero as $h \to 0$.

Let $\{e_i\}_{i=0}^\infty$ be an orthonormal basis of H. We have

$$\begin{aligned} \operatorname{Tr} Q_h D^2 P_t^{\mathcal{O}} \varphi(x) &= \operatorname{Tr} \Big[\int_0^h S(u) Q S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x)\, du \Big] \\ &= \sum_{i=0}^\infty \int_0^h \langle S(u) Q S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x) e_i, e_i \rangle\, du \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} &\sum_{i=0}^\infty \int_0^h |\langle S(u) Q S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x) e_i, e_i \rangle|\, du \\ &\quad = \int_0^h \sum_{i=0}^\infty |\langle S(u) Q S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x) e_i, e_i \rangle|\, du \\ &\quad \le \int_0^h \|S(u) Q S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x)\|_1\, du \le \|D^2 P_t^{\mathcal{O}} \varphi(x)\| \int_0^h \|S(u) Q^{1/2}\|_{\mathrm{HS}}^2\, du, \end{aligned}$$

which by (2.4) is finite. Thus interchanging summation and integration in (3.38) we get

$$\operatorname{Tr} Q_h D^2 P_t^{\mathcal{O}} \varphi(x) = \int_0^h \operatorname{Tr}[S(u) Q S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x)]\, du.$$

Using the fact that the operators $S(u)$ are nuclear, we get

$$\operatorname{Tr} Q_h D^2 P_t^{\mathcal{O}} \varphi(x) = \int_0^h \operatorname{Tr}[Q^{1/2} S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x) S(u) Q^{1/2}]\, du.$$

Thus for the last term in (3.37) we obtain the estimate

$$\begin{aligned} &\Big| \frac{1}{h} \operatorname{Tr} Q_h D^2 P_t^{\mathcal{O}} \varphi(x) - \operatorname{Tr} Q^{1/2} D^2 P_t^{\mathcal{O}} \varphi(x) Q^{1/2} \Big| \\ &\quad \le \frac{1}{h} \int_0^h |\operatorname{Tr} Q^{1/2} S^*(u) D^2 P_t^{\mathcal{O}} \varphi(x) S(u) Q^{1/2} - \operatorname{Tr} Q^{1/2} D^2 P_t^{\mathcal{O}} \varphi(x) Q^{1/2}|\, du. \end{aligned} \tag{3.39}$$

It suffices to prove that the integrand in (3.39) tends to zero as $h \to 0$. Let k_0 be as in Step 3, but assume that $|S(t/k)x - x| < r_0/(8M)$ for all $k > k_0$. We can use the expansion (3.12) of $D^2 P_t^{\mathcal{O}} \varphi(x)$ but starting from k_0 instead of 1. Both series are absolutely convergent. For the first one this is obvious since $S(u)Q^{1/2}$ is Hilbert–Schmidt and $\sum \|D^2(\psi_k - \psi_{k-1})\| < \infty$. For the second we have (3.14).

Thus we can estimate the integrand in (3.39) by

$$\begin{aligned} &\sum_{i=1}^\infty |\langle (Q^{1/2} S^*(u) D^2 \psi_{k_0}(x) S(u) Q^{1/2} - Q^{1/2} D^2 \psi_{k_0}(x) Q^{1/2}) e_i, e_i \rangle| \\ &\quad + \sum_{k=k_0}^\infty \sum_{i=1}^\infty |\langle [Q^{1/2} S^*(u) D^2(\psi_k - \psi_{k-1})(x) S(u) Q^{1/2} \\ &\quad - Q^{1/2} D^2 (\psi_k - \psi_{k-1})(x) Q^{1/2}] e_i, e_i \rangle| \end{aligned} \tag{3.40}$$

where $\{e_i\}_{i=1}^\infty$ is an arbitrary orthonormal basis of H. By (3.9) and the estimate $\|S(u)\| \le M$ we have

$$\begin{aligned} &|\langle (Q^{1/2} S^*(u) D^2 \psi_{k_0}(x) S(u) Q^{1/2} - Q^{1/2} D^2 \psi_{k_0}(x) Q^{1/2}) e_i, e_i \rangle| \\ &\quad \le C \int_H \Big\| D^2 P_{t/(2k_0)} \widehat{P}^{\mathcal{O}}_{(k_0-1)t/k_0} \varphi \Big(S\Big(\frac{t}{2k_0}\Big) x + y \Big) \Big\| N_{Q_{t/(2k_0)}}(dy) \\ &\qquad \times \Big| S\Big(\frac{t}{2k_0}\Big) Q^{1/2} e_i \Big|^2 \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} &|\langle [Q^{1/2} S^*(u) D^2 (\psi_k - \psi_{k-1})(x) S(u) Q^{1/2} \\ &\qquad - Q^{1/2} D^2 (\psi_k - \psi_{k-1})(x) Q^{1/2}] e_i, e_i \rangle| \\ &\quad \le C \int_H \Big\| D^2 \Psi_k \Big(S\Big(\frac{t}{2k}\Big) x + y \Big) \Big\| N_{Q_{t/(2k)}}(dy) \Big| S\Big(\frac{t}{2k}\Big) Q^{1/2} e_i \Big|^2. \end{aligned} \tag{3.42}$$

Since $\sum_{i=1}^{\infty}|S(t/(2k))Q^{1/2}e_i|^2 = \|S(t/(2k))Q^{1/2}\|_{\mathrm{HS}}^2$, the sum of the right hand sides of (3.41) and (3.42) is exactly the same as the one that has been estimated in Step 3, thus it is finite.

But for all $v\in H$, $|S(u)v-v|\to 0$ as $u\to 0$, hence each summand of (3.40) converges to zero as well. By the dominated convergence theorem, (3.40) converges to zero. Putting this together with (3.39) and (3.37) we see that

$$\lim_{h\to 0} I_{22}(h) = \tfrac12 \operatorname{Tr} Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}. \tag{3.43}$$

Applying (3.34), (3.36) and (3.43) to (3.33) we obtain

$$\lim_{h\to 0} I_2(h) = \tfrac12 \operatorname{Tr} Q^{1/2}D^2P_t^{\mathcal{O}}\varphi(x)Q^{1/2}. \tag{3.44}$$

We now show that $I_3(h)$ in (3.29) tends to 0 as $h\to 0$. First we look at

$$I_{31}(h) = \Big|\int_{|y|<r_0/(8M)} \Big\langle D^3P_t^{\mathcal{O}}\varphi(x+\theta_{hy})(S(h)x-x)(S(h)x-x), \frac1h A\int_0^h S(u)x\,du\Big\rangle N_{Q_h}(dy)\Big|. \tag{3.45}$$

From (2.6) it is easy to see that for $\phi\in C_{\mathrm{b}}^3(H)$ and $f,g,q\in H$,

$$\langle (D^3P_t\phi(x)f)g,q\rangle = \int_H \langle (S^*(t)D^3\phi(S(t)x+y)S(t)f)S(t)g,q\rangle\, N_{Q_t}(dy). \tag{3.46}$$

