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M-complete approximate identities in operator spaces

by

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Abstract. This work introduces the concept of an M-complete approximate identity (M-cai) for a given operator subspace X of an operator space Y. M-cai's generalize central approximate identities in ideals in C^*-algebras, for it is proved that if X admits an M-cai in Y, then X is a complete M-ideal in Y. It is proved, using "special" M-cai's, that if $\mathcal{J}$ is a nuclear ideal in a C^*-algebra $\mathcal{A}$, then $\mathcal{J}$ is completely complemented in Y for any (isomorphically) locally reflexive operator space Y with $\mathcal{J} \subset Y \subset \mathcal{A}$ and $Y/\mathcal{J}$ separable. (This generalizes the previously known special case where $Y = \mathcal{A}$, due to Effros–Haagerup.) In turn, this yields a new proof of the Oikhberg–Rosenthal Theorem that $\mathcal{K}$ is completely complemented in any separable locally reflexive operator superspace, where $\mathcal{K}$ is the C^*-algebra of compact operators on ℓ^2. M-cai's are also used in obtaining some special affirmative answers to the open problem of whether $\mathcal{K}$ is Banach-complemented in $\mathcal{A}$ for any separable C^*-algebra $\mathcal{A}$ with $\mathcal{K} \subset \mathcal{A} \subset B(\ell^2)$. It is shown that if, conversely, X is a complete M-ideal in Y, then X admits an M-cai in Y in the following situations: (i) Y has the (Banach) bounded approximation property; (ii) Y is 1-locally reflexive and X is λ-nuclear for some $\lambda \geq 1$; (iii) X is a closed 2-sided ideal in an operator algebra Y (via the Effros–Ruan result that then X has a contractive algebraic approximate identity). However, it is shown that there exists a separable Banach space X which is an M-ideal in $Y = X^{**}$, yet X admits no M-approximate identity in Y.

Contents

Introduction. Let $\mathcal{K}$ denote the C^*-algebra of compact operators on a separable infinite-dimensional Hilbert space H. Consider the following open

PROBLEM A. Let $X \subset Y$ be separable operator spaces, and let $T : X \to \mathcal{K}$ be a completely bounded (linear) operator. Does T admit a bounded linear

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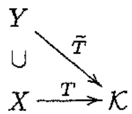
extension $\widetilde{T} : Y \to \mathcal{K}$? That is, can we find a bounded $\widetilde{T}$ completing the following diagram?

(0.1)



(See [BP], [ER2], [Pa], [Pi] or [Ro] for basic properties of operator spaces that we use here. For the definition and a brief sketch, see the beginning of our Section 1.) An interesting example due to E. Kirchberg [Ki] yields that one cannot, in general, complete this diagram with a completely bounded $\widetilde{T}$; by a result of [OR], it follows there are even locally reflexive separable operator spaces where this is the case (in fact, where $Y = C_1$, the space of trace class operators).

However, the following result is proved in [OR] (see Section 2 for the definition of locally reflexive operator spaces).

THEOREM 1. *Assume in* (0.1) *that* T *is a complete surjective isomorphism and* Y *is locally reflexive* (*with* Y *separable*). *Then there exists a completely bounded* $\widetilde{T}$ *completing the diagram* (0.1).

We give here a new proof of Theorem 1, using also another structural result obtained in [OR], as well as positive solutions to Problem A in special cases. Our methods involve the new concept of an M-complete approximate identity (an M-cai) for a given operator space X contained in another space Y; this is a uniformly bounded net (T_α) of (linear) operators from Y to X satisfying certain conditions (see Definition 1.1). For example, if X is an ideal in a C^*-algebra Y and (x_α) is a central approximate unit for X in Y consisting of positive contractions, and if we let $T_\alpha(y) = x_\alpha y$ for all $y \in \mathcal{A}$, then (T_α) is an M-cai for X in Y (see Proposition 1.4).

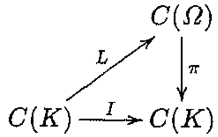
Theorem 1 may be regarded as a "quantized" version of a result discovered by A. Sobczyk in 1941:

SOBCZYK'S THEOREM [So]. *Let* $X \subset Y$ *be separable Banach spaces and* $T : X \to c_0$ *be a given bounded operator. There exists a bounded extension* $\widetilde{T} : Y \to c_0$ *of* T *with* $\|\widetilde{T}\| \le 2\|T\|$; *moreover*, 2 *is the best constant here for general* Y.

Many proofs have been given since [So] appeared, cf. [Pe], [V], [HWW], and [Ro]. We give yet another proof here, which perhaps explains why this isomorphic result (i.e., constant 2) really follows from the application of two isometric results whose quantized versions form the basis for our approach to Problem A and Theorem 1. One of these is a classical theorem concerning $C(\Omega)$, the space of continuous functions on a compact Hausdorff space, published in 1933, namely

BORSUK'S THEOREM [B]. *Let* Ω *be a compact Hausdorff space and* K *be a closed metrizable subset. There exists a norm-one linear operator* $L : C(K) \to C(\Omega)$ *so that* $\pi L(f) = f$ *for all* $f \in C(K)$, *where* $\pi f = f|_K$ *for all* $f \in C(\Omega)$. *That is, we have the diagram*

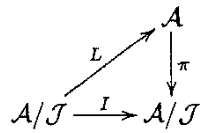
(0.2)



Now of course $C(\Omega)$ is a commutative unital C^*-algebra; if K is a closed subset of Ω and $\mathcal{J}_K = \{f \in C(\Omega) : f|_K = 0\}$, then $\mathcal{J}_K$ is an ideal in $C(\Omega)$ and every (closed) ideal is of this form. Moreover, K is metrizable iff $C(K) = C(\Omega)/\mathcal{J}_K$ is separable. In view of the Gelfand–Neumark Theorem [GN], Borsuk's result may then be reformulated as follows:

THEOREM. *Let* $\mathcal{A}$ *be a unital commutative* C^**-algebra and* $\mathcal{J}$ *a* (*closed*) *ideal in* $\mathcal{A}$ *with* $\mathcal{A}/\mathcal{J}$ *separable. Then there exists a contractive* (*linear*) *lift* $L : \mathcal{A}/\mathcal{J} \to \mathcal{A}$ *of* $I_{\mathcal{A}/\mathcal{J}}$; *that is, the following diagram holds*:

(0.3)



Let us note also that if L is a contractive linear map satisfying (0.3), then $\mathcal{J}$ *is contractively co-complemented in* $\mathcal{A}$ via the map $P = I - L\pi$; that is, P is a projection onto $\mathcal{J}$ such that $\|I - P\| = \|L\pi\| = 1$.

We now apply Borsuk's Theorem and the injectivity of ℓ^∞ to obtain a

Proof of Sobczyk's Theorem. Let X, Y and T be as in the statement, and regard $c_0 \subset \ell^\infty$; note that ℓ^∞ is a commutative C^*-algebra and c_0 is an ideal in ℓ^∞. Now ℓ^∞ is isometrically injective (by an easy application of the Hahn–Banach theorem). Hence we may choose a linear extension $\widetilde{\widetilde{T}} : Y \to \ell^\infty$ of T with

$$\|\widetilde{\widetilde{T}}\| = \|T\|. \tag{0.4}$$

Now let $\mathcal{A}$ be the C^*-subalgebra of ℓ^∞ generated by c_0, $\mathbf{1}$, and $\widetilde{\widetilde{T}}(Y)$. Then since $c_0 \subset \mathcal{A} \subset \ell^\infty$, c_0 is an ideal in $\mathcal{A}$ and of course $\mathcal{A}$ is commutative; hence by Borsuk's theorem, c_0 is co-contractively complemented in $\mathcal{A}$. Thus we may choose a projection P mapping $\mathcal{A}$ onto c_0 with $\|I - P\| = 1$. Thus

$$\|P\| \le 2. \tag{0.5}$$

Then $\widetilde{T} := P\widetilde{\widetilde{T}}$ is an extension of T satisfying

$$\|\widetilde{T}\| \le \|P\| \cdot \|\widetilde{\widetilde{T}}\| \le 2\|T\| \tag{0.6}$$

by (0.4) and (0.5). ∎

We obtain here the following "quantized" version of Borsuk's Theorem (see Theorem 2.1 and the Theorem of the Appendix).

THEOREM 2. *Let $\mathcal{J}$ be a nuclear (2-sided closed) ideal in a C^*-algebra $\mathcal{A}$. Let $\lambda \ge 1$ and let Y be a closed linear subspace of $\mathcal{A}$ with $\mathcal{J} \subset Y$ so that $Y/\mathcal{J}$ is separable and Y is λ-locally reflexive. Then for every $\varepsilon > 0$, there exists a completely bounded lift $L : Y/\mathcal{J} \to Y$ of $I_{Y/\mathcal{J}}$ with $\|L\|_{\mathrm{cb}} < \lambda + \varepsilon$. That is,*

$$\begin{array}{ccc} & & Y \\ & \overset{L}{\nearrow} & \downarrow \pi \\ Y/\mathcal{J} & \xrightarrow{\ I\ } & Y/\mathcal{J} \end{array} \tag{0.6}$$

holds, where π is the quotient map. Moreover, if $\lambda = 1$, L may be chosen to be a complete isometry.

E. Effros and U. Haagerup establish this result for the case $Y = \mathcal{A}$ in [EH] (when necessarily $\lambda = 1$). Although our proof uses a basic idea in their discussion, the latter is isometric, and does not adapt to the case $\lambda > 1$, which however is crucial in order to recapture Theorem 1, via the following result obtained in [OR] ($B(H)$ denotes the space of bounded linear operators on H).

THEOREM 3 (Theorem 1.1 of [OR]). *Let Y a separable operator space, X a subspace of Y, and $T : X \to B(H)$ a complete isomorphic injection of X be given. There exists a complete isomorphic injection $T' : Y \to B(H)$ extending T. That is, if $X' = T(X)$ and $Y' = T'(Y)$, then $X' \subset Y' \subset B(H)$ and T, T' are complete isomorphisms with*

$$\begin{array}{ccc} Y & \xrightarrow{\ T'\ } & Y' \\ \cup & & \cup \\ X & \xrightarrow{\ T\ } & X' \end{array} \tag{0.7}$$

We now obtain Theorem 1 in the same spirit as our proof of Sobczyk's Theorem.

Proof of Theorem 1. Let $X \subset Y$ be separable operator spaces with Y locally reflexive and let $T : X \to \mathcal{K}$ be a complete surjective isomorphism. Now $\mathcal{K} \subset B(H)$ ($H = \ell^2$, say), and $\mathcal{K}$ is an *ideal* in $B(H)$. Letting $X' = \mathcal{K}$, choose

$$Y' \supset \mathcal{K} \quad \text{with } Y' \subset B(H) \text{ and } \widetilde{T} : Y \to Y'$$

satisfying the conclusion of Theorem 3. Then of course Y' is separable *and* locally reflexive (since $\widetilde{T}$ is a complete isomorphism); Theorem 2 then yields a completely bounded lift $L : Y'/\mathcal{K} \to Y'$ of $I_{Y'/\mathcal{K}}$. It follows that $P := I_{Y'} - L\pi$ is a completely bounded projection of Y' onto $\mathcal{K}$, where $\pi : Y' \to Y'/\mathcal{K}$ is the quotient map, whence $\widetilde{T} := PT'$ is a completely bounded operator satisfying (0.1). ∎

REMARKS. (a) It is proved in [OR] that T' may be chosen satisfying (0.7) with

$$\|T'\|_{\mathrm{cb}} \le 3\|T\|_{\mathrm{cb}} \quad \text{and} \quad \|T'\|_{\mathrm{cb}}\|T'^{-1}\|_{\mathrm{cb}} \le 12\|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}} + 6. \tag{0.8}$$

The proof of Theorem 1 then yields the existence of absolute positive constants A and B (with $A \le 108$ and $B < 55$) so that if Y is λ-locally reflexive, then $\widetilde{T}$ may be chosen satisfying (0.1) with

$$\|\widetilde{T}\|_{\mathrm{cb}} \le (A\gamma + B\lambda)\|T\|_{\mathrm{cb}}, \quad \text{where } \gamma = \|T\|_{\mathrm{cb}}\|T^{-1}\|_{\mathrm{cb}}. \tag{0.9}$$

What are the optimal values of these constants? Our estimates (as well as the constants in (0.8)) are surely far from best possible. We must have, however, $A + B \ge 2$, even in the case where $\lambda = 1$ and T is a complete isometry. (Actually, it seems likely that in this case, $A + B \ge 3$, and also that there exists such a T so that Y' cannot be chosen 1-locally reflexive.)

(b) N. Ozawa [O] has also (independently) obtained another proof of Theorem 1, somewhat along the same lines as our argument.

Return now to Problem A, which is easily seen to be a special case of the open

PROBLEM B. Let $\mathcal{J} \subset \mathcal{A}$ be a (closed 2-sided) ideal in a C^*-algebra $\mathcal{A}$ with $\mathcal{A}/\mathcal{J}$ separable. Is $\mathcal{J}$ complemented in $\mathcal{A}$?

Again, this can be rephrased as a lifting problem, namely, does there exist a (bounded linear) lift $L : \mathcal{A}/\mathcal{J} \to \mathcal{A}$ of $I_{\mathcal{A}/\mathcal{J}}$? (This problem dates to at least 1974, when it appeared in [A].)

To see that Problem A is a special case, consider $\mathcal{K}$ as an ideal in $B(H)$ and let X, Y and T be as in the statement of Problem A. Now $B(H)$ is isometrically injective in the operator space category. Thus there exists a linear extension $T' : Y \to B(H)$ of T satisfying $\|T'\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$. Now let $\mathcal{A}$ denote the C^*-algebra generated by $\mathcal{K}$ and $T'(Y)$. Then $\mathcal{A}$ is separable; were P a bounded linear projection from $\mathcal{A}$ onto $\mathcal{K}$, $\widetilde{T} := PT$ would be a bounded extension of $\widetilde{T}$ satisfying (0.1). That is, *Problem A is equivalent to the special case of Problem B when $\mathcal{J} = \mathcal{K} \subset \mathcal{A} \subset B(H)$.*

Problem B has an affirmative answer if $\mathcal{A}/\mathcal{J}$ has the (Banach) bounded approximation property, by rather deep work of T. Ando ([An]; see also [CE2] and Theorem II.2.1 of [HWW]). Most of the affirmative known results actually yield that $I_{\mathcal{A}/\mathcal{J}}$ has a completely positive contractive lift L; cf.

[CE1], [EH], [ER2]. (The example in [Ki] does have a contractive lift but no completely bounded one.) The methods of the present paper recapture Ando's theorem in the special setting of Problem A, via the following result (see Theorem 2.8 and Corollary 2.9).

THEOREM 4. *Let $\mathcal{A}$ be a C^*-subalgebra of $B(H)$ with $\mathcal{K} \subset \mathcal{A}$ and assume for some $\lambda \ge 1$ that $(\mathcal{K}, \mathcal{A})$ has λ-extendable local liftings. Then for every $\varepsilon > 0$ and separable Y with $\mathcal{K} \subset Y \subset \mathcal{A}$, there exists a lift $L : Y/\mathcal{K} \to Y$ of $I_{Y/\mathcal{K}}$ with $\|L\| < \lambda + \varepsilon$. In particular, this holds if $\mathcal{A}/\mathcal{K}$ has the λ-bap or $\mathcal{A}$ is λ-extendably locally reflexive.*

(We say that $(\mathcal{K}, \mathcal{A})$ has λ-extendable local liftings (λ-ell's) if for all $\varepsilon > 0$, and finite-dimensional subspaces E of $\mathcal{A}/\mathcal{K}$, there exists a linear operator $L : \mathcal{A}/\mathcal{K} \to \mathcal{A}^{**}$ with $\|L\| < \lambda + \varepsilon$ so that $L(E) \subset \mathcal{A}$ and $L|E$ is a lift of the identity injection of E into $\mathcal{A}/\mathcal{K}$. See Propositions 2.6 and 2.7 for general permanence properties.)

COROLLARY. *If $(\mathcal{K}, B(H))$ has λ-ell's for some $\lambda \ge 1$, Problem A has an affirmative answer.*

(The case where $\mathcal{A}$ is extendably locally reflexive in Theorem 4 was previously obtained in [OR]. The corollary thus extends the consequence obtained there: Problem A has an affirmative answer if $B(H)$ is an extendably locally reflexive Banach space.)

We now discuss the methods and setting of our results. What is the Banach space technology which yields the theorems of Borsuk, Effros–Haagerup and our generalizations thereof? Why do Banach and operator space hypotheses intervene in the algebraic setting of our Theorem 2 and Theorem 4, and what is the appropriate operator space setting of these results? The answer to the first question lies in the concept of an M-ideal, as pioneered in [AE]; see [HWW] for a comprehensive reference.

We briefly recall the relevant notions.

DEFINITION 0.1. Let $X \subset Y$ be Banach/operator spaces.

(a) X is called an *M-summand* in Y if there exists a closed linear subspace Z of Y with

(0.10.i) $$X \oplus Z = Y$$

so that

(0.10.ii) $$\|x + z\| = \max\{\|x\|, \|z\|\} \quad \text{for all } x \in X \text{ and } z \in Z.$$

In case these are operator spaces, X is called a *complete M-summand* if Z satisfying (0.10.i) also satisfies

(0.10′.ii) $\|(x_{ij}+z_{ij})\| = \max\{\|(x_{ij})\|, \|(z_{ij})\|\}$ for all n and $n \times n$ matrices (x_{ij}) and (z_{ij}) of elements of X and Z respectively.

(b) X is called an *L-summand* if Z can be chosen satisfying (0.10.i) so that

(0.11) $$\|x + z\| = \|x\| + \|z\| \quad \text{for all } x \in X \text{ and } z \in Z.$$

(c) X is called an *M-ideal* (resp. *complete M-ideal*) in Y if $X^{**} = X^{\perp\perp}$ is an M-summand (resp. complete M-summand) in Y^{**}.

It turns out that M- (resp. L-) summands for X are unique; if $X \oplus Z$ is the corresponding M- (resp. L-) decomposition, the projection P from Y onto X with kernel Z is called the *M-* (resp. *L-*) *projection* onto X. Also, X is an M-ideal in Y if and only if $X^\perp$ is an L-summand in Y^* (see [HWW]; also see [ER2] for the case of complete M-ideals).

Now, M-summands and M-ideals are very special in the general Banach space setting. However, the following remarkable result shows their importance.

THEOREM. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{J}$ be a closed linear subspace. Then $\mathcal{J}$ is an M-ideal in $\mathcal{A}$ iff $\mathcal{J}$ is a (2-sided) ideal in $\mathcal{A}$ iff $\mathcal{J}$ is a complete M-ideal in $\mathcal{A}$.*

(See [HWW] for a proof and complete references; for the remarkable theorem that M-ideals are algebraic ideals, see [AE] and [SW].)

Why is an ideal in a C^*-algebra an M-ideal? The commutative case is rather transparent. For then $\mathcal{A} = C_0(\Omega)$ for some locally compact Hausdorff space Ω and $\mathcal{J} = \mathcal{J}_K$ for some closed subset K of Ω (and Borsuk's Theorem of course could be formulated in this possibly non-unital setting also). But then, identifying $\mathcal{A}^*$ with $M(\Omega)$, the space of regular complex Borel measures "on" Ω, we see that $\mathcal{J}^\perp = M(K)$, and if we let $Z = M(\Omega \sim K)$, then $\mathcal{J}^\perp \oplus Z$ is an L-decomposition of $M(K)$.

The non-commutative case is certainly not so transparent. The highly motivated case of $\mathcal{K}$ in $B(H)$ was established by Dixmier in 1950 [Di]. Note that it seems one must at least consider $\mathcal{K}^\perp \subset B(H)^*$, a rather huge object! Our approach here yields the general non-commutative case, via M-approximate identities, a notion defined only, in this setting, on the C^*-algebra $\mathcal{A}$ itself; one has no need to "look" at $\mathcal{A}^*$ or $\mathcal{A}^{**}$, to then "see" the M-ideal property.

The results of our paper are all cast in the general setting of Banach/operator spaces and (complete) M-ideals. We shall see that our complementation results also use the property of certain "special" M-complete approximable identities, and not just the general M-cai concept. Our methods may also be used to recapture several of the results given in the initial paper [Ro].

To more thoroughly answer the second of our "motivating" questions, we now proceed with a more detailed discussion and outline of our results.

Various refinements of the concept of an M-approximate identity are given in Definition 1.1. Theorem 1.1 then establishes that if X admits an M-ai (resp. M-cai) in Y, then X is an M-ideal (resp. complete M-ideal) in Y. Moreover, if X admits a strong M-cai (T_α), then T_α^{**} converges in the weak*-operator topology (W*-OT) on Y^{**} to the M-projection mapping Y^{**} onto X^{**}. We show in Proposition 1.4 that central approximate units yield strong contractive M-cai's for ideals $\mathcal{J}$ in C^*-algebras $\mathcal{A}$. A by-product of Corollary 1.5: the central approximate unit (u_α) may be chosen so that setting $U_\alpha a = u_\alpha a$ for all $a \in \mathcal{A}$, we get

$$\lim_\alpha(\|U_\alpha^* y^*\| + \|(I - U_\alpha^*)y^*\|) = \|y^*\| \quad \text{for all } y^* \in \mathcal{A}^*. \tag{0.12}$$

Section 1 concludes with a permanence property of "good" M-cai's (see Definition 1.3), which has the consequence: if X admits a good M-cai in Y, then $Z \otimes_{\mathrm{op}} X$ is a complete M-ideal in $Z \otimes_{\mathrm{op}} Y$ for all operator spaces Z (Proposition 1.7). It remains an open problem if this permanence property holds for general complete M-ideals.

In §2, we introduce the notion of a special M-cai for an operator space $X \subset Y$. We show in Proposition 2.3 that if X is an ideal in a C^*-algebra Y and (x_α) is a positive contractive central approximate unit for X in Y (the x_α's being positive contractions in X), and if we define $U_\alpha(y) = \sqrt{x_\alpha}\, y \sqrt{x_\alpha}$ for all α, then (U_α) is a special M-cai for X in Y. Thus Theorem 2 of this Introduction is a special case (for $\lambda > 1$) of our Theorem 2.4: *If X is an approximately injective subspace of a λ-locally reflexive subspace Y so that X admits a special M-cai in Y, with Y/X separable, then for all $\varepsilon > 0$, there is a lift $L : Y/X \to Y$ of $I_{Y/X}$ with $\|L\|_{\mathrm{cb}} < \lambda + \varepsilon$.*

In the Theorem of the Appendix, we obtain a complete isometric extension of the Effros–Haagerup lifting result, without the special M-cai assumption; namely, the lift L may be chosen completely contractive provided X is an approximately injective complete M-ideal in Y, when Y is 1-locally reflexive and Y/X is separable.

