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The exceptional set of Goldbach numbers (II)

by

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1. Introduction. A positive number which is a sum of two odd primes is called a *Goldbach number*. Let $E(x)$ denote the number of even numbers not exceeding x which cannot be written as a sum of two odd primes. Then the *Goldbach conjecture* is equivalent to proving that

$$E(x) = 2 \quad \text{for every } x \geq 4.$$

$E(x)$ is usually called the *exceptional set of Goldbach numbers*. In [8] H. L. Montgomery and R. C. Vaughan proved that $E(x) = O(x^{1-\Delta})$ for some positive constant $\Delta > 0$. In [3] Chen and Pan proved that one can take $\Delta > 0.01$. In [6], we proved that $E(x) = O(x^{0.921})$. In this paper we prove the following result.

THEOREM. *For sufficiently large* x,

$$E(x) = O(x^{0.914}).$$

Throughout this paper, ε always denotes a sufficiently small positive number that may be different at each occurrence. A is assumed to be sufficiently large, $A < Y$, and $D = Y^{1+\varepsilon}$.

2. Some lemmas. Let $A < q \leq Y$ and χ_q be a non-principal character mod q. Write $\alpha = 1 - \lambda/\log D$, and assume

$$\alpha \leq \sigma \leq 1, \qquad |t| \leq D/q. \tag{2.1}$$

Let χ (mod q) and χ_0 (mod q) be a character and a principal character mod q, and $\mathcal{L} = \log D$.

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LEMMA 1. *Let χ be a non-principal character modulo q, and let $\phi = 3/8$. Then for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that*

$$-\Re \frac{L'}{L}(s,\chi) \le -\sum_{|1+it-\varrho|\le\delta} \Re \frac{1}{s-\varrho} + \left(\frac{\phi}{2}+\varepsilon\right)H$$

uniformly for

$$1 + \frac{1}{H\log H} \le \sigma \le 1 + \frac{\log H}{H},$$

providing that q is sufficiently large; here $H = \log q(|t|+2)$.

This is Lemma 2.4 of [5].

LEMMA 2. *Suppose Y is sufficiently large. Then no function $L(s,\chi)$ with χ primitive modulo $q \le Y$, except for a possible exceptional one only, has a zero in the region*

$$\sigma \ge 1 - \frac{0.239}{\log Y}, \qquad q(|t|+1) \le Y^{1+\varepsilon}.$$

If the exceptional function exists, say $L(s,\widetilde{\chi})$, then $\widetilde{\chi}$ must be a real primitive character modulo $\widetilde{q} \le Y$, and $L(s,\widetilde{\chi})$ has a real simple zero $\widetilde{\beta}$ satisfying

$$1 - \frac{0.239}{\log Y} \le \widetilde{\beta} \le 1 - \frac{c}{\widetilde{q}^{10^{-8}}}.$$

This is Lemma 2.3 of [6].

For a real number a, let $a^* = a\mathcal{L}^{-1}$, and let $\varrho_j = 1-\lambda_j^*+i\gamma_j^*$, $j = 1,2,\ldots,$ denote the non-trivial zeros of $L(s,\chi)$ in (2.1), with λ_j in increasing order.

LEMMA 3. *Suppose χ is a real non-principal character mod $q \le Y$, and ϱ_1 is real. Then $\lambda_2 > 0.8$.*

Proof. Apply Lemma 3.2 of [5].

When $\chi^2 = \chi_0$ and ϱ_1 is complex, or $\chi^3 = \chi_0$, we follow Lemma 9.1 of [4]. Let a, k, ε be positive constants, and let $\phi = 3/8$, $P(x) = x + x^2 + \frac{2}{3}x^3$. Then

$$\left(k^2+\frac{1}{2}\right)\left\{P\left(\frac{a+\lambda_1}{a}\right) - P\left(\frac{a+\lambda_1}{a+\lambda_2}\right)\right\} - 2kP(1) + (a+\lambda_1)(\psi+\varepsilon) \ge 0,$$

where

$$\psi = \frac{\phi}{2}\left(k^2+3k+\frac{3}{2}\right),$$

providing that

$$k_0(a+\lambda_1)^{-3} + (a+\lambda_2)^{-3} \ge a^{-3} \quad \text{with} \quad k_0 = \min\left(k+\frac{3}{4k}, 4k\right).$$

Taking $a = 2.4$, $k = 0.88$, we see that if $\lambda_1 \le 0.618$, then $\lambda_2 > 0.618$.