The third derivative of $P_t^{\mathcal{O}}\varphi$ in the ball $B(x,r_0/(4M))$ can be written as $D^3\psi_{k_0}+\sum_{k=k_0+1}^{\infty}D^3(\psi_{k+1}-\psi_k)$. Applying (3.46) and then estimating the norm of the third derivative by Lemma 3.2 we get

$$\begin{aligned}(3.47)\quad &\Big|\int_{|y|<r_0/(8M)}\Big\langle [D^3P_{t/(k_0)}\widehat{P}^{\mathcal{O}}_{(k_0-1)t/k_0}\varphi(x+\theta_{hy})(S(h)x-x)](S(h)x-x), \frac1h A\int_0^h S(u)x\,du\Big\rangle N_{Q_h}(dy)\Big|\\ &\le \int_{|y|<r_0/(8M)}\int_H\Big|\Big\langle S^*\Big(\frac{t}{2k_0}\Big)\Big[D^3P_{t/(2k_0)}\widehat{P}^{\mathcal{O}}_{(k_0-1)/k_0}\varphi\Big(S\Big(\frac{t}{2k_0}\Big)(x+\theta_{hy})+z\Big)\\ &\qquad S\Big(\frac{t}{2k_0}\Big)(S(h)x-x)\Big]S\Big(\frac{t}{2k_0}\Big)(S(h)x-x), \frac1h A\int_0^h S(u)x\,du\Big\rangle\Big|\\ &\qquad N_{Q_{t/(2k_0)}}(dz)\,N_{Q_h}(dy)\end{aligned}$$

$$\le C\Big(\frac{2k_0}{t}\Big)^{4\delta}\|\varphi\|_{\mathcal{O}}|S(h)x-x|^2\Big|\frac1h\int_0^h S(u)x\,du\Big| \xrightarrow[h\to 0]{} 0.$$

Applying (3.46) to the terms containing $\psi_k-\psi_{k-1}$ we obtain

$$\begin{aligned}(3.48)\quad &\Big|\int_{|y|<r_0/(8M)}\Big\langle D^3(\psi_k-\psi_{k-1})(x+\theta_{hy})(S(h)x-x)(S(h)x-x), \frac1h A\int_0^h S(u)x\,du\Big\rangle N_{Q_h}(dy)\Big|\\ &\le C\Big(\frac{k}{t}\Big)^{\delta}|S(u)x-x|^2\Big|\frac1h\int_0^h S(h)x\,du\Big|\\ &\quad\times\int_{|y|<r_0/(8M)}\int_H\Big\|D^3\Psi_k\Big(S\Big(\frac{t}{2k}\Big)(x+\theta_{hy})+z\Big)\Big\|N_{Q_{t/(2k)}}(dz)\,N_{Q_h}(dy),\end{aligned}$$

were Ψ_k is defined by (3.15). Recall that $|\theta_{hy}|\le r_0/(4M)$, thus

$$\Big|S\Big(\frac{t}{2k}\Big)(x+\theta_{hy})-x\Big|\le\frac{r_0}{4}+\frac{r_0}{8}.$$

Processing the inner integral in (3.48) as in (3.19) and (3.21), but taking $|z|<r_0/8$ and $|z|\ge r_0/8$, we see that the expression in (3.48) can be estimated by

$$C|S(h)x-x|^2\Big|\frac1h\int_0^h S(h)x\,du\Big|\Big(\frac{k}{t}\Big)^{4\delta}e^{-(k/t)^{\alpha}/C}.$$

Hence the sum of these terms over k tends to zero. Since we also have (3.47), we see that for $I_{31}(h)$ defined in (3.45),

$$\lim_{h\to 0} I_{31}(h)=0. \tag{3.49}$$

As for $I_{31}(h)$ we get

$$\begin{aligned}(3.50)\quad I_{32}(h) &= \Big|\int_{|y|<r_0/(8M)}\Big\langle [D^3P_t^{\mathcal{O}}\varphi(x+\theta_{hy})y](S(h)x-x), \frac1h A\int_0^h S(u)x\,du\Big\rangle N_{Q_h}(dy)\Big|\\ &\le C|S(h)x-x|\Big|\frac1h\int_0^h S(h)x\,du\Big|\xrightarrow[h\to 0]{}0.\end{aligned}$$

There is a slight difference when estimating

$$I_{33}(h)=\Big|\int_{|y|<r_0/(8M)}\Big\langle [D^3P_t^{\mathcal{O}}\varphi(x+\theta_{hy})y]y, \frac1h A\int_0^h S(u)x\,du\Big\rangle N_{Q_h}(dy)\Big|.$$

Here instead of (3.48) we get

$$C\left(\frac{k}{t}\right)^{\delta}\left|\frac{1}{h}\int_0^h S(u)x\,du\right|$$
$$\times \int_{|y|<r_0/(8M)} |y|^2 \int_H \left\|D^3\Psi_k\left(S\left(\frac{t}{2k}\right)(x+\theta_{hy})+z\right)\right\| N_{Q_{t/(2k)}}(dz)\, N_{Q_h}(dy)$$
$$\le C\left(\frac{k}{t}\right)^{4\delta}\left|\frac{1}{h}\int_0^h S(u)x\,du\right| e^{-(k/t)^{\alpha}/C}\operatorname{Tr} Q_h.$$

The sum over k converges to zero because $\operatorname{Tr} Q_h \to 0$ by (2.4). Also the term with k_0 converges to zero. Consequently,

$$\lim_{h\to 0} I_{33}(h) = 0. \tag{3.51}$$

Now we need to estimate

$$I_{34}(h) = \frac{1}{h}\left|\int_{|y|<r_0/(8M)} \langle [D^3 P_t^{\mathcal{O}}\varphi(x+\theta_{hy})y]y, y\rangle\, N_{Q_h}(dy)\right|. \tag{3.52}$$

To do this we can write $D^3 P_t^{\mathcal{O}}\varphi$ as a sum and then applying (3.46) we obtain

$$\frac{1}{h}\int_{|y|<r_0/(8M)} |\langle [D^3 P_{t/k_0}\widehat{P}^{\mathcal{O}}_{(k_0-1)t/k_0}\varphi(x+\theta_{hy})y]y, y\rangle|\, N_{Q_h}(dy) \tag{3.53}$$
$$\le \frac{1}{h}\int_{|y|<r_0/(8M)}\int_H \left|\left\langle\left[S^*\left(\frac{t}{2k_0}\right) D^3 P_{t/(2k_0)}\widehat{P}^{\mathcal{O}}_{(k_0-1)t/k_0}\varphi\left(S\left(\frac{t}{2k_0}\right)(x+\theta_{hy})+z\right)\right.\right.\right.$$
$$\left.\left.\left. S\left(\frac{t}{2k_0}\right)y\right] S\left(\frac{t}{2k_0}\right)y, y\right\rangle\right| N_{Q_{t/(2k_0)}}(dz)\, N_{Q_h}(dy).$$