Our proof of Theorem 2.4 yields that its conclusion holds if we replace its hypothesis that Y is locally reflexive by the assumption that X is *locally complemented* in Y; that is, for some $\gamma \ge 1$, X is γ-completely complemented in Z for all $X \subset Z \subset Y$ with Z/X finite-dimensional. (See Theorem 2.4′ and the following Remark.) Thus it follows that *if $\mathcal{K} \subset Y \subset B(\ell^2)$ with Y separable, then $\mathcal{K}$ is completely complemented in Y provided $\mathcal{K}$ is locally complemented in Y.*

In Definition 2.2, we give the Banach space concept of *extendable local liftings* (ell's) for a pair of Banach spaces $X \subset Y$ (this is the same as the special case $(\mathcal{K}, \mathcal{A})$ stated above). We observe in Proposition 2.7 that (X, Y) has λ-ell's if (X, Y) has λ-local liftings and, e.g., Y/X has the λ-bap or Y is λ-extendably locally reflexive. Theorem 4 of this Introduction is then a special case of our Theorem 2.8, which yields that *if $X \subset Y$ are operator spaces with X approximately injective and Y/X separable, so that X admits a special M-cai in Y and (X, Y) has extendable local liftings, then X is complemented in Y.*

Section 3 gives further applications of the general complementation results in Section 2. The easy Proposition 3.2(a) yields that if $X_1, X_2, \ldots$ are given operator spaces and $X = (X_1 \oplus X_2 \oplus \ldots)_{c_0}$ and $Y = (X_1 \oplus X_2 \oplus \ldots)_{\ell^\infty}$, then X admits a (canonical) strong special M-cai in Y; moreover, if the X_j's are approximately injective, so is X. Corollary 3.3 then yields that if $X \subset Z \subset Y$ with Z/X separable and Z locally reflexive, X is completely complemented in Z. Moreover, if Z is 1-locally reflexive, X is completely co-contractively complemented in Z. This yields the discoveries in [Ro] that if the X_j's are all 1-injective Banach spaces, X has the 2-Separable Extension Property; in particular, $c_0(\ell^\infty)$ has the 2-SEP. If, moreover, the X_j's are all 1-injective operator spaces and $X \subset Z \subset Y$ with Z/X separable and Z λ-locally reflexive, then X is completely $(\lambda + \varepsilon)$-co-complemented in Z for all $\varepsilon > 0$.

We also recapture the main result obtained in [Ro] concerning the Complete Separable Extension Property (CSEP), namely that for all $n \ge 1$, $Z := c_0(M_{n,\infty} \oplus M_{\infty,n})$ has the 2-CSEP. That is, for all separable operator spaces $X \subset Y$ and completely bounded maps $T : X \to Z$, there is an extension $\widetilde{T} : Y \to Z$ with $\|\widetilde{T}\|_{\mathrm{cb}} \le 2\|T\|_{\mathrm{cb}}$. (This is a full quantized extension of Sobczyk's Theorem; see Corollary 3.8.) In turn, this is obtained via an interesting recent operator space extension of the Banach local reflexivity principle due to L. Ge and D. Hadwin [GH] and the following application (via an elementary result in [Ro]): For Z as above, Z^{**} is 1-locally reflexive (Proposition 3.7).

Section 3 also treats the question of when the converse of Theorem 1.1 holds. Precisely, *suppose $X \subset Y$ are Banach (resp. operator) spaces with X an M-ideal (resp. complete M-ideal) in Y. Does X admit an M-ai (resp. M-cai) in Y?*

In Theorem 3.1, we prove that this is indeed true in the case where X is an ideal (closed 2-sided) in a (possibly) non-self-adjoint operator algebra Y. Effros–Ruan prove in [ER1] that then X is an M-ideal in Y precisely when X has a contractive approximate identity. We show directly that then X admits a strong contractive M-cai in Y.

In Theorem 3.11, we obtain the (perhaps surprisingly general) result that (assuming X is a complete M-ideal in Y), *X admits an M-cai in Y provided Y has the Banach bounded approximation property. We also obtain the same conclusion if Y is 1-locally reflexive and X is a finitely injective operator space.* Moreover, we find out that the M-cai (T_α) may be chosen to consist of finite rank operators when Y has the bap or X is λ-nuclear and Y is 1-locally reflexive. (λ-finitely injective operator spaces are defined in Definition 3.2 (just before the statement of Theorem 3.11); these include λ-nuclear and λ-injective operator spaces.)

Theorem 3.11 uses an extension of the Banach local reflexivity principle due to S. Bellenot [Be], which we formulate and prove in the operator space setting in Lemma 3.12, as well as its consequence, an extension of the above-mentioned result of [GH], which we obtain in Lemma 3.13.

Finally, we give an example of a Banach space X which is an M-ideal in X^{**}, yet X admits no M-approximate identity in X^{**} (Proposition 3.16). The example is at the "surface" modulo some rather deep known results; namely, X is a subspace of c_0 failing the compact bounded approximation property.

We do not know of a separable pair (X, Y) forming a counterexample (as of this writing!). However, we conjecture that if X is as in Proposition 3.16, then there exists $X \subset Y \subset X^{**}$ with Y separable, yet X admits no M-ai in Y. This conjecture, however, appears to lie considerably below the surface of known results, unlike 3.16.

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1. Permanence properties of M-(complete) approximate identities. For the sake of completeness, we first recall the concept of an *operator space* X, by which we mean a closed linear subspace X of $B(H)$ for some Hilbert space H, endowed with its natural tensor product structure with $\mathcal{K}$. Given operator subspaces X_1 and X_2 of $B(H_1)$ and $B(H_2)$ for Hilbert spaces H_1 and H_2, $X_1 \otimes_{\rm op} X_2$ denotes the closed linear span in $B(H_1 \otimes_2 H_2)$ of $X_1 \otimes X_2$. A linear map $T : X_1 \to X_2$ is called *completely bounded* if $I_{\mathcal{K}} \otimes T$ is bounded, and we then set $\|T\|_{\rm cb} = \|I_{\mathcal{K}} \otimes T\|$; equivalently, $\|T\|_{\rm cb} = \sup_n\{\|(T(x_{ij}))\| : (x_{ij}) \text{ is in } \mathcal{M}_n(X_1)\}$ where we identify $\mathcal{M}_n(X_1)$ with $B(\ell_n^2) \otimes X_1$.

Remarkable axioms of Z. J. Ruan abstractly characterize the space $\mathcal{K} \otimes_{\rm op} X$ without reference to the ambient Hilbert space H. Any Banach space X can be regarded as an operator space in the so-called MIN structure (where $\|(x_{ij})\|_{\rm MIN} = \sup\{\|(x^*(x_{ij}))\| : x^* \in X^*,\ \|x^*\| = 1\}$, and (x_{ij}) ranges over $\mathcal{K}_{00}$, the space of all infinite matrices with only finitely many non-zero entries, regarded as a subspace of $\mathcal{K}$ acting on ℓ^2). MIN is then the smallest operator space structure one can place on X. There is also a largest structure, MAX, defined by $\|(x_{ij})\|_{\rm MAX} = \sup\{\|(T(x_{ij}))\| : T : X \to B(H)$ is a linear contraction for some $H\}$. For further basic facts and concepts, see [Pi] and the references cited in the Introduction.

We now give the definition of the basic concept studied in this work.

Definition 1.1. (A) Let $X \subset Y$ be Banach/operator spaces, and let $(T_\alpha)_{\alpha\in\mathcal{D}}$ be a uniformly bounded net in $B(Y)$. Then $(T_\alpha)_{\alpha\in\mathcal{D}}$ is an *M-approximate identity* (*M-ai*) for X in Y if

(i) $T_\alpha Y \subset X$ for all α,
(ii) $T_\alpha x \to x$ for all $x \in X$,
(iii) $\overline{\lim}_\alpha \|T_\alpha u + (I - T_\alpha)v\| \le \max\{\|u\|, \|v\|\}$ for all $u, v \in Y$.

$(T_\alpha)_{\alpha\in\mathcal{D}}$ is an *M-complete approximate identity* (M-cai) if (T_α) satisfies (i), (ii), and

(iii)′ for all n and all $n \times n$ matrices (u_{ij}), (v_{ij}) in Y,

$$\overline{\lim_\alpha} \|(T_\alpha u_{ij} + (I - T_\alpha)v_{ij})\| \le \max\{\|(u_{ij})\|, \|(v_{ij})\|\}.$$

(B) (T_α) is called a *strong M-ai* (resp. *strong M-cai*) if in addition we have

(iv) $T_\alpha^{**} x^{**} \xrightarrow{\rm w^*} x^{**}$ for all $x^{**} \in X^{**}$

and

(v) $\overline{\lim}_\alpha \|T_\alpha^* y^*\| \le \|y^*\|$ for all $y^* \in Y^*$

resp.

(v′) The T_α's are completely bounded and

$$\overline{\lim_\alpha} \|I_{\mathcal{K}} \otimes T_\alpha^*(\tau)\| \le \|\tau\| \quad \text{for all } \tau \in \mathcal{K} \otimes_{\rm op} Y^*.$$

(C) (T_α) is called a *contractive M-ai* (resp. *contractive M-cai*) if (T_α) is an M-ai (resp. M-cai) for X in Y so that $\|T_\alpha\| \le 1$ (resp. $\|T_\alpha\|_{\rm cb} \le 1$) for all α.

Remarks. 1. Approximate units in Banach spaces were studied by E. Oja [Oj] and W. Werner [We]. Part of Theorem 1.1 for Banach spaces appears in [Oj]; a converse for certain Banach algebras is given in [We]. It follows by a result of Å. Lima in [L2] that X is an M-ideal in Y precisely when there is a possibly unbounded net (T_α) satisfying (i)–(iii).

2. Evidently, if (T_α) is a contractive M-ai (resp. a contractive M-cai) satisfying (iv), then (T_α) is a strong M-ai (resp. strong M-cai) for X in Y.

3. All these concepts are hereditary in the following sense: if $X \subset Z \subset Y$ and (T_α) is an M-approximate identity of one of the various kinds, for X in Y, then also $(T_\alpha|Z)$ is an M-ai of the *same kind* for X in Z.

4. We do not know the answers to the following questions. Let $X \subset Y$ be Banach/operator spaces. If X admits an M-ai in Y, does X admit a contractive M-ai in Y? If X admits an M-cai in Y, does X admit an M-cai (T_α) with

(i) the T_α's completely bounded?
(ii) $\sup_\alpha \|T_\alpha\|_{\mathrm{cb}} < \infty$?
(iii) $\|T_\alpha\|_{\mathrm{cb}} \le 1$ for all α (i.e., so that (T_α) is a contractive M-cai)?

Our first result provides basic motivation for introducing the concept in Definition 1.1.

THEOREM 1.1. *Let $X \subset Y$ be Banach (resp. operator) spaces and assume X admits an M-ai (resp. M-cai) (T_α) in Y. Then X is an M-ideal (resp. complete M-ideal) in Y and in fact (T_α^*) converges in the W^*-OT to the L-projection P on Y^* with kernel $X^\perp$.*

If (T_α) is a strong M-ai (resp. M-cai), then (T_α^{**}) converges in the W^*-OT on Y^{**} to the M-projection P^* onto X^{**} (resp. $I_{\mathcal{K}} \otimes T_\alpha^{**}$ converges in the W^*-OT on $\mathcal{K} \otimes_{\mathrm{op}} Y^{**}$ to $I_{\mathcal{K}} \otimes P^*$).

REMARK. We note that in the operator space case, it is *not* assumed in the definition of an M-cai (T_α) that the T_α's are completely bounded. If, however, it is assumed that in fact $\sup_\alpha \|T_\alpha\|_{\mathrm{cb}} < \infty$ and (T_α) is an M-cai, it follows that $(I_{\mathcal{K}} \otimes T_\alpha^*)$ converges in the W*-OT on $\mathcal{K} \otimes_{\mathrm{op}} Y^*$ to $I_{\mathcal{K}} \otimes P$.

Proof of Theorem 1.1. We first deduce: There is an L-decomposition $X^\perp \oplus W$ for Y^*, and if P denotes the projection onto W with kernel $X^\perp$, then $T_\alpha^* \to P$ in the W*-OT. By first passing to a subnet, we may assume:

$$\text{w}^*\text{-}\lim_\alpha T_\alpha^* y^* \quad \text{exists for all } y^* \in Y^*. \tag{1.1}$$

(Later, we will show this is not needed.) Now we have

$$T_\alpha^* x^\perp = 0 \text{ and hence } (I - T_\alpha^*)x^\perp = x^\perp \text{ for all } \alpha \text{ and all } x^\perp \in X^\perp. \tag{1.2}$$

This is trivial, because $\langle T_\alpha^* x^\perp, y\rangle = \langle x^\perp, T_\alpha y\rangle = 0$ for all $y \in Y$. Now

$$\lim_\alpha (I - T_\alpha^*)y^* \in X^\perp \quad \text{for all } y^* \in Y^*. \tag{1.3}$$

Indeed, if $x \in X$, then

$$\lim_\alpha \langle (I - T_\alpha^*)y^*, x\rangle = \lim_\alpha \langle y^*, (I - T_\alpha)x\rangle = 0.$$

Since the T_α^*'s are *uniformly bounded*, the operator

$$Q = \lim_\alpha (I - T_\alpha^*) \quad \text{is bounded} \tag{1.4}$$

(where the net converges in the W*-OT). But we have: $y^* \in Y^* \Rightarrow Qy^* \in X^\perp$ by (1.3) and $y^* \in X^\perp \Rightarrow Qy^* = y^*$ by (1.2), hence Q is indeed a projection onto $X^\perp$. Let $P = I - Q$ and $W = PX^* = \ker Q$.

Now we prove: $X^\perp \oplus W$ is an L-decomposition of Y^*. Let $x^\perp \in X^\perp$, $w \in W$; $\varepsilon > 0$. We may choose norm-1 elements u, v of Y so that

$$\|x^\perp\| + \|w\| \le (1+\varepsilon)(\langle x^\perp, u\rangle + \langle w, v\rangle) \tag{1.5}$$

(and the right hand terms are actually non-negative real numbers). Now

$$\langle x^\perp, u\rangle + \langle w, v\rangle = \langle x^\perp, u\rangle + \lim_\alpha \langle T_\alpha^* w, v\rangle. \tag{1.6}$$

For all α,

$$\begin{aligned} &\langle x^\perp + w, (I - T_\alpha)u + T_\alpha v\rangle \\ &\quad = \langle x^\perp, u\rangle + \langle w, (I - T_\alpha)u\rangle + \langle w, T_\alpha v\rangle \quad \text{by (1.2)} \\ &\quad = \langle x^\perp, u\rangle + \langle (I - T_\alpha^*)w, u\rangle + \langle T_\alpha^* w, v\rangle. \end{aligned} \tag{1.7}$$

But $\lim_\alpha (I - T_\alpha)^* w = Qw = 0$, whence we have proved, by (1.6) and (1.7), that

$$\begin{aligned} \langle x^\perp, u\rangle + \langle w, v\rangle &= \lim_\alpha \langle x^\perp + w, (I - T_\alpha)u + T_\alpha(v)\rangle \\ &\le \|x^\perp + w\| \varlimsup_\alpha \|(I - T_\alpha)u + T_\alpha(y)\| \\ &\le \|x^\perp + w\| \quad \text{(by Definition 1.1(iii))}. \end{aligned} \tag{1.8}$$

Hence by (1.5), $\|x^\perp\| + \|w\| \le (1+\varepsilon)\|x^\perp + w\|$. Since $\varepsilon > 0$ is arbitrary, this shows: P is indeed an L-projection.

Thus by uniqueness of such, we conclude by the way that we did not need to take a subnet; and so our original net satisfies

$$P = \lim T_\alpha^* \quad \text{in the W}^*\text{-OT}. \tag{1.9}$$

We have now proved: X is indeed an M-ideal with $X^{**} \oplus W^\perp = Y^{**}$ the M-decomposition for Y^{**}.

It now follows immediately from results of Effros–Ruan that X is a complete M-ideal in Y in case (T_α) is an M-cai. Indeed, fixing n and then defining $T_\alpha^n = I_{M_n} \otimes T_\alpha$ on $M_n(Y)$, we infer that (T_α^n) is an M-ai for $M_n(X)$ in $M_n(Y)$, hence $M_n(X)$ is an M-ideal in $M_n(X)$, whence X is indeed a complete M-ideal in Y by [ER2].

We now proceed to the final assertion of the theorem. Assume then that (T_α) is a strong M-ai (resp. M-cai).

Let $w^\perp \in W^\perp$. We claim

$$T_\alpha^{**} w^\perp \to 0 \quad \text{in the w}^* \text{ topology}. \tag{1.10}$$

Now once (1.10) is proved, we deduce by (iv) that for all $x^{**} \in X^{**}$ and $w^\perp \in W^\perp$,

$$(1.11)\qquad \text{w}^*\text{-}\lim_\alpha T_\alpha^{**}(x^{**} + w^\perp) = x^{**},$$

whence $T_\alpha^{**} \to P^*$ in the W*-OT, and of course P^* is the M-projection onto X^{**}.

Suppose (1.10) were false. Then for some $y^* \in Y^*$,

$$(1.12)\qquad \overline{\lim_\alpha} |\langle y^*, T_\alpha^{**} w^\perp\rangle| \neq 0.$$

Now let $y^* = x^\perp + w$, $x^\perp \in X^\perp$, $w \in W$. But then trivially (since $T_\alpha^* x^\perp = 0$ for all α)

$$(1.13)\qquad \overline{\lim_\alpha} |\langle w, T_\alpha^{**} w^\perp\rangle| \neq 0.$$

By passing to a subnet if necessary and taking obvious normalizations, we may assume without loss of generality that $\|w\| = 1 = \|w^\perp\|$ and there is a $\delta > 0$ so that

$$(1.14)\qquad |\langle w, T_\alpha^{**} w^\perp\rangle| \geq \delta \quad \text{for all } \alpha.$$

Now choose $x \in X$, $\|x\| = 1$, so that

$$(1.15)\qquad |\langle w, x\rangle| > 1 - \delta/2.$$

Choose α_0 so that also

$$(1.16)\qquad |\langle T_\alpha^* w, x\rangle| > 1 - \delta/2 \quad \text{for all } \alpha > \alpha_0.$$

Thus combining (1.14) and (1.16), we find for such α that

$$(1.17)\qquad |\langle T_\alpha^* w, w^\perp\rangle| + |\langle T_\alpha^* w, x\rangle| > 1 + \delta/2.$$

Finally, for each such α, choose scalars θ_α and ψ_α of modulus one so that

$$(1.18)\qquad |\langle T_\alpha^* w, w^\perp\rangle| = \langle T_\alpha^* w, \theta_\alpha w^\perp\rangle \quad \text{and} \quad \langle T_\alpha^* w, x\rangle = \langle T_\alpha^* w, \psi_\alpha x\rangle.$$

Hence, by (1.17) and (1.18),

$$(1.19)\qquad 1+\delta/2 < |\langle T_\alpha^* w, \theta_\alpha w^\perp + \psi_\alpha x\rangle| \leq \|T_\alpha^* w\|\, \|\theta_\alpha w^\perp + \psi_\alpha x\| = \|T_\alpha^* w\|$$

because $x \in X^{**}$ and $X^{**} \oplus W^\perp$ is an M-decomposition.

Thus

$$(1.20)\qquad \overline{\lim_\alpha} \|T_\alpha^* w\| \geq 1 + \delta/2 \quad \text{but} \quad \|w\| = 1,$$

contradicting (v).

Finally, in the complete-category, it follows that for all $\mu \in \mathcal{K} \otimes Y^*$,

$$(1.21)\qquad (I_\mathcal{K} \otimes T_\alpha^*)(\mu) \to (I_\mathcal{K} \otimes P)(\mu) \quad \text{weakly.}$$

But $\mathcal{K} \otimes Y^*$ is a norm-dense linear subspace of $\mathcal{K} \otimes_{\text{op}} Y^*$, whence (v′) then indeed implies that (1.21) holds for all $\mu \in \mathcal{K} \otimes_{\text{op}} Y^*$, yielding the final assertion. ∎

Corollary 1.2. *Suppose that X admits a strong M-ai (resp. a strong M-cai) in Y. Then also X admits a strong M-ai (resp. M-cai) (U_α) such that if P is the L-projection with kernel $X^\perp$, then $U_\alpha^* \to P$ in the SOT on Y^*. In particular,*

$$(1.22)\qquad \overline{\lim_\alpha} \|U_\alpha^* y^*\| + \|(I - U_\alpha^*)y^*\| = \|y^*\| \quad \textit{for all } y^* \in Y^*.$$

If X admits a strong M-cai, then (U_α) may be chosen so that $I_\mathcal{K} \otimes U_\alpha^* \to I_\mathcal{K} \otimes P^*$ in the SOT on $\mathcal{K} \otimes_{\text{op}} Y^*$.

Proof. Let (T_α) be a strong M-ai (resp. M-cai) for X in Y. Theorem 1.1 shows that

$$(1.23)\qquad T_\alpha^* y^* \to P y^* \quad \text{weakly,} \quad \text{for all } y^* \in Y^*.$$

Hence there exists a net (U_α) of "far-out" convex combinations of the T_α's so that

$$(1.24)\qquad U_\alpha^* y^* \to P y^* \quad \text{in norm,} \quad \text{for all } y^* \in Y^*.$$

Of course, this yields that $U_\alpha^* \to P$ in the SOT. In particular, for $y^* \in Y^*$, since P is an L-projection, we see that for all $y^* \in Y^*$,

$$(1.25)\qquad \|y^*\| = \|Py^*\| + \|(I-P)y^*\| = \lim_\alpha (\|U_\alpha^* y^*\| + \|(I - U_\alpha^*)y^*\|),$$

proving (1.22).

Now it is easily seen that (U_α) is a strong M-ai (resp. M-cai). Finally, if (T_α) is a strong M-cai, then

$$(1.26)\qquad I_\mathcal{K} \otimes T_\alpha^* \to I_\mathcal{K} \otimes P \quad \text{in the WOT on } \mathcal{K} \otimes_{\text{op}} Y^*,$$

whence again the U_α's may be chosen as above to satisfy the final assertion of the corollary. ∎

Let $X \subset Y$ be Banach (resp. operator) spaces. An inspection of the proof of Theorem 1.1 shows that its conclusion holds under the following modified assumptions on the net of operators (T_α).

Definition 1.2. Let (T_α) be a uniformly bounded net of operators on Y. (T_α) is called a *weak M-ai* (*resp. weak M-cai*) if (i) and (ii) of Definition 1.1 hold, but in (iii) and (iii′), we restrict the u's (resp. the u_{ij}'s) to lie in X.

