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Extension maps in ultradifferentiable and ultraholomorphic function spaces

by

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Abstract. The problem of the existence of extension maps from 0 to $\mathbb{R}$ in the setting of the classical ultradifferentiable function spaces has been solved by Petzsche [9] by proving a generalization of the Borel and Mityagin theorems for C^∞-spaces. We get a Ritt type improvement, i.e. from 0 to sectors of the Riemann surface of the function log for spaces of ultraholomorphic functions, by first establishing a generalization to some nonclassical ultradifferentiable function spaces.

1. Introduction. A result of Borel [1] states that, for every sequence $(c_n)_{n\in\mathbb{N}_0}$ of complex numbers, there is $f \in C^\infty(\mathbb{R})$ such that $f^{(n)}(0) = c_n$ for every $n \in \mathbb{N}_0$. This result has been quite sharpened by Ritt [10] who proved as a corollary to his main result that the function f may moreover be supposed real-analytic outside the origin. In fact the main result of Ritt states that for every open sector S of angle $< 2\pi$ and every sequence $(c_n)_{n\in\mathbb{N}_0}$ of complex numbers, there is a holomorphic function on S having $\sum_{n=0}^{\infty} c_n z^n$ as asymptotic behaviour at 0. In [8], Mityagin has given another information about the Borel theorem: there is no extension map from ω into $C^\infty(\mathbb{R})$; in this paper, the word map stands for a continuous linear operator.

Petzsche [9] has considered the results of Borel and Mityagin in the setting of the classical ultradifferentiable function spaces of Beurling and of Roumieu types. The situation here is quite interesting: he has got characterizations of those spaces for which there is an extension map.

In this paper we investigate how the Ritt result can be adapted to this setting. We begin by studying some properties of the sequences $\boldsymbol{M} =$

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$(M_n)_{n\in\mathbb{N}_0}$ of positive numbers which are normalized and logarithmically convex, especially the ones related to the new condition (γ_r) with $r \in \mathbb{N}_0$ generalizing the condition (γ_1) of Petzsche. Given $r \in \mathbb{N}$, we then introduce some nonclassical ultradifferentiable function spaces. Their elements are C^∞-functions f on different intervals of the real line having 0 as left end point or interior point and such that $f^{(nr+j)}(0) = 0$ for every $n \in \mathbb{N}_0$ and $j \in \{1, \ldots, r-1\}$. They lead to results comparable to those of Petzsche: cf. Theorems 4.4 and 5.4. We then introduce spaces of what we call ultraholomorphic functions on a sector of the Riemann surface of the function $\log(z)$. Using the results obtained in the nonclassical ultradifferentiable case as well as some holomorphic arguments, we get the main Theorems 4.7 and 5.6 which are of Ritt's type: they provide necessary and sufficient conditions under which there are extension maps with values in these ultraholomorphic function spaces. Finally we get some surjectivity results in the case of the ultraholomorphic function spaces of the Roumieu type.

2. The sequences M and m

DEFINITIONS. Let $\boldsymbol{M} = (M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive numbers. The *corresponding sequence* $\boldsymbol{m} = (m_n)_{n\in\mathbb{N}_0}$ is then defined by $m_0 = 1$ and $m_n = M_n/M_{n-1}$ for every $n \in \mathbb{N}$.

The sequence $\boldsymbol{M}$ itself is

(a) *normalized* if $M_0 = 1$ and $M_n \geq 1$ for every $n \in \mathbb{N}$;
(b) *logarithmically convex* if $M_n^2 \leq M_{n-1}M_{n+1}$ for every $n \in \mathbb{N}$.

If $\boldsymbol{M}$ and $\boldsymbol{P} = (P_n)_{n\in\mathbb{N}_0}$ are two sequences of positive numbers, we say that $\boldsymbol{M}$ and $\boldsymbol{P}$, or equivalently the corresponding sequences $\boldsymbol{m}$ and $\boldsymbol{p}$, are *equivalent* if

$$0 < \inf_{n\in\mathbb{N}_0} \frac{m_n}{p_n} \leq \sup_{n\in\mathbb{N}_0} \frac{m_n}{p_n} < \infty.$$

Let us recall some of the conditions studied by H.-J. Petzsche in [9] and concerning sequences $(m_n)_{n\in\mathbb{N}_0}$ of positive numbers:

(α) $m_0 = 1$ and $m_n \uparrow \infty$,
(α_1) (α) and $m_n/n \uparrow \infty$,
(β_1) there is $p \in \mathbb{N}$ such that $\liminf_{n\to\infty} \frac{m_{pn}}{pm_n} > 1$,
(β_2) for every $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \left(\frac{M_{kn}}{M_n}\right)^{1/(n(k-1))} \frac{1}{m_{kn}} \leq \varepsilon,$$

(γ_1) $\displaystyle\sup_{n\in\mathbb{N}} \frac{m_n}{n} \sum_{k=n}^{\infty} \frac{1}{m_k} < \infty.$

In a private communication to one of the authors, H.-J. Petzsche:

(a) reported that the proof of Proposition 1.6.c of [9] has a bug;
(b) mentioned that a correct result is: *if the sequence $(m_n)_{n\in\mathbb{N}_0}$ of positive numbers satisfies (α_1) and (β_2), then it also satisfies (β_1)*: as it satisfies the condition (β_2^*) of [9], there are $k, n_0 \in \mathbb{N}$ such that $k \geq 2$ and

$$\left(\frac{M_{kn}^*}{M_n^*}\right)^{1/(n(k-1))} \frac{1}{m_{kn}^*} \leq \frac{1}{2}, \qquad \forall n \geq n_0;$$

hence the conclusion since, by condition (α_1),

$$\left(\frac{M_{kn}^*}{M_n^*}\right)^{1/(n(k-1))} \frac{1}{m_{kn}^*} \geq \frac{m_n^*}{m_{kn}^*};$$

(c) observed that therefore the hypothesis of Theorem 2.1(b) of [9] should read "(γ_1) and (β_2)" instead of "(γ_2)";
(d) wrote that in the same way in the statement of Theorem 3.1(a) of [9] one should read "(γ_1) and (β_2)" instead of "(β_2)" and mentioned that "consequently Theorem 3.4(c) must be used for the proof of 3.1(a)".

For $r \in \mathbb{N}$, let us now introduce the condition

$$(\gamma_r) \qquad \sup_{n\in\mathbb{N}} \frac{\sqrt[r]{m_n}}{n} \sum_{k=n}^{\infty} \frac{1}{\sqrt[r]{m_k}} < \infty$$

that will play an important role later on.

LEMMA 2.1. *Let $(m_n)_{n\in\mathbb{N}_0}$ be a sequence of positive numbers satisfying the condition (α). For every $r \in \mathbb{N}$, the following assertions are equivalent*:

(1) *the sequence $(m_n)_{n\in\mathbb{N}_0}$ satisfies the condition (γ_{r+1})*,
(2) *there is $p \in \mathbb{N}$ such that $\liminf_n m_{pn}/(p^{r+1}m_n) > 1$*,
(3) *the sequence $(m_n/n)_{n\in\mathbb{N}_0}$ satisfies the condition (γ_r)*.

Proof. (1)$\Leftrightarrow$(2) is an immediate consequence of Proposition 1.1(a) of [9]: as the sequence $(\sqrt[r+1]{m_n})_{n\in\mathbb{N}}$ satisfies (α), it satisfies (γ_1) if and only if it satisfies (β_1).

(1)$\Rightarrow$(3). As $(\sqrt[r+1]{m_n})_{n\in\mathbb{N}_0}$ satisfies (α) and (γ_1), Proposition 1.1(a) of [9] provides a sequence $(s_n)_{n\in\mathbb{N}_0}$ equivalent to it and satisfying (α_1). As $(q_n := s_n^{r+1})_{n\in\mathbb{N}_0}$ is then equivalent to $(m_n)_{n\in\mathbb{N}_0}$, it satisfies (γ_{r+1}) and the equivalence (1)$\Leftrightarrow$(2) leads to the existence of some $p \in \mathbb{N}$ such that

$$\liminf_n \frac{\sqrt[r]{\frac{q_{pn}}{pn}}}{p\sqrt[r]{\frac{q_n}{n}}} = \liminf_n \sqrt[r]{\frac{q_{pn}}{p^{r+1}q_n}} > 1.$$

Finally as the sequence $(t_n)_{n\in\mathbb{N}_0}$ defined by $t_0 = 1$ and $t_n = \sqrt[r]{q_n/n}$ for every $n \in \mathbb{N}$ satisfies (α) and (β_1), it also satisfies (γ_1), which proves the conclusion since $(m_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$ are equivalent.

(3)⇒(2). As we clearly have

$$\sup_n \frac{\sqrt[r]{m_n}}{n}\sum_{k=n}^{\infty}\frac{1}{\sqrt[r]{m_k}} \le \sup_n \frac{1}{n}\sqrt[r]{\frac{m_n}{n}}\sum_{k=n}^{\infty}\sqrt[r]{\frac{k}{m_k}},$$

the sequence $(\sqrt[r]{m_n})_{n\in\mathbb{N}_0}$ satisfies (α) and (γ_1). So up to substituting it by an equivalent sequence, we may very well suppose that it also satisfies (α_1). Therefore the sequence $(s_n)_{n\in\mathbb{N}_0}$ defined by $s_0 = 1$ and $s_n = \sqrt[r]{m_n/n}$ for every $n \in \mathbb{N}$ satisfies (α). As by hypothesis it satisfies (γ_1), we see that it also satisfies (β_1), i.e. there is $p \in \mathbb{N}$ such that

$$\liminf_n \frac{1}{p}\cdot\frac{\sqrt[r]{\frac{m_{pn}}{pn}}}{\sqrt[r]{\frac{m_n}{n}}} = \liminf_n \sqrt[r]{\frac{m_{pn}}{p^{r+1}m_n}} > 1. \ \blacksquare$$

IMPORTANT NOTE. From now on, $\boldsymbol{M}$ designates a sequence $(M_n)_{n\in\mathbb{N}_0}$ of positive numbers which is normalized and logarithmically convex. In particular, the sequences $\boldsymbol{M}$ and $\boldsymbol{m}$ are increasing.

We will need the following information about some sequences associated with $\boldsymbol{M}$.

LEMMA 2.2. *Let $\boldsymbol{M}^* = (M_n^*)_{n\in\mathbb{N}_0}$ be the sequence defined by $M_0^* = 1$ and $M_n^* = M_n/n!$ for every $n \in \mathbb{N}$.*

(a) *If the sequence $\boldsymbol{m}$ satisfies the condition (α_1), then $\boldsymbol{M}^*$ is normalized and logarithmically convex. If moreover $\boldsymbol{m}$ satisfies the condition (γ_{r+1}) for some $r \in \mathbb{N}$, then the corresponding sequence $\boldsymbol{m}^*$ satisfies the condition (γ_r).*

(b) *The sequence $\boldsymbol{m}^*$ satisfies the condition (β_2) if and only if the sequence $\boldsymbol{m}$ does.*

Proof. (a) Everything is trivial if one notes for the last assertion that $m_n^* = m_n/n$ for every $n \in \mathbb{N}$ and applies the previous lemma.

(b) Just note that, for every $k, n \in \mathbb{N}$ with $k \ge 2$, we have

$$\left(\frac{M_{kn}}{(kn)!}\frac{n!}{M_n}\right)^{1/(n(k-1))}\frac{kn}{m_{kn}} = \left(\frac{M_{kn}}{M_n}\right)^{1/(n(k-1))}\frac{1}{m_{kn}}\left(\frac{n!}{(kn)!}\right)^{1/(n(k-1))} kn$$

and, by the Stirling formula,

$$\lim_{n\to\infty}\left(\frac{n!}{(kn)!}\right)^{1/(n(k-1))} kn = ek^{-1/(k-1)}. \ \blacksquare$$

LEMMA 2.3. *For $r \in \mathbb{N}$, let $\boldsymbol{P} = (P_n)_{n\in\mathbb{N}_0}$ be the r-interpolating sequence defined by*

$$P_{kr+j} = \sqrt[r]{M_k^{r-j}M_{k+1}^j}, \quad \forall k \in \mathbb{N}_0,\ \forall j \in \{0,\ldots,r\}.$$

Then the sequence $\boldsymbol{P}$ is normalized and logarithmically convex. Moreover $P_{nr} = M_n$ for every $n \in \mathbb{N}_0$ as well as $p_{nr+j} = \sqrt[r]{m_{n+1}}$ for every $n \in \mathbb{N}_0$ and $j \in \{1,\ldots,r\}$.

(a) *The sequence $\boldsymbol{m}$ satisfies the condition (γ_r) if and only if the corresponding sequence $\boldsymbol{p}$ satisfies the condition (γ_1).*

(b) *The sequence $\boldsymbol{m}$ satisfies the condition (β_2) if and only if the corresponding sequence $\boldsymbol{p}$ does.*

Proof. Everything is trivial except (a) and (b).