Now suppose χ does not have order 2 or 3. Let

$$\mathcal{L}^{-1}\sum_k a_k \frac{(a+0.239)^k}{(k-1)!}\sum_{n=1}^{\infty}\Lambda(n)\,\Re\bigg(\frac{\chi(n)}{n^s}\bigg)\bigg(\frac{\log n}{\mathcal{L}}\bigg)^{k-1} = \Sigma(s,\chi). \tag{2.2}$$

Again we follow Lemma 9.1 of [4] with a, k, ε, ϕ and $P(x)$ as above. Then

$$\bigg(k^2+\frac{1}{2}\bigg)\bigg\{P\bigg(\frac{a+0.239}{a}\bigg)-P\bigg(\frac{a+0.239}{a+\lambda_2}\bigg)\bigg\} \\ -2kP\bigg(\frac{a+0.239}{a+\lambda_1}\bigg)+(a+0.239)(\psi+\varepsilon)\ge 0.$$

Taking $a = 2.21, k = 0.89$, we see that if $\lambda_1 \le 0.575$, then $\lambda_2 > 0.575$.

Now we consider λ_3. Our starting point is the inequality

$$\prod_{j=1}^{3}(1+\Re(\chi(n)n^{-i\gamma_j^*}))\ge 0. \tag{2.3}$$

Let $P(x) = \sum a_k x^k = x + x^2 + \frac{2}{3}x^3$ and

$$\mathcal{L}^{-1}\sum_k a_k \frac{a^k}{(k-1)!}\sum_{n=1}^{\infty}\Lambda(n)\,\Re\bigg(\frac{\chi(n)}{n^s}\bigg)\bigg(\frac{\log n}{\mathcal{L}}\bigg)^{k-1} = \Sigma(s,\chi). \tag{2.4}$$

Then

$$\Sigma(\sigma,\chi_0)+\sum_{j=1}^{3}\Sigma(\sigma+i\gamma_j^*,\chi)+\frac{1}{2}\sum\nolimits_2+\frac{1}{4}\sum\nolimits_3\ge 0, \tag{2.5}$$

with

$$\sum\nolimits_2 = \sum_{1\le j<k\le 3}\{\Sigma(\sigma+i\gamma_j^*+i\gamma_k^*,\chi^2)+\Sigma(\sigma+i\gamma_j^*-i\gamma_k^*,\chi_0)\}$$

and

$$\sum\nolimits_3 = \Sigma(\sigma+i\gamma_1^*+i\gamma_2^*+i\gamma_3^*,\chi^3)+\Sigma(\sigma+i\gamma_1^*+i\gamma_2^*-i\gamma_3^*,\chi) \\ +\Sigma(\sigma+i\gamma_1^*-i\gamma_2^*+i\gamma_3^*,\chi)+\Sigma(\sigma+i\gamma_1^*-i\gamma_2^*-i\gamma_3^*,\widetilde{\chi}).$$

Let $s = \sigma + it$, $\sigma = 1 + a\mathcal{L}^{-1}$. We now observe that

$$\Re\bigg(P\bigg(\frac{a}{(s-\varrho)\mathcal{L}}\bigg)\bigg)\ge 0$$

for all zeros ϱ, since $\Re P(1/z) \ge 0$ for $\Re z \ge 1$. Moreover, if $|1 + it - \varrho| \ge \delta$, then

$$\Re\bigg(P\bigg(\frac{a}{(s-\varrho)\mathcal{L}}\bigg)\bigg)=O(\mathcal{L}^{-1}).$$

Here we follow Lemma 8.3 of [4]. Thus

$$\Sigma(\sigma,\chi_0)\le P(1)+\varepsilon,$$

$$\sum_{1\le j<k\le 3} \Sigma(\sigma+i\gamma_j^*-i\gamma_k^*,\chi_0) \le \sum_{1\le j<k\le 3} \Re\Big\{P\Big(\frac{a}{a+i(\gamma_j-\gamma_k)}\Big)\Big\}+\varepsilon,$$

$$\Sigma(\sigma+i\gamma_1^*,\chi) \le -P\Big(\frac{a}{a+\lambda_1}\Big)-\Re\Big\{P\Big(\frac{a}{a+\lambda_2+i(\gamma_1-\gamma_2)}\Big)\Big\} - \Re\Big\{P\Big(\frac{a}{a+\lambda_3+i(\gamma_1-\gamma_3)}\Big)\Big\}+a\Big(\frac{\phi}{2}+\varepsilon\Big),$$

$$\Sigma(\sigma+i\gamma_2^*,\chi) \le -P\Big(\frac{a}{a+\lambda_2}\Big)-\Re\Big\{P\Big(\frac{a}{a+\lambda_3+i(\gamma_2-\gamma_3)}\Big)\Big\}+a\Big(\frac{\phi}{2}+\varepsilon\Big),$$

$$\Sigma(\sigma+i\gamma_3^*,\chi) \le -P\Big(\frac{a}{a+\lambda_3}\Big)+a\Big(\frac{\phi}{2}+\varepsilon\Big),$$

$$\sum_{1\le j<k\le 3} \Sigma(\sigma+i\gamma_j^*-i\gamma_k^*,\chi^2) \le a\Big(\frac{3}{2}\phi+\varepsilon\Big),$$