Theorem 1.1′. *The conclusion of Theorem* 1.1 *holds provided X admits a weak contractive M-ai* (*resp. a weak contractive M-cai*) *in Y.*

Proof. We may argue as in the proof of Theorem 1.1, which essentially proceeds from first principles. Alternatively, we may use the following characterization of M-ideals given in Theorem I.2.2 of [HWW] (the "restricted 3-ball property", which was introduced by Å. Lima [L1]): *X is an M-ideal in Y provided for any $y \in \operatorname{Ba} Y$, x_1, x_2, x_3 in $\operatorname{Ba} X$ and $\varepsilon > 0$, there exists* *an* $x \in X$ *with*

$$\|x_i + y - x\| \le 1 + \varepsilon \quad \text{for all } 1 \le i \le 3.$$

Now letting (T_α) be a weak M-ai for X in Y, simply choose α so that

$$\|x_i + (I - T_\alpha)(y)\| \le 1 + \varepsilon \quad \text{for all } 1 \le i \le 3.$$

Since the T_α's map Y into X, $x := T_\alpha y$ satisfies the above criterion, whence X is an M-ideal in Y. It may then be directly verified that $T_\alpha^* \to P$ in the W*-OT, where P is the L-projection with kernel $X^\perp$. Assuming that (T_α) is a weak M-cai, we infer that for all n, $(I_n \otimes T_\alpha)$ is a weak contractive M-ai on $M_n \otimes Y$ (where I_n is the identity on M_n), whence $M_n(X)$ is an M-ideal in $M_n(Y)$ and so as before, via the result in [ER2], X is a complete M-ideal in Y. Finally, assuming the additional hypotheses that $T_\alpha^{**} x^* \to x^{**}$ weak* for all $x^{**} \in X^{**}$, we obtain the final conclusion of Theorem 1.1. ■

We have chosen the stronger concept given in Definition 1.1 in the contractive case, since this is what occurs when X is an ideal in a C^*-algebra Y. Before dealing with this remarkable special case, we briefly consider when our methods yield that a Banach/operator space X is an M-ideal/complete M-ideal in X^{**}.

COROLLARY 1.3. *Let X be a Banach (resp. operator) space and let (T_α) be a uniformly bounded net of weakly compact operators on X. Suppose that*

(i) *$T_\alpha x \to x$ for all $x \in X$,*

(ii) *$\overline{\lim}_\alpha \|T_\alpha x + (I - T_\alpha^{**}) y^{**}\| \le \max\{\|x\|, \|y^{**}\|\}$ for all $x \in X$ and $y^{**} \in Y^{**}$, or*

(ii′) *in the operator-space setting, for all n and $n \times n$ matrices (x_{ij}) and (y_{ij}^{**}) in $M_n(X)$ and $M_n(Y^{**})$ respectively,*

$$\overline{\lim_\alpha} \|(T_\alpha x_{ij} + (I - T_\alpha^{**}) y_{ij}^{**})\| \le \max\{\|(x_{ij})\|, \|(y_{ij}^{**})\|\}.$$

Then X is an M-ideal (resp. complete M-ideal) in X^{**}. Moreover, then $T_\alpha^* \to I_{X^*}$ in the WOT.

REMARKS. 1. If (T_α^{**}) is an M-ai (resp. M-cai) for X in X^{**}, then (T_α) satisfies the hypotheses of 1.3.

2. It follows immediately that if (T_α) satisfies these hypotheses, then also there exists a net $(\widetilde{T}_\alpha)$ of weakly compact operators on X satisfying them, so that in addition $\widetilde{T}_\alpha^* \to I_{X^*}$ in the SOT.

In particular, we recover the facts (in this setting) that if X is separable and satisfies these hypotheses, X^* is separable; if the T_α's are all compact, then X^* has the bounded compact approximation property; and finally, if the T_α's are finite rank, then X^* has the bounded approximation property.

In particular, if X is separable and the T_α's are of finite rank, then X^* is separable with the bounded (and hence metric) approximation property.

This suggests the conjecture: if X admits a weak M-ai in X^{**} consisting of finite rank operators, then also X admits a weak M-ai in X^{**} consisting of contractive finite rank operators.

Proof of Corollary 1.3. Actually, everything but the final statement follows immediately from Theorem 1.1′. (Simply note that since the T_α's are weakly compact, $T_\alpha^{**} X^{**} \subset X$ for all α.)

Now, if $Y := X^{**}$ and $S_\alpha := T_\alpha^{**}$, then $S_\alpha^* \to P$ in the W*-OT, where P is the L-projection on Y^* with kernel $X^\perp$. But in this case, we must have $P(Y^*) = X^*$, i.e., the L-decomposition of $Y^* = X^{***} = X^\perp \oplus X^*$ (see [HWW]). But then, after taking the various identities into account, we simply find that if $x^* \in X^* \subset X^{***}$, then $T_\alpha^{***} x^* = T_\alpha^* x^*$, whence $T_\alpha^{***} x^* \to x^*$ weak* simply means that $\langle x^{**}, T_\alpha^* x^* \rangle \to \langle x^{**}, x^* \rangle$ for all $x^{**} \in X^{**}$; i.e., $T_\alpha^* x^* \to x^*$ weakly. ■

We now pass to the strongly motivating case of ideals in C^*-algebras. The proof uses certain standard arguments in C^*-algebras, for which we nevertheless give details for the sake of completeness.

PROPOSITION 1.4. *Let $\mathcal{J}$ be an ideal in a C^*-algebra $\mathcal{A}$. Then there is a strong contractive M-cai (T_α) for $\mathcal{J}$ in $\mathcal{A}$.*

Proof. We may assume that $\mathcal{A}$ is unital, by simply adjoining an identity, denoted by 1. For once the result is proved here, its hereditary character yields the non-unital case. Choose a net $(x_\alpha)_{\alpha \in \gamma}$ of elements of $\mathcal{J}$ with the following properties:

$$0 \le x_\alpha \le 1 \quad \text{for all } \alpha, \tag{1.27}$$

$$x_\alpha x \to x \quad \text{for all } x \in \mathcal{J}, \tag{1.28}$$

$$x_\alpha y - y x_\alpha \to 0 \quad \text{for all } y \in \mathcal{A}. \tag{1.29}$$

(Such a net $(x_\alpha)_{\alpha \in \gamma}$ is called a *central approximate unit* for $\mathcal{J}$. For the existence of such nets, see W. B. Arveson [Ar] and C. Akemann and G. Pedersen [AP]. See also our proof of Theorem 3.1 below.)

Now define $T_\alpha : \mathcal{A} \to \mathcal{A}$ by $T_\alpha(y) = x_\alpha y$ for all $y \in \mathcal{A}$ and α. We claim that (T_α) is the desired net. Now (i), (ii), and (v) of Definition 1.1 are immediate, since $\mathcal{J}$ is an ideal, (1.27) holds, and T_α is a complete contraction for all α. Using standard facts about C^*-algebras, we also have (iv).

Indeed, $\mathcal{A}^{**}$ is in fact a von Neumann algebra acting on a certain Hilbert space $\mathcal{H}$, in which the w*-topology on bounded sets (w.r.t. $\mathcal{A}^*$) coincides with the weak operator topology with respect to $B(\mathcal{H})$. Now letting e denote the unit element of $\mathcal{J}^{**}$ (which exists since $\mathcal{J}$ is a von Neumann algebra), it follows that $x_\alpha \to e$ weak*, i.e., $x_\alpha \to e$ in the WOT on $B(\mathcal{H})$, whence given $x^{**} \in \mathcal{J}^{**}$, also $x_\alpha \cdot x^{**} \to e \cdot x^{**} = x^{**}$ in the WOT, i.e., $T_\alpha^{**}(x^{**}) = x_\alpha \cdot x^{**} \to x^{**}$ in the weak* topology, proving (iv).

Now the results in [Ar] yield that for all $a \in \mathcal{A}$,

$$\sqrt{x_\alpha}a - a\sqrt{x_\alpha} \to 0 \quad\text{and}\quad \sqrt{1-x_\alpha}a - a\sqrt{1-x_\alpha} \to 0. \tag{1.30}$$

(This also follows immediately from the known inequality $\|\sqrt{x}\,a - a\sqrt{x}\,\| \le 2\sqrt{\|a\|}\,\|xa - ax\|^{1/2}$ for all x, a in a C^*-algebra with $x \ge 0$; cf. [D, page 73].) It then follows that for any $a \in \mathcal{A}$,

$$x_\alpha a - \sqrt{x_\alpha}\,a\sqrt{x_\alpha} \to 0 \tag{1.31.i}$$

and

$$(1-x_\alpha)a - \sqrt{1-x_\alpha}a\sqrt{1-x_\alpha} \to 0 \tag{1.31.ii}$$

Indeed, by (1.25), $\sqrt{x_\alpha}(\sqrt{x_\alpha}a - a\sqrt{x_\alpha}) \to 0$, and $\sqrt{(1-x_\alpha)}(\sqrt{1-x_\alpha}a - a\sqrt{1-x_\alpha}) \to 0$, yielding (1.31).

Now for each α, define U_α and V_α on $\mathcal{A}$ by

$$U_\alpha a = \sqrt{x_\alpha}a\sqrt{x_\alpha}, \qquad V_\alpha a = \sqrt{1-x_\alpha}a\sqrt{1-x_\alpha} \qquad \text{for all } a \in \mathcal{A}. \tag{1.32}$$

Then note that U_α and V_α are also complete contractions, and moreover, $U_\alpha\mathcal{A} \subset \mathcal{J}$ and $(I - V_\alpha)\mathcal{A} \subset \mathcal{J}$ for all α. (See Remark 1 below for the last assertion.)

Next, define $S : \mathcal{A} \oplus \mathcal{A} \to \mathcal{A}$ by $S(u \oplus v) = u + v$ for all $u, v \in \mathcal{A}$. Then endowing $\mathcal{A} \oplus \mathcal{A}$ with the L^∞-direct sum norm, and fixing α, we claim that

$$S \circ (U_\alpha \oplus V_\alpha) \text{ is a complete contraction.} \tag{1.33}$$

This follows immediately from the matrix formula: for $u, v \in \mathcal{A}$ and $0 \le x \le 1$,

$$U_\alpha u + V_\alpha v = (\sqrt{x_\alpha} \quad \sqrt{1-x_\alpha}) \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \sqrt{x_\alpha} \\ \sqrt{1-x_\alpha} \end{pmatrix} \tag{1.34}$$

and the easily seen fact that

$$\left\| \begin{pmatrix} \sqrt{x_\alpha} \\ \sqrt{1-x_\alpha} \end{pmatrix} \right\| = \|(\sqrt{x_\alpha} \quad \sqrt{1-x_\alpha})\| = 1.$$

It now follows that (T_α) is a strong M-cai, i.e. (iii′) of Definition 1.1 holds (since we have verified all the other conditions). Indeed, rephrasing (1.31), for all $a \in \mathcal{A}$ we get

$$T_\alpha a - U_\alpha a \to 0 \quad\text{and}\quad (I - T_\alpha)a - V_\alpha a \to 0. \tag{1.35}$$

Hence for any n and $(a_{ij}), (b_{ij})$ in $M_n(\mathcal{A})$, we have

$$\begin{aligned} \varlimsup_\alpha \|(T_\alpha(a_{ij}) + (I - T_\alpha)(b_{ij}))\| &= \varlimsup_\alpha \|(U_\alpha(a_{ij}) + V_\alpha(b_{ij}))\| \\ &\le \max\{\|(a_{ij})\|, \|(b_{ij})\|\} \end{aligned} \tag{1.36}$$

by (1.33). ■

REMARKS. 1. It is trivial that $U_\alpha\mathcal{A} \subset \mathcal{J}$ for all α, since if $0 \le x \le 1$ belongs to $\mathcal{J}$, then $\sqrt{x} \in \mathcal{J}$, whence $\sqrt{x}a\sqrt{x} \in \mathcal{J}$ for any $a \in \mathcal{A}$ since $\mathcal{J}$ is an ideal. A less trivial fact: also $(I - V_\alpha)\mathcal{A} \subset \mathcal{J}$ for all α. This is so because for x as above, if $a \in \mathcal{A}$, then also $a - \sqrt{1-x}a\sqrt{1-x} \in \mathcal{J}$.

Here is a simple proof of this fact: It suffices to show that

$$\sqrt{1-x}a - a\sqrt{1-x} \in \mathcal{J}. \tag{$*$}$$

For then it follows that

$$\sqrt{1-x}(\sqrt{1-x}a - a\sqrt{1-x}) = (1-x)a - \sqrt{1-x}a\sqrt{1-x} \in \mathcal{J}. \tag{$**$}$$

But $xa \in \mathcal{J}$, whence $a - \sqrt{x}a\sqrt{1-x} \in \mathcal{J}$ as desired. But in fact, for any continuous function $f : [0,1] \to \mathbb{C}$, we have $f(x)a - af(x) \in \mathcal{J}$! Indeed, the family $\mathcal{F}$ of all such functions f is clearly a closed linear subspace of $C([0,1])$ which contains the constants and all powers of t, $t \mapsto t^n$, whence $\mathcal{F}$ contains all polynomials, so $\mathcal{F} = C([0,1])$ by the Weierstrass approximation theorem.

2. In Proposition 2.2 of the next section, we obtain a stronger form of M-cai's in this context.

Applying Corollary 1.2, we thus obtain

COROLLARY 1.5. *Let $\mathcal{J}$ be an ideal in a C^*-algebra $\mathcal{A}$. There exists a central approximation unit (u_α) in $\mathcal{J}$ with $0 \le u_\alpha \le 1$ for all α so that setting $U_\alpha a = u_\alpha a$ for all a in $\mathcal{A}$, and letting P be the L-projection on $\mathcal{A}^*$ with kernel $\mathcal{J}^\perp$, we get $U_\alpha^* \to P$ in the SOT; in particular*, (1.22) *holds* (*where $Y = \mathcal{A}$*).

We may crystallize some of the ideas in the above proof via the following notion.

DEFINITION 1.3. Let $X \subset Y$ be operator spaces and $(U_\alpha)_{\alpha\in\mathcal{D}}$ be a net of operators on Y. Say that (U_α) is a *good* M-cai if the following conditions hold:

(i) $U_\alpha Y \subset X$ for all α.
(ii) $U_\alpha x \to x$ for all $x \in X$.
(iii) There exists a net $(V_\alpha)_{\alpha\in\mathcal{D}}$ of operators on Y so that
  (a) $U_\alpha(y) + V_\alpha(y) \to y$ for all $y \in Y$,
  (b) $(I - V_\alpha)Y \subset X$ for all α,
  (c) $S \circ (U_\alpha \oplus V_\alpha)$ is a complete contraction for all α, where $S : Y \oplus Y \to Y$ denotes the sum operator $S(u \oplus v) = u + v$ and $Y \oplus Y$ is a complete L^∞-direct sum.

REMARK. Condition (b) of 1.3 yields that good M-cai's (U_α) are hereditary; that is, if Z is an operator space with $X \subset Z \subset Y$, then also $V_\alpha Z \subset Z$ for all α and hence $(U_\alpha|Z)$ is a good M-cai for X in Z.

Now the proof of Proposition 1.4 yields that X admits a good M-cai in Y if Y is a C^*-algebra and X is an ideal in Y. It also easily yields

COROLLARY 1.6. *Suppose (U_α) is a good M-cai for X in Y. Then (U_α) is a contractive M-cai.*

We conclude this section with a permanence property for good M-cai's. Its motivation comes from the following open problem. If $X \subset Y$ are operator spaces with X a complete M-ideal in Y, is $Z \otimes_{\rm op} X$ a complete M-ideal in $Z \otimes_{\rm op} Y$, for all operator spaces Z? In the Banach space category, a result due to D. Werner shows that Proposition 1.7 is true without any approximate identity assumption (see Proposition VI.3.1 of [HWW] and [W1]).

PROPOSITION 1.7. *Let X, Y and Z be operator spaces with $X \subset Y$ and suppose X admits a good M-cai in Y. Then $Z \otimes_{\rm op} X$ admits a good M-cai in $Z \otimes_{\rm op} Y$. Hence $Z \otimes_{\rm op} X$ is a complete M-ideal in $Z \otimes_{\rm op} Y$.*

Proof. Let $(U_\alpha)_{\alpha\in\mathcal{D}}$ be a good M-cai for X in Y. We claim that $(I_Z \otimes U_\alpha) =: (\widetilde{U}_\alpha)$ is then a good M-cai for $Z \otimes_{\rm op} X$ in $Z \otimes Y$. Let $(V_\alpha)_{\alpha\in\mathcal{D}}$ satisfy 1.3(iii) and set $\widetilde{V}_\alpha = I_Z \otimes V_\alpha$ for all α. Now it is immediate that if $\widetilde{X} = Z \otimes_{\rm op} X$ and $\widetilde{Y} = Z \otimes_{\rm op} Y$, then $(\widetilde{U}_\alpha)$ and $(\widetilde{V}_\alpha)$ satisfy (i) and (iii)(b) of Definition 1.3. It also follows that (iii)(c) holds. Indeed, let $\widetilde{S} : \widetilde{Y} \oplus \widetilde{Y} \to \widetilde{Y}$ be the sum operator. Then

$$S \circ (\widetilde{U}_\alpha \oplus \widetilde{V}_\alpha) = I_Z \otimes S \circ (U_\alpha \oplus V_\alpha),$$

hence $\widetilde{S} \circ (\widetilde{U}_\alpha \oplus \widetilde{V}_\alpha)$ is a complete contraction since $S \circ (U_\alpha \oplus V_\alpha)$ has this property.

It remains to check the approximation conditions (ii) and (iii)(a). But we easily see that for $x' \in Z \otimes X$ (the algebraic tensor product), $\widetilde{U}_\alpha x' \to x'$, whence since $Z \otimes X$ is dense in $Z \otimes_{\rm op} X$ and the $\widetilde{U}_\alpha$'s are complete contractions, (ii) holds. The identical density argument establishes (iii)(c) (since again $\widetilde{S} \circ (\widetilde{U}_\alpha \oplus \widetilde{V}_\alpha)$ is a complete contraction for all α). ■

REMARK. We may also introduce a weaker version of good M-cai's and obtain a similar permanence property. Given spaces $X \subset Y$, let us say that (U_α) satisfies $(*)$ if (U_α) satisfies (i), (ii), and (iii)(a),(b) of 1.3, but instead of (iii)(c), we have

(c′) $S \circ (U_\alpha | X \oplus V_\alpha) : X \oplus Y \to Y$ is a complete contraction for all α.

Again, if (U_α) satisfies $(*)$, then (U_α) is a weak contractive M-cai for X in Y, hence X is a complete M-ideal in Y. Moreover, if Z is an arbitrary operator space, since X admits a family satisfying $(*)$, so does $Z \otimes_{\rm op} X \subset Z \otimes_{\rm op} Y$, whence again, $Z \otimes_{\rm op} X$ is a complete M-ideal in $Z \otimes_{\rm op} Y$.

2. Complementation results. The main (motivating) result of this section is as follows.

THEOREM 2.1. *Let $\mathcal{J} \subset Y \subset \mathcal{A}$ with $\mathcal{J}$ an approximately injective ideal in a C^*-algebra $\mathcal{A}$ and Y a λ-locally reflexive operator space with $Y/\mathcal{J}$ separable. Then for every $\varepsilon > 0$, there exists a completely bounded lift $L : Y/\mathcal{J} \to Y$ of $I_{Y/\mathcal{J}}$ with $\|L\|_{\rm cb} < \lambda + \varepsilon$.*

When $Y = \mathcal{A}$, $\lambda = 1$ (necessarily); our result then generalizes (up to ε) the theorem of E. Effros and U. Haagerup [EH], which yields that then, assuming $\mathcal{A}$ is unital, there exists a completely positive lift $L : \mathcal{A}/\mathcal{J} \to Y$ of $I_{\mathcal{A}/\mathcal{J}}$. We give an isometric operator-space generalization of the Effros–Haagerup lifting theorem in the Appendix.

We now recall two definitions. An operator space Y is *λ-locally reflexive* if for every finite-dimensional operator space E and operator $u : E \to Y^{**}$, there exists a family of operators $u_\alpha : E \to Y$ satisfying $\|u_\alpha\|_{\rm cb} \le \lambda \|u\|_{\rm cb}$ and $u_\alpha(x) \to u(x)$ in the w*-topology for every $x \in E$. An operator space X is *approximately injective* if for all finite-dimensional operator spaces $E \subset F$, operators $u : E \to X$, and $\varepsilon > 0$, there exists an operator $v : F \to X$ satisfying $\|v\|_{\rm cb} \le (1+\varepsilon)\|u\|_{\rm cb}$ and $v|_E = u$.

To prove Theorem 2.1, we use the stronger properties of the M-cai's for ideals in C^*-algebras obtained in the proof of Proposition 1.4.

DEFINITION 2.1. Let $X \subset Y$ be operator spaces and $(U_\alpha)_{\alpha\in\mathcal{D}}$ a net of operators on Y be given. Say that (U_α) is a *special M-cai* if the following conditions hold:

(i) $U_\alpha Y \subset X$ for all α.

(ii) $U_\alpha x \to x$ for all $x \in X$.

(iii) For all $y \in Y$, $U_\alpha^{n+1}(y) - U_\alpha^n(y) \to 0$ as $n \to \infty$, uniformly in α.

(iv) For every positive integer k, there exists a net $(V_\alpha^{(k)})_{\alpha\in\mathcal{D}}$ of operators on Y so that

(a) $U_\alpha^k(y) + V_\alpha^{(k)}(y) \to y$ for all $y \in Y$,

(b) $(I - V_\alpha^{(k)})Y \subset X$ for all α,

(c) $S \circ (U_\alpha^k \oplus V_\alpha^{(k)})$ is a complete contraction for all α, where $S : Y \oplus Y \to Y$ denotes the sum operator, $S(u \oplus v) = u + v$ for all $u, v \in Y$, and $Y \oplus Y$ is a complete L^∞-decomposition.

Our next result yields that (U_α) is a special M-cai precisely when (U_α) satisfies (iii) and all powers (U_α^k) of (U_α) are good M-cai's as given in Definition 1.3.

PROPOSITION 2.2. *Let $X \subset Y$ be operator spaces and (U_α) a net of operators on Y.*

(A) *If* (U_α) *satisfies conditions* (i), (ii) *and* (iv) *of Definition* 2.1, *then for all* k, (U_α^k) *is a contractive M-cai for* X *in* Y.

(B) *If* (U_α) *is a special M-cai, then also* (U_α^k) *is a special M-cai for all* k.

(C) *If* Z *is a closed subspace of* Y *with* $X \subset Z$, *then* $(U_\alpha|_Z)$ *is a special M-cai for* X *in* Z.