(a) The condition is necessary since

$$\begin{aligned}\sup_{n\in\mathbb{N}}\frac{p_n}{n}\sum_{k=n}^{\infty}\frac{1}{p_k} &= \sup_{\substack{l\in\mathbb{N}_0\\ j\in\{1,\ldots,r\}}}\frac{p_{lr+j}}{lr+j}\sum_{k=lr+j}^{\infty}\frac{1}{p_k}\\ &\le \sup_{l\in\mathbb{N}_0}\frac{\sqrt[r]{m_{l+1}}}{lr+1}\sum_{k=l}^{\infty}\sum_{j=1}^{r}\frac{1}{p_{kr+j}}\\ &\le r\sup_{l\in\mathbb{N}_0}\frac{\sqrt[r]{m_{l+1}}}{lr+1}\sum_{k=l+1}^{\infty}\frac{1}{\sqrt[r]{m_k}}\\ &\le (1+r)\sup_{l\in\mathbb{N}}\frac{\sqrt[r]{m_l}}{l}\sum_{k=l}^{\infty}\frac{1}{\sqrt[r]{m_k}}.\end{aligned}$$

The condition is sufficient since for every $n \in \mathbb{N}$,

$$\frac{\sqrt[r]{m_n}}{n}\sum_{k=n}^{\infty}\frac{1}{\sqrt[r]{m_k}} \le r\frac{p_{nr}}{nr}\sum_{k=n}^{\infty}\frac{1}{p_{kr}} \le r\frac{p_{nr}}{nr}\sum_{k=nr}^{\infty}\frac{1}{p_k}.$$

(b) We first prove the necessity. By Petzsche's Lemma 1.5 of [9], the sequence $\boldsymbol{m}$ satisfies the condition (iii): for every $\varepsilon > 0$, there are $n_0 \in \mathbb{N}$ and $\beta \in\ ]0,1[$ such that, for every integer $n \ge n_0$, we have

$$\max_{j\le\beta n}\frac{M_n}{M_j}\cdot\frac{1}{m_n^{n-j}} \le \varepsilon^{rn}.$$

To conclude by the same Lemma 1.5, we are going to prove that for every $\varepsilon > 0$, if we set $n_0' = r\max\{n_0+1, 1+4/\beta\}$ and $\beta' = \beta/2$, then for every integer $n \ge n_0'$, we have

$$\max_{j\le\beta' n}\frac{P_n}{P_j}\cdot\frac{1}{p_n^{n-j}} \le \varepsilon^{n}.$$

Indeed, for every integer $n \ge n_0'$, there are unique integers $k \in \mathbb{N}_0$ and $l \in \{1,\ldots,r\}$ such that $n = kr + l$, hence $k+1 \ge k \ge n_0$ as well as $k \ge 4/\beta$ since $kr + l \ge n_0' \ge 4r/\beta + r$. Moreover for every positive integer $j \le \beta' n$, there are unique integers $h \in \mathbb{N}_0$ and $i \in \{1,\ldots,r\}$ such that $j = hr + i$, hence

$$h \le h+1 \le \frac{j}{r}+1 \le \frac{\beta}{2}\cdot\frac{kr+l}{r}+1 \le \frac{\beta}{2}k+2 \underset{(*)}{\le} \beta k \le \beta(k+1)$$

(the inequality $(*)$ comes from the fact that $k \ge 4/\beta$). Now we check the quotient $P_n/(P_j p_n^{n-j})$.

(1) If $j=0$, it is equal to

$$\left(M_k^{r-l}M_{k+1}^l\frac{1}{m_{k+1}^{rk+l}}\right)^{1/r} \le \left(\frac{M_k}{M_0}\cdot\frac{1}{m_k^k}\right)^{(r-l)/r}\cdot\left(\frac{M_{k+1}}{M_0}\cdot\frac{1}{m_{k+1}^{k+1}}\right)^{l/r} \le \varepsilon^n.$$

(2) If $j \ge 1$, it is equal to

$$\left(\frac{M_k^{r-l}M_{k+1}^l}{M_h^{r-i}M_{h+1}^i}\cdot\frac{1}{m_{k+1}^{rk+l-rh-i}}\right)^{1/r}$$

and one checks directly that it is $\le \varepsilon^n$ in each of the following cases: $i=l$, $i<l$ and $i>l$.

The condition is sufficient since

$$\left(\frac{M_{kn}}{M_n}\right)^{1/(n(k-1))}\frac{1}{m_{kn}} = \left(\left(\frac{P_{krn}}{P_{rn}}\right)^{1/(rn(k-1))}\frac{1}{p_{krn}}\right)^r. \ \blacksquare$$

DEFINITION. Let us say that a sequence $(b_n)_{n\in\mathbb{N}_0}$ of positive numbers is *quasi-increasing* if there is $n_0\in\mathbb{N}$ such that the sequence $(b_n)_{n=n_0}^\infty$ is increasing.

LEMMA 2.4. *For every $r\in\mathbb{N}_0$, if the sequence $\boldsymbol{m}$ satisfies the condition (β_2) and $(m_n/n^r)_{n\in\mathbb{N}}$ is quasi-increasing, then $\boldsymbol{m}$ satisfies the condition (γ_r).*

Proof. The sequence $\boldsymbol{Q}=(Q_n := \sqrt[r]{M_n})_{n\in\mathbb{N}_0}$ is clearly normalized and logarithmically convex. It is also straightforward to check that the corresponding sequence $\boldsymbol{q}$ satisfies (β_2) and hence the equivalent condition (β_2^*) of [9]; so there are $k, n_0\in\mathbb{N}$ such that $k\ge 2$ and

$$\left(\frac{Q_{kn}^*}{Q_n^*}\right)^{1/(n(k-1))}\frac{1}{q_{kn}^*} \le \frac{1}{2}, \quad \forall n\ge n_0.$$

If n_0 is such that the sequence $(m_n/n^r)_{n=n_0}^\infty$ increases, we get

$$\frac{q_n^*}{q_{kn}^*} \le \left(\frac{Q_{kn}^*}{Q_n^*}\right)^{1/(n(k-1))}\frac{1}{q_{kn}^*} \le \frac{1}{2}, \quad \forall n\ge n_0.$$

In particular $\boldsymbol{q}$ satisfies (β_1). To conclude, one has to proceed as in the proof of (ii)⇒(i) of Proposition 1.1(a) of [9]. ■

3. Some Fréchet and (LB)-spaces

(a) Let us first introduce for reference all the Fréchet spaces used throughout the Beurling type case.

DEFINITIONS. *The classical Fréchet space $\Lambda_{(\boldsymbol{M})}$.* Its elements are the sequences $\boldsymbol{a}=(a_n)_{n\in\mathbb{N}_0}$ of complex numbers such that, for every $m\in\mathbb{N}$,

$$\|\boldsymbol{a}\|_m := \sup_{n\in\mathbb{N}_0}\frac{m^n|a_n|}{M_n} < \infty.$$

It is clearly a vector subspace of ω and we endow it with the countable fundamental family of norms $\{\|\cdot\|_m : m\in\mathbb{N}\}$.

The Fréchet space $\mathcal{D}_{r,(\boldsymbol{M})}$ (resp. $\mathcal{L}_{r,(\boldsymbol{M})}$; $\mathcal{E}_{r,(\boldsymbol{M})}$; $\mathcal{N}_{r,(\boldsymbol{M})}$) for $r\in\mathbb{N}$. Its elements are the complex-valued functions $f\in C^\infty(\mathbb{R})$ with compact support contained in $[-1,1]$ (resp. $f\in C^\infty([0,\infty[)$ with support contained in $[0,1]$; $f\in C^\infty([0,1])$; $f\in C^\infty([0,\infty[)$) and such that

(1) $f^{(nr+j)}(0)=0$ for every $n\in\mathbb{N}_0$ and $j\in\{1,\ldots,r-1\}$;

(2) for every $m\in\mathbb{N}$,

$$|f|_m := \sup_{n\in\mathbb{N}_0}\sup_{x\in A}\frac{m^n|f^{(nr)}(x)|}{M_n} < \infty$$

with $A=\mathbb{R}$ (resp. $[0,\infty[$; $[0,1]$; $[0,\infty[$). It is clearly a vector space and we endow it with the countable fundamental system of norms $\{|\cdot|_m : m\in\mathbb{N}\}$.

The Fréchet space $\mathcal{K}_{\alpha,(\boldsymbol{M})}$ for $\alpha>0$. Let us designate by Σ the Riemann surface of definition of the function $\log(z)$ and set

$$S_\alpha := \{z\in\Sigma : -\alpha\pi/2 \le \arg(z) \le \alpha\pi/2\}\cup\{0\}.$$

Its elements are the holomorphic functions f on $\operatorname{int}(S_\alpha)$ such that:

(1) for every $n\in\mathbb{N}_0$, $f^{(n)}$ has a continuous extension to S_α, which we continue to denote by $f^{(n)}$;

(2) for every $m\in\mathbb{N}$,

$$|f|_m := \sup_{n\in\mathbb{N}_0}\sup_{z\in S_\alpha}\frac{m^n|f^{(n)}(z)|}{M_n} < \infty.$$

Of course it is a vector space and we endow it with the countable fundamental system of norms $\{|\cdot|_m : m\in\mathbb{N}\}$.

If $\boldsymbol{M}$ and $\boldsymbol{P}$ are two equivalent sequences of real numbers which are normalized and logarithmically convex, we clearly have $\Lambda_{(\boldsymbol{M})}=\Lambda_{(\boldsymbol{P})}$ and $\mathcal{D}_{r,(\boldsymbol{M})}=\mathcal{D}_{r,(\boldsymbol{P})}$, $\mathcal{L}_{r,(\boldsymbol{M})}=\mathcal{L}_{r,(\boldsymbol{P})}$, $\mathcal{E}_{r,(\boldsymbol{M})}=\mathcal{E}_{r,(\boldsymbol{P})}$, $\mathcal{N}_{r,(\boldsymbol{M})}=\mathcal{N}_{r,(\boldsymbol{P})}$ for every $r\in\mathbb{N}$ as well as $\mathcal{K}_{\alpha,(\boldsymbol{M})}=\mathcal{K}_{\alpha,(\boldsymbol{P})}$ for every $\alpha>0$.

(b) Let us now introduce all the Banach and Hausdorff (LB)-spaces used throughout the Roumieu type case.

DEFINITIONS. *The classical* (LB)*-space* $\Lambda_{\{M\}}$. Its elements are the sequences $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}_0}$ of complex numbers for which there is $m \in \mathbb{N}$ such that

$$|\boldsymbol{a}|_m := \sup_{n\in\mathbb{N}_0} \frac{|a_n|}{m^n M_n} < \infty.$$

For every $m \in \mathbb{N}$, we designate by $\Lambda^m_{\{M\}}$ the subset of $\Lambda_{\{M\}}$ of elements $\boldsymbol{a}$ such that $|\boldsymbol{a}|_m < \infty$. It is immediate that $\Lambda^m_{\{M\}}$ is a vector subspace of $\Lambda_{\{M\}}$ on which $|\cdot|_m$ is a norm. The corresponding normed space is indeed a Banach space, also denoted by $\Lambda^m_{\{M\}}$. Of course we have $\Lambda^m_{\{M\}} \subset \Lambda^{m+1}_{\{M\}}$, the canonical injection being continuous, and $\Lambda_{\{M\}} = \bigcup_{m=1}^{\infty} \Lambda^m_{\{M\}}$. So we can define the locally convex space $\Lambda_{\{M\}}$ as the inductive limit of the spaces $\Lambda^m_{\{M\}}$; it turns out to be a Hausdorff (LB)-space.

We proceed in the same way to define the following spaces.

The (LB)*-space* $\mathcal{D}_{r,\{M\}}$ (resp. $\mathcal{L}_{r,\{M\}}$; $\mathcal{N}_{r,\{M\}}$) *for* $r \in \mathbb{N}$. Its elements are the complex-valued functions $f \in C^\infty(\mathbb{R})$ such that $\operatorname{supp}(f) \subset [-1,1]$ (resp. $f \in C^\infty([0,\infty[)$ such that $\operatorname{supp}(f) \subset [0,1]$; $f \in C^\infty([0,\infty[)$) such that

(1) $f^{(nr+j)}(0) = 0$ for every $n \in \mathbb{N}_0$ and $j \in \{1,\ldots,r-1\}$,

(2) there is $m \in \mathbb{N}$ such that

$$|f|_m := \sup_{n\in\mathbb{N}_0} \sup_{x\in\mathbb{R}} \frac{|f^{(nr)}(x)|}{m^n M_n} < \infty.$$

For every $m \in \mathbb{N}$, $\mathcal{D}^m_{r,\{M\}}$ (resp. $\mathcal{L}^m_{r,\{M\}}$; $\mathcal{N}^m_{r,\{M\}}$) is the Banach space obtained by endowing the vector subspace of elements f of $\mathcal{D}_{r,\{M\}}$ (resp. $\mathcal{L}_{r,\{M\}}$; $\mathcal{N}_{r,\{M\}}$) such that $|f|_m < \infty$ with the norm $|\cdot|_m$. The space $\mathcal{D}_{r,\{M\}}$ (resp. $\mathcal{L}_{r,\{M\}}$; $\mathcal{N}_{r,\{M\}}$) then is of course the Hausdorff inductive limit of these Banach spaces.