We estimate $D^3 P_{t/(2k_0)}\widehat{P}^{\mathcal{O}}_{(k_0-1)t/k_0}\varphi$ using Lemma 3.2. Hence (3.53) is less than

$$C\left(\frac{k_0}{t}\right)^{3\delta}\|\varphi\|_{\mathcal{O}}\frac{1}{h}\int_H \left|S\left(\frac{t}{2k_0}\right)y\right|^2 |y| N_{Q_h}(dy)$$
$$\le C\left(\frac{k_0}{t}\right)^{3\delta}\|\varphi\|_{\mathcal{O}}\frac{1}{h}\left[\int_H \left|S\left(\frac{t}{2k_0}\right)y\right|^4 N_{Q_h}(dy)\right]^{1/2}\left[\int_H |y|^2 N_{Q_h}(dy)\right]^{1/2}.$$

We again use Corollary 2.17 of [10] to get

$$\frac{1}{h}\int_{|y|<r_0/(8M)} |\langle [D^3 P_{t/k_0}\widehat{P}^{\mathcal{O}}_{(k_0-1)t/k_0}\varphi(x+\theta_{hy})y]y, y\rangle|\, N_{Q_h}(dy) \tag{3.54}$$
$$\le C\left(\frac{k_0}{t}\right)^{3\delta}\|\varphi\|_{\mathcal{O}}\frac{1}{h}\operatorname{Tr}\left[Q_h S^*\left(\frac{t}{2k_0}\right)S\left(\frac{t}{2k_0}\right)\right]\sqrt{\operatorname{Tr} Q_h}.$$

Moreover

$$\left|\frac{1}{h}\operatorname{Tr} Q_h S^*\left(\frac{t}{2k_0}\right)S\left(\frac{t}{2k_0}\right)\right| \le \left\|\frac{1}{h}S\left(\frac{t}{2k_0}\right)Q_h S^*\left(\frac{t}{2k_0}\right)\right\|_1$$
$$\le \frac{1}{h}\int_0^h \left\|S\left(\frac{t}{2k_0}\right)S(u)QS^*(u)S^*\left(\frac{t}{2k_0}\right)\right\|_1 du$$
$$\le M^2\frac{1}{h}\int_0^h \left\|S\left(\frac{t}{2k_0}\right)Q^{1/2}\right\|_{\mathrm{HS}}^2 du$$
$$= M^2\left\|S\left(\frac{t}{2k_0}\right)Q^{1/2}\right\|_{\mathrm{HS}}^2.$$

This shows that (3.54) tends to zero, since $\operatorname{Tr} Q_h$ does.

The term with the difference $\psi_k - \psi_{k-1}$ can be estimated by

$$C\frac{1}{h}\int_{|y|<r_0/(8M)} \left|S\left(\frac{t}{2k}\right)y\right|^2 |y| \tag{3.55}$$
$$\times \int_H \left\|D^3\Psi_k\left(S\left(\frac{t}{2k}\right)(x+\theta_{hy})+z\right)\right\| N_{Q_{t/(2k)}}(dz)\, N_{Q_h}(dy).$$

Again, if z is small $D^3\Psi_k$ can be estimated by Lemma 3.1, (2.12) and (3.5), otherwise we use Lemma 3.2, the assumption (2.12) and notice that the measure of the set $|z| \ge r_0/8$ is small. That is, we get the following estimate for (3.55):

$$C\left(\frac{k}{t}\right)^{4\delta}\|\varphi\|_{\mathcal{O}}\frac{1}{h}\operatorname{Tr}\left[Q_h S^*\left(\frac{t}{2k}\right)S\left(\frac{t}{2k}\right)\right]e^{-(k/t)^{\alpha}/C}\sqrt{\operatorname{Tr} Q_h}$$
$$\le C\left(\frac{k}{t}\right)^{4\delta}\|\varphi\|_{\mathcal{O}}\left\|S\left(\frac{t}{2k}\right)Q^{1/2}\right\|_{\mathrm{HS}}^2 e^{-(k/t)^{\alpha}/C}\sqrt{\operatorname{Tr} Q_h},$$

and a sum over k converges to 0 as $h \to 0$ since $\operatorname{Tr} Q_h \to 0$. Thus

$$\lim_{h\to 0} I_{34}(h) = 0. \tag{3.56}$$

Since $I_3(h)$ in (3.29) can be estimated by a sum of terms of the form I_{31}, I_{32}, I_{33} and I_{34}, it follows from (3.49)–(3.51) and (3.56) that $I_3(h)$ converges to zero. Combining this with (3.32), (3.44) and (3.29) we get (3.26).

Step 5. To finish the proof of the theorem we need to show that $P_t^{\mathcal{O}}\varphi(x)$ is differentiable with respect to t for $t > 0$.

Fix $x \in \mathcal{O}$. First notice that by Step 3 for arbitrary $0 < t_1 < T < \infty$ the right hand side of (2.1) is uniformly bounded for $t \in [t_1, T]$. Thus $\frac{\partial^+}{\partial t}P_t^{\mathcal{O}}\varphi(x)$ is bounded for $t \in [t_1, T]$.

It is easy to see that estimates (3.24) and the argument from Step 4 give the uniform convergence of $(P^{\mathcal{O}}_{t+h}\varphi(x) - P^{\mathcal{O}}_t\varphi(x))/h$ with respect to $t \in [t_1, T]$ as $h \to 0$. Therefore there exists h_0 such that for each $0 < h < h_0$,

$$\sup_{t\in[t_1,T]}\left|\frac{P^{\mathcal{O}}_{t+h}\varphi(x) - P^{\mathcal{O}}_t\varphi(x)}{h}\right| \le C + \sup_{t\in[t_1,T]}\left|\frac{\partial^+}{\partial t}P^{\mathcal{O}}_t\varphi(x)\right|,$$

hence $P^{\mathcal{O}}_t\varphi(x)$ is continuous in (t_1, T). Since on compact subsets of (t_1, T), $\frac{\partial^+}{\partial t}P^{\mathcal{O}}_t\varphi(x)$ is the limit of a uniformly convergent sequence of uniformly continuous functions, it follows that $\frac{\partial^+}{\partial t}P^{\mathcal{O}}_t\varphi(x)$ is continuous in (t_1, T).

Fix $t \in (t_1, T)$. For $h > 0$ and $u > 0$ such that $u \le (t - t_1)/2$ we have

$$\begin{aligned}
&\left|\frac{P^{\mathcal{O}}_t\varphi(x) - P^{\mathcal{O}}_{t-h}\varphi(x)}{h} - \frac{\partial^+}{\partial t}P^{\mathcal{O}}_t\varphi(x)\right| \\
&\le \left|\frac{P^{\mathcal{O}}_{t-u+h}\varphi(x) - P^{\mathcal{O}}_{t-u}\varphi(x)}{h} - \frac{\partial^+}{\partial t}P^{\mathcal{O}}_{t-u}\varphi(x)\right| + \left|\frac{\partial^+}{\partial t}P^{\mathcal{O}}_{t-u}\varphi(x) - \frac{\partial^+}{\partial t}P^{\mathcal{O}}_t\varphi(x)\right| \\
&\quad + \left|\frac{P^{\mathcal{O}}_{t-u}\varphi(x) - P^{\mathcal{O}}_{t-h}\varphi(x)}{h}\right| + \left|\frac{P^{\mathcal{O}}_{t-u+h}\varphi(x) - P^{\mathcal{O}}_t\varphi(x)}{h}\right|.
\end{aligned}$$

The first term converges to zero as $h \to 0$ uniformly with respect to u. It follows from the continuity of $P^{\mathcal{O}}_t\varphi(x)$ and $\frac{\partial^+}{\partial t}P^{\mathcal{O}}_t\varphi(x)$ that $P^{\mathcal{O}}_t\varphi(x)$ has a continuous derivative with respect t. This finishes the proof of (iii) and (iv) of Definition 2.1. ∎

For the proof of Theorem 2.3 we need a result on regular points of the boundary of $\mathcal{O}$.