Proof. Condition (iv)(b) implies that the U_α's are complete contractions, hence so are the U_α^k's. Clearly, (U_α^k) satisfies (i) for any k. We easily see that (U_α^k) satisfies (ii) by induction on k. The case $k = 1$ follows by definition. Assuming the claim for k, for $x \in X$ we get

$$\begin{aligned}\|U_\alpha^{k+1}x - x\| &= \|U_\alpha^{k+1}(x) - U_\alpha^k(x) + U_\alpha^k(x) - x\| \\ &\le \|U_\alpha^k\|\,\|U_\alpha(x) - (x)\| + \|U_\alpha^k(x) - x\|.\end{aligned}$$

Hence $\lim_\alpha \|U_\alpha^{k+1}(x) - x\| = 0$ as desired.

To finish proving (A), we need only verify that (U_α^k) satisfies condition (iii′) of Definition 1.1. If we choose $V_\alpha^{(k)}$ as in 2.1(iv), then given n and $n\times n$ matrices (u_{ij}), (v_{ij}) in Y, we have

$$\lim_\alpha \|((I - U_\alpha^k)(v_{ij}) - V_\alpha^{(k)}(v_{ij}))\| = 0 \quad \text{by (iv)(a)},$$

whence

$$\text{(2.1)}\quad \begin{aligned}\overline{\lim_\alpha} \|(U_\alpha^k(u_{ij}) + (I - U_\alpha^k)(v_{ij}))\| &= \overline{\lim_\alpha} \|(U_\alpha^k(u_{ij}) + V_\alpha^{(k)}(v_{ij}))\| \\ &\le \max\{\|(u_{ij})\|, \|(v_{ij})\|\}.\end{aligned}$$

Now to prove (B), we only need to verify that (U_α^k) satisfies (iii). We first observe that if (U_α) is special, then for any positive integer k and $y \in Y$,

$$\text{(2.2)}\qquad (U_\alpha^{n+k} - U_\alpha^n)(y) \to 0 \quad \text{as } n \to \infty, \qquad \text{uniformly in } \alpha.$$

To see this by induction: the case $k = 1$ follows by definition. Assuming the claim valid for k, then for $y \in Y$,

$$\text{(2.3)}\quad U_\alpha^{n+k+1}(y) - U_\alpha^n(y) = [U_\alpha^{n+k+1}(y) - U_\alpha^{n+k}(y)] + [U_\alpha^{n+k}(y) - U_\alpha^n(y)].$$

But $(U_\alpha^{n+k+1} - U_\alpha^{n+k})(y) \to 0$ as $n \to \infty$, uniformly in α, by definition, and $U_\alpha^{n+k}(y) - U_\alpha^n(y) \to 0$ as $n \to \infty$, uniformly in α, by the induction hypothesis.

Now it follows directly that (U_α^k) satisfies (iii) for all k. Indeed, given $y \in Y$, for all α we have

$$\text{(2.4)}\quad (U_\alpha^k)^{n+1}(y) - (U_\alpha^k)^n(y) = U_\alpha^{kn+k}(y) - U_\alpha^{kn}(y) \to 0 \quad \text{as } n \to \infty$$

uniformly in α, by (2.2). Thus (B) is proved.

(C) is immediate from (iv)(c), since the latter implies $V_\alpha^{(k)} Z \subset Z$ for all α and k. ■

An inspection of the proof of Proposition 1.4 now yields our main example of this phenomenon.

PROPOSITION 2.3. *Let* $\mathcal{J}$ *be an ideal in a* C^*-*algebra* $\mathcal{A}$. *Then there is a special M-cai* $(U_\alpha)_{\alpha\in\mathcal{D}}$ *for* $\mathcal{J}$ *in* $\mathcal{A}$.

Proof. As before, we may assume that $\mathcal{A}$ is unital. For if, say, $\mathcal{J} \subset \mathcal{A}_0 \subset \mathcal{A}$, with $\mathcal{J}$ an ideal in $\mathcal{A}_0$ non-unital and $\mathcal{A}$ just $\mathcal{A}_0$ with unit adjoined, $\mathcal{J}$ is an ideal in $\mathcal{A}$ and then by 2.2(C), $(U_\alpha|_{\mathcal{A}_0})$ serves as the special M-cai for $\mathcal{J}$ in $\mathcal{A}_0$.

For any $0 \le y \le 1$ in $\mathcal{A}$, define the operator T_y on $\mathcal{A}$ by

$$\text{(2.5)}\qquad T_y(A) = yAy \quad \text{for all } A \in \mathcal{A}.$$

Next, let (x_α) be a central approximate unit for $\mathcal{J}$ (i.e., (x_α) satisfies (1.27)–(1.29)). Define U_α by (1.32); that is,

$$\text{(2.6)}\qquad U_\alpha = T_{\sqrt{x_\alpha}} \quad \text{for all } \alpha.$$

We claim that (U_α) is a special M-cai for $\mathcal{J}$ in $\mathcal{A}$.

We first note that for any positive integer k,

$$\text{(2.7)}\qquad (x_\alpha^k) \text{ is a central approximate unit for } \mathcal{J}.$$

This follows easily by induction on k. The case $k = 1$ is simply the definition. Suppose the claim proved for k. Then of course $0 \le x_\alpha^{k+1} \le 1$ for all α. Given $x \in \mathcal{J}$,

$$\begin{aligned}x_\alpha^{k+1}x - x &= x_\alpha^{k+1}x - x_\alpha^k x + x_\alpha^k x - x \\ &= x_\alpha^k(x_\alpha x - x) + x_\alpha^k x - x \\ &\to 0 \quad \text{by induction hypothesis.}\end{aligned}$$

Similarly, given $y \in Y$,

$$\begin{aligned}x_\alpha^{k+1}y - yx_\alpha^{k+1} &= x_\alpha(x_\alpha^k y - yx_\alpha^k) + (x_\alpha y - yx_\alpha)x_\alpha^k \\ &\to 0 \quad \text{by induction hypothesis.}\end{aligned}$$

Now we verify (i)–(iv) of Definition 2.1 for (U_α). (i) holds via Remark 1 following the proof of Proposition 1.4, (ii) holds by (1.31.i) and the fact that (x_α) is an approximate identity in $\mathcal{J}$. Now fix a positive integer k, and define

$$\text{(2.8)}\qquad V_\alpha^{(k)} = T_{(1-x_\alpha^k)^{1/2}} \quad \text{for all } \alpha.$$

Now (1.31.i), (1.31.ii) (applied to $x_\alpha = x_\alpha^k$) yield that (iv)(a) holds, while (iv)(b) holds via (1.32) (again applied to $x_\alpha = x_\alpha^k$ for all α). Also, (iv)(c) holds, again via Remark 1 following the proof of 1.4.

It remains to verify (iii). In fact, we have the stronger condition

$$\text{(2.9)}\qquad \|U_\alpha^{n+1} - U_\alpha^n\| \to 0 \quad \text{as } n \to \infty, \qquad \text{uniformly in } \alpha.$$

Indeed, for any $0 \le y \le 1$ in $\mathcal{A}$ we have, for all $A \in \mathcal{A}$,

$$\text{(2.10)}\qquad T_y^n(A) = T_{y^n}(A) = y^n A y^n.$$

Hence

$$T_y^{n+1}(A) - T_y^n(A) = y^{n+1}A(y^{n+1} - y^n) + (y^{n+1} - y^n)Ay^n. \tag{2.11}$$

Now thanks to the (elementary) operational calculus, we have

$$\lim_{n\to\infty} \|y^{n+1} - y^n\| = 0, \qquad \text{uniformly over } y \in \mathcal{A} \text{ with } 0 \le y \le 1. \tag{2.12}$$

Thus, given $\varepsilon > 0$, we may choose n so that

$$\|y^{n+1} - y^n\| \le \varepsilon/2 \qquad \text{for all } 0 \le y \le 1,\ y \in \mathcal{A}, \tag{2.13}$$

But then

$$\|T_y^{n+1}(A) - T_y^n(A)\| \le \|y^{n+1}\|\,\|A\|\,\|y^{n+1} - y^n\| + \|y^{n+1} - y^n\|\,\|A\|\,\|y^n\| \le \varepsilon\|A\|. \tag{2.14}$$

Hence

$$\|T_y^{n+1} - T_y^n\| \le \varepsilon \qquad \text{for all } y \in \mathcal{A},\ 0 \le y \le 1. \tag{2.15}$$

But of course $U_\alpha^n = T^n_{x_\alpha^{1/2}}$ for all α, proving (2.9). ■

REMARK. Buried in the proof of this result, as well as the proof of Proposition 1.4, is the following elementary fact: For any $0 \le y \le 1$ in $\mathcal{A}$, $S \circ (T_{\sqrt{y}} \oplus T_{\sqrt{1-y}})$ is a completely positive contraction.

(The complete positivity is evident, upon explicitly writing this map as

$$u \oplus v \mapsto \sqrt{y}\, u\sqrt{y} + \sqrt{1-y}\, v\sqrt{1-y}.$$

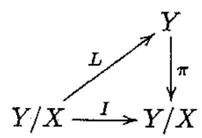
Hence, assuming that $\mathcal{A}$ is unital with identity I, we need only compute $\|S \circ (T_{\sqrt{y}} \oplus T_{\sqrt{I-y}})(I \oplus I)\|$; but of course this equals

$$\|\sqrt{y}\sqrt{y} + \sqrt{I-y}\sqrt{I-y}\| = \|y + I - y\| = 1.)$$

We may now give the proof of Theorem 2.1, via the following more general result.

THEOREM 2.4. *Let $\lambda \ge 1$, and let $X \subset Y$ be operator spaces with X approximately injective, Y λ-locally reflexive, and Y/X separable. Assume that X admits a special M-cai in Y. Then for all $\varepsilon > 0$, there exists a completely bounded lift $L : Y/X \to Y$ of $I_{Y/X}$ with $\|L\|_{\mathrm{cb}} < \lambda + \varepsilon$.*

REMARKS. 1. L is called a *lift* of $I = I_{Y/X}$ if, for $\pi : Y \to Y/X$ the quotient map, the following diagram commutes:



We note that the conclusion is equivalent to the assertion that for all $\varepsilon > 0$, X is completely $(\lambda + \varepsilon)$-co-complemented in Y; that is, there exists a linear projection P mapping Y onto X with $\|I - P\|_{\mathrm{cb}} \le \lambda + \varepsilon$.

2. An operator space X is defined to be *approximately injective* if for all finite-dimensional operator spaces $E \subset F$, linear maps $T : E \to X$, and $\varepsilon > 0$, there exists a linear map $\widetilde{T} : F \to X$ extending T with $\|\widetilde{T}\|_{\mathrm{cb}} < \|T\|_{\mathrm{cb}} + \varepsilon$. (This is equivalent to the definition given in [EH] when X is a C^*-algebra.) We recall that nuclear operator spaces are approximately injective, while approximately injective 1-locally reflexive operators spaces are nuclear (cf. [EOR]).

3. Theorem 2.4 immediately yields the following generalization. *Let λ, X and Y satisfy the assumptions of 2.4, but delete the hypothesis that Y/X is separable. Then for all $\varepsilon > 0$, separable operator spaces Z and completely bounded maps $T : Z \to Y/X$, there is a lift $\widetilde{T} : Z \to Y$ of T with $\|\widetilde{T}\|_{\mathrm{cb}} \le (\lambda + \varepsilon)\|T\|_{\mathrm{cb}}$.* Indeed, let Y_0 equal the closed linear span of X and $T(Z)$. Then Y_0 is also λ-locally reflexive and, since special M-cai's are hereditary (Proposition 2.2(C)), X admits a special M-cai in Y_0. Thus by Theorem 2.4, there exists a lift $L : Y_0/X \to Y_0$ of $I_{Y_0/X}$ with $\|L\|_{\mathrm{cb}} < \lambda + \varepsilon$. But then $\widetilde{T} := L \circ T$ is a lift of T with $\|\widetilde{T}\| \le (\lambda + \varepsilon)\|T\|_{\mathrm{cb}}$. ■

The following lemma is the crucial tool for the proof.

LEMMA 2.5. *Let X and Y satisfy the hypotheses of Theorem 2.4. Let $Y_1 \subset Y_2$ be linear subspaces of Y with $X \subset Y_i$ and $E_i := Y_i/X$ finite-dimensional, $i = 1, 2$. Let $L_1 : E_1 \to Y_1$ be a lift of I_{E_1} and set $\gamma = \|L_1\|_{\mathrm{cb}}$. Then given $\varepsilon > 0$, there exists a lift $L_2 : E_2 \to Y_2$ of I_{E_2} with*

$$\|L_2|_{E_1} - L_1\| < \varepsilon \tag{2.16}$$

and

$$\|L_2\|_{\mathrm{cb}} < \max\{\gamma, \lambda\} + \varepsilon. \tag{2.17}$$

Proof. Let $\varepsilon > 0$. We first note that there exists a lift $L : E_2 \to Y_2$ of I_{E_2} with

$$\|L\|_{\mathrm{cb}} < \lambda + \varepsilon. \tag{2.18}$$

Indeed, the hypotheses yield that X is a complete M-ideal in Y, and hence X^{**} is completely co-contractively complemented in Y_2^{**}. Then Proposition 2.6(ii) yields that X is completely $(\lambda + \varepsilon)$-co-complemented in Y (see also Sublemma 3.11 of [Ro]). Now choosing a projection P from Y_2 onto X with $\|I - P\|_{\mathrm{cb}} < \lambda + \varepsilon$ and letting $G = (I - P)X$, we deduce that $\pi|_G$ maps G one-to-one onto Y/X and $\|(\pi|_G)^{-1}\|_{\mathrm{cb}} < \lambda + \varepsilon$, whence $(\pi|_G)^{-1}$ is the desired lift L.

Let $(U_\alpha)_{\alpha\in\mathcal{D}}$ be a special M-cai for X in Y. Now, choose k so that

$$\|(U_\alpha^{k+1} - U_\alpha^k)L_1\| < \varepsilon/3 \quad \text{for all } \alpha. \tag{2.19}$$

We may do this, since $L_1(E_1)$ is a finite-dimensional subspace of Y, using (iii) of Definition 2.1. Now choose $(V_\alpha^{(k)})$ as in (iv) of Definition 2.1.

Since L_1 and $L|_{E_1}$ lift I_{E_1}, we have

$$(L_1 - L)(E_1) \subset X. \tag{2.20}$$

Hence since (U_α^k) is a special M-cai, choose an α so that

$$\|(L_1 - L|_{E_1}) - U_\alpha^k(L_1 - L|_{E_1})\| < \varepsilon/3 \tag{2.21}$$

and

$$\|V_\alpha^{(k)}L|_{E_1} + U_\alpha^k L|_{E_1} - L|_{E_1}\| < \varepsilon/3 \quad \text{(by (iv)(b))}. \tag{2.22}$$

Now note that $U_\alpha L_1(E_1) \subset X$. Hence by the approximate injectivity of X and the complete contractivity of U_α, we may choose an extension $\theta : E_2 \to X$ of $U_\alpha L_1$ with

$$\|\theta\|_{\mathrm{cb}} < \gamma + \varepsilon. \tag{2.23}$$

Define $L_2 : E_2 \to Y$ by

$$L_2 = V_\alpha^{(k)} L + U_\alpha\theta. \tag{2.24}$$

Then (iv)(b) yields that

$$\|L_2\|_{\mathrm{cb}} \le \max\{\|L\|_{\mathrm{cb}}, \|\theta\|_{\mathrm{cb}}\} \le \max\{\lambda + \varepsilon, \gamma + \varepsilon\} \tag{2.25}$$

by (2.18) and (2.23).

To see that L_2 is a lift of I_{E_2}, let $e \in E_2$. Then

$$\begin{aligned} \pi L_2(e) &= \pi(V_\alpha^{(k)}L(e) + U_\alpha\theta(e)) \\ &= \pi V_\alpha^{(k)} L(e) && \text{by 2.1(i)} \\ &= \pi L(e) && \text{by 2.1(iv)(b)} \\ &= e && \text{since } L \text{ is a lift.} \end{aligned}$$

Finally, we must estimate the norm of $L_1 - L_2|_{E_1}$. Now we have

$$\begin{aligned} L_2|_{E_1} &= V_\alpha^{(k)}L|_{E_1} + U_\alpha^{k+1}L_1 \\ &= V_\alpha^{(k)}L|_{E_1} + U_\alpha^k L_1 + R_1 \quad \text{where } \|R_1\| < \varepsilon/3 \end{aligned} \tag{2.26}$$

by (2.19). But also

$$V_\alpha^{(k)}L|_{E_1} = L|_{E_1} - U_\alpha^k L|_{E_1} + R_2 \quad \text{where } \|R_2\| < \varepsilon/3 \tag{2.27}$$

by (2.22). Next,

$$L_1 = L|_{E_1} - U_\alpha^k L|_{E_1} + U_\alpha^k L_1 + R_3 \quad \text{where } \|R_3\| < \varepsilon/3 \tag{2.28}$$

by (2.21). Hence

$$\begin{aligned} L_2|_{E_1} - L_1 &= L|_E - U_\alpha^k L|_{E_1} + R_2 + U_\alpha^k L_1 \\ &\quad + R_1 - L|_{E_1} + U_\alpha^k L|_{E_1} - U_\alpha^k L_1 - R_3 \\ &= R_1 + R_2 - R_3. \end{aligned} \tag{2.29}$$

Finally,

$$\|L_2|_{E_1} - L_1\| \le \|R_1\| + \|R_2\| + \|R_3\| < \varepsilon. \ \blacksquare \tag{2.30}$$

We are now prepared for the

Proof of Theorem 2.4. Let $0 < \varepsilon < 1$ and choose finite-dimensional spaces $E_1 \subset E_2 \subset \ldots$ in Y/X with

$$\overline{\bigcup_j E_j} = Y/X.$$

As in the first step of Lemma 2.5, choose a lift $L_1 : E_1 \to Y/X$ of I_{E_1} with

$$\|L_1\|_{\mathrm{cb}} < \lambda + \varepsilon/2.$$

Let $n \ge 1$ and suppose $L_n : E_n \to Y/X$ has been chosen, lifting I_{E_n} with

$$\|L_n\|_{\mathrm{cb}} < \lambda + \sum_{j=1}^{n} \frac{\varepsilon}{2^j}. \tag{2.31}$$

Then by Lemma 2.5, we may choose $L_{n+1} : E_{n+1} \to Y/X$ lifting $I_{E_{n+1}}$ with

$$\|L_{n+1}\|_{\mathrm{cb}} < \|L_n\|_{\mathrm{cb}} + \frac{\varepsilon}{2^{n+1}} \le \lambda + \sum_{j=1}^{n+1} \frac{\varepsilon}{2^j} \tag{2.32.i}$$

and

$$\|L_{n+1}|_{E_n} - L_n\| < \varepsilon/2^n. \tag{2.32.ii}$$

Now it follows that if we set $Z = \bigcup_{j=1}^\infty E_j$, then (L_n) converges pointwise to a lift L of I_Z satisfying

$$\|L\|_{\mathrm{cb}} \le \lambda + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \lambda + \varepsilon. \tag{2.33}$$

To see this, let $z \in E_k$ for some k. Then for any $k \le m < n$,

$$\begin{aligned} \|L_n(z) - L_m(z)\| &= \Big\| \sum_{j=m}^{n-1} (L_{j+1}(z) - L_j(z)) \Big\| \\ &\le \sum_{j=m}^{n-1} \|L_{j+1}|_{E_j} - L_j\| \, \|z\| \\ &\le \frac{\varepsilon}{2^{m-1}} \to 0 \quad \text{as } m \to \infty, \end{aligned}$$

hence (L_n) indeed converges to a linear operator L on Z. But we also see that if we fix k, then since $\|L_n|_{E_k}\|_{\mathrm{cb}} \le \lambda+\varepsilon$ for all n, by (2.31), also $\|L|_{E_k}\|_{\mathrm{cb}} \le \lambda+\varepsilon$. Moreover, since $L_n|_{E_k}$ lifts I_{E_k}, so does L. Hence L indeed lifts I_Z. It now remains to simply extend L to all of Y/X by continuity. ■

REMARK. Say that a net (U_α) of operators on Y is a *weak special M-cai* provided (U_α) satisfies all the conditions of Definition 2.1 except that we replace (iv)(c) by

(iv)(c′) $S\circ(U_\alpha^k|_X \oplus V_\alpha^{(k)})$ is a complete contraction from $X\oplus Y$ to Y for all α.

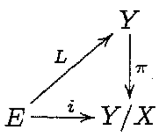
(In other words, for all k, (U_α^k) satisfies condition (∗) given in Remark 1 at the end of Section 1, and also (U_α) satisfies condition (iii) of Definition 2.1.) The proof of Theorem 2.4 yields that its conclusion holds provided we assume instead that X admits a weak special M-cai in Y.

We next take up the problem of ensuring that X is *complemented* in Y, when $X\subset Y$ are operator spaces with Y/X separable and X approximately injective. (It apparently remains an open question if this is always the case in this setting.) Note, however, that X need not be completely complemented. A remarkable example by E. Kirchberg yields a non-exact separable C^* algebra $\mathcal{A}$ and an ideal $J\subset\mathcal{A}$ with $\mathcal{J}$ nuclear and $\mathcal{A}/\mathcal{J}$ exact [Ki]. Were $\mathcal{J}$ completely complemented, $\mathcal{A}$ would be λ-exact for some λ; but then, since $\mathcal{A}$ is a C^*-algebra, $\mathcal{A}$ would be exact (cf. [Pi]). Another example [OR], due to T. Oikhberg and the second author of the present paper, yields an example with X completely isometric to $\mathcal{K}$ and Y/X completely isometric to c_0.

We introduce several new concepts for our investigation.

DEFINITION 2.2. Let $X\subset Y$ be Banach/operator spaces and $\lambda\ge 1$. Let π denote the quotient map from Y onto Y/X. We consider the following diagram, for a general finite-dimensional subspace E of Y/X and the identity injection $i: E\to Y/X$:

(∗)



That is, L is a lift of I_E to Y.

(i) (X,Y) is said to have *λ-local liftings* (*λ-ll's*) if for all such E and $\varepsilon>0$, there exists a map L satisfying (∗) with $\|L\|<\lambda+\varepsilon$.

(ii) (X,Y) is said to have *λ-complete local liftings* (*λ-cll's*) if for all such E and $\varepsilon>0$, there exists a map L satisfying (∗) with $\|L\|_{\mathrm{cb}}<\lambda+\varepsilon$.

(iii) (X,Y) is said to have *λ-extendable local liftings* (*λ-ell's*) if for all such E and $\varepsilon>0$, there exists a map $T: Y/X\to Y^{**}$ with $\|T\|<\lambda+\varepsilon$ so that $L=T|_E$ satisfies (∗).

Finally, (X,Y) is said to have *local liftings* (resp. *complete local liftings*, resp. *extendable local liftings*), if there exists a $\lambda\ge 1$ so that (X,Y) has λ-ll's (resp. λ-cll's resp. λ-ell's).

As we show below, if X^{**} is completely complemented in Y^{**} and Y is locally reflexive, then (X,Y) has complete local liftings. The following is thus a strengthening of Theorem 2.4 (cf. the Remark following its proof for the definition of weak special M-cai's).