The (LB)*-space* $\mathcal{H}_{\alpha,\{M\}}$ *for* $\alpha > 0$. Let Σ and S_α be as in the definition of the space $\mathcal{K}_{\alpha,(M)}$. The elements of $\mathcal{H}_{\alpha,\{M\}}$ are the holomorphic functions f on $\operatorname{int}(S_\alpha)$ such that

(1) for every $n \in \mathbb{N}_0$, $f^{(n)}$ has a continuous extension to S_α, which we still denote by $f^{(n)}$,

(2) there is $m \in \mathbb{N}$ such that

$$|f|_m := \sup_{n\in\mathbb{N}_0} \sup_{z\in S_\alpha} \frac{|f^{(n)}(z)|}{m^n M_n} < \infty.$$

For every $m \in \mathbb{N}$, $\mathcal{H}^m_{\alpha,\{M\}}$ is the Banach space obtained by endowing the vector subspace of elements f of $\mathcal{H}_{\alpha,\{M\}}$ such that $|f|_m < \infty$ with the norm $|\cdot|_m$. The space $\mathcal{H}_{\alpha,\{M\}}$ is of course the Hausdorff inductive limit of the Banach spaces $\mathcal{H}^m_{\alpha,\{M\}}$.

If $\boldsymbol{M}$ and $\boldsymbol{P}$ are two equivalent sequences of real numbers which are normalized and logarithmically convex, we clearly have $\Lambda_{\{M\}} = \Lambda_{\{P\}}$ and $\mathcal{D}_{r,\{M\}} = \mathcal{D}_{r,\{P\}}$, $\mathcal{L}_{r,\{M\}} = \mathcal{L}_{r,\{P\}}$, $\mathcal{N}_{r,\{M\}} = \mathcal{N}_{r,\{P\}}$ for every $r \in \mathbb{N}$ as well as $\mathcal{H}_{\alpha,\{M\}} = \mathcal{H}_{\alpha,\{P\}}$ for every $\alpha > 0$.

4. The Beurling case

4.1. *Case of the spaces* $\mathcal{D}$, $\mathcal{L}$, $\mathcal{E}$ *and* $\mathcal{N}$. The proof of the next proposition follows an idea of H.-J. Petzsche (cf. [9], Theorem 3.4, (c)⇒(a)). We use the Lemma 1.3.6 of [6] in the following setting: *let* $(p_k)_{k\in\mathbb{N}}$ *be a non-decreasing sequence of positive numbers and choose an infinite subset* N *of* $\mathbb{N}$ *containing* $\{k \in \mathbb{N} : p_k < p_{k+1}\}$. *Then for every real-valued* $f \in C^\infty(\mathbb{R})$ *vanishing identically on* $]-\infty, 0]$ *and every* $x \in\,]0, \sum_{k=1}^\infty 1/p_k]$, *one has*

$$|f(x)| \le \sum_{k\in N} \frac{4^k}{p_1\cdots p_k} \sup_{0\le y\le x} |f^{(k)}(y)|.$$

PROPOSITION 4.1. *For every* $r \in \mathbb{N}$, *if the restriction map*

$$R : \mathcal{L}_{r,(M)} \to \Lambda_{(M)}, \qquad f \mapsto (f^{(nr)}(0))_{n\in\mathbb{N}_0},$$

is surjective, then the sequence $\boldsymbol{m}$ *satisfies the condition* (γ_r).

Proof. As R is a continuous linear surjection between Fréchet spaces, it is open. Therefore there are $C > 0$ and $m \in \mathbb{N}$ such that, for every $\boldsymbol{a} \in \Lambda_{(M)}$, there is $f \in \mathcal{L}_{r,(M)}$ satisfying $Rf = \boldsymbol{a}$ and $|f|_1 \le C\|\boldsymbol{a}\|_m$. So, for every $p \in \mathbb{N}$, there is a real-valued function $\varphi_p \in \mathcal{L}_{r,(M)}$ such that

$$\begin{cases} \varphi_p^{(pr)}(0) = 1, \\ \varphi_p^{(jr)}(0) = 0, \quad \forall j \in \mathbb{N}_0,\ j \ne p, \\ |\varphi_p|_1 \le C\|(\varphi_p^{(nr)}(0))_{n\in\mathbb{N}_0}\|_m = Cm^p/M_p. \end{cases}$$

For every $p \in \mathbb{N}$, we have $\varphi_p^{(pr)}(0) = 1$ and $\varphi_p^{(pr)}(1) = 0$, hence

$$\alpha_p := \inf\{x \in [0,1] : \varphi_p^{(pr)}(x) < 1/2\}$$

satisfies $0 < \alpha_p < 1$. In particular, $\varphi_{2p}^{(2pr)}(x) \ge 1/2$ for every $x \in [0, \alpha_{2p}]$. So integrating pr times leads to

$$\varphi_{2p}^{(pr)}(x) \ge \frac{x^{pr}}{2(pr)!}, \qquad \forall x \in [0, \alpha_{2p}],$$

hence

$$\begin{aligned} \alpha_{2p}^{pr} &\le 2(pr)!\varphi_{2p}^{(pr)}(\alpha_{2p}) \le 2(pr)^{pr}|\varphi_{2p}|_1 M_p \\ &\le 2(pr)^{pr} C m^{2p} \frac{M_p}{M_{2p}} \le 2C(pr)^{pr} \frac{m^{2p}}{m_p^p}. \end{aligned}$$

Let us next check that if

$$\exists p_0 \in \mathbb{N} \text{ and } h > 0: \quad \alpha_{2p} \geq \sum_{k=4p}^{\infty} \frac{h}{\sqrt[r]{m_k}}, \quad \forall p \geq p_0, \tag{1}$$

then we can conclude at once. Indeed, in such a case, for every integer $p \geq p_0$, we get

$$\sum_{k=p}^{\infty} \frac{1}{\sqrt[r]{m_k}} = \sum_{k=p}^{4p-1} \frac{1}{\sqrt[r]{m_k}} + \sum_{k=4p}^{\infty} \frac{1}{\sqrt[r]{m_k}} \leq \frac{3p}{\sqrt[r]{m_p}} + \frac{\alpha_{2p}}{h}$$
$$\leq \left(3 + \frac{r}{h} \sqrt[pr]{2C}\sqrt[r]{m^2}\right) \frac{p}{\sqrt[r]{m_p}}.$$

Finally we prove that the condition (1) holds. In fact it is enough to establish that if we take $h \in]0, 1/(4m^2)[$, then the set

$$P = \left\{p \in \mathbb{N} : \alpha_{2p} < \sum_{k=4p}^{\infty} \frac{h}{\sqrt[r]{m_k}}\right\}$$

is finite. For this purpose, for every $p \in P$, we set

$$\frac{1}{p_k} = \begin{cases} \dfrac{h}{\sqrt[r]{m_{4p}}} & \text{if } k \in \mathbb{N} \text{ satisfies } k \leq pr, \\ \dfrac{h}{\sqrt[r]{m_{4p+l+1}}} & \text{if } k \in \mathbb{N} \text{ and } l \in \mathbb{N}_0 \text{ satisfy } (p+l)r < k \leq (p+l+1)r, \end{cases}$$

and clearly we get $\alpha_{2p} < \sum_{k=1}^{\infty} 1/p_k$. So, for every $p \in P$, an application of Lemma 1.3.6 of [6] to the function $\varrho_p \in C^{\infty}(\mathbb{R})$ defined by

$$\varrho_p(x) = \begin{cases} 0 & \text{if } x < 0, \\ \varphi_{2p}^{(2pr)}(x) - 1 & \text{if } x \geq 0, \end{cases}$$

leads to

$$\frac{1}{2} = |\varrho_p(\alpha_{2p})| \leq \sum_{k=0}^{\infty} \frac{4^{(p+k)r} h^{(p+k)r}}{m_{4p}^p m_{4p+1} \cdots m_{4p+k}} \sup\{|\varphi_{2p}^{((3p+k)r)}(x)| : x \geq 0\}$$
$$\leq \sum_{k=0}^{\infty} \frac{(4h)^{(p+k)r}}{m_{4p}^p m_{4p+1} \cdots m_{4p+k}} |\varphi_{2p}|_1 M_{3p+k}$$
$$\leq \sum_{k=0}^{\infty} (4h)^{(p+k)r} C m^{2p} \leq C \frac{(4m^2 h)^{pr}}{1-(4h)^r}.$$

As we chose $h \in]0, 1/(4m^2)[$, this last inequality is only valid for a finite number of integers $p \in \mathbb{N}$. ∎

In the next proof we use the following inequality of Gorny and Cartan (cf. [3], [4] or statement 6.4.IV of [7]): *if* $f \in C^r([-1,1])$ *and*

$$Q_0 := \sup_{x \in [-1,1]} |f(x)| \quad \text{and} \quad Q_r := \sup_{x \in [-1,1]} |f^{(r)}(x)|,$$

then

$$\sup_{x \in [-1,1]} |f^{(j)}(x)| \leq (8er/j)^j \max\{Q_0^{1-j/r} Q_r^{j/r}, (r/2)^j Q_0\}$$

for every $j \in \{1, \ldots, r-1\}$.

Proposition 4.2. *For every* $r \in \mathbb{N}$, *if the restriction map*

$$S : \mathcal{E}_{r,(M)} \to \Lambda_{\{M\}}, \quad f \mapsto (f^{(nr)}(0))_{n \in \mathbb{N}_0},$$

is surjective and if $\mathcal{E}_{r,(M)}$ *contains a nonzero element* f *such that* $f^{(n)}(0) = 0$ *for every* $n \in \mathbb{N}_0$, *then the restriction map* R *of Proposition* 4.1 *is also surjective and hence the sequence* $\boldsymbol{m}$ *satisfies the condition* (γ_r).

Proof. We use the r-interpolating sequence $\boldsymbol{P}$ of Lemma 2.3.

We first prove that $\mathcal{E}_{r,(M)}$ is a vector subspace of $\mathcal{E}_{1,(P)}$. Let $g \in \mathcal{E}_{r,(M)}$. For every $m \in \mathbb{N}$, we of course have

$$|g^{(nr)}(x)| \leq |g|_m m^{-n} M_n, \quad \forall x \in [0,1], \ \forall n \in \mathbb{N}_0.$$

Now we apply the Gorny–Cartan inequality to $h(t) := g^{(nr)}((t+1)/2) \in C^{\infty}([-1,1])$. For every $t \in [-1,1]$, as

$$|h(t)| = |g^{(nr)}((t+1)/2)| \leq |g|_{m^r} m^{-nr} M_n =: Q_0 \tag{2}$$

and

$$|h^{(r)}(t)| = 2^{-r}|g^{((n+1)r)}((t+1)/2))| \leq 2^{-r}|g|_{m^r} m^{-(n+1)r} M_{n+1} =: Q_r,$$

for every $j \in \{1, \ldots, r-1\}$ we get

$$2^{-j}|g^{(nr+j)}((t+1)/2)| = |(g^{(nr)}((t+1)/2))^{(j)}| = |h^{(j)}(t)|$$
$$\leq \left(\frac{8er}{j}\right)^j \max\left\{\left(\frac{|g|_{m^r} M_n}{m^{nr}}\right)^{1-j/r} \left(\frac{|g|_{m^r} M_{n+1}}{2^r m^{(n+1)r}}\right)^{j/r}, \left(\frac{r}{2}\right)^j \frac{|g|_{m^r} M_n}{m^{nr}}\right\}$$
$$\leq \left(\frac{4er^2}{j}\right)^j \frac{|g|_{m^r} P_{nr+j}}{m^{nr}} \leq A \frac{|g|_{m^r} P_{nr+j}}{m^{nr}}$$

for $A := \max\{(4er^2/j)^j : j = 1, \ldots, r-1\}$, a constant indeed. Consequently, for every $x \in [0,1]$, we get

$$|g^{(nr+j)}(x)| \leq 2^r m^r A |g|_{m^r} m^{-(nr+j)} P_{nr+j}. \tag{3}$$

It is then straightforward to check that the inequalities (2) and (3) lead to

$$|g|_{\mathcal{E}_{1,(P)},m} \leq 2^r m^r A |g|_{m^r},$$

hence the conclusion.

In particular as $f \in \mathcal{E}_{r,(M)}$ satisfies $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}_0$ and is not identically 0, the Denjoy–Carleman–Mandelbrojt theorem provides $\sum_{n=1}^{\infty} P_{n-1}/P_n < \infty$, yielding an element φ of $\mathcal{D}_{1,(P)}$ taking the value 1 on a neighbourhood of 0.