LEMMA 3.6. *Assume that* (2.4) *and* (2.8) *hold. If x is a regular point of $\partial\mathcal{O}$ then for $t > 0$,*

$$\lim_{\substack{y\in\mathcal{O}\\ y\to x}} P(\tau^y_{\mathcal{O}} > t) = 0. \tag{3.57}$$

Proof. A nonnegative measurable function f on H is called *α-excessive* if

$$\forall_{t\ge 0}\quad e^{-\alpha t}P_t f \le f, \tag{3.58}$$

$$\forall_{x\in H}\quad \lim_{t\to 0} e^{-\alpha t}P_t f(x) = f(x). \tag{3.59}$$

α-excessive functions were studied in [1] on locally compact spaces, which is not our case, but some arguments are similar.

Define

$$f(x) = Ee^{-\tau^x_{\mathcal{O}}}. \tag{3.60}$$

This is a Borel measurable function, since $\tau^x_{\mathcal{O}}$ is measurable with respect to x and ω. The function f is 1-excessive. The latter fact is an analogue of a special case of Proposition II.2.8 of [1].

By the Markov property,

$$e^{-t}P_t f(x) = Ee^{-\tau^{x,t}_{\mathcal{O}}},$$

where $\tau^{x,t}_{\mathcal{O}} = \inf\{s > t : X^x_s \in \mathcal{O}^c\}$. It is obvious that $\tau^{x,t}_{\mathcal{O}} \ge \tau^x_{\mathcal{O}}$ and for $t \to 0$, $\tau^{x,t}_{\mathcal{O}} \to \tau^x_{\mathcal{O}}$. Thus (3.58) and (3.59) follow with $\alpha = 1$.

A bounded α-excessive function f is lower semicontinuous. By (3.58),

$$f(y) - f(x) \ge (e^{-\alpha t}P_t f(y) - e^{-\alpha t}P_t f(x)) + (e^{-\alpha t}P_t f(x) - f(x)). \tag{3.61}$$

By (2.4) and (2.8) the semigroup P_t has the smoothing property and $P_t f$ is continuous. We also have (3.59), thus f is lower semicontinuous.

The function f defined in (3.60) is bounded by 1, is lower semicontinuous, and $f(x) = 1$ for a regular point $x \in \partial\mathcal{O}$. Therefore if y tends to x then for each $t > 0$, $P(\tau^y_{\mathcal{O}} > t)$ tends to 0. ∎

Proof of Theorem 2.3. In view of Theorem 2.6 it remains to show that $u(t,x) = P^{\mathcal{O}}_t\varphi(x)$ is jointly continuous on $[0,\infty) \times \mathcal{O}$ and (v) and (vi) hold.

$P^{\mathcal{O}}_t\varphi(x)$ is jointly continuous on $(0,\infty) \times \mathcal{O}$, since $P^{\mathcal{O}}_t\varphi(x)$ has locally bounded space and time derivatives.

Let $x_n \in \mathcal{O}$, $x_n \to x \in \mathcal{O}$ as $n \to \infty$, and $t_n > 0$, $t_n \to 0$. Fix $\varepsilon > 0$. By the continuity at x of the initial function φ and by Lemma 3.4 there exist positive constants r, r_0, t_0 such that $0 < r_0 < r < \operatorname{dist}(x, \mathcal{O}^c)/2$, and

$$\forall_{y\in B(x,r)}\quad |\varphi(y) - \varphi(x)| < \varepsilon/2,$$

$$\forall_{t\le t_0}\quad \sup_{y\in B(x,r_0)} P(\tau^y_{B(x,r)} \le t) \le Ce^{-1/(Ct^\alpha)}.$$

Let n_0 be such that for all $n \ge n_0$ we have $|x - x_n| < r_0$, $t_n \le t_0$ and $2\|\varphi\|_{\mathcal{O}}Ce^{-1/(Ct_n^\alpha)} < \varepsilon/2$. Then for all $n \ge n_0$,

$$\begin{aligned}
&|E\varphi(X^{x_n}_{t_n})1_{\tau^{x_n}_{\mathcal{O}} > t_n} - \varphi(x)| \\
&\quad \le E|\varphi(X^{x_n}_{t_n}) - \varphi(x)|1_{\tau^{x_n}_{B(x,r)} > t_n} + \|\varphi\|_{\mathcal{O}}P(\tau^{x_n}_{\mathcal{O}} > t_n,\ \tau^{x_n}_{B(x,r)} \le t_n) \\
&\qquad + \|\varphi\|_{\mathcal{O}}P(\tau^{x_n}_{B(x,r)} \le t_n) \\
&\quad \le \frac{\varepsilon}{2} + 2\|\varphi\|_{\mathcal{O}}P(\tau^{x_n}_{B(x,r)} \le t_n) \le \varepsilon.
\end{aligned}$$

Thus we obtain the continuity of u on $[0,\infty) \times \mathcal{O}$ and (v).

Let t_n be a sequence of positive numbers converging to $t > 0$ and $\{x_n\} \subset \mathcal{O}$ converge to some regular $x \in \partial\mathcal{O}$. Then for n large enough, $t_n > t/2$ and

$$|E\varphi(X^{x_n}_{t_n})1_{\tau^{x_n}_{\mathcal{O}} > t_n}| \le \|\varphi\|_{\mathcal{O}}P(\tau^{x_n}_{\mathcal{O}} > t/2).$$

By Lemma 3.6 the right hand side converges to zero as n tends to infinity, and we get property (vi). We conclude that u is a strong solution of (2.1) and (2.2). ∎

4. Proof of the uniqueness theorem. We now pass to the proof of Theorem 2.7.

Fix $x \in \mathcal{O}$ and $t > 0$, $t < T$. In the proof we will always consider processes starting from x, therefore we omit the superscript x in X^x. It will also be more convenient to write $X(t)$ instead of X_t.

STEP 1. Let $\varepsilon > 0$, $0 < \gamma < t$, $i \in \mathbb{N}$. First we construct sets $U_{\varepsilon,\gamma,i}$ that increase as i increases and such that $|Du(s,y)|$, $\|D^2u(s,y)\|$, $\|Q^{1/2}D^2u(s,y)Q^{1/2}\|_1$ and $|A^*Du(s,y)|$ are bounded on $[\gamma,T]\times\overline{U}_{\varepsilon,\gamma,i}$ and that the exit times of $X(t)$ from $U_{\varepsilon,\gamma,i}$ converge to $\tau^x_{\mathcal{O}}$ when i goes to infinity, on a subset of Ω with probability grater than $1-\varepsilon$.