THEOREM 2.4′. *Let $X\subset Y$ be operator spaces. The conclusion of Theorem 2.4 holds if one replaces in its hypotheses the assumption that Y is λ-locally reflexive by the assumption that (X,Y) has λ-complete local liftings, and that instead X admits a weak special M-cai in Y.*

In fact, the proof of Theorem 2.4 gives this immediately; one only needs to observe that the local reflexivity assumption on Y is used solely to produce the lift L in the proof of Lemma 2.5, satisfying (2.16). As we show below, the existence of this map follows directly from the assumption that (X,Y) has λ-complete local liftings.

REMARK. Let $X\subset Y$ be operator spaces. Then (X,Y) has complete local liftings if and only if X is *locally complemented* in Y; that is, there exists a $\beta\ge 1$ so that X is β-completely complemented in Z for all $X\subset Z\subset Y$ with Z/X finite-dimensional (one then says X is β-locally complemented in Y). In fact, it is easily seen that if (X,Y) has λ-cll's, then X is $(\lambda+1+\varepsilon)$-locally complemented in Y for all $\varepsilon>0$, while if X is β-locally complemented in Y, then (X,Y) has $(\beta+1)$-local liftings (cf. [Ro], [OR] for certain consequences of local complementability). The quantitative cll concept is more appropriate in the context of the present work.

The next two results list several easily proved permanence properties of the concepts introduced in Definition 2.2.

PROPOSITION 2.6. *Let $X\subset Y$ be Banach/operator spaces and let $\lambda\ge 1$.*

(i) *(X,Y) has λ-ll's if and only if*

(∗∗) *there exists a lift $L: Y^{**}/X^{**}\to Y^{**}$ of the identity map on Y^{**}/X^{**} so that $\|L\|\le\lambda$.*

(ii) *If (X,Y) has λ-cll's, (∗∗) holds with $\|L\|_{\mathrm{cb}}\le\lambda$. If (∗∗) holds with $\|L\|_{\mathrm{cb}}\le\lambda$ and Y is β-locally reflexive, then (X,Y) has λβ-cll's.*

(iii) *If (X,Y) has λ-cll's, then (X,Z) has λ-cll's for all operator spaces Z with $X\subset Z\subset Y$.*

PROPOSITION 2.7. *Let X,Y be as in 2.6. Assume that (X,Y) has local liftings. Then (X,Y) has extendable local liftings under any of the following hypotheses:*

(a) *Y/X has the bounded approximation property (the bap).*
(b) *Y^{**} is an (isomorphically) injective Banach space.*
(c) *Y is extendably locally reflexive.*
(d) *Y is an operator space with Y^{**} an (isomorphically) injective operator space and (X,Y) has complete local liftings.*

In fact, let $\lambda,\beta\ge 1$ and assume that (X,Y) has β-ll's. Then (X,Y) has $(\beta\lambda)$-ell's provided any of the following holds:

(a′) *Y/X has the λ-bap.*
(b′) *Y^{**} is λ-injective.*
(c′) *Y is λ-extendably locally reflexive.*
(d′) *(X,Y) has β-cll's and Y^{**} is a λ-injective operator space.*

REMARK. We recall (cf. [OR]) that a Banach space Y is called *λ-extendably locally reflexive* (*λ-elr*) provided for all $\varepsilon>0$ and all finite-dimensional subspaces $G\subset Y^{**}$ and $F\subset Y^*$, there exists a linear operator $T:Y^{**}\to Y^{**}$ with $\|T\|<\lambda+\varepsilon$, $TG\subset Y$, and $\langle Tg,f\rangle=\langle g,f\rangle$ for all $g\in G$ and $f\in F$.

Proof of Proposition 2.6. Let $\pi:Y\to Y/X$ be the quotient map.

(i) Suppose first that there exists a lift L satisfying $(**)$. Let E be a finite-dimensional subspace of Y/X and let $\varepsilon>0$ be given. Regarding $Y/X\subset Y^{**}/X^{**}$, let $G=L(E)$. Let $Y_0=\pi^{-1}(E)$. Then of course $X\subset Y_0$, $Y_0/X=E$, and $G\subset Y^{**}$. Let F be $X^\perp$ relative to Y_0^{**}. Then F is finite-dimensional and for all $x^\perp\in X^\perp$ there exists an $f\in F$ with $x^\perp|_{Y_0^{**}}=f|_{Y_0^{**}}$.

By the local reflexivity principle, choose $T:G\to Y$ with $\|T\|<1+\varepsilon/\lambda$ and

$$\|T\|<1+\varepsilon/\lambda, \tag{2.34}$$
$$\langle Tg,f\rangle=\langle g,f\rangle \quad \text{for all } g\in G \text{ and } f\in F. \tag{2.35}$$

Then $T\circ L|_E$ is our desired lift of I_E. Indeed, $\|T\circ L\|<(1+\varepsilon/\lambda)\lambda=\lambda+\varepsilon$, and for all $e\in E$,

$$Le-TLe\in X^{**} \tag{2.36}$$

thanks to the definition of F. But then

$$\begin{aligned}\pi TLe&=\pi^{**}TLe=\pi^{**}Le &&\text{by (2.35)}\\ &=e &&\text{since } L \text{ is a lift of } I_{Y^{**}/X^{**}}.\end{aligned}$$

Now suppose (X,Y) has λ-ll's and let $\mathcal{D}$ be the following directed set: $\mathcal{D}=\{(E,\varepsilon): E$ is a finite-dimensional subspace of Y/X and $\varepsilon>0\}$, where $(E,\varepsilon)\le(E',\varepsilon')$ if $E\subset E'$ and $\varepsilon\ge\varepsilon'$. For each $\alpha=(E,\varepsilon)\in\mathcal{D}$, choose a lift $L_\alpha:E\to Y$ of I_E with $\|L_\alpha\|<\lambda+\varepsilon$. By the Tikhonov theorem, we may choose a subnet $(L_{\alpha_\beta})_{\beta\in\mathcal{D}'}$ of (L_α) such that for all $e\in Y/X$,

$$\lim_\beta L_{\alpha_\beta}(e)=:L(e)$$

exists weak* in Y^{**}. We easily verify that then $L:Y/X\to Y^{**}$ is a linear operator with

$$\|L\|\le\lambda \quad\text{and}\quad \pi^{**}\circ L=\chi \tag{2.37}$$

where $\chi:Y/X\to(Y/X)^{**}$ is the canonical injection. It follows that if we let $P:Y^{****}\to Y^{**}$ be the canonical projection, then $P\circ L^{**}$ is the desired lift of $(Y/X)^{**}$ into Y^{**}.

(ii) The first assertion follows immediately by the argument given in (i), and indeed so does the second: we just choose T as in that argument, but so that

$$\|T\|_{\rm cb}<\beta+\lambda/\varepsilon. \tag{2.38}$$

Then $\|T\circ L\|_{\rm cb}<(\beta+\lambda/\varepsilon)\lambda=\lambda\beta+\varepsilon$ as desired.

(iii) Let $\varepsilon>0$, let E be a finite-dimensional subspace of Z/X, regarded as a subspace of Y/X, and let $L:E\to Y$ be a lift of I_E with $\|L\|_{\rm cb}<\lambda+\varepsilon$. But then $L(E)\subset Z$! Indeed, for $e\in E$, we must have $\pi\circ L(e)=e$, which means there exists a $z\in Z$ so that $L(e)-z\in X$. But this says that $Le\in X+Z=Z$. This completes the proof of 2.6. ■

Proof of Proposition 2.7. Of course, we just need to prove the quantitative assertions. Let $\lambda,\beta\ge1$ and assume that (X,Y) has β-ll's. We show that (X,Y) has $\beta\lambda$-ell's under any of (a′)–(d′). Let E be a finite-dimensional subspace of Y/X and let $\varepsilon>0$. If (a′) holds, we may choose a finite-rank operator $T:Y/X\to Y/X$ with $\|T\|<\lambda+\varepsilon/\beta$ and $T|_E=I_E$. By Proposition 2.6, we may choose a lift $L:Y/X\to Y^{**}$ of $\chi:Y/X\to(Y/X)^{**}$ with $\|L\|\le\beta$. Then $L\circ T:Y/X\to Y^{**}$ satisfies

$$\|L\circ T\|\le\|L\|\,\|T\|<\beta(\lambda+\varepsilon/\beta)=\beta\lambda+\varepsilon. \tag{2.39}$$

But $L|_E=I_E$ (regarding $Y/X\subset(Y/X)^{**}$), hence $(L\circ T)|_E=I_E$, and case (a′) is thus proved.

Now assume (b′) holds, and choose $L:E\to Y$ a lift of I_E with $\|L\|<\beta+\varepsilon/\lambda$. Since Y^{**} is λ-injective, choose an extension $\widetilde L:Y/X\to Y^{**}$ of L with $\|\widetilde L\|\le\lambda\|L\|<\lambda\beta+\varepsilon$. This proves case (b′). Similarly, if (d′) holds, we choose L as above so that instead $\|L\|_{\rm cb}<\beta+\lambda/\varepsilon$, and then choose $\widetilde L$ with $\|\widetilde L\|_{\rm cb}\le\lambda\|L\|_{\rm cb}<\lambda\beta+\varepsilon$.

Finally, suppose (c′) holds. By Proposition 2.6, choose a lift $L:Y^{**}/X^{**}\to Y^{**}$ of the identity on Y^{**}/X^{**} with $\|L\|\le\beta$. Let $W=L(Y^{**}/X^{**})$ and $P=L\circ\pi^{**}$. It follows easily that P is a projection from Y^{**} onto W with kernel equal to X^{**}. Let $G=L(E)$, and let F be a finite-dimensional subspace of $X^\perp$. Now by the definition of λ-extendable local reflexivity, choose a linear operator $T:Y^{**}\to Y^{**}$ so that

$$\|T\|<\lambda+\varepsilon/\beta, \tag{2.40}$$

(2.41) $$TG \subset Y$$

and

(2.42) $$\langle Tg, f\rangle = \langle g, f\rangle \quad \text{for all } f \in F \text{ and } g \in G.$$

Now let $T_F = \widetilde{T} := T \circ L|_{Y/X}$. Then

$$\|\widetilde{T}\| \le \|T\|\,\|L\| < (\lambda + \varepsilon/\beta)\beta = \lambda\beta + \varepsilon.$$

By (2.42), identifying $(Y/X)^*$ with $X^\perp$ and using the fact that L is a lift, we have

(2.43) $$\langle e - \pi T_F e, f\rangle = 0 \quad \text{for all } f \in F.$$

Now let $\mathcal{D}$ be the set of finite-dimensional subspaces of $X^\perp$, directed by inclusion. We then deduce by (2.43) that the net $((\pi T_F)|_E)_{F\in\mathcal{D}}$ converges to I_E in the WOT. Since E is finite-dimensional, convex combinations of this net converge to I_E in norm. Thus given $\eta > 0$, there exists a convex combination S of the T_F's such that $\|I_E - \pi S\| < \eta$. Of course, $S(E) \subset Y$ and $\|S\| < \lambda\beta + \varepsilon$ also. Now suppose $k = \dim E$; choose a normalized Auerbach basis (e_i) for E; so there also exist normalized f_i's in $X^\perp$ with $f_i(e_j) = \delta_{ij}$ and $e = \sum f_i(e)e_i$ for all $e \in E$. Now $\|\pi S e_i - e_i\| < \eta$ for all i. Hence for each i, we may choose $y_i \in Y$ with $\pi y_i = e_i$ and $\|Se_i - y_i\| < \eta$. Finally, define $\widetilde{S} : Y/X \to Y^{**}$ by

$$\widetilde{S}(z) = S(z) + \sum f_i(z)(y_i - Se_i).$$

Then clearly $\|\widetilde{S}\| < \lambda\beta + \varepsilon + k\eta$, $\widetilde{S}(E) \subset Y$, and if $e \in E$, then

$$\pi\widetilde{S}(e) = S(e) + \sum f_i(e)e_i - \sum f_i(e)S(e_i) = e.$$

Thus $\widetilde{S}|_E$ is indeed a lift of I_E, completing the proof (since $\eta > 0$ is arbitrary). ■

REMARK. The proof of 2.7 case (c) yields that the assumption that (X, Y) has extendable local liftings is considerably weaker than the joint assumption that (X, Y) has local liftings and Y is extendably locally reflexive.

Indeed the proof yields the following result: *Suppose that $X \subset Y$ are Banach spaces with X^{**} β-co-complemented in Y^{**}, and let $P : Y^{**} \to Y^{**}$ be a projection with kernel equal to X^{**}, $\|P\| \le \beta$. Now assume that for all finite-dimensional subspaces $E \subset Y$, $F \subset X^\perp$, and $\varepsilon > 0$, if we set $G = PE$, then there exists an operator $T : Y^{**} \to Y^{**}$ with $TG \subset Y$, $\|T\| < \lambda + \varepsilon$, and $\langle Tg, f\rangle = \langle g, f\rangle$ for all $g \in G$, $f \in F$. Then (X, Y) has $\lambda\beta$-extendable local liftings.*

Thus if e.g., Y is separable, we need only find "extensions" of local-reflexivity operators on a certain countable family of finite-dimensional subspaces of Y^{**}, rather than all its finite-dimensional subspaces, as in the definition of elr.

We are now prepared for the second main result of this paper. The following qualitative special case provides its main motivation: *Suppose $\mathcal{J}$ is an approximately injective ideal in a C^*-algebra $\mathcal{A}$ with $\mathcal{A}/\mathcal{J}$ separable. Then $\mathcal{J}$ is complemented in $\mathcal{A}$ provided $(\mathcal{J}, \mathcal{A})$ has extendable local liftings.*

THEOREM 2.8. *Let $X \subset Y$ be operator spaces with X approximately injective. Assume that*

(a) *(X, Y) has λ-extendable local liftings.*
(b) *X admits a weak special M-cai in Y.*

Then for all operator spaces Z with $X \subset Z \subset Y$ with Z/X separable, and for all $\varepsilon > 0$, there exists a lift $L : Z/X \to Z$ of $I_{Z/X}$ with $\|L\| < \lambda + \varepsilon$.

NOTE. Weak special M-cai's are defined in the Remark following the proof of Theorem 2.4.

Proof of Theorem 2.8. Let $(U_\alpha)_{\alpha\in\mathcal{D}}$ be a weak special M-cai for X in Y. We shall define a NEW operator space structure on Y with the following properties (where (X, OLD) denotes the given operator space structure on X, and $(X, Y)_{\text{NEW}}$ denotes the pair (X, Y) in the NEW structure).

(2.44) The identity injection $i : (Y, \text{NEW}) \to (Y, \text{OLD})$ is a semi-isometry.
(2.45) $(X, \text{NEW}) = (X, \text{OLD})$.
(2.46) (U_α) is also a weak special M-cai for X in (Y, NEW).
(2.47) $(X, Y)_{\text{NEW}}$ has λ-complete local liftings.

(Recall that if Z and W are operator spaces, $T : Z \to W$ is a *semi-isometry* provided T is a norm-preserving complete contraction.) Once this is accomplished, the conclusion of Theorem 2.8 follows immediately from Theorem 2.4′, Proposition 2.2 and Proposition 2.6. Indeed, X is now an approximately injective subspace of (Y, NEW) satisfying (2.46) and (2.47), hence by Proposition 2.6(iii), $(X, Z)_{\text{NEW}}$ has λ-cll's, and also (by the proof of Proposition 2.2(C)), $(U_\alpha|_Z)$ is a weak special M-cai for X in Z. Hence by Theorem 2.4′, for all $\varepsilon > 0$, there exists a lift $L : (Z/X)_{\text{NEW}} \to (Z, \text{NEW})$ of $I_{Z/X}$ with $\|L\|_{\text{cb}} < \lambda + \varepsilon$. But then of course $\|L\| < \lambda + \varepsilon$, and since (2.44) holds, the Banach norm of L is the same in the NEW and OLD structures.

We define NEW as follows (where $\pi : Y \to Y/X$ is the quotient map): For $\tau \in \mathcal{K} \otimes Y$, set

(2.48) $$\|\tau\|_{\text{NEW}} = \max\{\|\tau\|, \|(I_{\mathcal{K}} \otimes \pi)(\tau)\|_{\text{MAX}}\}.$$

We first note that $\|\cdot\|_{\text{NEW}}$ is indeed an operator space structure on Y, and moreover, for all $\tau \in \mathcal{K} \otimes Y^{**}$,

(2.49) $$\|\tau\|_{\text{NEW}} = \max\{\|\tau\|, \|(I_{\mathcal{K}} \otimes \pi^{**})(\tau)\|_{\text{MAX}}\}.$$

(This fact uses only the definition of NEW; none of the other assumptions on (X,Y) are needed.) To see this, define $T: Y \to Y \oplus Y/X$ (ℓ^∞-direct sum) by $Ty = y \oplus \pi y$ for all $y \in Y$, and set $Y' = T(Y)$. It is trivial that $T: Y \to Y'$ is a surjective isometry. Now simply endow $Y \oplus Y/X$ with the ℓ^∞-direct sum operator space structure of (Y, OLD) and $(Y/X)_{\mathrm{MAX}}$, and call this $(Y \oplus Y/X, \mathrm{NEW})$. Now (Y, NEW) is nothing but the operator-space structure induced on Y' by $(Y \oplus Y/X, \mathrm{NEW})$. Hence, since $(Y \oplus Y/X, \mathrm{NEW})$ is an operator space, so is (Y, NEW). But furthermore,

$$((Y \oplus Y/X)^{**}, \mathrm{NEW}) = (Y^{**}, \mathrm{OLD}) \oplus (Y^{**}/X^{**}, \mathrm{MAX}), \tag{2.50}$$

and so again (Y^{**}, NEW) is nothing but the operator space structure induced on $(Y')^{**}$ in $(Y^{**} \oplus Y^{**}/X^{**}, \mathrm{NEW})$, which is of course given by (2.49).

Now it is trivial that (2.44) and (2.45) hold; it remains to verify (2.46). Since the Banach norms in (Y, OLD) and (Y, NEW) coincide, all of the norm properties of (U_α) remain valid in (Y, NEW), so in fact we only need to verify that given k and $V_\alpha^{(k)}$ satisfying "weak" (iv) of Definition 2.1 for (Y, OLD), also "weak" (iv)(d) holds in (Y, NEW). Precisely, $S \circ (U_\alpha^k|_X \oplus V_\alpha^{(k)})$ is a complete contraction on $(X \oplus Y, \mathrm{OLD})$, and we must verify the same for $(X \oplus Y, \mathrm{NEW})$.

Now by Theorem 1.1, X is a complete M-ideal in Y; let then W be the (w*-closed) linear subspace of Y^{**} such that $X^{**} \oplus W$ is a complete M-decomposition of (Y^{**}, OLD) and let R be the projection from Y^{**} onto W with kernel X^{**}. It follows that π^{**} is a complete surjective isometry from W onto Y^{**}/X^{**}. But then (2.49) yields that

$$(Y^{**}, \mathrm{NEW}) = (X^{**}, \mathrm{OLD}) \oplus (W, \mathrm{MAX}) \tag{2.51}$$

(where we take the complete ℓ^∞-direct sum norm in this decomposition). Now to verify (iv)(a), it is enough to show that for all α,

(2.52) $S^{**} \circ (U_\alpha^{k**}|_{X^{**}} \oplus V_\alpha^{(k)**})$ is a complete contraction from $(X^{**} \oplus Y^{**}, \mathrm{NEW})$ to (Y^{**}, NEW).

Take then n and $n \times n$ matrices $(x_{ij}^{**}), (y_{ij}^{**})$ of elements of X^{**} and Y^{**} respectively; choose unique $\overline{x}_{ij}^{**}$'s and w_{ij}'s in X^{**} and W respectively so that

$$(y_{ij}^{**}) = (\overline{x}_{ij}^{**}) \oplus (w_{ij}). \tag{2.53}$$

Then

$$\begin{aligned} \|(U_\alpha^{k**} x_{ij}^{**} + V_\alpha^{(k)**} y_{ij}^{**})\|_{\mathrm{NEW}} &= \max\{\|(U_\alpha^{k**} x_{ij}^{**} + V_\alpha^{(k)**} \overline{x}_{ij}^{**})\|, \|(RV_\alpha^{(k)**} w_{ij})\|_{\mathrm{MAX}}\} \\ &\le \max\{\|(x_{ij}^{**})\|, \|(\overline{x}_{ij}^{**})\|, \|RV_\alpha^{(k)**}\|\,\|(w_{ij})\|_{\mathrm{MAX}}\} \\ &= \max\{\|(x_{ij}^{**})\|_{\mathrm{NEW}}, \|(y_{ij}^{**})\|_{\mathrm{NEW}}\}. \end{aligned} \tag{2.54}$$

Here, in the above inequality, we have used the fact that U_α^k and $V_\alpha^{(k)}$ satisfy "weak (iv)(a)", plus the crucial observation that since $(W, \mathrm{NEW}) = (W, \mathrm{MAX})$,

$$\|RV_\alpha^{(k)**}|_W\|_{\mathrm{cb}} = \|RV_\alpha^{(k)**}|_W\| = 1.$$

The latter holds since the Banach norm of $V_\alpha^{(k)**}$ in the NEW and OLD structures is the same, namely equal to one, since weak (iv)(b) implies $V_\alpha^{(k)}$ is a complete contraction in OLD and hence a contraction.

Of course, (2.54) now yields that (2.52) holds, completing the proof that (U_α) is a weak special M-cai in (Y, NEW).

It remains to prove that (2.47) holds. Let E be a finite-dimensional subspace of Y/X, and let $\varepsilon > 0$. Since (X,Y) has λ-ell's, we may choose a linear operator $T: Y/X \to Y^{**}$ so that

$$\|T\| < \lambda + \varepsilon, \tag{2.55.i}$$

$$T(E) \subset Y \tag{2.55.ii}$$

and

(2.55.iii) $T|_E$ is a lift of I_E.