To conclude we just have to check that

$$T : \mathcal{E}_{r,(M)} \to \mathcal{L}_{r,(M)}, \quad h \mapsto \varphi h,$$

is well defined, continuous, linear and such that $(\varphi h)^{(n)}(0) = h^{(n)}(0)$ for every $n \in \mathbb{N}_0$, which clearly implies that the map R of Proposition 4.1 is surjective. ■

PROPOSITION 4.3. *For every $r \in \mathbb{N}$, if the restriction map*

$$T : \mathcal{N}_{r,(M)} \to \Lambda_{\{M\}}, \quad f \mapsto (f^{(nr)}(0))_{n\in\mathbb{N}_0},$$

is surjective, then the sequence $\boldsymbol{m}$ satisfies the condition (γ_r).

Proof. As the map S of Proposition 4.2 is obviously surjective, we just have to prove the existence of some nonzero element f of $\mathcal{E}_{r,(M)}$ such that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}_0$.

As the sequence $\boldsymbol{e} = (\delta_{1,n})_{n\in\mathbb{N}_0}$ clearly belongs to $\Lambda_{(M)}$, there is $\varphi \in \mathcal{N}_{r,(M)}$ such that $\varphi^{(nr)}(0) = \delta_{1,n}$ for every $n \in \mathbb{N}_0$. Since φ is bounded on $[0,\infty[$, $\psi(x) := \varphi(x) - x^r/r!$ defines a nonzero C^∞-function on $[0,\infty[$. If ψ is not identically 0 on $[0,1]$, we denote by k its restriction to $[0,1]$. If ψ is identically 0 on $[0,1]$, we set $x_0 := \sup\{x \geq 0 : \psi(t) = 0, \ \forall t \in [0,x]\}$ and denote by k the restriction of $\psi(x + x_0)$ to $[0,1]$. In both cases, k is an element of $\mathcal{E}_{r,(M)}$ such that $k^{(n)}(0) = 0$ for every $n \in \mathbb{N}_0$. ■

DEFINITION. For $r \in \mathbb{N}$, an *extension map* T from $\Lambda_{(M)}$ into $\mathcal{D}_{r,(M)}$ [resp. $\mathcal{L}_{r,(M)}$; $\mathcal{N}_{r,(M)}$] is a map such that $(T\boldsymbol{a})^{(nr)}(0) = a_n$ for every $\boldsymbol{a} \in \Lambda_{(M)}$ and $n \in \mathbb{N}_0$.

THEOREM 4.4. *For each $r \in \mathbb{N}$, the following assertions are equivalent*:

(1) *the sequence $\boldsymbol{m}$ satisfies the condition (γ_r)*,
(2) *there is an extension map from $\Lambda_{(M)}$ into $\mathcal{D}_{r,(M)}$*,
(3) *there is an extension map from $\Lambda_{(M)}$ into $\mathcal{L}_{r,(M)}$*,
(4) *there is an extension map from $\Lambda_{(M)}$ into $\mathcal{N}_{r,(M)}$*.

Proof. (1)⇒(2). Let $\boldsymbol{P}$ be the r-interpolating sequence of Lemma 2.3. As the corresponding sequence $\boldsymbol{p}$ satisfies the conditions (α) and (γ_1), Theorem 2.1(a) of [9] gives the existence of an extension map S from $\Lambda_{(P)}$ into $\mathcal{E}_{(P)}([-1,1])$. Moreover the Denjoy–Carleman–Mandelbrojt theorem provides a function $\varphi \in \mathcal{D}_{1,(P)}$ which is identically 1 on a neighbourhood of 0. It is then straightforward to check that

$$U : \Lambda_{(P)} \to \mathcal{D}_{1,(P)}, \quad \boldsymbol{a} \mapsto \varphi S\boldsymbol{a} \text{ on } [-1,1],$$

is well defined, continuous, linear and such that $(U\boldsymbol{a})^{(n)}(0) = a_n$ for every $\boldsymbol{a} \in \Lambda_{(P)}$ and $n \in \mathbb{N}_0$. Moreover

$$V : \Lambda_{(M)} \to \Lambda_{(P)}, \quad \boldsymbol{a} \mapsto \boldsymbol{b},$$

with $b_{nr} = a_n$ and $b_{nr+j} = 0$ for every $n \in \mathbb{N}_0$ and $j \in \{1,\ldots,r-1\}$ is of course well defined, continuous and linear.

Now on the one hand the image of the map UV is contained in the topological vector subspace E of $\mathcal{D}_{1,(P)}$, whose elements satisfy $f^{(nr+j)}(0) = 0$ for every $n \in \mathbb{N}_0$ and $j \in \{1,\ldots,r-1\}$. On the other hand, it is clear that E is a vector subspace of $\mathcal{D}_{r,(M)}$ such that the canonical injection $W : E \to \mathcal{D}_{r,(M)}$ is continuous. This implies that WUV is an extension map from $\Lambda_{(M)}$ into $\mathcal{D}_{r,(M)}$.

(2)⇒(3) and (3)⇒(4) are trivial.

(4)⇒(1) is known by Proposition 4.3. ■

4.2. *Case of the spaces $\mathcal{K}_{\alpha,(M)}$*

THEOREM 4.5. *For every $r \in \mathbb{N}$, if there is an extension map S from $\Lambda_{(M)}$ into $\mathcal{D}_{r+1,(M)}$, then for every $\alpha \in]0,r[$ there is also an extension map from $\Lambda_{(M)}$ into $\mathcal{K}_{\alpha,(M)}$.*

Proof. We first establish the existence of an extension map U from $\Lambda_{(M^*)}$ into $\mathcal{D}_{r,(M^*)}$ where $\boldsymbol{M}^*$ is the sequence of Lemma 2.2. By Theorem 4.4, the sequence $(\sqrt[r+1]{m_n})_{n\in\mathbb{N}_0}$ satisfies (γ_1). As it also satisfies (α), Proposition 1.1(a) of [9] tells that up to substituting it by an equivalent sequence, we may suppose that it also satisfies (α_1). Hence we may assume that $(m_n/n)_{n\in\mathbb{N}_0}$ increases to ∞. By Lemma 2.2, $(\sqrt[r]{m_n^*})_{n\in\mathbb{N}_0}$ then satisfies (γ_1) and we get the existence of the desired map U by Theorem 4.4.

The link with the space $\Lambda_{(M)}$ is provided by the isomorphism

$$V : \Lambda_{(M)} \to \Lambda_{(M^*)}, \quad \boldsymbol{a} \mapsto (a_n/n!)_{n\in\mathbb{N}_0}.$$

Now for elements of $\mathcal{D}_{r,(M^*)}$ we use a method appearing in [2], [11] and [12]. With any $f \in \mathcal{D}_{r,(M^*)}$, we associate the function φ defined on $\{z \in \mathbb{C} : \Re z > 0\}$ by

$$\varphi(z) := \frac{1}{z}\int_0^1 e^{-t/z} f(t)\,dt.$$

For any $z \in \mathbb{C}$ such that $\Re z > 0$, integrating by parts leads directly to

$$\varphi(z) - \sum_{j=0}^{n-1} f^{(j)}(0) z^j = z^{n-1} \int_0^1 e^{-t/z} f^{(n)}(t)\,dt, \tag{4}$$

hence

$$\left|\frac{\varphi(z)-\sum_{j=0}^{n-1} f^{(j)}(0)z^j}{z^n} - f^{(n)}(0)\right| = \left|\int_0^1 e^{-t/z} f^{(n+1)}(t)\,dt\right|$$

for every $n \in \mathbb{N}$. So if we choose $\beta \in \,]\alpha, r[$ and set

$$L := \left\{z \in \mathbb{C} : z \neq 0,\ -\frac{\beta}{r}\cdot\frac{\pi}{2} \le \arg(z) \le \frac{\beta}{r}\cdot\frac{\pi}{2}\right\},$$

a direct application of the dominated convergence theorem gives

$$\lim_{\substack{z\to 0\\ z\in L}} \frac{\varphi(z)-\sum_{j=0}^{n-1} f^{(j)}(0)z^j}{z^n} = f^{(n)}(0), \quad \forall n \in \mathbb{N}. \tag{5}$$

Now we fix an integer $s > 1$ such that for every $z \in S_\alpha \setminus \{0\}$, the circle γ of centre z and radius $|z|/s$ is contained in $S_\beta \setminus \{0\}$. Then (4) and an easy calculation provide the existence of a constant A [for instance $A := 1/\cos(\frac{\beta}{r}\cdot\frac{\pi}{2})$] such that, for every $m \in \mathbb{N}$,

$$\begin{aligned}\sup_{z\in L}\left|\frac{\varphi(z)-\sum_{j=0}^{nr-1} f^{(j)}(0)z^j}{z^{nr}}\right| &= \sup_{z\in L}\left|\frac{1}{z}\int_0^1 e^{-t/z} f^{(nr)}(t)\,dt\right| \\ &\le |f|_{2sm}\frac{M_n^*}{(2sm)^n}\sup_{z\in L}\frac{1}{|z|}\int_0^1 e^{-t\Re(1/z)}\,dt \\ &\le A|f|_{2sm}\frac{M_n^*}{(2sm)^n}.\end{aligned}$$

Now let us define the function g on

$$\{z \in \Sigma : z \neq 0,\ -r\pi/2 < \arg(z) < r\pi/2\}.$$

by $g(z) = \varphi(\sqrt[r]{z})$. For every $z \in S_\alpha \setminus \{0\}$ and $n \in \mathbb{N}$, the Cauchy representation formula gives

$$\begin{aligned} g^{(n)}(z) &= \Big(g(z) - \sum_{j=0}^{n-1} f^{(rj)}(0)z^j\Big)^{(n)} \\ &= \frac{n!}{2\pi i}\int_\gamma \frac{\varphi(\sqrt[r]{u}) - \sum_{j=0}^{nr-1} f^{(j)}(0)(\sqrt[r]{u})^j}{(u-z)^{n+1}}\,du.\end{aligned}$$

Therefore on the one hand, for every $n \in \mathbb{N}_0$, we get

$$\begin{aligned} g^{(n)}(z) &= \frac{n!}{2\pi i}\int_\gamma\left(\frac{\varphi(\sqrt[r]{u}) - \sum_{j=0}^{nr-1} f^{(j)}(0)(\sqrt[r]{u})^j}{(\sqrt[r]{u})^{nr}} - f^{(nr)}(0)\right)\frac{u^n}{(u-z)^{n+1}}\,du \\ &\quad + \frac{n!}{2\pi i}\int_\gamma f^{(nr)}(0)\frac{u^n}{(u-z)^{n+1}}\,du\end{aligned}$$

and this leads directly to

$$\lim_{\substack{z\to 0\\ z\in S_\alpha\setminus\{0\}}} g^{(n)}(z) = n!f^{(nr)}(0).$$

On the other hand, for every $n \in \mathbb{N}_0$, we obtain

$$\begin{aligned}|g^{(n)}(z)| &\le \frac{n!s^n}{|z|^n}\sup_{u\in\gamma}\left|\frac{\varphi(\sqrt[r]{u}) - \sum_{j=0}^{nr-1} f^{(j)}(0)(\sqrt[r]{u})^j}{(\sqrt[r]{u})^{nr}}\right||u|^n \\ &\le A|f|_{2sm}M_n m^{-n},\end{aligned}$$

hence $|g|_m \le A|f|_{2sm}$.

So if we set $g^{(n)}(0) := n!f^{(nr)}(0)$ for every $n \in \mathbb{N}_0$, then

$$W : \mathcal{D}_{r,(M^*)} \to \mathcal{K}_{\alpha,(M)}, \quad f \mapsto g|_{S_\alpha},$$

is well defined, linear (by construction) and continuous.

To conclude, we just have to check that WUV is an extension map, which is straightforward. ■

Definition. Let us say that the sequence $\boldsymbol{m}$ *satisfies the condition* (δ) if the space $\mathcal{E}_{(M)}(\mathbb{R})$ is stable by derivation, i.e. if there are constants a, $A > 0$ such that $M_{n+1} \le Aa^n M_n$ for every $n \in \mathbb{N}_0$.

Theorem 4.6. *For every $r \in \mathbb{N}$, if the sequence $\boldsymbol{m}$ satisfies the condition (δ) and if there is an extension map S from $\Lambda_{(M)}$ into $\mathcal{K}_{r,(M)}$, then there is also an extension map from $\Lambda_{(M)}$ into $\mathcal{D}_{r+1,(M)}$.*

Proof. We first prove that we may suppose that the sequence $\boldsymbol{m}$ satisfies the condition (α_1). Indeed, as S is an extension map from $\Lambda_{(M)}$ into $\mathcal{K}_{r,(M)}$, it is clear that the restriction map

$$R : \mathcal{N}_{1,(M)} \to \Lambda_{(M)}, \quad f \mapsto (f^{(n)}(0))_{n\in\mathbb{N}_0},$$

is surjective. By Proposition 4.3, this implies that $\boldsymbol{m}$ satisfies (γ_1); as it also satisfies (α), we may, up to a substitution by an equivalent sequence, suppose that it also satisfies (α_1).