For each $\varepsilon > 0$ there exists a compact set L_ε in H such that

$$P(X(s) \in L_\varepsilon,\ \forall_{s\in[0,T]}) \ge 1-\varepsilon \tag{4.1}$$

(see Proposition 2 of [24]). Let $i_0 \in \mathbb{N}$ be such that $\operatorname{dist}(x,\mathcal{O}^c)/4 \ge 1/i_0$. For $i \ge i_0$ we define an open set $(\mathcal{O}^c)_i = \bigcup_{y\in\mathcal{O}^c} B(y,2/i)$. Then $x \notin (\mathcal{O}^c)_i$ and $\operatorname{dist}(\mathcal{O}^c,(\mathcal{O}^c)_i^c) \ge 2/i$. Let $K_{\varepsilon,i} = L_\varepsilon \cap (\mathcal{O}^c)_i^c$; it is a compact subset of $\mathcal{O}$. By properties (i) and (ii) of a strong solution, for each $y \in K_{\varepsilon,i}$ there exists $r_y > 0$ such that $|Du(s,y)|$, $\|D^2u(s,y)\|$, $\|Q^{1/2}D^2u(s,y)Q^{1/2}\|_1$ and $|A^*Du(s,y)|$ are bounded on $[\gamma,T]\times\overline{B(y,r_y)}$. By compactness of $K_{\varepsilon,i}$ we can choose a finite covering $\{B(y_j,r_{y_j}\wedge 1/i)\}_{j=1}^{n_{\varepsilon,\gamma,i}}$ of $K_{\varepsilon,i}$. Let $V_{\varepsilon,\gamma,i} = \bigcup_{j=1}^{n_{\varepsilon,\gamma,i}} B(y_j,r_{y_j}\wedge 1/i)$. We define sets $U_{\varepsilon,\gamma,i}$ for $i \ge i_0$ inductively, taking $U_{\varepsilon,\gamma,i_0} = V_{\varepsilon,\gamma,i_0}$ and $U_{\varepsilon,\gamma,i} = U_{\varepsilon,\gamma,i-1}\cup V_{\varepsilon,\gamma,i}$.

For each $i \ge i_0$ we have $U_{\varepsilon,\gamma,i} \subset U_{\varepsilon,\gamma,i+1}$. Moreover, $\overline{U}_{\varepsilon,\gamma,i} \subset \mathcal{O}$, $x \in U_{\varepsilon,\gamma,i}$ and $|Du(s,y)|$, $\|D^2u(s,y)\|$, $|A^*Du(s,y)|$ and $\|Q^{1/2}D^2u(s,y)Q^{1/2}\|_1$ are bounded on $[\gamma,T]\times\overline{U}_{\varepsilon,\gamma,i}$.

Denote by $\tau_{\varepsilon,\gamma,i}$ the first exit time of X from $U_{\varepsilon,\gamma,i}$, that is, $\tau_{\varepsilon,\gamma,i} = \inf\{s>0 : X(s)\in U^c_{\varepsilon,\gamma,i}\}$. We show that

$$\lim_{i\to\infty}\tau_{\varepsilon,\gamma,i}\wedge T = \tau^x_{\mathcal{O}}\wedge T \quad \text{on } \{\omega : X(s,\omega)\in L_\varepsilon,\ \forall_{s\in[0,T]}\}. \tag{4.2}$$

Let ω be such that $X(s,\omega)\in L_\varepsilon$ for all $s\in[0,T]$. It is obvious that $\tau_{\varepsilon,\gamma,i}(\omega) \le \tau^x_{\mathcal{O}}(\omega)$ with a strict inequality if $\tau^x_{\mathcal{O}}(\omega) \ne \infty$. Consequently, if $\lim_{i\to\infty}\tau_{\varepsilon,\gamma,i}(\omega)\wedge T = T$ then (4.2) holds. Suppose that $\lim_{i\to\infty}\tau_{\varepsilon,\gamma,i}(\omega) = \sigma(\omega) < T$. We observe that $X(\tau_{\varepsilon,\gamma,i}(\omega)) \in (\mathcal{O}^c)_i$. Indeed, $X(\tau_{\varepsilon,\gamma,i}(\omega)) \in U^c_{\varepsilon,\gamma,i}$ and $K_{\varepsilon,i} = L_\varepsilon\cap(\mathcal{O}^c)_i^c \subset U_{\varepsilon,\gamma,i}$. Thus $X(\tau_{\varepsilon,\gamma,i}(\omega))\in(\mathcal{O}^c)_i$ or $X(\tau_{\varepsilon,\gamma,i}(\omega)) \in L^c_\varepsilon$ but our assumption on ω excludes the second case. Hence, by the construction of $(\mathcal{O}^c)_i$ we see that $\operatorname{dist}(X(\tau_{\varepsilon,\gamma,i}(\omega)),\mathcal{O}^c) < 2/i$. By the continuity of X and the dist function we obtain

$$\operatorname{dist}(X(\sigma(\omega)),\mathcal{O}^c) = \lim_{i\to\infty}\operatorname{dist}(X(\tau_{\varepsilon,\gamma,i}(\omega)),\mathcal{O}^c) = 0.$$

Since $\mathcal{O}^c$ is closed, $X(\sigma(\omega))\in\mathcal{O}^c$, which means that $\sigma(\omega) = \tau^x_{\mathcal{O}}(\omega)$.

STEP 2. We now show that

$$Eu(t-(t-\gamma)\wedge\tau_{\varepsilon,\gamma,i}, X((t-\gamma)\wedge\tau_{\varepsilon,\gamma,i})) = u(t,x). \tag{4.3}$$

We would like to apply Itô's formula to $u(t-s,X(s))$. Since in general $X(s)$ is not a strong solution to (2.3) we will apply Itô's formula to approximating processes.

Let $\{e_l\}_{l=1}^\infty$ be an orthonormal basis of H and let $\{\beta_l\}$ be a sequence of independent one-dimensional standard Wiener processes. Write $W_n(t) = \sum_{l=1}^n e_l\beta_l(t)$ and $W(t) = \sum_{l=1}^\infty e_l\beta_l(t)$. Let A_k be the Yosida approximations of A,

$$A_k = AkR(k,A), \quad \text{where} \quad R(k,A) = \int_0^\infty e^{-ks}S(s)\,ds.$$

Let $X_{n,k}$, X_n and X be the mild solutions to the following equations:

$$dX_{n,k}(t) = A_kX_{n,k}(t)dt + Q^{1/2}dW_n(t), \qquad X_{n,k}(0) = x, \tag{4.4}$$

$$dX_n(t) = AX_n(t)dt + Q^{1/2}dW_n(t), \qquad X_n(0) = x, \tag{4.5}$$

$$dX(t) = AX(t)dt + Q^{1/2}dW(t), \qquad X(0) = x, \tag{4.6}$$

respectively. Of course the process X has the same law as the process satisfying (2.3) and $X_{n,k}$ is a strong solution to (4.4).