We claim that

$$L := RT|_E \tag{2.56}$$

is the desired lift. Of course, L is a lift of $I_{Y/X}$; the crucial point is to compute its cb-norm. But as we have pointed out above, $(Y/X, \mathrm{NEW}) = (Y/X, \mathrm{MAX})$. Hence

$$\|L\|_{\mathrm{cb}} \le \|RT\|_{\mathrm{cb}} = \|RT\| < \lambda + \varepsilon \tag{2.57}$$

as desired. This completes the proof of Theorem 2.8. ■

COROLLARY 2.9. *Let $\mathcal{J}$ be an approximately injective ideal in a C^*-algebra $\mathcal{A}$ and let Y be a closed linear subspace of $\mathcal{A}$ with $\mathcal{J} \subset Y$ and $Y/\mathcal{J}$ separable. Then $\mathcal{J}$ is Banach-complemented in Y provided $(\mathcal{J}, \mathcal{A})$ has extendable local liftings. In particular, for a given $\lambda \ge 1$ and every $\varepsilon > 0$, there exists a lift $L: Y/\mathcal{A} \to Y$ of $I_{Y/\mathcal{A}}$ with $\|L\| < \lambda + \varepsilon$ provided any of the following holds:*

(i) *$\mathcal{A}/\mathcal{J}$ has the λ-bounded approximation property.*
(ii) *$Y/\mathcal{J}$ has the λ-bounded approximation property.*
(iii) *Y^{**} is a λ-injective Banach space.*
(iv) *Y is λ-extendably locally reflexive.*
(v) *$\mathcal{A}$ is λ-extendably locally reflexive.*

Proof. This is an immediate consequence of our previous work. First of all, $\mathcal{J}$ has a special M-cai in $\mathcal{A}$, by Proposition 2.3, and of course such is then a weak special M-cai. Secondly, since $\mathcal{J}$ is a (complete) M-ideal in $\mathcal{A}$, $(\mathcal{J},\mathcal{A})$ has 1-ll's by Proposition 2.6(ii), whence also $(\mathcal{J},Y)$ has this property. Thus cases (i) and (v) yield that $(\mathcal{J},\mathcal{A})$ has λ-ell's by (a′) and (c′) of Proposition 2.7, while cases (ii)–(iv) yield that $(\mathcal{J},Y)$ has λ-ell's by (a′)–(c′) of 2.7. Thus Theorem 2.8 yields the conclusion of the corollary. ■

COROLLARY 2.10. *If $(\mathcal{K},B(\ell^2))$ has extendable local liftings, then $\mathcal{K}$ is Banach complemented in Y for any separable operator space Y with $\mathcal{K}\subset Y$.*

3. Examples and complements. We first consider the case of (closed two-sided) ideals $\mathcal{J}$ in non-self-adjoint operator algebras $\mathcal{A}$. We say that a net (u_α) in $\mathcal{J}$ is a *contractive approximate identity* for $\mathcal{J}$ if $\|u_\alpha\|\le 1$ for all α and both $u_\alpha x\to x$ and $xu_\alpha\to x$ for all $x\in\mathcal{J}$. A remarkable result of Effros–Ruan yields that *a closed linear subspace $\mathcal{J}$ of an operator algebra $\mathcal{A}$ is an M-ideal in $\mathcal{A}$ iff $\mathcal{J}$ is an ideal in $\mathcal{A}$ which admits a contractive approximate identity* [ER1]. (The same equivalences were established earlier by R. Smith [S] in the case of *uniform* algebras $\mathcal{A}$.) The discussion in [ER1] easily yields that when this happens, $\mathcal{J}$ is a complete M-ideal in $\mathcal{A}$.

We obtain the additional information that these conditions are *equivalent* to $\mathcal{J}$ having an M-cai in $\mathcal{A}$; in fact, we obtain the direct generalization of Proposition 1.4 to the non-self-adjoint case.

THEOREM 3.1. *Let $\mathcal{J}$ be an ideal in an operator algebra $\mathcal{A}$ such that $\mathcal{J}$ has a contractive approximate identity. Then $\mathcal{J}$ admits a strong contractive M-cai in $\mathcal{A}$.*

Proof. We assume that $\mathcal{A}$ is a (closed) subalgebra of $B(H)$ for some Hilbert space H. We may easily reduce to the case where $\mathcal{A}$ is unital (then $\mathcal{A}$ may be assumed to be a unital subalgebra of $B(H)$). Indeed, if $\mathcal{A}$ is non-unital, simply let $\widetilde{\mathcal{A}}$ be $\mathcal{A}$ with I adjoined (where $\mathcal{A}$ is a non-unital closed subalgebra of $B(H)$). Then $\mathcal{J}$ remains an ideal in $\widetilde{\mathcal{A}}$; if (U_α) is a strong contractive M-cai for $\mathcal{J}$ in $\widetilde{\mathcal{A}}$, $(U_\alpha|_{\mathcal{A}})$ is such for $\mathcal{J}$ in $\mathcal{A}$. Let then (u_α) be a contractive (algebraic) approximate identity for $\mathcal{J}$ in $\mathcal{A}$. By passing to a subnet, and regarding $\mathcal{A}\subset\mathcal{A}^{**}\subset B(H)^{**}$, we may assume that

(3.1) $(u_\alpha)_{\alpha\in\mathcal{D}}$ converges weak* to an element e^{**} of $\mathcal{A}^{**}$.

This is nothing but the first step of the proof by Effros–Ruan that the stated hypotheses yield that $\mathcal{J}$ is an M-ideal in $\mathcal{A}$ (p. 919 of [ER1]). Now it is proved in [ER1] that then

(3.2) e^{**} is a self-adjoint idempotent in the center of $\mathcal{A}^{**}$, with $e^{**}x=x$ for all $x\in\mathcal{J}^{**}$.

Of course, this uses remarkable properties of C^*-algebras such as the fact that $\mathcal{A}^{**}$ is a subalgebra of $B(H)^{**}$, a von Neumann algebra. We also make use of the fact that if (v_α) is a bounded net in $B(H)^{**}$ with w*-$\lim_\alpha v_\alpha=v$ ($v\in B(H)^{**}$), then for any $w\in B(H)^{**}$,

$$v_\alpha w\xrightarrow{\mathrm{w}^*}vw\quad\text{and}\quad wv_\alpha\xrightarrow{\mathrm{w}^*}wv.\tag{3.3}$$

Finally, the set of positive elements in $\mathrm{Ba}\,B(H)$ is w*-dense in the set of positive elements of $\mathrm{Ba}(B(H))^{**}$; hence we may choose a net (e_α) in $B(H)$ with

$$0\le e_\alpha\le 1\quad\text{for all }\alpha\text{ and }e_\alpha\xrightarrow{\mathrm{w}^*}e^{**}.\tag{3.4}$$

It then follows that we may choose an appropriate directed set $\mathcal{D}$ and "re-labeled" new nets $(u_\alpha)_{\alpha\in\mathcal{D}}$ and $(e_\alpha)_{\alpha\in\mathcal{D}}$ such that

$$\lim_{\alpha\in\mathcal{D}}u_\alpha=e^{**}=\lim_{\alpha\in\mathcal{D}}e_\alpha.\tag{3.5}$$

Hence it follows that

$$u_\alpha-e_\alpha\to 0\quad\text{weakly.}\tag{3.6}$$

But then we may find a new net of "far out" convex combinations of $(u_\alpha,e_\alpha)_{\alpha\in\mathcal{D}}$, say $(\widetilde{u}_\alpha,\widetilde{e}_\alpha)_{\alpha\in\widetilde{\mathcal{D}}}$, with

$$\|\widetilde{u}_\alpha-\widetilde{e}_\alpha\|\to 0.\tag{3.7}$$

Of course, the $\widetilde{u}_\alpha$'s remain a contractive approximate identity in $\mathcal{J}$ and still $0\le\widetilde{e}_\alpha\le 1$ for all α with $\widetilde{e}_\alpha\to e^{**}$ weak*. Thus, by re-labeling again we may assume without loss of generality that

$$\lim_{\alpha\in\mathcal{D}}\|u_\alpha-e_\alpha\|=0\quad\text{and}\quad\text{w*-}\lim_{\alpha\in\mathcal{D}}e_\alpha=e^{**}=\text{w*-}\lim_{\alpha\in\mathcal{D}}u_\alpha.\tag{3.8}$$

Now moreover we have, for any $a\in\mathcal{A}$,

$$\lim_\alpha u_\alpha a=e^{**}a\quad\text{and}\quad\lim_\alpha au_\alpha=ae^{**}\tag{3.9}$$

(using (3.3)). But since e^{**} is central in $\mathcal{A}^{**}$, for all $a\in\mathcal{A}$ we obtain

$$\lim_\alpha(u_\alpha a-au_\alpha)=0\quad\text{weakly in }\mathcal{A}.\tag{3.10}$$

Finally, by again taking far out convex combinations in our net $(u_\alpha,e_\alpha)_{\alpha\in\mathcal{D}}$, we may assume without loss of generality that

$$\lim_{\alpha\in\mathcal{D}}\|u_\alpha a-au_\alpha\|=0\quad\text{for all }a\in\mathcal{A}\tag{3.11}$$

and still that (3.8) holds.

Define then $U_\alpha:\mathcal{A}\to\mathcal{A}$ by

$$U_\alpha(a)=u_\alpha a\quad\text{for all }a\in\mathcal{A}.\tag{3.12}$$

We shall now prove that $(U_\alpha)_{\alpha\in\mathcal{D}}$ is a strong contractive M-cai for $\mathcal{J}$ in $\mathcal{A}$ (by essentially the same argument as the proof of Proposition 1.4). It is trivial that U_α is a complete contraction for all α, since $0 \le \|u_\alpha\| \le 1$.

Since the u_α's lie in $\mathcal{J}$, it is trivial that $U_\alpha\mathcal{A} \subset \mathcal{J}$ for all α, and of course $U_\alpha(x) \to x$ for all $x \in \mathcal{J}$ since (u_α) is an approximate identity. We deduce that (iv) of Definition 1.1 holds just as in the proof of Proposition 1.4. Indeed, for any $x^{**} \in \mathcal{J}^{**}$,

$$U_\alpha^{**}(x^{**}) = u_\alpha x^{**} \xrightarrow{\text{w}^*} e^{**}x^{**} = x^{**} \quad \text{by (3.2) and (3.3).}$$

To complete the proof, it remains to verify condition (iii′) of Definition 1.1. Now by (3.8) and (3.11),

$$(3.13) \qquad \|e_\alpha a - a e_\alpha\| \to 0 \quad \text{for all } a \in \mathcal{A}.$$

Thus we obtain

$$(3.14) \qquad \sqrt{e_\alpha}a - a\sqrt{e_\alpha} \to 0, \quad \sqrt{1-e_\alpha}a - a\sqrt{1-e_\alpha} \to 0 \quad \text{for all } a \in \mathcal{A}.$$

(See the comment following (1.30).)

Then just repeating the proof of (1.31.i,ii), for all $a \in \mathcal{A}$ we obtain

$$(3.15.\text{i}) \qquad e_\alpha a - \sqrt{e_\alpha}a\sqrt{e_\alpha} \to 0$$

and

$$(3.15.\text{ii}) \qquad (1-e_\alpha)a - \sqrt{1-e_\alpha}a\sqrt{1-e_\alpha} \to 0.$$

Now define operators $\widetilde{U}_\alpha$ and $\widetilde{V}_\alpha$ on $B(H)$ by

$$(3.16) \qquad \widetilde{U}_\alpha y = \sqrt{e_\alpha}y\sqrt{e_\alpha}, \quad \widetilde{V}_\alpha y = \sqrt{1-e_\alpha}y\sqrt{1-e_\alpha} \quad \text{for all } y \in B(H).$$

Then by (3.8) and (3.15), for all $a \in \mathcal{A}$,

$$(3.17) \qquad U_\alpha a - \widetilde{U}_\alpha a \to 0 \quad \text{and} \quad (I-U_\alpha)a - \widetilde{V}_\alpha a \to 0.$$

But as we showed in the proof of Proposition 1.4,

$$(3.18) \qquad S \circ (\widetilde{U}_\alpha \oplus \widetilde{V}_\alpha) : B(H) \oplus B(H) \to B(H) \text{ is a complete contraction,}$$

where $S(y \oplus z) = y + z$ for all $y, z \in B(H)$. Thus, given n and (a_{ij}), (b_{ij}) in $M_n(\mathcal{A})$,

$$(3.19) \qquad \overline{\lim_\alpha}\, \|U_\alpha(a_{ij}) + (I-U_\alpha)(b_{ij})\|$$
$$= \overline{\lim_\alpha}\, \|\widetilde{U}_\alpha(a_{ij}) + \widetilde{V}_\alpha(b_{ij})\| \quad \text{by (3.17)}$$
$$\le \max\{\|(a_{ij})\|, \|(b_{ij}\|\} \quad \text{by (3.18).} \ \blacksquare$$

REMARK. See [DP] for an application, in a different direction, of algebraic approximate identities and M-ideal theory to non-self-adjoint operator algebras.

We next deal with c_0 sums of operator spaces. Our methods yield many previously obtained results, in this setting, via the following very simple result.

PROPOSITION 3.2. (a) *Let $X_1, X_2, \ldots$ be given operator spaces and let $X = (X_1 \oplus X_2 \oplus \ldots)_{c_0}$, $Y = (X_1, \oplus X_2 \oplus \ldots)_{\ell^\infty}$. Then X admits a strong special M-cai in Y.*

(b) *If the X_j's are all approximately injective, so is X.*

Proof. (a) Define $T_n : Y \to X$ by $T_n(y) = x_1 \oplus \ldots \oplus x_n$ if $y = (x_j)_{j=1}^\infty$, regarding $X_1 \oplus \ldots \oplus X_n$ as canonically embedded in X. It is then essentially immediate that (T_n) is the desired strong M-cai for X in Y. Indeed, fixing n, we see, since T_n is a projection, that for any $k \ge 1$, $T_n^k = T_n$, $(I-T_n)^k = I - T_n$, and in fact $S \circ (T_n \oplus I - T_n) : Y \oplus Y \to Y$ is a complete contraction, where $S : Y \oplus Y \to Y$ is the sum operator. Moreover, $T_n^{k+1} - T_n^k \equiv 0$, $T_n(Y) \subset X$, and $T_n x \to x$ for all $x \in X$. Finally, we have $X^{**} = (X_1^{**} \oplus X_2^{**} \oplus \ldots)_{\ell^\infty}$ and $X^* = (x_1^* \oplus x_2^* \oplus \ldots)_{\ell^1}$. Thus if $x^{**} \in X^{**}$, $x^{**} = (x_j^{**})_{j=1}^\infty$, then $T_n^{**}(x^{**}) = x_1^{**} \oplus \ldots \oplus x_n^{**} \to x^{**}$ weak*, completing the proof of (a).

To prove (b), let $P_n = (T_n - T_{n-1})X$ for all $n \ge 1$ (where $T_0 = 0$), i.e., P_n is just the canonical projection onto the nth coordinate. Of course, P_n is completely contractive for all n. Now suppose $E \subset F$ are finite-dimensional operator spaces, $S : E \to X$ is a given linear map, and $\varepsilon > 0$ is given. For each n, choose an extension $\widetilde{S}_n : F \to X_n$ of $S_n := P_n S$ so that

$$(3.20) \qquad \|\widetilde{S}_n\|_{\text{cb}} \le (1+\varepsilon)\|S_n\|_{\text{cb}} \le (1+\varepsilon)\|S\|_{\text{cb}}.$$

Now simply define $\widetilde{S} : F \to Y$ by $\widetilde{S}(f) = (\widetilde{S}_n(f))_{n=1}^\infty$, $f \in F$. Let us first note: If $f \in F$, $\widetilde{S}(f) \in X$. We obtain that *because* E is finite-dimensional and $\|S_n e\| \to 0$ for all $e \in E$,

$$(3.21) \qquad \|S_n\|_{\text{cb}} \to 0.$$

Hence

$$(3.22) \qquad \|\widetilde{S}_n(f)\| \le \|\widetilde{S}_n\|\,\|f\| \le (1+\varepsilon)\|S_n\|_{\text{cb}} \to 0 \quad \text{as } n \to \infty.$$

Thus $S(F) \subset X$. Moreover,

$$(3.23) \qquad \|\widetilde{S}\|_{\text{cb}} = \sup_n \|\widetilde{S}_n\|_{\text{cb}} \le (1+\varepsilon)\|S\|_{\text{cb}}$$

and of course $\widetilde{S}|_E = S$. ■

An example by A. M. Davie (as refined by W. Lusky) yields a sequence of finite-dimensional Banach spaces $(X_1, X_2, \ldots)$ and a separable Z with $X \subset Z \subset X^{**}$ so that X is uncomplemented in Z, where $X = (X_1 \oplus X_2 \oplus \ldots)_{c_0}$ (cf. Proposition II.2.3 of [HWW] for an exposition and the relevant references). Thus although X admits a strong special M-cai in Z and Z is 1-locally reflexive (using the MIN structure), there is *no* bounded lift $L : Z/X \to Y$

of $I_{Z/X}$. When the X_j's are all approximately injective and Z is locally reflexive, however, we do obtain a (completely) bounded lift via the next result.

COROLLARY 3.3. *Let $X_1, X_2, \ldots, X$, and Y be as in Proposition* 3.2(a), *and suppose $X \subset Z \subset Y$ with Z/X separable.*

(a) *X is a complete M-ideal in Y. Hence if the X_n's are all reflexive, X is a complete M-ideal in X^{**}.*

(b) *If the X_j's are all approximately injective and Z is λ-locally reflexive, or more generally, if (X,Z) admits λ-complete local liftings, then X is completely $(\lambda+\varepsilon)$-co-complemented in Z for all $\varepsilon > 0$. Moreover, if Z is* 1*-locally reflexive, X is completely co-contractively complemented in Z.*

(c) *If (X,Y) or (X,Z) admits λ-extendable local liftings and the X_j's are approximately injective, X is $(\lambda+\varepsilon)$-co-complemented in Z for all $\varepsilon > 0$. In particular, this is the case if Y or Z is λ-extendably locally reflexive, or Y^{**} or Z^{**} is a λ-injective Banach space, or Y/X or Z/X has the λ-bap.*

Proof. (a) follows immediately from Theorem 1.1. (b) follows immediately from Proposition 3.2(b) together with Theorem 2.4 in case $\lambda > 1$ and Z is λ-locally reflexive, or the Theorem in the Appendix in case $\lambda = 1$ (since then X is approximately injective and 1-locally reflexive, hence X is nuclear). Theorem 2.8 yields (a) under the λ-cll's hypothesis. Finally, (c) follows immediately from Theorem 1.1 and Theorem 2.8 and the (elementary) Proposition 2.7. ∎

In turn, Corollary 3.3 yields as special cases certain theorems discovered by the second author of the present paper.

COROLLARY 3.4 [Ro]. *Let $X_1, X_2, \ldots$ be* 1*-injective Banach spaces, and $X = (X_1 \oplus X_2 \oplus \ldots)_{c_0}$, $Y = (X_1 \oplus X_2 \oplus \ldots)_{\ell^\infty}$.*

(a) *X has the 2-SEP. In particular, $c_0(\ell^\infty)$ has the 2-SEP.*

(b) *Suppose the X_j's are all* 1*-injective operator spaces, and let Z be an operator space with $X \subset Z \subset Y$ and Z/X separable. Suppose that Z is λ-locally reflexive, or more generally, that (X,Z) admits λ-complete local liftings. Then X is completely $(\lambda+\varepsilon)$-cocomplemented in Z for all $\varepsilon > 0$. If Z is* 1*-locally reflexive, X is completely co-contractively complemented in Z.*

Proof. Part (b) follows immediately from Corollary 3.3, since the X_j's are thus all approximately injective.

Part (a) follows from the last statement in (b). Indeed, let $\widetilde{X} \subset \widetilde{Y}$ be separable Banach spaces and $T : \widetilde{X} \to X$ be a bounded linear operator. Then Y is a 1-injective Banach space, hence there exists $\widetilde{T} : \widetilde{Y} \to Y$ extending T. Let Z denote the closed linear span of X and $\widetilde{T}(\widetilde{X})$. Then Z satisfies the hypothesis of the final statement in (b), since of course we have Z endowed with MIN, which is thus 1-locally reflexive. Hence there is a linear projection $P : Z \to X$ with $\|I - P\| \le 1$, so $\|P\| \le 2$; $P \circ \widetilde{T}$ is thus an extension of T with $\|T\| \le 2\|T\|$. ∎

REMARK. Corollary 3.4(a) is obtained (up to $\varepsilon > 0$) as Theorem 1.1 of [Ro]. Corollary 3.4(b) is a special case of Theorem 3.4 of [Ro]; the quantitative result obtained there is not as good. However, the full qualitative result in [Ro] is more general than 3.4(b), for it is assumed in [Ro] that the X_j's are λ-injective operator spaces, for some $\lambda \ge 1$. Thus if $\lambda > 1$, the X_j's need not be approximately injective, and hence the methods of the present paper do not apply.

We next recapture the main results in [Ro] concerning the CSEP (using also some recent work of L. Ge and P. Hadwin [GH]). We first recall a concept introduced in [Ro].

DEFINITION 3.1. Let $C \ge 1$. A family $\mathcal{Z}$ of operator spaces is said to be of *C-finite matrix type* if for any finite-dimensional operator space G, there is an $n = \mathbf{n}(G)$ so that

$$\|T\|_{\rm cb} \le C\|T\|_n \text{ for all linear operators } T : G \to Z \text{ and all } Z \in \mathcal{Z}. \tag{3.24}$$

Briefly, we say that $\mathcal{Z}$ is *C-finite with function* $\mathbf{n}$; a single space Z is called *C-finite* provided $\{Z\}$ is C-finite.

(Recall that for operator spaces X and Y and $T : X \to Y$ a bounded linear map, $\|T\|_n = \|I_n \otimes T\|$, where I_n denotes the identity map on M_n.)

C-finite operator spaces are C-locally reflexive, thanks to the following interesting operator space extension of the Banach local reflexivity principle, due to L. Ge and P. Hadwin.

LEMMA 3.5 [GH]. *Let Y be an arbitrary operator space, $\varepsilon > 0$, $n \ge 1$, and F, G be finite-dimensional subspaces of Y^* and Y^{**} respectively. Then there exists a linear operator $T : G \to Y$ satisfying the following*:

(3.25)
(i) $\|T\|_n < 1 + \varepsilon$.
(ii) $\langle Tg, f\rangle = \langle g, f\rangle$ *for all $g \in G$ and $f \in F$.*
(iii) $T|_{G\cap Y} = I|_{G\cap Y}$.
(iv) *T is* 1-1 *and* $\|T^{-1}|_{T(G)}\|_n < 1 + \varepsilon$.

REMARK. The case $n = 1$ is precisely the Banach local reflexivity principle as formulated in [JRZ]. (We only use (3.25)(i)–(iii) in our discussion here.) We obtain an extension of Lemma 3.5 in Lemma 3.13 below.

We may now easily obtain the following permanence properties for C-finite families.

PROPOSITION 3.6. *Let $C, \lambda \ge 1$.*

(a) *Let $X_1, X_2, \ldots$ be operator spaces such that $\{X_1, X_2, \ldots\}$ is C-finite. Then $(X_1 \oplus X_2 \oplus \ldots)_{\ell^\infty}$ is C-finite.*

(b) *Let X be an operator space which is C-finite for all $C > \lambda$. Then X is λ-locally reflexive.*

Proof. (a) This is a simple consequence of Definition 3.1. Let G be a finite-dimensional operator space, and $n = \mathbf{n}(G)$ be the n which works for the family $\{X_1, X_2, \ldots\}$. Let $T : G \to (X_1 \oplus X_2 \oplus \ldots)_\infty$ be a linear operator and let P_j be the canonical projection onto X_j for all j. Then

$$
\begin{aligned}
(3.26) \qquad \|T\|_{\mathrm{cb}} &= \sup_j \|P_j T\|_{\mathrm{cb}} \\
&\le \sup_j C\|P_j T\|_n \qquad \text{by } C\text{-finiteness} \\
&= C\|T\|_n
\end{aligned}
$$

where the first and last equalities follow by the definition of the ℓ^∞-operator sum.