By Lemma 2.2, $\boldsymbol{M}^*$ is normalized and logarithmically convex. It is then a trivial matter to check that the map

$$V : \Lambda_{(M^*)} \to \Lambda_{(M)}, \quad \boldsymbol{a} \mapsto (n!a_n)_{n\in\mathbb{N}_0},$$

is an isomorphism.

The next step is to construct a map U from $\mathcal{K}_{r,(M)}$ into $\mathcal{E}_{r,(M^*)}$ such that USV is an extension map from $\Lambda_{(M^*)}$ into $\mathcal{E}_{r,(M^*)}$. Let ψ be any element of $\mathcal{K}_{r,(M)}$. We then have $|\psi(z)| \le |\psi|_m M_0$ and, by the Taylor formula,

$$\left|\psi(z) - \sum_{k=0}^{n-1}\frac{\psi^{(k)}(0)}{k!}z^k\right| \le \frac{|z|^n}{n!}\sup_{t\in[0,1]}|\psi^{(n)}(tz)| \le |\psi|_m\frac{M_n}{n!}\cdot\frac{|z|^n}{m^n} \tag{6}$$

for every $z \in \operatorname{int}(S_r)$ and $m, n \in \mathbb{N}$. Consider the function φ defined on $\{z \in \mathbb{C} : \Re z > 0\}$ by

$$\varphi(z) := \psi(z^{-r}) - \psi(0) - \psi^{(1)}(0) z^{-r}.$$

Using (6) for $m = 1$ and $n = 2$, we get the existence of a constant $C_2 > 0$ such that $|\varphi(z)| \le C_2 |z|^{-2r}$. Therefore

$$f(t) := \frac{1}{2\pi i} \int_{1-\infty i}^{1+\infty i} e^{tz} \varphi(z)\, dz$$

is a well defined function on $\mathbb{R}$ belonging to $\mathrm{C}^{2r-2}(\mathbb{R})$. In fact a lot more can be said. For every integer $n \ge 3$, by a classical formula, we have

$$f(t) - \sum_{j=2}^{n-1} \frac{\psi^{(j)}(0)}{j!} \cdot \frac{t^{rj-1}}{(rj-1)!} = \frac{1}{2\pi i} \int_{1-\infty i}^{1+\infty i} e^{tz} \Big(\varphi(z) - \sum_{j=2}^{n-1} \frac{\psi^{(j)}(0)}{j!} \cdot \frac{1}{z^{rj}} \Big)\, dz.$$

Moreover by (6) again, there is a constant $C_n > 0$ such that the absolute value of the integrand is $\le C_n e^t |z|^{-nr}$. Therefore

$$f(t) - \sum_{j=2}^{n-1} \frac{\psi^{(j)}(0)}{j!} \cdot \frac{t^{rj-1}}{(rj-1)!}$$

is a $\mathrm{C}^{nr-2}(\mathbb{R})$-function whose derivatives vanish at 0 (to check this last fact, just evaluate the integral $\int_{1-mi}^{1+mi} = \int_{\mathcal{C}_m}$ at $t = 0$, where $\mathcal{C}_m$ is the arc of circumference of centre 1, going from $1 - mi$ to $1 + mi$ and passing through $1 + m$). Altogether this means that f belongs to $\mathrm{C}^\infty(\mathbb{R})$ and satisfies

$$\begin{cases} f^{(0)}(0) = \ldots = f^{(2r-2)}(0) = 0, \\ f^{(nr-1)}(0) = \psi^{(n)}(0)/n!, \quad \forall n \ge 2, \\ f^{(nr-1+j)}(0) = 0, \quad \forall n \ge 2,\ \forall j \in \{1, \ldots, r-1\}. \end{cases}$$

Therefore the function g defined on $[0,1]$ by

$$g(t) := \int_0^t f(u)\, du + \psi(0) + \psi^{(1)}(0) \frac{t^r}{r!}$$

belongs to $\mathrm{C}^\infty([0,1])$ and satisfies

$$\begin{cases} g^{(nr)}(0) = \psi^{(n)}(0)/n!, \quad \forall n \in \mathbb{N}_0, \\ g^{(nr+j)}(0) = 0, \quad \forall n \in \mathbb{N}_0,\ \forall j \in \{1, \ldots, r-1\}. \end{cases}$$

Let us now prove that $g \in \mathcal{E}_{r,(M^*)}$. Let $n \in \mathbb{N}_0$. Then for every $m \in \mathbb{N}$ and $t \in [0,1]$, we have:

(a) for $n = 0$,

$$\begin{aligned} |g^{(0)}(t)| &\le |\psi(0)| + |\psi^{(1)}(0)| + \frac{1}{2\pi} \int_0^1 \Big| \int_{1-\infty i}^{1+\infty i} e^{uz} \varphi(z)\, dz \Big|\, du \\ &\le |\psi|_m \Big(M_0 + \frac{M_1}{m} + \frac{eM_2}{4\pi m^2} \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^r} \Big), \end{aligned}$$

(b) for $n = 1$,

$$\begin{aligned} |g^{(r)}(t)| &\le |\psi^{(1)}(0)| + \frac{1}{2\pi} \Big| \int_{1-\infty i}^{1+\infty i} z^{r-1} e^{tz} \varphi(z)\, dz \Big| \\ &\le |\psi|_m \Big(\frac{M_1}{m} + \frac{eM_2}{4\pi m^2} \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^{(r+1)/2}} \Big), \end{aligned}$$

(c) for $n \ge 2$,

$$g^{(nr)}(t) - \frac{\psi^{(n)}(0)}{n!} = \Big(f(t) - \sum_{j=2}^{n} \frac{\psi^{(j)}(0)}{j!} \cdot \frac{t^{rj-1}}{(rj-1)!} \Big)^{(nr-1)},$$

hence

$$g^{(nr)}(t) = \frac{1}{2\pi i} \int_{1-\infty i}^{1+\infty i} z^{nr-1} e^{tz} \Big(\varphi(z) - \sum_{j=2}^{n} \frac{\psi^{(j)}(0)}{j!} \cdot \frac{1}{z^{rj}} \Big)\, dz + \frac{\psi^{(n)}(0)}{n!}$$

and therefore

$$|g^{(nr)}(t)| \le |\psi|_m \frac{M_n}{n! m^n} \Big(1 + \frac{e}{2\pi} \frac{Aa^n}{(n+1)m} \int_{-\infty}^{\infty} \frac{dy}{(1+y^2)^{(r+1)/2}} \Big)$$

for some constants A, $a > 0$, as the sequence $\boldsymbol{m}$ satisfies the condition (δ).

Since the construction of g depends linearly on ψ, these inequalities lead to the fact that

$$U : \mathcal{K}_{r,(M)} \to \mathcal{E}_{r,(M^*)}, \quad \psi \mapsto g,$$

is a map such that $USV : \Lambda_{(M^*)} \to \mathcal{E}_{r,(M^*)}$ is an extension map.

In particular, the unit vector $\boldsymbol{e} = (\delta_{1,n})_{n \in \mathbb{N}_0}$ certainly belongs to $\Lambda_{(M)}$. So if we successively set

$$\eta = Se \quad (\eta \text{ corresponds to } \psi),$$

$$h(t) = \frac{1}{2\pi i} \int_{1-\infty i}^{1+\infty i} e^{tz} (\eta(z^{-r}) - z^{-r})\, dz \quad (h \text{ corresponds to } f),$$

then the function

$$\lambda(t) := \int_0^t h(u)\, du \quad (\text{corresponding to } g(t) - \psi(0) - \psi^{(1)}(0) t^r / r!)$$

belongs to $\mathcal{E}_{r,(M^*)}$, is not identically 0 and satisfies $\lambda^{(nr)}(0) = 0$ for every $n \in \mathbb{N}$.

Therefore by Proposition 4.2, the sequence $\boldsymbol{m}^*$ satisfies the condition (γ_r). This proves the conclusion by using Lemma 2.1 and Theorem 4.4. ∎

In particular we have the following result.

THEOREM 4.7. *If the sequence $\boldsymbol{m}$ satisfies the condition (δ), then the following conditions are equivalent*:

(1) *for every $\alpha > 0$, there is an extension map from $\Lambda_{(M)}$ into $\mathcal{K}_{\alpha,(M)}$*,
(2) *for every $r \in \mathbb{N}$, there is an extension map from $\Lambda_{(M)}$ into $\mathcal{D}_{r+1,(M)}$*,
(3) *for every $r \in \mathbb{N}$, the sequence $\boldsymbol{m}$ satisfies the condition (γ_{r+1})*. ∎

5. The Roumieu case

5.1. *Case of the spaces $\mathcal{D}$, $\mathcal{L}$, $\mathcal{N}$*

PROPOSITION 5.1. *For every $r \in \mathbb{N}$, if the restriction map*

$$R : \mathcal{L}_{r,\{M\}} \to \Lambda_{\{M\}}, \qquad f \mapsto (f^{(nr)}(0))_{n\in\mathbb{N}_0},$$

is surjective, then the sequence $\boldsymbol{m}$ satisfies the condition (γ_r).

Proof. By Theorem A of Grothendieck ([5], p. 16), there are $m \in \mathbb{N}$ and $C > 0$ such that $R\mathcal{L}^m_{r,\{M\}} \supset \Lambda^1_{\{M\}}$ and, for every $\boldsymbol{a} \in \Lambda^1_{\{M\}}$, there is $g \in \mathcal{L}^m_{r,\{M\}}$ such that

$$Rg = \boldsymbol{a} \quad\text{and}\quad |g|_m \le C|\boldsymbol{a}|_1 = C \sup_{n\in\mathbb{N}_0} \frac{|a_n|}{M_n}.$$

In particular, for every $p \in \mathbb{N}$, there is a real-valued function $\varphi_p \in \mathcal{L}^m_{r,\{M\}}$ such that

$$\begin{cases} \varphi_p^{(pr)}(0) = 1, \\ \varphi_p^{(jr)}(0) = 0, \quad \forall j \in \mathbb{N}_0,\ j \neq p, \\ |\varphi_p|_m \le C/M_p. \end{cases}$$

Now we proceed as in the proof of Proposition 4.1 with $|\varphi_p|_m$ instead of $|\varphi_p|_1$. This leads to

$$\alpha_{2p} \le r\sqrt[pr]{2C}\,\sqrt[r]{m}\,\frac{p}{\sqrt[r]{m_p}},$$

hence

$$\sum_{k=p}^{\infty} \frac{1}{\sqrt[r]{m_k}} \le \left(3 + \frac{r}{h}\sqrt[pr]{2C}\,\sqrt[r]{m}\right)\frac{p}{\sqrt[r]{m_p}}$$

if the condition (1) (of that proof) holds. To prove that (1) holds, we consider the set P for $h \in\,]0, 1/(4m^{3/r})[$ and obtain

$$\frac{1}{2} \le C\frac{((4h)^r m^3)^p}{1-(4h)^r m}. \ ∎$$

PROPOSITION 5.2. *For every $r \in \mathbb{N}$, if the restriction map*

$$S : \mathcal{N}_{r,\{M\}} \to \Lambda_{\{M\}}, \qquad f \mapsto (f^{(nr)}(0))_{n\in\mathbb{N}_0},$$

is surjective, then the restriction map R of Proposition 5.1 is also surjective and hence the sequence $\boldsymbol{m}$ satisfies the condition (γ_r).

Proof. We use the r-interpolating sequence $\boldsymbol{P}$ of Lemma 2.3.

We first prove that for every $g \in \mathcal{N}_{r,\{M\}}$, the restriction $g|_{[0,1]}$ belongs to the space $\mathcal{E}_{\{P\}}([0,1])$ introduced and studied in [9]. Let g be such a function; there are then $m \in \mathbb{N}$ and $A > 0$ such that

$$|g^{(nr)}(x)| \le Am^{nr}M_n, \quad \forall n \in \mathbb{N}_0,\ \forall x \in [0,\infty[.$$

We now proceed as in the proof of Proposition 4.2: from $Q_0 = Am^{nr}M_n$ and $Q_r = 2^{-r}Am^{(n+1)r}M_{n+1}$, we deduce

$$2^{-j}|g^{(nr+j)}((t+1)/2)| \le (4er^2/j)^j Am^{nr+j}P_{nr+j}$$

for every $t \in [-1,1]$ and $j \in \{1,\ldots,r-1\}$, which suffices.

Next we prove the existence of an element of $\mathcal{D}_{1,\{P\}}$ which is identically 1 on a neighbourhood of 0. Of course the sequence $\boldsymbol{e} = (e_n)_{n\in\mathbb{N}_0}$ defined by $e_1 = 1$ and $e_n = 0$ otherwise belongs to $\Lambda_{\{M\}}$. Therefore there is $\varphi \in \mathcal{N}_{r,\{M\}}$ such that $\varphi^{(r)}(0) = 1$ and $\varphi^{(j)}(0) = 0$ for every $j \in \mathbb{N}_0$ distinct from r. As φ is bounded on $[0,\infty[$, the function ψ defined on $[0,\infty[$ by $\psi(x) := \varphi(x) - x^r/r!$ is not identically 0 but $\psi^{(j)}(0) = 0$ for every $j \in \mathbb{N}_0$. Two cases are possible: either

(1) the restriction k of ψ to $[0,1]$ is not identically 0, or
(2) the restriction of ψ to $[0,1]$ is identically 0. We then set

$$x_0 := \sup\{x \in [0,\infty[\ : \psi(t) = 0,\ \forall t \in [0,x]\}$$

and use $\psi(x+x_0)$ instead of $\psi(x)$ to define k.