By Theorem 2.2.6 of [7] we have

$$\lim_{k\to\infty}\sup_{s\le T}E|X_{n,k}(s)-X_n(s)|^2 = 0,$$
$$\lim_{n\to\infty}\sup_{s\le T}E|X_n(s)-X(s)|^2 = 0.$$

Let us fix temporarily ε, γ and i and define exit times

$$\tau_{n,k} = \inf\{s>0 : X_{n,k}(s)\in U^c_{\varepsilon,\gamma,i}\},$$
$$\tau_n = \inf\{s>0 : X_n(s)\in U^c_{\varepsilon,\gamma,i}\},$$
$$\tau = \inf\{s>0 : X(s)\in U^c_{\varepsilon,\gamma,i}\}.$$

Set $\sigma = \tau_{n,k}\wedge\tau_n\wedge\tau$. Since $X_{n,k}$ is a strong solution to (4.4), by Itô's formula we get

$$\begin{aligned}(4.7)\quad & u(t-(t-\gamma)\wedge\sigma, X_{n,k}((t-\gamma)\wedge\sigma))\\ &= u(t,x) - \int_0^{(t-\gamma)\wedge\sigma}\frac{\partial}{\partial t}u(t-r,X_{n,k}(r))\,dr\\ &\quad + \int_0^{(t-\gamma)\wedge\sigma}\langle Du(t-r,X_{n,k}(r)), AkR(k,A)X_{n,k}(r)\rangle\,dr\end{aligned}$$

$$+\int_0^{(t-\gamma)\wedge\sigma}\langle Du(t-r,X_{n,k}(r)),Q^{1/2}dW_n(r)\rangle$$
$$+\frac{1}{2}\int_0^{(t-\gamma)\wedge\sigma}\operatorname{Tr}D^2u(t-r,X_{n,k}(r))(Q^{1/2}I_n)(Q^{1/2}I_n)^*\,dr,$$

where I_n denotes the orthogonal projection on $\operatorname{span}\{e_1,\ldots,e_n\}$. The fact that u satisfies (2.1) allows us to write (4.7) in the form

(4.8) $$u(t-(t-\gamma)\wedge\sigma,X_{n,k}((t-\gamma)\wedge\sigma))$$
$$=u(t,x)+\int_0^{(t-\gamma)\wedge\sigma}\langle A^*Du(t-r,X_{n,k}(r)),kR(k,A)X_{n,k}(r)-X_{n,k}(r)\rangle\,dr$$
$$+\int_0^{(t-\gamma)\wedge\sigma}\langle Du(t-r,X_{n,k}(r)),Q^{1/2}dW_n(r)\rangle$$
$$+\frac{1}{2}\int_0^{(t-\gamma)\wedge\sigma}\operatorname{Tr}Q^{1/2}D^2u(t-r,X_{n,k}(r))Q^{1/2}[I_n-I]\,dr.$$

After taking the expectation of both sides of (4.8) the term with the stochastic integral disappears since $|Du(s,y)|$ is bounded on $[\gamma,T]\times\overline{U}_{\varepsilon,\gamma,i}$. Hence we get

(4.9) $$Eu(t-(t-\gamma)\wedge\sigma,X_{n,k}((t-\gamma)\wedge\sigma))=u(t,x)$$
$$+E\int_0^{(t-\gamma)\wedge\sigma}\langle A^*Du(t-r,X_{n,k}(r)),kR(k,A)X_{n,k}(r)-X_{n,k}(r)\rangle\,dr$$
$$+E\frac{1}{2}\int_0^{(t-\gamma)\wedge\sigma}\operatorname{Tr}Q^{1/2}D^2u(t-r,X_{n,k}(r))Q^{1/2}[I_n-I]\,dr.$$

To pass in (4.9) to the limit as $k\to\infty$ and $n\to\infty$ we show first that

(4.10) $$\tau_n\wedge\tau\xrightarrow[n\to\infty]{\text{a.s.}}\tau\quad\text{and}\quad\tau_{n,k}\wedge\tau_n\xrightarrow[k\to\infty]{\text{a.s.}}\tau_n.$$

Consider first $\tau_n\wedge\tau$. It is enough to show that $P(\liminf_{n\to\infty}\tau_n<\tau)=0$. Suppose that it does not hold. Then

$$P(\liminf_{n\to\infty}\tau_n<\tau)=P(\exists_{a\ \text{rational}}\liminf_{n\to\infty}\tau_n<a<\tau)>0.$$

Hence there exists an a such that

(4.11) $$P(\liminf_{n\to\infty}\tau_n<a<\tau)>0.$$

On the set $a<\tau$,

(4.12) $$\operatorname{dist}(X,U^c_{\varepsilon,\gamma,i})=\inf_{s\in[0,a]}\{\operatorname{dist}(X(s),U^c_{\varepsilon,\gamma,i})\}>0$$

and

(4.13) $$|X_n(\tau_n)-X(\tau_n)|1_{\tau_n<a<\tau}\ge\inf_{s\in[0,a]}\{\operatorname{dist}(X(s),U^c_{\varepsilon,\gamma,i})\}1_{\tau_n<a<\tau}.$$

Thus we get

$$\liminf_{n\to\infty}E|X_n(\tau_n\wedge a)-X(\tau_n\wedge a)|^2\ge\liminf_{n\to\infty}E|X_n(\tau_n)-X(\tau_n)|^2 1_{\tau_n<a<\tau}.$$

By (4.13) and Fatou's lemma

$$\liminf_{n\to\infty}E|X_n(\tau_n\wedge a)-X(\tau_n\wedge a)|^2\ge\liminf_{n\to\infty}E[\operatorname{dist}(X,U^c_{\varepsilon,\gamma,i})]^2 1_{\tau_n<a<\tau}$$
$$\ge E[\operatorname{dist}(X,U^c_{\varepsilon,\gamma,i})]^2\liminf_{n\to\infty}1_{\tau_n<a<\tau}.$$

But $\liminf_{n\to\infty}1_{\tau_n<a<\tau}\ge 1_{\liminf_{n\to\infty}\tau_n<a<\tau}$ and

(4.14) $$\liminf_{n\to\infty}E|X_n(\tau_n\wedge a)-X(\tau_n\wedge a)|^2\ge E[\operatorname{dist}(X,U^c_{\varepsilon,\gamma,i})]^2 1_{\liminf_{n\to\infty}\tau_n<a<\tau}.$$

By (4.11) and (4.12) the right hand side of (4.14) is positive. But on the other hand

$$E|X_n(\tau_n\wedge a)-X(\tau_n\wedge a)|^2=E\Big|\int_0^{\tau_n\wedge a}S(\tau_n\wedge a-s)Q^{1/2}\,d\widetilde{W}_n(s)\Big|^2$$

where $\widetilde{W}_n=W-W_n$. Thus we see that

$$E|X_n(\tau_n\wedge a)-X(\tau_n\wedge a)|^2$$
$$=E\int_0^{\tau_n\wedge a}\operatorname{Tr}S(\tau_n\wedge a-s)Q^{1/2}(I-I_n)Q^{1/2}S^*(\tau_n\wedge a-s)\,ds$$
$$=E\int_0^{\tau_n\wedge a}\operatorname{Tr}S(s)Q^{1/2}(I-I_n)Q^{1/2}S^*(s)\,ds\xrightarrow[n\to\infty]{}0,$$

which is a contradiction. Thus we have $P(\liminf_{n\to\infty}\tau_n\ge\tau)=1$, and consequently $\tau_n\wedge\tau$ converges to τ almost surely as n goes to infinity.