(b) Let X satisfy the given hypothesis and let G and F be finite-dimensional subspaces of X^{**} and X^* respectively, and let $\varepsilon > 0$ be given. Let $n = \mathbf{n}(G)$ be the C-finiteness function for X. Choose C with $\lambda < C < \lambda + \varepsilon$, and then choose ε' with $C + \varepsilon' C < \lambda + \varepsilon$. Now by Lemma 3.5 (i.e., [GH]), choose $T : G \to X$ satisfying (3.25) (for $\varepsilon = \varepsilon'$). Then

$$
\begin{aligned}
(3.27) \qquad \|T\|_{\mathrm{cb}} &\le C\|T\|_n \qquad \text{since } X \text{ is } C\text{-finite} \\
&\le C(1 + \varepsilon') \qquad \text{by (3.25)} \\
&< \lambda + \varepsilon. \ \blacksquare
\end{aligned}
$$

We are mainly interested here in the case $C = 1$ in 3.6, to obtain the following

PROPOSITION 3.7. *Let $n \ge 1$. Then $\ell^\infty(M_{n,\infty} \oplus M_{\infty,n})$ is 1-locally reflexive.*

Proof. It is shown in [Ro, Proposition 2.6] that for all $j \ge 1$, $M_{\infty,j}$ and $M_{j,\infty}$ are 1-finite, with function $\mathbf{n}(G) = j \dim G$, where G ranges over all finite-dimensional operator spaces. Thus by Proposition 3.6(a), $\ell^\infty(M_{j,\infty} \oplus M_{\infty,j})$ is also 1-finite with the *same* function $\mathbf{n}$, and hence is 1-locally reflexive by Proposition 3.6(b). $\blacksquare$

The following is the main result concerning the CSEP, obtained in [Ro, Corollary 2.7].

COROLLARY 3.8. *$c_0(M_{n,\infty} \oplus M_{\infty,n})$ has the 2-CSEP for all $n \ge 1$.*

Proof. Let $X \subset Y$ be separable operator spaces and $T : X \to Z$ be a completely bounded map, where $Z = c_0(M_{n,\infty} \oplus M_{\infty,n})$. Then $Z^{**} = \ell^\infty(M_{n,\infty} \oplus M_{\infty,n})$ is an isometrically injective operator space, hence there exists a completely bounded extension $\widetilde{T} : Y \to Z^{**}$ of T with $\|\widetilde{T}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$. Set $\widetilde{Y} = [\widetilde{T}(Y), Z]$. Then Z^{**} is 1-locally reflexive by Proposition 3.7, hence so is $\widetilde{Y}$. So Z is an approximately injective complete ideal in $\widetilde{Y}$ by Proposition 3.2 and Theorem 1.1, whence Z is completely co-contractively complemented in $\widetilde{Y}$ by the Theorem in the Appendix. Thus choosing a projection $P : \widetilde{Y} \to Z$ with $\|I - P\| \le 1$, we obtain that $P \circ \widetilde{T}$ is an extension of T satisfying $\|P \circ \widetilde{T}\|_{\mathrm{cb}} \le 2\|T\|_{\mathrm{cb}}$. $\blacksquare$

REMARK. It is proved in [Ro] (see Corollary 2.5 and Proposition 2.15 therein) that if $Z_1, Z_2, \ldots$ are operator spaces so that for some λ and C, $Z_1, Z_2, \ldots$ have the λ-CSEP and $\{Z_1, Z_2, \ldots\}$ is C-finite, then again $(Z_1 \oplus Z_2 \oplus \ldots)_{c_0}$ has the CSEP (in fact the $(C\lambda^2 + \lambda + \varepsilon)$-CSEP for all $\varepsilon > 0$). This result does not follow from the methods of the present paper, in part because the Z_j's may not be approximately injective. However, if the Z_j's have this property, the next result yields better quantitative information than the cited results in [Ro].

COROLLARY 3.9. *Let $C \ge 1$ and let $X_1, X_2, \ldots$ be approximately injective operator spaces with $\{X_1, X_2, \ldots\}$ C-finite; let $X = (X_1 \oplus X_2 \oplus \ldots)_{c_0}$ and $Y = (X_1 \oplus X_2 \oplus \ldots)_\infty$. Then if $X \subset Z \subset Y$ with Z separable, X is completely $(C + \varepsilon)$-co-complemented in Z for all $\varepsilon > 0$.*

Proof. Y is C-locally reflexive by Proposition 3.6(b), hence the Corollary follows by Corollary 3.3(b). $\blacksquare$

We now continue with further study of the converse of Theorem 1.1. We shall show that this is valid in the setting of the Appendix, namely the case where X is a nuclear complete M-ideal in a 1-locally reflexive operator space Y. In fact our result here holds in the more general situation of λ-finitely injective complete M-ideals; these include the λ-nuclear ones (cf. [KR] for some results concerning the latter).

DEFINITION 3.2. Let $\lambda \ge 1$ and an operator space X be given.

(a) X is called *λ-finitely injective* if for all operator spaces Y, finite-dimensional subspaces G, $\varepsilon > 0$, and linear maps $T : G \to X$, there exists a linear extension $\widetilde{T} : Y \to X$ with $\|\widetilde{T}\|_{\mathrm{cb}} \le (\lambda + \varepsilon)\|T\|_{\mathrm{cb}}$. In case $\widetilde{T}$ can always be chosen finite rank, we shall call X *λ-finite rank injective*.

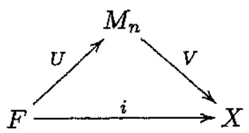
(b) X is called *λ-nuclear* if for all finite-dimensional subspaces F of X and all $\varepsilon > 0$, there exists an n and linear maps $U : F \to M_n$ and $V : M_n \to X$ with

$$(3.28.\mathrm{i}) \qquad \|U\|_{\mathrm{cb}}\|V\|_{\mathrm{cb}} < \lambda + \varepsilon$$

and

$$(3.28.\mathrm{ii}) \qquad V \circ U = i, \text{ where } i : F \to X \text{ is the identity injection.}$$

That is, we have the diagram



Thus, X is 1-nuclear precisely when X is nuclear.

REMARK. (a) It can be proved that in the Banach space category, the λ-nuclear operator spaces coincide with the $\mathcal{L}_\infty$-spaces. Precisely, (X, MIN) is λ-nuclear for some λ if and only if X is an $\mathcal{L}_\infty$-space (iff X^{**} is an isomorphically injective Banach space), while (X, MIN) is nuclear iff X is an L^1-predual (i.e., X^{**} is isometrically injective).

(b) Results in [Pi] yield that $B(H)$ is *not* λ-finite rank injective for any λ (we are indebted to T. Oikhberg for this fact). Note however that trivially, if X is λ-injective, X is λ-finitely injective.

The next simple result shows the connection between the concepts given in parts (a) and (b) of Definition 3.2.

LEMMA 3.10. *Let $\lambda \ge 1$ and X be a λ-nuclear operator space. Then X is λ-finite rank injective.*

Proof. Let $F = T(G)$ and choose n and $U : T(G) \to M_n$, $V : M_n \to X$ satisfying (3.28). Next, using the 1-injectivity of M_n, choose an extension $S : Y \to M_n$ of UT with

$$\|S\|_{\mathrm{cb}} = \|UT\|_{\mathrm{cb}}.$$

Finally, let $\widetilde{T} = VS$. By this equality,

$$\|\widetilde{T}\|_{\mathrm{cb}} \le \|V\|_{\mathrm{cb}}\|U\|_{\mathrm{cb}}\|T\|_{\mathrm{cb}} < (\lambda+\varepsilon)\|T\|_{\mathrm{cb}} \quad \text{by (3.28.i)}$$

and $\widetilde{T}$ is an extension of T by (3.28.ii). ∎

The following gives a converse to Theorem 1.1 in the case of λ-finitely injective complete M-ideals in 1-locally reflexive operator spaces, or in the case of arbitrary ones in spaces with the bap.

THEOREM 3.11. *Let $X \subset Y$ be operator spaces with X a complete M-ideal in Y, and let $\lambda \ge 1$. Then X admits an M-cai (T_α) in Y if either of the following holds:*

(a) *Y has the Banach λ-bap.*
(b) *Y is 1-locally reflexive and X is λ-finitely injective.*

In case (a), *the T_α's may be chosen to be finite rank operators with $\|T_\alpha\| \le \lambda$ for all α. In case* (b), *the T_α's may be chosen with $\|T_\alpha\|_{\mathrm{cb}} \le \lambda$ for all α; moreover, if X is λ-nuclear, again the T_α's may be chosen to be finite rank.*

REMARK. Thus if X is 1-nuclear and Y is 1-locally reflexive, Theorem 3.11 yields that X admits a contractive M-cai in Y, consisting of finite rank operators.

We first require an extension of the local reflexivity concept (due to S. Bellenot [Be] in the Banach space category).

LEMMA 3.12. *Let $X \subset Y$ be operator spaces with Y λ-locally reflexive, and let G and F be finite-dimensional subspaces of Y^{**} and Y^* respectively. Then for all $\varepsilon > 0$, there exists a linear operator $T : G \to Y$ satisfying the following*:

(i) $\|T\|_{\mathrm{cb}} \le \lambda + \varepsilon$.
(ii) $\langle Tg, f\rangle = \langle g, f\rangle$ *for all* $g \in G$, $f \in F$.
(iii) $T|_{G\cap Y} = I|_{G\cap Y}$.
(iv) $T(G \cap X^{**}) \subset X$.

COMMENTS. 1. λ-local reflexivity may be *defined* as the existence of T's satisfying (i) and (ii) only. (iii) is known as folklore. (iv) is the new extension.

2. We only use Lemma 3.12 here in the case $\lambda = 1$.

Proof of Lemma 3.12. We first obtain (i), (ii) and (iv). By the basic known equivalences, we may alternatively express λ-local reflexivity as follows:

For all finite-dimensional operator spaces G,

$$\mathrm{Ba}\,\mathrm{cb}(G, Y^{**}) \subset \lambda \overline{\mathrm{Ba}\,\mathrm{cb}(G, Y)}^{w^*}. \tag{3.29}$$

(Since $\mathrm{Ba}\,\overline{\mathrm{cb}(G,Y)}^{w^*} = \mathrm{Ba}\,\mathrm{cb}(G,Y)^{**}$ by Goldstein's theorem, this is equivalent to the identity map $i : \mathrm{cb}(G, Y^{**}) \to \mathrm{cb}(G,Y)^{**}$ having the property that $\|i\|_{\mathrm{cb}} \le \lambda$; note that $\|i^{-1}\|_{\mathrm{cb}} = 1$ always). Now fixing $G \subset Y^{**}$ finite-dimensional, set $H = G \cap X^{**}$ and $W = \{T \in \mathrm{cb}(G, Y^{**}) : T(H) \subset X^{**}\}$. So of course we may just *identify* $\mathrm{cb}(G, Y^{**})$ with $G^* \otimes_{\mathrm{op}} Y^{**} = G^* \otimes Y^{**}$ algebraically. Since G is finite-dimensional, we have

$$W = H^\perp \otimes Y^{**} + G^* \otimes X^{**}. \tag{3.30}$$

Then

$$W_\perp = H \otimes X^\perp, \tag{3.31}$$

hence

$$W_{\perp\perp} = H^\perp \otimes Y + G^* \otimes X. \tag{3.32}$$

But then $W = \overline{W_{\perp\perp}}^{w^*}$ and

$$W_{\perp\perp} = \{T \in \mathrm{cb}(G, Y) : T(H) \subset X\}. \tag{3.33}$$

Now $I_G \in \mathrm{Ba}\, W$, so given $\varepsilon > 0$, since $W = (W_{\perp\perp})^{**}$, again applying Goldstein's theorem, and the fact that $I_G \in \overline{\lambda(\mathrm{Ba}\, W_{\perp\perp})}^{*}$ by (3.29), for each finite-dimensional subspace α of Y^* with $\alpha \supset F$, we may choose $T_\alpha \in W_{\perp\perp}$ satisfying (i) and (ii) for "ε" $= \varepsilon/2$; of course, (iv) holds by (3.33) (where $T = T_\alpha$ in (i)–(iv)).

Now let $\mathcal{D}$ be the family of all finite-dimensional subspaces α of Y^* with $\alpha \supset F$, directed by inclusion. But then it follows that

$$(3.34) \qquad T_\alpha g \to g \quad \text{weakly} \quad \text{for all } g \in G \cap Y.$$

Hence we may choose a net $(\widetilde{T}_\alpha)$ of far-out convex combinations of the T_α's so that

$$(3.35) \qquad \widetilde{T}_\alpha g \to g \quad \text{strongly} \quad \text{for all } g \in G \cap Y.$$

Of course it follows trivially that each $\widetilde{T}_\alpha$ still satisfies (i), (ii), and (iv) (for $T = \widetilde{T}_\alpha$).

Finally, a standard perturbation argument yields that for one of these $\widetilde{T}_\alpha$'s, there exists a perturbation T of $\widetilde{T}_\alpha$ satisfying (i)–(iv). ∎

We next obtain an extension of the result of Ge–Hadwin (stated as Lemma 3.5 above).

LEMMA 3.13. *Let $X \subset Y$ be operator spaces, $\varepsilon > 0$, $n \ge 1$ and F, G be finite-dimensional subspaces of Y^* and Y^{**} respectively. There exists a linear operator $T : G \to Y$ satisfying* (3.25) *so that*

$$(3.36) \qquad T(G \cap X^{**}) \subset X.$$

Proof. Let $\mathcal{G}$ be the group of $n\times n$ unitary matrices and m its associated Haar measure. Let $T_n = M_n^*$ (with elements regarded as $n \times n$ matrices) and let F, G be finite-dimensional subspaces of Y^* and Y^{**} respectively. Since $M_n(G)$ is finite-dimensional, we may assume (by enlarging F if necessary) that $T_n(F)$ $(1+\varepsilon)$-norms $M_n(G)$; that is, for all $\widetilde{g}$ in $M_n(G)$, there exists an $\widetilde{f} \in T_n(F)$ with $\|\widetilde{f}\| = 1$ and

$$(3.37) \qquad \|\widetilde{g}\| < (1+\varepsilon)|\langle \widetilde{g}, \widetilde{f}\rangle|.$$

(We only need this to obtain (3.25)(iv), which we did not really need in our subsequent discussion.)

Now let $\widetilde{Y} = M_n(Y)$, $\widetilde{X} = M_n(X)$, $\widetilde{G} = M_n(G)$, and $\widetilde{F} = T_n(F)$. By Lemma 3.12 applied in the Banach space category, we obtain a linear operator $\widetilde{T} : \widetilde{G} \to \widetilde{Y}$ satisfying (i)–(iv) of 3.12 (where $Y = \widetilde{Y}$, $X = \widetilde{X}$, etc., and $\widetilde{Y}^{**}$, $\widetilde{X}^{**}$ are identified with $M_n(Y^{**})$ and $M_n(X^{**})$ respectively). Now define a linear operator S on $\widetilde{G}$ by

$$(3.38) \qquad S(\widetilde{y}) = \int_{\mathcal{G}} \widetilde{T}(\widetilde{y}u)u^*\, dm(u),$$

for all $\widetilde{y} \in \widetilde{G}$. Now it follows that also S satisfies (i)–(iv) of 3.12 (for $T = S$, $Y = \widetilde{Y}$, etc.). For example, to see that (ii) holds, let $\widetilde{g} \in \widetilde{G}$, $\widetilde{f} \in \widetilde{F}$. Then

$$(3.39) \qquad \langle S\widetilde{g}, \widetilde{f}\rangle = \int_{\mathcal{G}} \langle \widetilde{T}(\widetilde{g}u), u^*\widetilde{f}\rangle\, dm(u) = \int_{\mathcal{G}} \langle \widetilde{g}u, u^*\widetilde{f}\rangle\, dm(u) = \int_{\mathcal{G}} \langle \widetilde{g}uu^*, \widetilde{f}\rangle\, dm(u) = \langle \widetilde{g}, \widetilde{f}\rangle.$$

The first equality follows since the pairing between $M_n(Y^{**})$ and $T_n(Y^*)$ is taken so that $\langle \widetilde{g}b, \widetilde{f}\rangle = \langle \widetilde{g}, b\widetilde{f}\rangle$ for all $\widetilde{g} \in M_n(Y^{**})$, $\widetilde{f} \in T_n(Y^*)$, and $b \in M_n$. The second equality follows since for each $u \in \mathcal{G}$, $\widetilde{g} \in \widetilde{G}$, and $\widetilde{f} \in \widetilde{F}$, we have $\widetilde{g}u \in \widetilde{G}$ and $u^*\widetilde{f} \in \widetilde{F}$.

It then follows moreover that

$$(3.40) \qquad S \text{ is 1-1} \quad \text{and} \quad \|(S|_{\widetilde{G}})^{-1}\| < 1+\varepsilon.$$

Indeed, if $\widetilde{g} \ne 0$ in $\widetilde{G}$, choose $\widetilde{f} \in \widetilde{F}$ of norm one, satisfying (3.37); then

$$(3.41) \qquad \|\widetilde{S}\widetilde{g}\| > |\langle S\widetilde{g}, \widetilde{f}\rangle| > (1+\varepsilon)^{-1}\|\widetilde{g}\|$$

by (3.37), which yields (3.40).

Now, moreover, we have for all $u_0 \in \mathcal{G}$ and $\widetilde{y} \in \widetilde{G}$,

$$(3.42) \qquad S(\widetilde{y}u_0) = \int \widetilde{T}(\widetilde{y}u_0u)u^*\, dm(u) = \int \widetilde{T}(\widetilde{y}v)v^*u_0\, dm(v) = S(\widetilde{y})u_0$$

where the second equality holds by left translation invariance of Haar measure. But then

$$(3.43) \qquad S(\widetilde{y}A) = S(\widetilde{y})A \quad \text{for all } A \in M_n.$$

In turn, (3.43) yields there is a unique linear operator $T : G \to Y$ with $S = I_n \otimes T$ (where $I_n = I_{M_n}$). It now follows immediately that T satisfies the conclusion of Lemma 3.13, since S satisfies (i)–(iv) of Lemma 3.14 and also satisfies (3.40). ∎

The main part of the proof of Theorem 3.11 is conveniently isolated in the following result.

LEMMA 3.14. *Let λ, X and Y be as in the hypotheses of Theorem* 3.11, *let G be a given finite-dimensional subspace of Y with $G \cap X \ne \{0\}$, and let $n \ge 1$, $\varepsilon > 0$. Let $\alpha = (G, n, \varepsilon)$. There exists an operator $T_\alpha : Y \to X$ satisfying the following*:

(i) $\|T_\alpha\| < \lambda + \varepsilon$ *if* (a) *of Theorem* 3.11 *holds, or*
(i′) $\|T_\alpha\|_{\mathrm{cb}} < \lambda + \varepsilon$ *if* (b) *of* 3.11 *holds.*
(ii) $(T_\alpha)|_{G\cap X} = I|_{G\cap X}$.
(iii) $\|T_\alpha(u_{ij}) + (I - T_\alpha)(v_{ij})\| \le (1+\varepsilon)\max\{\|(u_{ij})\|, \|(v_{ij})\|\}$ *for all* (u_{ij}) *and* $(v_{ij}) \in M_n(G)$.

(iv) *T_α is finite rank in case* 3.11(a) *holds, or* 3.11(b) *holds with X λ-nuclear.*

Proof. Let $X^{**} \oplus W$ be the complete M-decomposition of Y^{**} and P the projection onto X^{**} with kernel W. Let $G_1 = PG$ and $G_2 = (I-P)G$, and set $\widetilde{G} = G_1 \oplus G_2$. Then evidently

$$G \subset \widetilde{G}. \tag{3.44}$$

Now first assume 3.11(b). By Lemma 3.13 (applied to $\widetilde{G}$) we may choose a linear operator $T : \widetilde{G} \to Y$ so that

$$\|T\|_{\mathrm{cb}} < 1 + \varepsilon/\lambda, \tag{3.45}$$

$$T|_{\widetilde{G}\cap Y} = I|_{\widetilde{G}\cap Y} \tag{3.46}$$

and

$$T(G_1) \subset X. \tag{3.47}$$

Since P is a complete contraction, we have

$$\|TP|_G\|_{\mathrm{cb}} < 1 + \varepsilon/\lambda. \tag{3.48}$$

Hence by the definition of λ-finite injectivity, since $\lambda(1+\varepsilon/\lambda) = \lambda + \varepsilon$, we may choose a linear extension $T_\alpha : Y \to X$ of $P|_G$ satisfying (i′) of 3.14, which is moreover finite rank if X is λ-nuclear, by Lemma 3.10.

Suppose now that 3.11(a) holds. Let $\varepsilon' > 0$, to be decided, and choose a finite rank operator $A : Y \to Y$ such that

$$A|_G = I|_G \tag{3.49}$$

and

$$\|A\| < \lambda + \varepsilon'. \tag{3.50}$$

Now set $G' = A(Y)$, $G_1' = PG'$, $G_2 = (I-P)G'$, and $\widetilde{G}' = G_1' \oplus G_2'$. Of course, $G' \supset G$ by (3.49). Then by our extension of the Ge–Hadwin result, namely Lemma 3.13, we may choose a linear operator $T : \widetilde{G}' \to Y$ so that

$$\|T\|_n < 1 + \varepsilon' \tag{3.51}$$

and again, (3.46) and (3.47) hold (in fact, they hold on replacing $\widetilde{G}$ by $\widetilde{G}'$ and G_1 by G_1'). Now define T_α by

$$T_\alpha = TPA. \tag{3.52}$$

Thus

$$\|T_\alpha\| < (1+\varepsilon')(\lambda+\varepsilon') < \lambda + \varepsilon \tag{3.53}$$

if ε' is chosen with $\varepsilon'\lambda + \varepsilon' + (\varepsilon')^2 < \varepsilon$. It follows that if $x \in G \cap X$, then $Tx = x$, and so in case (b), $T_\alpha x = TPx = x$, while in case (a),

$T_\alpha(x) = TPAx = x$; thus (ii) of 3.14 holds. Now note that if $v \in G$, then by (3.44) and (3.46),

$$Tv = v. \tag{3.54}$$

Hence for any such v,

$$(I - TP)(v) = (T - TP)(v). \tag{3.55}$$

Finally, let (u_{ij}) and (v_{ij}) be elements of $M_n(G)$. Then

$$\begin{aligned} \|T_\alpha u_{ij} + (I - T_\alpha)v_{ij})\| &= \|(TPu_{ij}) + ((T-TP)v_{ij})\| \\ &\qquad \text{(by (3.55) and the definition on } T_\alpha) \\ &\le (1+\varepsilon)\|Pu_{ij} + (I-P)v_{ij}\| \\ &\qquad \text{(by (3.45) in case (b), (3.51) in case (a))} \\ &= (1+\varepsilon)\max\{\|(Pu_{ij})\|, \|(I-P)v_{ij}\|\} \\ &\le (1+\varepsilon)\max\{\|(u_{ij})\|, \|(v_{ij})\|\}. \end{aligned} \tag{3.56}$$

(The last equality holds because $X^{**} \oplus W$ is a complete M-decomposition of Y^{**}, while the last inequality holds because P and $I-P$ are complete contractions.) Thus 3.14(iii) holds. ■

We are finally prepared for the

Proof of Theorem 3.11. Let $\mathcal{D}$ be the directed set consisting of all $\alpha = (G, n, \varepsilon)$ where G is a finite-dimensional subspace of Y with $G \cap X \ne \{0\}$, $n \ge 1$, and $\varepsilon > 0$. Of course $(G,n,\varepsilon) \le (G',n',\varepsilon')$ if $G \subset G'$, $n \le n'$ and $\varepsilon \ge \varepsilon'$. For each such α, choose T_α satisfying the conclusion of Lemma 3.14.