In both cases, k belongs to $\mathcal{E}_{\{P\}}([0,1])$, is not identically 0 and satisfies $k^{(j)}(0) = 0$ for every $j \in \mathbb{N}_0$. Therefore the Denjoy–Carleman–Mandelbrojt theorem shows that $\sum_{n=1}^{\infty} P_{n-1}/P_n < \infty$ and there is an element φ of $\mathcal{D}_{1,\{P\}}$ which is identically 1 on a neighbourhood of 0.

Now we can complete the proof. For every $\boldsymbol{a} \in \Lambda_{\{M\}}$, the hypothesis provides an element g of $\mathcal{N}_{r,\{M\}}$ such that $g^{(nr)}(0) = a_n$ for every $n \in \mathbb{N}_0$. As the function f defined on $[0,\infty[$ by $f(x) := \varphi(x)g(x)$ if $x \in [0,1]$ and $f(x) := 0$ otherwise clearly belongs to $C^\infty([0,\infty[)$ and satisfies $f^{(nr)}(0) = a_n$ for every $n \in \mathbb{N}_0$, we just have to check that it belongs to $\mathcal{L}_{r,\{M\}}$. This is straightforward since $\varphi \in \mathcal{D}_{1,\{P\}}$ and $g|_{[0,1]} \in \mathcal{E}_{\{P\}}([0,1])$. ∎

DEFINITION. For every $r \in \mathbb{N}$, an *extension map* T from $\Lambda_{\{M\}}$ into $\mathcal{D}_{r,\{M\}}$ [resp. $\mathcal{L}_{r,\{M\}}$; $\mathcal{N}_{r,\{M\}}$] is a map such that $(T\boldsymbol{a})^{(nr)}(0) = a_n$ for every $\boldsymbol{a} \in \Lambda_{\{M\}}$ and $n \in \mathbb{N}_0$.

PROPOSITION 5.3. *For every $r \in \mathbb{N}$, if there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{N}_{r,\{M\}}$, then the sequence $\boldsymbol{m}$ satisfies the condition (β_2).*

Proof. Let T be an extension map from $\Lambda_{\{M\}}$ into $\mathcal{N}_{r,\{M\}}$. By the localization theorem, there is an integer $m \in \mathbb{N}$ such that T is also an extension map from $\Lambda^1_{\{M\}}$ into $\mathcal{N}^m_{r,\{M\}}$. So there is $C > 0$ such that

$$|T\boldsymbol{a}|_m \le C|\boldsymbol{a}|_1 = C \sup_{n\in\mathbb{N}_0} \frac{|a_n|}{M_n}, \quad \forall \boldsymbol{a} \in \Lambda^1_{\{M\}}.$$

Therefore for every $p \in \mathbb{N}_0$, there is a real function $\varphi_p \in \mathcal{N}^m_{r,\{M\}}$ such that

$$\begin{cases} \varphi_p^{(pr)}(0) = 1, \\ \varphi_p^{(jr)}(0) = 0, \quad \forall j \in \mathbb{N}_0,\ j \ne p, \\ |\varphi_p|_m \le C/M_p. \end{cases}$$

The Taylor formula gives for every $y > 0$ a number $\theta \in \,]0,1[$ such that

$$\begin{aligned} |\varphi_p^{(pr)}(y) - 1| &\le \frac{y^{pr}}{(pr)!}|\varphi_p^{(2pr)}(\theta y)| \le \frac{y^{pr}}{(pr)!}|\varphi_p|_m m^{2p} M_{2p} \\ &\le C\frac{y^{pr}}{(pr)!} m^{2p} m_{p+1}\ldots m_{2p} \le C\frac{y^{pr}}{(pr)!} m^{2p} m_{2p}^p. \end{aligned}$$

Fix $A \in \,]0, m^{-2/r}[$. Then for every $y \in I := [0, A\sqrt[pr]{(pr)!}/\sqrt[r]{m_{2p}}]$, we have $|\varphi_p^{(pr)}(y) - 1| \le C(A^r m^2)^p$, hence $\varphi_p^{(pr)}(y) \ge 1/2$ for all p larger than some integer p_0. So for every integer $p \ge p_0$ and $j \in \{0,\ldots,p\}$, integrating $(p-j)r$ times provides

$$|\varphi_p^{(jr)}(y)| \ge \frac{1}{2} \cdot \frac{y^{(p-j)r}}{((p-j)r)!}, \quad \forall y \in I,$$

and hence by inserting the end point of I,

$$\frac{1}{2}\frac{A^{(p-j)r}}{((p-j)r)!} \cdot \frac{((pr)!)^{(p-j)/p}}{(m_{2p})^{p-j}} \le \sup_{x\in[0,\infty[} |\varphi_p^{(jr)}(x)|.$$

As clearly $((pr)!)^{(p-j)r} \ge (((p-j)r)!)^{pr}$, this leads finally to

$$(7) \qquad \frac{1}{2}\left(\frac{A}{\sqrt[r]{m_{2p}}}\right)^{(p-j)r} \le \sup_{x\in[0,\infty[} |\varphi_p^{(jr)}(x)|, \quad \forall p \ge p_0,\ \forall j \in \{0,\ldots,p\}.$$

Given $\varepsilon > 0$, fix $s \in \mathbb{N}$ such that $8s^{-1} < \varepsilon A^r$. If we proceed as at the beginning of the proof, we get a positive integer q and a constant $B > 1$ such that $T\Lambda^s_{\{M\}} \subset \mathcal{N}^q_{r,\{M\}}$ and

$$|T\boldsymbol{a}|_q \le B \sup_{n\in\mathbb{N}_0} \frac{|a_n|}{s^n M_n}, \quad \forall \boldsymbol{a} \in \Lambda^s_{\{M\}},$$

hence in particular $|\varphi_p|_q \le B/(s^p M_p)$ for every $p \in \mathbb{N}_0$. On the other hand, as

$$\sup_{x\in[0,\infty[} |\varphi_p^{(jr)}(x)| \le |\varphi_p|_q q^j M_j \le Bq^j \frac{M_j}{s^p M_p},$$

the use of (7) leads to

$$\frac{1}{2}\left(\frac{A}{\sqrt[r]{m_{2p}}}\right)^{(p-j)r} \le Bq^j \frac{M_j}{s^p M_p}$$

and hence

$$(8) \qquad \left(\frac{M_p}{M_j}\right)^{1/(p-j)} \frac{1}{m_{2p}} \le (2B)^{1/(p-j)} A^{-r} q^{j/(p-j)} s^{-p/(p-j)}$$

for every $p \ge p_0$ and $j \in \{0,\ldots,p\}$.

Now fix $\beta \in \,]0,1/2[$ such that $q^{4\beta} < 2$ and $s^{-1/(1-\beta)} < 2s^{-1}$ as well as an integer $p_1 > \sup\{4, p_0\}$ such that $(2B)^{4/p_1} < 2$ and $q^{4\beta+4/p_1} < 2$. For every integer $p \ge p_1$, set $j_p := [\beta p] + 1$ where $[\beta p]$ designates the largest integer $\le \beta p$. Then of course $\beta p < j_p < 1 + p/2 < p$. On the one hand, from $j_p > \beta p$, we get $p - j_p < p(1-\beta)$, hence $p/(p-j_p) > 1/(1-\beta)$; on the other hand, from $j_p \le \beta p + 1$, we get $p - j_p \ge p(1-\beta) - 1 \ge p/4$, hence $j_p/(p-j_p) < 4j_p/p \le 4\beta + 4/p$. The inequality (8) then leads to

$$\left(\frac{M_p}{M_{j_p}}\right)^{1/(p-j_p)} \frac{1}{m_{2p}} \le (2B)^{4/p} A^{-r} q^{4\beta+4/p} s^{-1/(1-\beta)} \le \frac{8}{A^r s} < \varepsilon,$$

i.e. the sequence $\boldsymbol{m}$ satisfies the condition (ii) of Lemma 1.5(b) of [9] with $l = 2$, a condition equivalent to (β_2). ■

THEOREM 5.4. *For $r \in \mathbb{N}$, the following conditions are equivalent:*

(1) *the sequence $\boldsymbol{m}$ satisfies the conditions (β_2) and (γ_r),*
(2) *there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{D}_{r,\{M\}}$,*
(3) *there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{L}_{r,\{M\}}$,*
(4) *there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{N}_{r,\{M\}}$.*

Proof. (1)⇒(2). Let $\boldsymbol{P}$ be the r-interpolating sequence of Lemma 2.3. As $\boldsymbol{m}$ satisfies (β_2) and (γ_r), the corresponding sequence $\boldsymbol{p}$ satisfies (γ_1) and (β_2).

So Theorem 2.1(b) of [9] provides the existence of an extension map S from $\Lambda_{\{P\}}$ into $\mathcal{E}_{\{P\}}([-1,1])$. Now we choose $\varphi \in \mathcal{E}_{\{P\}}([-1,1])$ which is identically 1 on a neighbourhood of 0 and $\varphi^{(n)}(-1) = \varphi^{(n)}(1) = 0$ for every $n \in \mathbb{N}_0$, and introduce the extension map

$$U : \Lambda_{\{P\}} \to \mathcal{D}_{1,\{P\}}, \qquad \boldsymbol{a} \mapsto \begin{cases} \varphi(x)S\boldsymbol{a}(x) & \text{if } x \in [-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

In order to get a link with the sequence $\boldsymbol{M}$, we first remark that

$$V : \Lambda_{\{M\}} \to \Lambda_{\{P\}}, \quad \boldsymbol{a} \mapsto \boldsymbol{b},$$

defined by

$$\begin{cases} b_{nr} = a_n, & \forall n \in \mathbb{N}_0, \\ b_{nr+j} = 0, & \forall n \in \mathbb{N}_0,\ \forall j \in \{1, \ldots, r-1\}, \end{cases}$$

is well defined, linear and continuous. Next we check that the image of the map $UV : \Lambda_{\{M\}} \to \mathcal{D}_{1,\{P\}}$ is contained in the topological vector subspace E of $\mathcal{D}_{1,\{P\}}$ of elements f such that $f^{(nr+j)}(0) = 0$ for every $n \in \mathbb{N}_0$ and $j \in \{1, \ldots, r-1\}$. Finally we remark that a direct application of the Gorny–Cartan inequality proves that

$$W : \mathcal{D}_{r,\{M\}} \to E, \quad f \mapsto f,$$

is well defined, injective, linear and continuous. As W is also surjective, the linear operator

$$T := W^{-1}UV : \Lambda_{\{M\}} \to \mathcal{D}_{r,\{M\}}$$

has a closed graph between (LF)-spaces and hence it is continuous, an extension map indeed.

(2)⇒(3) and (3)⇒(4) are obvious.

(4)⇒(1) is an immediate consequence of Propositions 5.2 and 5.3. ∎

5.2. *Link with the spaces $\mathcal{H}$*

REMARK. Let α belong to $]0,\infty[$. If there is an extension map T from $\Lambda_{\{M\}}$ into $\mathcal{H}_{\alpha,\{M\}}$, then $(T\cdot)|_{[0,\infty[}$ is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{N}_{1,\{M\}}$. By Proposition 5.3, this implies that the sequence $\boldsymbol{m}$ satisfies the condition (β_2). If moreover $(m_n/n^r)_{n\in\mathbb{N}}$ is quasi-increasing, Lemma 2.4 implies that we are in a position to apply Theorem 5.4.

Let us consider a converse of this result.

THEOREM 5.5. *For every $r \in \mathbb{N}$, if there is an extension map S from $\Lambda_{\{M\}}$ into $\mathcal{D}_{r+1,\{M\}}$, then, for every $\alpha \in]0,r[$, there is also an extension map from $\Lambda_{\{M\}}$ into $\mathcal{H}_{\alpha,\{M\}}$.*

Proof. Let $\boldsymbol{M}^*$ be the sequence of Lemma 2.2 and $\boldsymbol{m}^*$ be its corresponding sequence.

We first prove that, up to substituting $\boldsymbol{M}$ by an equivalent sequence, we may suppose that $\boldsymbol{m}$ satisfies (α_1), which is equivalent to saying that $\boldsymbol{m}^*$ satisfies (α). Indeed, by Theorem 5.4, $\boldsymbol{m}$ satisfies (γ_{r+1}), i.e. the sequence $(\sqrt[r+1]{m_n})_{n\in\mathbb{N}_0}$ satisfies (γ_1). As the latter sequence also satisfies (α), Proposition 1.1(a) of [9] tells us that there is an equivalent sequence satisfying (α_1) and (β_1^0), hence (α_1), (γ_1) and (β_1). So up to substituting $(\sqrt[r+1]{m_n})_{n\in\mathbb{N}_0}$ by an equivalent sequence, we may suppose that $(\sqrt[r+1]{m_n}/n)_{n\in\mathbb{N}}$ is increasing to ∞ and finally that $\boldsymbol{m}^*$ is increasing to ∞.