Similar arguments prove the second convergence in (4.10):

(4.15) $$E|X_{n,k}(\tau_{n,k}\wedge a)-X_n(\tau_{n,k}\wedge a)|^2$$
$$\le 2E|S_k(\tau_{n,k}\wedge a)x-S(\tau_{n,k}\wedge a)x|^2$$
$$+2E\Big|\int_0^{\tau_{n,k}\wedge a}[S_k(\tau_{n,k}\wedge a-s)-S(\tau_{n,k}\wedge a-s)]Q^{1/2}dW_n(s)\Big|^2$$

$$\leq 2\sup_{s\leq a}|S_k(s)x - S(s)x|$$

$$+2E\int_0^{\tau_{n,k}\wedge a}\operatorname{Tr}[S_k(s)-S(s)]Q^{1/2}I_nQ^{1/2}[S_k(s)-S(s)]^*\,ds \xrightarrow[k\to\infty]{} 0.$$

Thus $\tau_{n,k}\wedge\tau_n \to \tau_n$ as $k\to\infty$ and the proof of (4.10) is complete.

Now we are ready to pass to the limit in (4.9). To do this note that $X_{n,k}((t-\gamma)\wedge\tau_{n,k}\wedge\tau_n\wedge\tau)$ converges in L^2 to $X_n((t-\gamma)\wedge\tau_n\wedge\tau)$, since

$$(4.16)\quad E|X_{n,k}((t-\gamma)\wedge\tau_{n,k}\wedge\tau_n\wedge\tau) - X_n((t-\gamma)\wedge\tau_n\wedge\tau)|^2$$
$$\leq 2E|X_{n,k}((t-\gamma)\wedge\tau_{n,k}\wedge\tau_n\wedge\tau) - X_n((t-\gamma)\wedge\tau_{n,k}\wedge\tau_n\wedge\tau)|^2$$
$$+2E|X_n((t-\gamma)\wedge\tau_{n,k}\wedge\tau_n\wedge\tau) - X_n((t-\gamma)\wedge\tau_n\wedge\tau)|^2.$$

As in (4.15) we can show that the first term on the right hand side of (4.16) converges to zero as k goes to infinity. So does the second term, since $U_{\varepsilon,\gamma,i}$ is a bounded set, X_n is a continuous process and $\tau_{n,k}\wedge\tau_n$ converges to τ_n. The function $u(s,y)$ is jointly continuous on $[0,T]\times\mathcal{O}$. Thus

$$(4.17)\quad \lim_{k\to\infty} Eu(t-(t-\gamma)\wedge\tau_{n,k}\wedge\tau_n\wedge\tau, X_{n,k}((t-\gamma)\wedge\tau_{n,k}\wedge\tau_n\wedge\tau))$$
$$= Eu(t-(t-\gamma)\wedge\tau_n\wedge\tau, X_n((t-\gamma)\wedge\tau_n\wedge\tau)),$$

and similarly, by the first part of (4.10) and the continuity of X we have

$$(4.18)\quad \lim_{n\to\infty} Eu(t-(t-\gamma)\wedge\tau_n\wedge\tau, X_n((t-\gamma)\wedge\tau_n\wedge\tau))$$
$$= Eu(t-(t-\gamma)\wedge\tau, X((t-\gamma)\wedge\tau)).$$

Now we show that the integrals on the right hand side of (4.9) converge to zero. Since $|A^*Du(s,y)|$ is bounded on $[\gamma,T]\times\overline{U}_{\varepsilon,\gamma,i}$ we obtain

$$(4.19)\quad \Big|E\int_0^{(t-\gamma)\wedge\sigma}\langle A^*Du(t-r,X_{n,k}(r)), kR(k,A)X_{n,k}(r) - X_{n,k}(r)\rangle\,dr\Big|$$
$$\leq CE\int_0^{t-\gamma}|kR(k,A)X_{n,k}(r) - X_{n,k}(r)|\,dr$$
$$\leq C\int_0^{t-\gamma}(E|kR(k,A)X_{n,k}(r) - X_{n,k}(r)|^2)^{1/2}\,dr$$
$$\leq C\int_0^{t-\gamma}(\sup_{r\in[0,t-\gamma]}E|X_{n,k}(r) - X_n(r)|^2)^{1/2}\,dr$$
$$+C\int_0^{t-\gamma}(E|kR(k,A)X_n(r) - X_n(r)|^2)^{1/2}\,dr \xrightarrow[k\to 0]{} \infty.$$

For the last term in (4.9) we have

$$(4.20)\quad \Big|E\int_0^{(t-\gamma)\wedge\sigma}[\operatorname{Tr}Q^{1/2}D^2u(t-r,X_{n,k}(r))Q^{1/2}(I-I_n)]\,dr\Big|$$
$$\leq \int_0^{t-\gamma}E\|Q^{1/2}D^2u(t-r,X_{n,k}(r))Q^{1/2}1_{r\leq\tau_{n,k}\wedge\tau_n\wedge\tau}$$
$$- Q^{1/2}D^2u(t-r,X_n(r))Q^{1/2}1_{r\leq\tau_n\wedge\tau}\|_1\,dr$$
$$+\int_0^{t-\gamma}E\|Q^{1/2}D^2u(t-r,X_n(r))Q^{1/2}(I-I_n)1_{r\leq\tau_n\wedge\tau}\|_1\,dr.$$

By assumption (ii) and (4.10) the first term on the right hand side of (4.20) converges to zero as k tends to infinity. For the second term we have

$$(4.21)\quad \int_0^{t-\gamma}E\|Q^{1/2}D^2u(t-r,X_n(r))Q^{1/2}1_{r\leq\tau_n\wedge\tau}(I-I_n)\|_1\,dr$$
$$\leq \int_0^{t-\gamma}E\|Q^{1/2}D^2u(t-r,X_n(r))Q^{1/2}1_{r\leq\tau_n\wedge\tau}$$
$$- Q^{1/2}D^2u(t-r,X(r))Q^{1/2}1_{r\leq\tau}\|_1\,dr$$
$$+\int_0^{t-\gamma}E\|Q^{1/2}D^2u(t-r,X(r))Q^{1/2}(I-I_n)1_{r\leq\tau}\|_1\,dr.$$

Again, by (ii) and (4.10) the first term on the right hand side converges to zero as n goes to zero. So does the second term since for any $r\leq(t-\gamma)\wedge\tau$ the operator $Q^{1/2}D^2u(t-r,X(r))Q^{1/2}$ is nuclear. Hence $\|Q^{1/2}D^2u(t-r, X(r))Q^{1/2}(I-I_n)\|_1$ converges to zero and

$$\int_0^{t-\gamma}E\|Q^{1/2}D^2u(t-r,X(r))Q^{1/2}1_{r\leq\tau}\|_1\,dr<\infty.$$

Thus we obtain

$$(4.22)\quad \lim_{n\to\infty}\lim_{k\to\infty}E\int_0^{(t-\gamma)\wedge\sigma}[\operatorname{Tr}Q^{1/2}D^2u(t-r,X_{n,k}(r))Q^{1/2}(I-I_n)]\,dr = 0.$$

Recall the notation $\tau_{\varepsilon,\gamma,i}=\tau$. Letting first $k\to\infty$ and then $n\to\infty$ in (4.9), by (4.17)–(4.19) and (4.22) we obtain (4.3).