Finally, let $\widetilde{T}_\alpha = T_\alpha(1+\varepsilon/\lambda)^{-1}$. Then trivially

$$\|\widetilde{T}_\alpha\|_{\mathrm{cb}} \le (\lambda+\varepsilon)(1+\varepsilon/\lambda)^{-1} = \lambda \quad \text{(by 3.14(i))}. \tag{3.56}$$

Since also trivially

$$\lim_\alpha \|T_\alpha - \widetilde{T}_\alpha\| = 0, \tag{3.57}$$

it suffices to prove that $(T_\alpha)_{\alpha\in\mathcal{D}}$ is an M-cai, for then (3.57) yields that $(\widetilde{T}_\alpha)_{\alpha\in\mathcal{D}}$ is an M-cai. Now fix n, and $x \in X$ with $x \ne 0$, and $(u_{ij}), (v_{ij}) \in M_n(Y)$. If we let $G = \mathrm{sp}\{x, u_{ij}, v_{ij} : 1 \le i,j \le n\}$ and $\varepsilon > 0$, then $\alpha = (G,n,\varepsilon) \in \mathcal{D}$; hence for any $\beta \in \mathcal{D}$ with $\beta \ge \alpha$, $T_\beta x = x$ and (iii) of Lemma 3.14 holds. This completes the proof of the theorem. ■

We next briefly discuss the case when $\mathcal{K}(X)$ is an M-ideal in $B(X)$. Here, X is a fixed Banach space, $\mathcal{K}(X)$ denotes the space of compact operators on X, and $B(X)$ denotes the space of bounded operators on X. For a comprehensive survey of known facts, see Section VI.4 of [HWW]. These results yield immediately (via our Theorem 1.1) that *$\mathcal{K}(X)$ is an M-ideal in $B(X)$ if and only if $\mathcal{K}(X)$ admits a (contractive) M-ai in $B(X)$*. (See Theorem VI.4.17 of [HWW] and [W].)

It is known that if $X = c_0$ or ℓ^p, $1 < p < \infty$, then $\mathcal{K}(X)$ is an M-ideal in $B(X)$ (generalizing the $p = 2$ case), but this is not necessarily true for subspaces of these spaces. Indeed, C.-M. Cho and W. B. Johnson proved that for $X \subset \ell^p$, $1 < p < \infty$, $\mathcal{K}(X)$ is an M-ideal in $B(X)$ iff X has the compact metric approximation property [CJ]. (The same result for c_0 is obtained in [W2]. See also [KW] for a remarkable generalization.) In fact, fundamental work of P. Harmand and Å. Lima [HL], and later of N. J. Kalton [K], yields that if $\mathcal{K}(X)$ is an M-ideal in $B(X)$, then X has a shrinking contractive approximation to the identity (K_α), consisting of compact operators, so that in fact, if we let $T_\alpha \in B(B(X))$ be defined as $T_\alpha(T) = K_\alpha T$ for all α, then (T_α) is an M-ai for $\mathcal{K}(X)$ in $B(X)$. It is also known that X is then an M-ideal in X^{**}. We now pose the following question (which certainly seems accessible via the technology given in [HWW]). Does X then admit an M-ai in X^{**}? In fact, can the shrinking compact approximation to the identity (K_α) be chosen as above, so that additionally (K_α^{**}) is an M-ai for X in X^{**} (in case X is non-reflexive)? Note that this question is simply equivalent to: can (K_α) be chosen so that in addition,

$$\overline{\lim_\alpha} \|K_\alpha^{**}x^{**} + (I - K_\alpha^{**})y^{**}\| \le \max\{\|x^{**}\|\,\|(y^{**})\|\} \quad \text{for all } x^{**}, y^{**} \in X^{**}?$$

(By Theorem VI.5.3.b of [HWW] (see also [PW]), this is indeed so if $\mathcal{K}(X \oplus X)$ is an M-ideal in $\mathcal{L}(X \oplus X)$ (L^∞-direct sum), for then (K_α) may be chosen so that $\overline{\lim}_\alpha \|S \circ (K_\alpha \oplus (I - K_\alpha))\| = 1$, where $S : X \oplus X \to X$ is the sum operator.)

We conclude with an example of a Banach space X which is an M-ideal in X^{**}, but so that there does not exist either an M-ai or a weak contractive M-ai for X in X^{**}. The example follows quickly from known (but rather non-trivial!) results; the same example (for a different purpose) is given in [JO].

We first need a standard fact (also given in [JO]).

LEMMA 3.15. *Let X be a closed linear subspace of c_0.*

(a) *Let $T : X \to X$ be a given (bounded linear) operator. Then either T is compact or there is a subspace Y of X with Y isomorphic to c_0 so that $T|_Y$ is an isomorphism.*

(b) *Let $T : X^{**} \to X$ be a given operator. Then $T|_X$ is compact.*

Proof. (a) Any semi-normalized weakly null sequence in c_0 contains a subsequence equivalent to the usual c_0 basis. This implies X^* has the Schur property (i.e., weak and norm sequential convergence coincide on X^*), whence T weakly compact implies T compact. But if T is not weakly compact, there exists a bounded sequence (u_j) in c_0 so that (Tu_j) has no weakly convergent subsequence. We may then pass to a subsequence (u_{n_j}) of (u_j) so that *both* (u_{n_j}) and (Tu_{n_j}) are equivalent to the summing basis of c_0, which implies that $Y := [u_{n_j}]$ is isomorphic to c_0 and $T|_Y$ is an isomorphism.

(b) Suppose that $T|_X$ were not compact. Then by part (a), there exists a subspace Y of X so that Y is isomorphic to c_0 and $T|_Y$ is an isomorphism. Hence Y^{**} is isomorphic to ℓ^∞ and $P := (T|_Y)^{-1}T|_{Y^{**}}$ is a projection from Y^{**} onto Y, which contradicts the fact that c_0 is uncomplemented in ℓ^∞. ■

Finally, we recall a rather deep result of A. Szankowski (cf. [LT, Theorem 2.a.7]):

(3.58) *there exists a subspace X of c_0 failing the compact bounded approximation property.*

PROPOSITION 3.16. *Let X satisfy* (3.58). *Then X is an M-ideal in X^{**}, but X has no M-ai or weak contractive M-ai in X^{**}.*

Proof. Since c_0 is an M-ideal in $\ell^\infty = c_0^{**}$, X is an M-ideal in X^{**} (cf. [HWW, page 111] and [HL]). If $(T_\alpha)_{\alpha\in\mathcal{D}}$ were either an M-ai or a weak contractive M-ai for X in X^{**}, we would have

(3.59) (T_α) is uniformly bounded,

(3.60) $T_\alpha : X^{**} \subset X$ for all α,

and

(3.61) $T_\alpha x \to x$ for all $x \in X$.

But then by Lemma 3.15(b), $T_\alpha|_X$ is compact for all α, hence by (3.59) and (3.61), X has the compact bounded approximation property, a contradiction. ■

We conjecture that one may also find a separable situation in which there are no M-ai's for a certain M-ideal. Precisely,

CONJECTURE. *Let X satisfy* (3.58). *Then there exists a separable Y with $X \subset Y \subset X^{**}$ so that there is no uniformly bounded sequence (T_n) of operators satisfying*

(i) *$T_nY \subset X$ for all n;*
(ii) *$T_nx \to x$ for all $x \in X$;*
(iii) *$\overline{\lim}_n \|T_nx + (I - T_n)y\| \le \max\{\|x\|, \|y\|\}$ for all $x \in X$ and $y \in Y$.*

Of course, if X satisfies (3.58) and $X \subset Y \subset X^{**}$, then X is an M-ideal in Y; if X satisfied the Conjecture, X would admit no M-ai or weak contractive M-ai in Y. The Conjecture, however, appears to lie much further below the surface (modulo known results) than Proposition 3.16.

Appendix. An isometric lifting theorem. We obtain here an operator space generalization of the Effros–Haagerup lifting result.

THEOREM. *Let $X \subset Y$ be operator spaces with X nuclear, Y 1-locally reflexive, and Y/X separable. Assume that X is a complete M-ideal in Y. Then there exists a completely contractive lift $L : Y/X \to Y$ of $I_{Y/X}$.*

In the classical case (i.e., MIN operator structures), a nuclear operator space X is an L^1-predual, and since all Banach spaces are 1-locally reflexive, the Theorem reduces to Ando's result [An] that one always has contractive liftings of the identity on Y/X when Y/X is separable and X is an L^1-predual which is an M-ideal in Y. In fact, our proof of the Theorem is the quantized version of Ando's argument, as expressed in [HWW, page 58].

We first assemble the facts needed to obtain the isometric assertions of the Theorem. Throughout, we assume that $X \subset Y$ are Banach spaces with X an M-ideal in Y; $\pi : Y \to Y/X$ denotes the quotient map.

LEMMA A1. *Given $e_0 \in Y/X$, there exists a $y_0 \in Y$ with $\|e_0\| = \|y_0\|$ and $\pi y_0 = e_0$.*

Proof. Let $y \in Y$ be such that $\pi y = e_0$. Then $d(y, X) = \|e_0\|$. By Proposition II.1.1 of [HWW] (the proximality of M-ideals) there exists an $x_0 \in X$ with $\|y - x_0\| = d(y, X)$. Then $y_0 := y - x_0$ is the desired element of Y. ■

LEMMA A2. *Let V be a closed linear subspace of X and $L \in Y$. Suppose $d(L, V) := \inf\{\|L - v\| : v \in V\} = 1$. Then for all $\varepsilon > 0$, there exists $V_\varepsilon \in V$ and $L_\varepsilon \in \mathrm{Ba}\, Y$ so that*

$$\|(L - V_\varepsilon) - L_\varepsilon\| \le \varepsilon \quad \textit{and} \quad (L - V_\varepsilon) - L_\varepsilon \in X. \tag{A1}$$

Proof. This follows from Lemma 2.5 of [HWW]. We sketch a proof for completeness. Since X is an M-ideal, besides its proximality, X also has the "strict 2-ball" property: *given closed balls B_1, B_2 in Y with $\mathrm{Int}(B_1 \cap B_2) \neq \emptyset$ and $B_i \cap X \neq \emptyset$ for $i = 1, 2$, we have $B_1 \cap B_2 \cap X \neq \emptyset$.*

First choose $V_\varepsilon \in V$ with

$$\|L'\| < 1 + \varepsilon, \quad \text{where} \quad L' = L - V_\varepsilon. \tag{A2}$$

Now let $B_1 = B(L', 1)$. Of course, $d(L', V) = d(L, V) = 1$, so by the proximality of X, $B_1 \cap X \neq \emptyset$. Now set $B_2 = B(0, \varepsilon)$ $(= \varepsilon\, \mathrm{Ba}\, Y)$. Since $\|L'\| \ge 1$, (A2) yields that $\mathrm{Int}(B_1 \cap B_2) \neq \emptyset$. Hence choosing $x \in B_1 \cap B_2 \cap X$, and letting $L_\varepsilon = L' - x$, we get

$$\|L_\varepsilon\| \le 1 \quad (\text{since } x \in B(L', 1)) \tag{A3}$$

and of course

$$(L - V_\varepsilon) - L_\varepsilon = L' - L_\varepsilon = x \in X. \ ■$$

A simple compactness argument yields the next result.

LEMMA A3. *If X is a nuclear operator space, then X^{**} is an isometrically injective operator space.*

REMARK. In fact, it is proved in [EOR] that X is nuclear iff X is locally reflexive and X^{**} is 1-injective.

LEMMA A4. *Let E be a finite-dimensional operator space, Y an operator space, and let $\mathrm{cb}(E, Y)$ be the operator space of completely bounded maps from E to Y. Then if Y is 1-locally reflexive, $\mathrm{cb}(E, Y)^{**}$ is (canonically isometric to) $\mathrm{cb}(E, Y^{**})$.*

Proof (sketch). Of course, $\mathrm{cb}(E, Y)$ is nothing but the space of linear maps T from E to Y, endowed with $\|T\|_{\mathrm{cb}}$. Thus $\mathrm{cb}(E, Y) = E^* \otimes_{\mathrm{op}} Y$. But Y is 1-locally reflexive iff $F \otimes_{\mathrm{op}} Y^{**} = (F \otimes_{\mathrm{op}} Y)^{**}$ isometrically for all finite-dimensional operator spaces F (cf. [EH]). ■

The next result is again a quantization of an observation in [HWW] (page 62).

LEMMA A5. *Assume that X and Y are operator spaces with Y 1-locally reflexive and X a complete M-ideal in Y, and let E be a finite-dimensional operator space. Then $\mathrm{cb}(E, X)$ is a complete M-ideal in $\mathrm{cb}(E, Y)$.*

Proof. Let W be the (weak*-closed) subspace of Y^{**} so that $Y^{**} = X^{**} \oplus W$ is a complete L^∞-decomposition for Y^{**}, and set $F = E^*$. Then $(F \otimes_{\mathrm{op}} X^{**}) \oplus (F \otimes_{\mathrm{op}} W)$ is a complete L^∞-decomposition for $F \otimes_{\mathrm{op}} Y^{**}$. The result now follows upon identifying $F \otimes_{\mathrm{op}} X^{**}$ with $\mathrm{cb}(E, X)^{**}$ and $F \otimes_{\mathrm{op}} Y^{**}$ with $\mathrm{cb}(E, Y)^{**}$, via Lemma A4. ■

Finally, we are prepared for the fundamental lemma yielding the proof of the Theorem.

LEMMA A6 (The Crucial Lemma). *Let $X \subset Y$ be operator spaces with X nuclear and Y 1-locally reflexive. Let $E_1 \subset E$ be finite-dimensional subspaces of Y/X, and let $L_1 : E_1 \to Y$ be a completely contractive lift of I_{E_1}. Then given $\varepsilon > 0$, there exists a completely contractive lift $L_2 : E \to Y$ of I_E with $\|L_2|_{E_1} - L_1\| \le \varepsilon$.*

A6 is a simple consequence of the results already assembled and the basic

SUBLEMMA. *Assuming the hypotheses of A6, there exists a lift $L_\varepsilon : E \to Y$ of I_E with $L_\varepsilon|_{E_1} = L_1$ and $\|L_\varepsilon\|_{\mathrm{cb}} < 1 + \varepsilon$.*

Proof. Let P be the M-projection from Y^{**} onto X^{**} and $L : Y^{**}/X^{**} \to Y^{**}$ the completely contractive lift of the identity on Y^{**}/X^{**} induced by P. Since $L|_{E_1}$ and L_1 are then both lifts of I_{E_1} into Y^{**}, $L(e) - L_1(e) \in X^{**}$ for all $e \in E_1$, hence

$$(I - P)(L|_{E_1} - L_1) = 0. \tag{A4}$$

Since X^{**} is 1-injective by Lemma A3, we may choose a complete contraction $\theta : E_2 \to X^{**}$ with

$$\theta|_{E_1} = PL_1. \tag{A5}$$

Next, define $\widetilde{L} : E_2 \to Y^{**}$ by

$$\widetilde{L} = \theta + (I - P)L|_{E_2}. \tag{A6}$$

Then if $e \in E_2$,

$$\pi^{**}\widetilde{L}(e) = \pi^{**}(I - P)L(e) = \pi^{**}L(e) = e. \tag{A7}$$

Thus, $\widetilde{L}$ is a lift of I_{E_2} into Y^{**}. If $e \in E_1$, then

$$\begin{aligned} \widetilde{L}(e) &= PL_1(e) + (I - P)L(e) \\ &= PL_1(e) + (I - P)L_1(e) \quad \text{(by (A4))} \\ &= L_1(e). \end{aligned} \tag{A8}$$

Since P is an M-projection, $X^{**} \oplus (I - P)Y^{**}$ is a complete M-decomposition of Y^{**}, whence by (A6)–(A8), $\widetilde{L}$ is a completely contractive lift of I_{E_2} extending L_1.

Of course, $\widetilde{L}$ lifts into Y^{**}, not Y. To get a lift into Y, we apply our extended local reflexivity principle for operator spaces, Lemma 3.12. First let $\chi : E \to Y$ be a linear lift of I_E. Now let $G = \widetilde{L}(E) + \chi(E)$. Choose $T : G \to Y$ satisfing (i), (iii) and (iv) of Lemma 3.12 (for $\lambda = 1$). Define $L_\varepsilon : E \to Y$ by

$$L_\varepsilon = T\widetilde{L}. \tag{A9}$$

Then if $e \in E_1$, we have $\widetilde{L}(e) = L_1(e) = L_\varepsilon(e)$, where the last equality holds by 3.12(iii) since $L_1(e) \in Y$.

Next $\|L_\varepsilon\|_{\text{cb}} < 1 + \varepsilon$ by (A9) and Lemma 3.12(i), since $\widetilde{L}$ is a complete contraction. Finally, if $e \in E$,

$$T\widetilde{L}(e) = T(\widetilde{L}(e) - \chi(e)) + T\chi(e) = T(\widetilde{L}(e) - \chi(e)) + \chi(e) \tag{A10}$$

since $T|_{G\cap Y} = I|_{G\cap Y}$. But then

$$\pi T\widetilde{L}(e) = \pi\chi(e) = e, \tag{A11}$$

since $\widetilde{L}(e) - \chi(e) \in X^{**} \Rightarrow T(\widetilde{L}(e) - \chi(e)) \in X$ by 3.12(iv). Thus L_ε satisfies the conclusion of the Sublemma. ■

Proof of Lemma A6. Let $\widetilde{X} = \text{cb}(E, X)$ and $\widetilde{Y} = \text{cb}(E, Y)$, and let $V = \{T \in \widetilde{X} : \ker T \supset E_1\}$. Let L be a lift of I_E so that $L|_{E_1} = L_1$, and let $\varepsilon > 0$. Let L_ε be a lift of I_E satisfying the conclusion of the Sublemma. Then $L - L_\varepsilon \in V$. This proves that

$$d(L, V) = 1. \tag{A12}$$

We now apply Lemma A2 to $\widetilde{X} \subset \widetilde{Y}$; $\widetilde{X}$ is an M-ideal in $\widetilde{Y}$ by Lemma A5. Thus, choose $V_\varepsilon \in V$ and $L_\varepsilon \in \text{Ba}\,\widetilde{Y}$ satisfying (A1), and set $L_2 = L_\varepsilon$. Now it is trivial that $L - V_\varepsilon$ is a lift of I_E, and moreover, $(L - V_\varepsilon)|_{E_1} = L_1$. Since $(L - V_\varepsilon) - L_2 \in \widetilde{X}$, $[(L - V_\varepsilon) - L_2](E) \subset X$, whence $\pi(L - V_\varepsilon) = \pi L_2 = I_E$, i.e., L_2 is indeed a lift of I_E, which is completely contractive. Finally,

$$\|L_1 - L_2|_{E_1}\| = [(L - V_\varepsilon) - L_\varepsilon)|_{E_1}\| \le \|(L - V_\varepsilon) - L_\varepsilon\| \le \varepsilon \quad \text{by (A1);} \tag{A13}$$

thus L_2 satisfies the conclusion of the lemma. ■

Proof of the Theorem. Let $e_0 \in E_0$ with $\|e_0\| = 1$. Choose $y_0 \in Y$ with $\pi y_0 = e_0$ and $\|y_0\| = \|e_0\|$ (by Lemma A1) and set $E_0 = [e_0]$. Choose finite-dimensional spaces $E_0 \subset E_1 \subset \ldots$ of Y/X with $\bigcup_{j=1}^\infty E_j$ dense in Y/X. Define $L_0 : E_0 \to Y$ by $L_0(\lambda e_0) = \lambda y_0$ for all scalars λ. It is trivial that L_0 is a completely contractive lift of I_{E_0}. Now let $\varepsilon > 0$ and suppose $i \ge 0$ and a completely contractive lift $L_i : E_i \to Y$ of I_{E_i} has been chosen. By the Crucial Lemma, we may choose a completely contractive lift $L_{i+1} : E_{i+1} \to Y$ of $I_{E_{i+1}}$ with

$$\|L_{i+1}|_{E_i} - L_i\| \le \varepsilon/2^{i+1}. \tag{A14}$$

Then it follows (as in the proof of Theorem 2.4) that if we set $Z = \bigcup_{i=0}^\infty E_i$, then $\lim_{i\to\infty} L_i(z) =: L(z)$ exists for all $z \in Z$, and L extends to a completely contractive lift of $I_{Y/X}$. ■

REMARK. In the above argument, we also have

$$\|L(e_0) - y_0\| \le \sum_{i=1}^\infty \|(L_{i+1} - L_i)(y_0)\| \le \sum_{i=0}^\infty \frac{\varepsilon}{2^{i+1}} = \varepsilon. \tag{A15}$$

That is, we have also proved that *for given $\varepsilon > 0$, $e_0 \in Y/X$ and $y_0 \in Y$ with $\|e_0\| = 1 = \|y_0\|$ and $\pi y_0 = e_0$, the lift L may also be chosen so that* $\|Le_0 - y_0\| \le \varepsilon$. At this level of generality, however, it is impossible to ensure that L may be chosen with $Le_0 = y_0$. Indeed, one may give an extreme point obstruction by constructing X and Y satisfying the hypotheses of the Theorem with e_0, y_0 as above, y_0 an extreme point of $\text{Ba}\,Y$, but so that e_0 is *not* an extreme point of $\text{Ba}\,Y/X$. Since L is an isometry, Le_0 cannot be an extreme point of $\text{Ba}\,Y$.

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