It is then straightforward to check that $\boldsymbol{M}^*$ is normalized and logarithmically convex. Moreover by Theorem 5.4, $\boldsymbol{m}$ satisfies (β_2) and hence so does $\boldsymbol{m}^*$ by Lemma 2.2. Finally, as $\boldsymbol{m}$ satisfies (γ_{r+1}), Lemma 2.1 shows that $\boldsymbol{m}^*$ satisfies (γ_r). Therefore Theorem 5.4 gives the existence of an extension map U from $\Lambda_{\{M^*\}}$ into $\mathcal{D}_{r,\{M^*\}}$.

Now we go on as in the proof of Theorem 4.5: we introduce the isomorphism

$$V : \Lambda_{\{M^*\}} \to \mathcal{D}_{r,\{M^*\}}, \quad \boldsymbol{a} \mapsto (a_n/n!)_{n\in\mathbb{N}_0},$$

and with every $f \in \mathcal{D}_{r,\{M^*\}}$, we associate the very same function φ and proceed as before up to formula (5). Now we fix $m \in \mathbb{N}$ such that $f \in \mathcal{D}^m_{r,\{M^*\}}$ and get

$$\sup_{z\in L} \left| \frac{\varphi(z) - \sum_{j=0}^{nr-1} f^{(j)}(0) z^j}{z^{nr}} \right| \le A|f|_m m^n M_n^*$$

for some constant $A > 0$, $A = 1/\cos\left(\frac{\beta}{r} \cdot \frac{\pi}{2}\right)$ for instance. Then in order to study the function g, we fix an integer $s > 0$ such that for every $z \in S_\alpha \setminus \{0\}$, the circle γ of centre z and radius $|z|/s$ is contained in $S_\beta \setminus \{0\}$. For every $z \in S_\alpha \setminus \{0\}$, the Cauchy representation formula leads to

$$|g^{(n)}(z)| \le A|f|_m (2ms)^n M_n$$

and

$$g^{(n)}(0) = n! f^{(nr)}(0), \quad \forall n \in \mathbb{N}_0.$$

Therefore if we set $Wf := g|_{S_\alpha}$, we obtain $Wf \in \mathcal{H}^{2ms}_{\alpha,\{M\}}$ and

$$|Wf|_{2ms} = \sup_{n\in\mathbb{N}_0} \sup_{z\in S_\alpha} \frac{|g^{(n)}(z)|}{(2ms)^n M_n} \le A|f|_m.$$

So we have constructed a map W from $\mathcal{D}_{r,\{M^*\}}$ into $\mathcal{H}_{\alpha,\{M\}}$.

Finally it is straightforward to check that $T := WUV : \Lambda_{\{M\}} \to \mathcal{H}_{\alpha,\{M\}}$ is an extension map. ∎

THEOREM 5.6. *If, for every $r \in \mathbb{N}$, the sequence $(m_n/n^r)_{n\in\mathbb{N}}$ is quasi-increasing, then the following assertions are equivalent:*

(1) *for every $\alpha > 0$, there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{H}_{\alpha,\{M\}}$,*
(2) *for some $\alpha > 0$, there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{H}_{\alpha,\{M\}}$,*
(3) *for every $r \in \mathbb{N}$, there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{D}_{r,\{M\}}$,*
(4) *for some $r \in \mathbb{N}$, there is an extension map from $\Lambda_{\{M\}}$ into $\mathcal{D}_{r,\{M\}}$,*
(5) *the sequence $\boldsymbol{m}$ satisfies the condition (β_2).*

Proof. (1)⇒(2) and (3)⇒(4) are trivial.

(2)⇒(3) has been developed in the remark at the beginning of this subsection.

(4)⇒(5) is an immediate consequence of Theorem 5.4.

(5)⇒(1). Let $r \in \mathbb{N}$ belong to $]\alpha, \infty[$. The sequence $\boldsymbol{m}$ satisfies (β_2) and the sequence $(m_n/n^{r+1})_{n\in\mathbb{N}}$ is quasi-increasing, hence Lemma 2.4 says that $\boldsymbol{m}$ satisfies (γ_{r+1}). So Theorem 5.4 provides the existence of an extension map from $\Lambda_{\{M\}}$ into $\mathcal{D}_{r+1,\{M\}}$ and we conclude by Theorem 5.5. ∎

5.3. *Surjectivity of the restriction $R : \mathcal{D} \to \Lambda$.* Let us first state the following result as a starting point.

PROPOSITION 5.7. *The following conditions are equivalent:*

(1) *the restriction map $R : \mathcal{D}_{1,\{M\}} \to \Lambda_{\{M\}}$ is surjective,*

(2) *the sequence $\boldsymbol{m}$ satisfies the condition (γ_1),*

(3) *there is a positive integer d such that, for every $m \in \mathbb{N}$, there is an extension map from $\Lambda^m_{\{M\}}$ into $\mathcal{D}^{dm}_{1,\{M\}}$.*

Proof. (1)⇒(2) and (2)⇒(3) are known (cf. [9], Theorems 3.5 and 3.6). (3)⇒(1) is trivial. ∎

THEOREM 5.8. *Let $r \in \mathbb{N}$ and $\alpha \in]0, r[$. If the restriction map*

$$S : \mathcal{D}_{r+1,\{M\}} \to \Lambda_{\{M\}}, \quad f \mapsto (f^{(n(r+1))}(0))_{n\in\mathbb{N}_0},$$

is surjective, then there is a positive integer d such that, for every $m \in \mathbb{N}$, there is an extension map from $\Lambda^m_{\{M\}}$ into $\mathcal{H}^{dm}_{\alpha,\{M\}}$.

Proof. We are going to establish the existence of a positive integer d such that, for every $m \in \mathbb{N}$, there is an extension map from $\Lambda^{m^r}_{\{M\}}$ into $\mathcal{H}^{dm^r}_{\alpha,\{M\}}$. If we had introduced the spaces $\Lambda^h_{\{M\}}$ and $\mathcal{H}^h_{\alpha,\{M\}}$ for every $h > 0$ to define the spaces $\Lambda_{\{M\}}$ and $\mathcal{H}_{\alpha,\{M\}}$, the same proof would apply with m replaced by $\sqrt[r]{m}$ to give the stated result.

We first claim that we may suppose that $\boldsymbol{m}$ satisfies (α_1). Indeed, as S is surjective, Proposition 5.1 implies that the sequence $(\sqrt[r+1]{m_n})_{n\in\mathbb{N}_0}$ satisfies (γ_1). As it also satisfies (α), Lemma 1.1(a) of [9] provides an equivalent sequence satisfying (α_1), proving our claim.

Now as $\boldsymbol{m}$ satisfies (α_1), the sequence $\boldsymbol{M}^*$ of Lemma 2.2 is normalized and logarithmically convex. So by Lemma 2.3 the r-interpolating sequence $\boldsymbol{P}^*$ of $\boldsymbol{M}^*$ is normalized and logarithmically convex. Moreover the corresponding sequence $\boldsymbol{p}^*$ satisfies (γ_1): one has to proceed just as in the proof of Theorem 5.4 and use Lemma 2.1.

Proposition 5.1 then affords a positive integer d_1 such that, for every $m \in \mathbb{N}$, there is an extension map U_m from $\Lambda^m_{\{P^*\}}$ into $\mathcal{D}^{d_1 m}_{1,\{P^*\}}$.

Of course, on the one hand, the mapping

$$V_m : \Lambda^{m^r}_{\{M^*\}} \to \Lambda^m_{\{P^*\}}, \quad \boldsymbol{a} \mapsto \boldsymbol{b},$$

defined by

$$\begin{cases} b_{nr} = a_n, & \forall n \in \mathbb{N}_0, \\ b_{nr+j} = 0, & \forall n \in \mathbb{N}_0, \forall j \in \{1, \ldots, r-1\}, \end{cases}$$

is well defined, continuous and linear. On the other hand,

$$L_m : \Lambda^{m^r}_{\{M\}} \to \Lambda^{m^r}_{\{M^*\}}, \quad \boldsymbol{a} \mapsto (a_n/n!)_{n\in\mathbb{N}_0},$$

is an isomorphism. So $U_m V_m L_m$ is a map from $\Lambda^{m^r}_{\{M\}}$ into $\mathcal{D}^{d_1 m}_{1,\{P^*\}}$.

The image of this map is contained in the topological vector subspace E_m of $\mathcal{D}^{d_1 m}_{1,\{P^*\}}$ of elements f such that $f^{(nr+j)}(0) = 0$ for every $n \in \mathbb{N}_0$ and $j \in \{1, \ldots, r-1\}$. It is then straightforward to check that

$$W_m : E_m \to \mathcal{D}^{(d_1 m)^r}_{r,\{M^*\}}, \quad f \mapsto f,$$

is well defined, linear and continuous.

To go on further, we choose $\beta \in]\alpha, r[$ and fix $s \in \mathbb{N}$ such that, for every $z \in S_\alpha \setminus \{0\}$, the circle γ of centre z and radius $|z|/s$ is contained in $S_\beta \setminus \{0\}$. With every $f \in \mathcal{D}^{(d_1 m)^r}_{r,\{M^*\}}$, we then associate the function φ defined on $L := \{z \in \mathbb{C} : \Re z > 0\}$ by $\varphi(z) := \frac{1}{z}\int_0^1 e^{-t/z} f(t)\,dt$ and proceed as in the proof of Theorem 5.5. We end up with a continuous linear map N_m from $\mathcal{D}^{(d_1 m)^r}_{r,\{M^*\}}$ into $\mathcal{H}^{2s(d_1 m)^r}_{\alpha,\{M\}}$ such that, for $d = 2sd_1^r$,

$$N_m W_m U_m V_m L_m : \Lambda^{m^r}_{\{M\}} \to \mathcal{H}^{dm^r}_{\alpha,\{M\}}$$

is an extension map. ∎

From now on we work in the setting of Gevrey classes of order $\gamma > 1$. So the sequence $\boldsymbol{M}$ we are considering is $(n!^\gamma)_{n\in\mathbb{N}_0}$, which we abbreviate to $\boldsymbol{n!^\gamma}$. In this case, the sequence $\boldsymbol{m}$ satisfies the condition (α_1), so as in the previous proof the r-interpolating sequence $\boldsymbol{P}^*$ is normalized and logarithmically convex. Moreover if $\gamma > r+1$ with $r \in \mathbb{N}$, the corresponding sequence $\boldsymbol{p}^*$ clearly satisfies (α) and it is straightforward to check that it also satisfies (β_1) for $p = 2r$, and hence (γ_1). Under those circumstances, for every $\alpha \in]0, r[$, we may reproduce the previous proof to get the following result.

PROPOSITION 5.9. *If $r \in \mathbb{N}$, $\alpha \in]0, r[$ and $\gamma \in]r+1, \infty[$, then there is a positive integer d such that, for every $m \in \mathbb{N}$, there is an extension map from $\Lambda^m_{\{n!^\gamma\}}$ into $\mathcal{H}^{dm}_{\alpha,\{n!^\gamma\}}$.* ∎

The following theorem improves this last result. Its proof relies on asymptotic expansions as developed in [2] and [11], as well as on ideas of Borel and Mittag-Leffler about analytic extension (cf. [12], p. 500).