STEP 3. We now show that if $i\to\infty$, $\gamma\to 0$ and finally $\varepsilon\to 0$, then the left hand side of (4.3) converges to $E\varphi(X(t))1_{\tau_{\mathcal{O}}^x>t}$, which is the desired conclusion.

We have

$$(4.23)\quad |Eu(t-(t-\gamma)\wedge\tau_{\varepsilon,\gamma,i}, X((t-\gamma)\wedge\tau_{\varepsilon,\gamma,i}))-E\varphi(X(t))1_{\tau^x_{\mathcal{O}}>t}|$$
$$\le |Eu(\gamma, X(t-\gamma))1_{\tau_{\varepsilon,\gamma,i}\ge t-\gamma}1_{X\in L_\varepsilon}-E\varphi(X(t))1_{\tau^x_{\mathcal{O}}>t}1_{X\in L_\varepsilon}|$$
$$+|Eu(t-\tau_{\varepsilon,\gamma,i}, X(\tau_{\varepsilon,\gamma,i}))1_{\tau_{\varepsilon,\gamma,i}<t-\gamma}1_{X\in L_\varepsilon}|+2\|u\|_{[0,T]\times\mathcal{O}}P(X\notin L_\varepsilon)$$
$$=I_1(\varepsilon,\gamma,i)+I_2(\varepsilon,\gamma,i)+I_3(\varepsilon).$$

Observe that

$$(4.24)\quad I_1(\varepsilon,\gamma,i)\le E|u(\gamma,X(t-\gamma))-\varphi(X(t))|1_{\tau^x_{\mathcal{O}}>t}$$
$$+\|u\|_{[0,T]\times\mathcal{O}}E|1_{\tau_{\varepsilon,\gamma,i}\ge t-\gamma}-1_{\tau^x_{\mathcal{O}}>t}|1_{X\in L_\varepsilon}.$$

By (4.2) and the fact that $\tau_{\varepsilon,\gamma,i}<\tau^x_{\mathcal{O}}$ if $\tau^x_{\mathcal{O}}<\infty$ we see that

$$(4.25)\quad \limsup_{i\to\infty} E|1_{\tau_{\varepsilon,\gamma,i}\ge t-\gamma}-1_{\tau^x_{\mathcal{O}}>t}|1_{X\in L_\varepsilon}$$
$$\le \limsup_{i\to\infty} E|1_{\tau_{\varepsilon,\gamma,i}\ge t-\gamma}-1_{\tau^x_{\mathcal{O}}>t-\gamma}|1_{X\in L_\varepsilon}+P(t\ge\tau^x_{\mathcal{O}}>t-\gamma)$$
$$=P(t\ge\tau^x_{\mathcal{O}}>t-\gamma).$$

Moreover, by the assumption on $\tau^x_{\mathcal{O}}$,

$$(4.26)\quad \lim_{\gamma\to0}P(t\ge\tau^x_{\mathcal{O}}>t-\gamma)=P(\tau^x_{\mathcal{O}}=t)=0.$$

By (4.25) and (4.26) the second term in (4.24) converges to zero when we let $i\to\infty$ and then $\gamma\to0$. The first term on the right hand side of (4.24) does not depend on i and converges to zero as $\gamma\to0$ by the joint continuity and property (v) of the strong solution. Thus from (4.24) we obtain

$$(4.27)\quad \limsup_{\gamma\to0}\limsup_{i\to\infty}I_1(\varepsilon,\gamma,i)=0.$$

By (4.2) and property (vi),

$$(4.28)\quad \lim_{i\to\infty}I_2(\varepsilon,\gamma,i)=0.$$

We also have

$$(4.29)\quad \lim_{\varepsilon\to0}I_3(\varepsilon)\le\lim_{\varepsilon\to0}C\varepsilon=0.$$

By (4.27)–(4.29) applied to (4.23) we obtain

$$\limsup_{\varepsilon\to0}\limsup_{\gamma\to0}\limsup_{i\to\infty}|Eu(t-(t-\gamma)\wedge\tau_{\varepsilon,\gamma,i},X((t-\gamma)\wedge\tau_{\varepsilon,\gamma,i}))$$
$$-E\varphi(X_t)1_{\tau^x_{\mathcal{O}}>t}|=0.$$

Consequently, from (4.3) it follows that $u(t,x)=E\{\varphi(X(t))1_{\tau^x_{\mathcal{O}}>t}\}$. ∎

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M-complete approximate identities in operator spaces

by

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Abstract. This work introduces the concept of an M-complete approximate identity (M-cai) for a given operator subspace X of an operator space Y. M-cai's generalize central approximate identities in ideals in C^*-algebras, for it is proved that if X admits an M-cai in Y, then X is a complete M-ideal in Y. It is proved, using "special" M-cai's, that if $\mathcal{J}$ is a nuclear ideal in a C^*-algebra $\mathcal{A}$, then $\mathcal{J}$ is completely complemented in Y for any (isomorphically) locally reflexive operator space Y with $\mathcal{J} \subset Y \subset \mathcal{A}$ and $Y/\mathcal{J}$ separable. (This generalizes the previously known special case where $Y = \mathcal{A}$, due to Effros–Haagerup.) In turn, this yields a new proof of the Oikhberg–Rosenthal Theorem that $\mathcal{K}$ is completely complemented in any separable locally reflexive operator superspace, where $\mathcal{K}$ is the C^*-algebra of compact operators on ℓ^2. M-cai's are also used in obtaining some special affirmative answers to the open problem of whether $\mathcal{K}$ is Banach-complemented in $\mathcal{A}$ for any separable C^*-algebra $\mathcal{A}$ with $\mathcal{K} \subset \mathcal{A} \subset B(\ell^2)$. It is shown that if, conversely, X is a complete M-ideal in Y, then X admits an M-cai in Y in the following situations: (i) Y has the (Banach) bounded approximation property; (ii) Y is 1-locally reflexive and X is λ-nuclear for some $\lambda \ge 1$; (iii) X is a closed 2-sided ideal in an operator algebra Y (via the Effros–Ruan result that then X has a contractive algebraic approximate identity). However, it is shown that there exists a separable Banach space X which is an M-ideal in $Y = X^{**}$, yet X admits no M-approximate identity in Y.

Contents

Introduction. Let $\mathcal{K}$ denote the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space H. Consider the following open

PROBLEM A. Let $X \subset Y$ be separable operator spaces, and let $T : X \to \mathcal{K}$ be a completely bounded (linear) operator. Does T admit a bounded linear

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