THEOREM 5.10. *For all $\alpha, \gamma \in \mathbb{R}$ such that $0 < \alpha < \gamma - 1$, there is $d \in \mathbb{N}$ such that, for every $m \in \mathbb{N}$, there is an extension map T_m from $\Lambda^m_{\{n!^\gamma\}}$ into $\mathcal{H}^{dm}_{\alpha,\{n!^\gamma\}}$.*

Proof. Let us first set up some notation. In order to get a better appearance of some formulas, we set $k := 1/(\gamma-1)$. We also fix $\eta \in]\alpha, \gamma-1[$, then choose $r > 0$ such that, for every $z \in S_\alpha \setminus \{0\}$, the circle $\gamma(z)$ of centre z and radius $r|z|$ is contained in $\mathrm{int}(S_\eta)$, and finally set

$$\zeta := \frac{\eta}{\gamma-1} \cdot \frac{\pi}{2}.$$

Next, we need a comparison of $\Gamma(1+n(\gamma-1))$ with $n!^{\gamma-1}$; this can easily be derived from the Stirling formula:

(a) on the one hand, there is $B > 0$ such that

$$\frac{n!^{\gamma-1}}{\Gamma(1+n(\gamma-1))} \le B(2k)^{n(\gamma-1)}, \quad \forall n \in \mathbb{N}_0, \tag{9}$$

since

$$\frac{(e^{-n}n^n\sqrt{2\pi n})^{\gamma-1}}{e^{-n(\gamma-1)}(n(\gamma-1))^{n(\gamma-1)}\sqrt{2\pi n(\gamma-1)}} \le \frac{(2\pi)^{(\gamma-1)/2}}{\sqrt{\gamma-1}} \cdot \frac{n^{(\gamma-1)/2}}{((\gamma-1)^{\gamma-1})^n}$$

with $\sqrt{n} \le 2^n$ for every $n \in \mathbb{N}$;

(b) on the other hand, there is $b > 0$ such that

$$\Gamma(1+n(\gamma-1)) \le b^n n!^{\gamma-1}, \quad \forall n \in \mathbb{N}_0, \tag{10}$$

since

$$\begin{aligned} &e^{-n(\gamma-1)}(n(\gamma-1))^{n(\gamma-1)}\sqrt{2\pi n(\gamma-1)} \\ &\quad = \sqrt{2\pi(\gamma-1)}((\gamma-1)^{\gamma-1})^n \frac{\sqrt{n}}{(\sqrt{2\pi n})^{\gamma-1}}(e^{-n}n^n\sqrt{2\pi n})^{\gamma-1} \end{aligned}$$

with $\sqrt{n}/(\sqrt{2\pi n})^{\gamma-1} \le \sqrt{n} \le 2^n$ for every $n \in \mathbb{N}$. We will also use the identity

$$z^n\Gamma(1+n/k) = \frac{k}{z^k}\int_0^\infty e^{-t^k/z^k}t^{n+k-1}\,dt \quad \text{on } \mathrm{int}(S_{\gamma-1}) \tag{11}$$

for every $n \in \mathbb{N}_0$. To see this, we first note that, for $v > 0$ fixed, the change of variable $u = t^k/v^k$ in $\Gamma(1+n/k) = \int_0^\infty e^{-u}u^{n/k}\,du$ leads to

$$v^n\Gamma(1+n/k) = \frac{k}{v^k}\int_0^\infty e^{-t^k/v^k}t^{n+k-1}\,dt.$$

Since the last integral is a holomorphic function on $\mathrm{int}(S_{\gamma-1})$, (11) follows at once from the analytic extension theorem.

Now we start with the construction of the map T_m.

For every $\boldsymbol{a} \in \Lambda^m_{\{n!^\gamma\}}$, the inequality (9) provides

$$\frac{|a_n|}{n!\Gamma(1+n/k)} \le |\boldsymbol{a}|_m m^n \frac{n!^{\gamma-1}}{\Gamma(1+n/k)} \le |\boldsymbol{a}|_m B m^n (2k)^{n(\gamma-1)}, \quad \forall n \in \mathbb{N}_0.$$

So if we set $s := (2m(2k)^{\gamma-1})^{-1}$, we obtain

$$\frac{|a_n|}{n!\Gamma(1+n/k)}s^n \le 2^{-n}|\boldsymbol{a}|_m B, \quad \forall n \in \mathbb{N}_0,$$

which implies that the function

$$\varphi(z) := \sum_{n=0}^\infty \frac{a_n}{n!\Gamma(1+n/k)}z^n$$

is holomorphic on $\{z \in \mathbb{C} : |z| < s\}$ and continuous on $[-s, s]$. So

$$f(z) := \frac{k}{z^k}\int_0^s e^{-t^k/z^k}t^{k-1}\varphi(t)\,dt$$

defines a holomorphic function on $\mathrm{int}(S_{\gamma-1})$. In particular for every $z \in S_\eta \setminus \{0\}$ and $n \in \mathbb{N}_0$, the equality (11) leads directly to

$$\begin{aligned} f(z) - \sum_{j=0}^{n-1}\frac{a_j}{j!}z^j &= \frac{k}{z^k}\int_0^s e^{-t^k/z^k}t^{k-1}\sum_{j=n}^\infty \frac{a_j}{j!\Gamma(1+j/k)}t^j\,dt \\ &\quad - \frac{k}{z^k}\int_s^\infty e^{-t^k/z^k}t^{k-1}\sum_{j=0}^{n-1}\frac{a_j}{j!\Gamma(1+j/k)}t^j\,dt, \end{aligned}$$

hence

$$\begin{aligned} &\Big|f(z) - \sum_{j=0}^{n-1}\frac{a_j}{j!}z^j\Big| \\ &\quad \le \frac{k}{|z|^k}\int_0^s e^{-t^k\cos(\zeta)/|z|^k}t^{n+k-1}\frac{1}{s^n}\sum_{j=0}^\infty \frac{|a_j|}{j!\Gamma(1+j/k)}s^j\,dt \\ &\quad\quad + \frac{k}{|z|^k}\int_s^\infty e^{-t^k\cos(\zeta)/|z|^k}t^{n+k-1}\frac{1}{s^n}\sum_{j=0}^\infty \frac{|a_j|}{j!\Gamma(1+j/k)}s^j\,dt. \end{aligned}$$

So if we set

$$M := \sum_{j=0}^\infty \frac{|a_j|}{j!\Gamma(1+j/k)}s^j,$$

we obtain

$$\Big|f(z) - \sum_{j=0}^{n-1}\frac{a_j}{j!}z^j\Big| \le \frac{2M}{s^n}\cdot\frac{k}{|z|^k}\int_0^\infty e^{-t^k\cos(\zeta)/|z|^k}t^{n+k-1}\,dt.$$

Now we note that the change of variable $u^k = t^k\cos(\zeta)$ leads to

$$\frac{k}{|z|^k}\int_0^\infty e^{-t^k\cos(\zeta)/|z|^k}t^{n+k-1}\,dt$$
$$= \frac{1}{\cos(\zeta)\cos^{n(\gamma-1)}(\zeta)}\cdot\frac{k}{|z|^k}\int_0^\infty e^{-u^k/|z|^k}u^{n+k-1}\,du$$
$$\le \frac{1}{\cos(\zeta)\cos^{n(\gamma-1)}(\zeta)}|z|^n b^n n!^{\gamma-1},$$

the last inequality being obtained by use of (11) and (10). So we finally get

$$\left|f(z)-\sum_{j=0}^{n-1}\frac{a_j}{j!}z^j\right| \le \frac{2M}{\cos(\zeta)}\left(\frac{b}{s\cos^{\gamma-1}(\zeta)}\right)^n |z|^n n!^{\gamma-1}. \tag{12}$$

Let us derive from this last inequality that, for some positive integer d independent of m, the function f belongs to $\mathcal{H}^{dm}_{\alpha,\{n!^\gamma\}}$.

On the one hand, for every $z\in S_\alpha\setminus\{0\}$ and $n\in\mathbb{N}_0$, the equality

$$f^{(n)}(z) = \left(f(z)-\sum_{j=0}^{n-1}\frac{a_j}{j!}z^j\right)^{(n)} = \frac{n!}{2\pi i}\int_{\gamma(z)}\frac{f(u)-\sum_{j=0}^{n-1}\frac{a_j}{j!}u^j}{(u-z)^{n+1}}\,du$$

leads to

$$|f^{(n)}(z)| \le \frac{n!}{2\pi}\cdot\frac{2M}{\cos(\zeta)}\left(\frac{b}{s\cos^{\gamma-1}(\zeta)}\right)^n\frac{2\pi r|z|}{(r|z|)^{n+1}}|z|^n(1+r)^n n!^{\gamma-1}$$
$$\le \frac{2M}{\cos(\zeta)}\left(\frac{b(1+1/r)}{s\cos^{\gamma-1}(\zeta)}\right)^n n!^\gamma.$$

On the other hand, for every $z\in S_\eta\setminus\{0\}$ and $n\in\mathbb{N}_0$, the inequality (12) leads to

$$\left|\frac{f(z)-\sum_{j=0}^{n-1}\frac{a_j}{j!}z^j}{z^n}-\frac{a_n}{n!}\right| \le \frac{2M}{\cos(\zeta)}\left(\frac{b}{s\cos^{\gamma-1}(\zeta)}\right)^{n+1}(n+1)!^{\gamma-1}|z|$$

and hence, uniformly in S_η,

$$\lim_{\substack{z\to 0\\ z\in S_\eta\setminus\{0\}}}\frac{f(z)-\sum_{j=0}^{n-1}\frac{a_j}{j!}z^j}{z^n} = \frac{a_n}{n!}.$$

So if we proceed as in the proof of Theorem 5.5, we obtain

$$\lim_{\substack{z\to 0\\ z\in S_\eta\setminus\{0\}}} f^{(n)}(z) = a_n.$$

It is now straightforward to check that the map T_m can be defined by $T_m a = f|_{S_\alpha}$ for every $a\in\Lambda^m_{\{n!^\gamma\}}$. ■

THEOREM 5.11. *Let $\alpha,\gamma>0$. If there is a function $f\in\mathcal{H}_{\alpha,\{n!^\gamma\}}$ such that $f^{(n)}(0)=\delta_{1,n}$ for every $n\in\mathbb{N}_0$, then $\alpha<\gamma-1$.*

Proof. Fix $m\in\mathbb{N}$ such that $f\in\mathcal{H}^m_{\alpha,\{n!^\gamma\}}$. As f is a bounded function on S_α, the function $\varphi(z):=f(z)-z$ is not identically 0 on S_α. Moreover the Taylor formula provides a constant $A>0$ such that, for every $z\in S_\alpha$ satisfying $0<|z|\le 1$ and $n\in\mathbb{N}_0$, we have

$$\left|\frac{\varphi(z)}{z^n}\right| \le A\frac{m^n n!^\gamma}{n!} = Am^n n!^{\gamma-1}.$$

So if we introduce $u^\alpha=z$, we get $|\varphi(u^\alpha)|\le Am^n n!^{\gamma-1}|u|^{\alpha n}$ for every $u\in\mathbb{C}$ such that $\Re u\ge 0$ and $|u|\le 1$, and every $n\in\mathbb{N}_0$. Finally we set $v=1/u$ and get a non-identically 0 function $\psi(v)=\varphi(1/v^\alpha)$ holomorphic on $\{z\in\mathbb{C}:\Re z>0\}$ such that

$$|\psi(v)|\le\frac{Am^n n!^{\gamma-1}}{|v|^{\alpha n}}\quad\text{if }\Re v\ge 1.$$

As $(n!)^{1/n}\to\infty$, Theorem 2.4.III of [7] implies

$$\sum_{n=1}^\infty\left(\frac{Am^n n!^{\gamma-1}}{Am^{n+1}(n+1)!^{\gamma-1}}\right)^{1/\alpha} = \frac{1}{m^{1/\alpha}}\sum_{n=1}^\infty\frac{1}{(n+1)^{(\gamma-1)/\alpha}}<\infty. \ ■$$

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The Heisenberg group and the group Fourier transform of regular homogeneous distributions

by

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Abstract. We calculate the group Fourier transform of regular homogeneous distributions defined on the Heisenberg group, $\mathbf{H}^n$. All such distributions can be written as an infinite sum of terms of the form $f(\theta)\overline{w}^{-k}P(z)$, where $(z,t)\in\mathbb{C}^n\times\mathbb{R}$, $w=|z|^2-it$, $\theta=\arg(\overline{w}/w)$ and $P(z)$ is an element of an orthonormal basis for the spherical harmonics. The formulas derived give the Fourier transform of the distribution in terms of a smooth kernel of the variable θ and the Weyl correspondent of P.

1. Introduction. In this paper we derive formulas for the group Fourier transform of regular homogeneous distributions on the Heisenberg group, $\mathbf{H}^n$. (We use coordinates $(z,t)\in\mathbb{C}^n\times\mathbb{R}$ on $\mathbf{H}^n$). It can be shown that all such distributions can be expressed as an infinite sum $\sum f_i(\theta)\overline{w}^{-k_i}P_i(z)$. Here, $w=|z|^2-it$, $\theta=\arg(\overline{w}/w)$ and the $P_i(z)$ are elements of an orthonormal basis for the spherical harmonics.

The group Fourier transform is a map from $L^1(\mathbf{H}^n)$ into the space of families of bounded operators defined on a Hilbert space. In many applications the Hilbert space is taken to be $L^2(\mathbb{R}^n)$. The domain of definition of the transform can be extended to include tempered distributions on $\mathbf{H}^n$. The group Fourier transform is of interest because it extends to a unitary map from $L^2(\mathbf{H}^n)$ to the space of families of Hilbert–Schmidt operators. Also, the group Fourier transform (which we will denote by $\hat{}_H$) behaves nicely with respect to convolution defined by the group multiplication on $\mathbf{H}^n$. That is, $(f*g)^\wedge_H=\widehat{f}_H\cdot\widehat{g}_H$, where the multiplication on the right is composition of operators.

The group Fourier transform is closely related to the Weyl correspondence. In fact, the formula we present gives the group Fourier transform of a regular homogeneous distribution in terms of the Weyl correspondent of P_i. This correspondent will be denoted by $\mathcal{W}(P_i)$. The set

$$\{\mathcal{W}(P)\mid P\text{ is a homogeneous harmonic polynomial}\}$$

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