$$\Sigma(\sigma+i\gamma_1^*+i\gamma_2^*-i\gamma_3^*,\chi) \le -\Re\Big\{P\Big(\frac{a}{a+\lambda_1+i(\gamma_2-\gamma_3)}\Big)\Big\} - \Re\Big\{P\Big(\frac{a}{a+\lambda_2+i(\gamma_1-\gamma_3)}\Big)\Big\}+a\Big(\frac{\phi}{2}+\varepsilon\Big),$$

$$\Sigma(\sigma+i\gamma_1^*-i\gamma_2^*+i\gamma_3^*,\chi) \le -\Re\Big\{P\Big(\frac{a}{a+\lambda_3+i(\gamma_1-\gamma_2)}\Big)\Big\}+a\Big(\frac{\phi}{2}+\varepsilon\Big),$$

$$\Sigma(\sigma+i\gamma_1^*+i\gamma_2^*+i\gamma_3^*,\chi^3) \le a(\phi/2+\varepsilon),$$

$$\Sigma(\sigma+i\gamma_1^*-i\gamma_2^*-i\gamma_3^*,\widetilde{\chi}) \le a(\phi/2+\varepsilon).$$

Hence

$$\begin{aligned}
P(1)-P\Big(\frac{a}{a+\lambda_1}\Big)&-P\Big(\frac{a}{a+\lambda_2}\Big)-P\Big(\frac{a}{a+\lambda_3}\Big)+a\Big(\frac{11}{4}\phi+\varepsilon\Big)\\
&+\frac{1}{2}\Re\Big\{P\Big(\frac{a}{a+i(\gamma_1-\gamma_2)}\Big)-2P\Big(\frac{a}{a+\lambda_2+i(\gamma_1-\gamma_2)}\Big)\\
&-\frac{1}{2}P\Big(\frac{a}{a+\lambda_3+i(\gamma_1-\gamma_2)}\Big)\Big\}\\
&+\frac{1}{2}\Re\Big\{P\Big(\frac{a}{a+i(\gamma_1-\gamma_3)}\Big)-2P\Big(\frac{a}{a+\lambda_3+i(\gamma_1-\gamma_3)}\Big)\\
&-\frac{1}{2}P\Big(\frac{a}{a+\lambda_2+i(\gamma_1-\gamma_3)}\Big)\Big\}\\
&+\frac{1}{2}\Re\Big\{P\Big(\frac{a}{a+i(\gamma_2-\gamma_3)}\Big)-2P\Big(\frac{a}{a+\lambda_3+i(\gamma_2-\gamma_3)}\Big)\\
&-\frac{1}{2}P\Big(\frac{a}{a+\lambda_1+i(\gamma_2-\gamma_3)}\Big)\Big\}\ge 0.
\end{aligned}$$

Providing that

$$a^{-3} \le \tfrac{5}{2}(a+\lambda_3)^{-3}$$

we have

$$P(1) - P\left(\frac{a}{a+\lambda_1}\right) - P\left(\frac{a}{a+\lambda_2}\right) - P\left(\frac{a}{a+\lambda_3}\right) + a\left(\frac{11}{4}\phi + \varepsilon\right) \ge 0.$$

Taking $a = 2$, we have $\lambda_3 \ge 0.68$.

LEMMA 4. *Suppose χ is a non-principal character mod $q \le Y$, and $\varrho_1, \varrho_2, \varrho_3$ are the zeros of $L(s,\chi)$. Then*

$$\lambda_2 > 0.575, \qquad \lambda_3 > 0.618.$$

LEMMA 5. *Suppose $\chi \ne \chi_0$ is a character mod $q \le Y$. Let n_0, n_1, n_2 denote the numbers of zeros of $L(s,\chi)$ in the rectangles*

$$R_0 : 1 - \mathcal{L}^{-1} \le \sigma \le 1,\ |t - t_0| \le 5.8\mathcal{L}^{-1},$$
$$R_1 : 1 - 5\mathcal{L}^{-1} \le \sigma \le 1,\ |t - t_1| \le 23.4\mathcal{L}^{-1},$$
$$R_2 : 1 - \lambda_+\mathcal{L}^{-1} \le \sigma \le 1,\ |t - t_2| \le 23.4\mathcal{L}^{-1},$$

where t_0, t_1, t_2 are real numbers satisfying $|t_i| \le T$, and $5 < \lambda_+ \le \log\log\mathcal{L}$. Then

$$n_0 \le 3, \quad n_1 \le 10, \quad n_2 \le 0.2292(\lambda_+ + 42.9).$$

Proof. It is well known that

$$-\frac{\zeta'}{\zeta}(\sigma) - \Re\frac{L'}{L}(s,\chi) \ge 0;$$

here $\sigma = \Re s$.

(i) We consider the rectangle R_0. Let $s = \sigma + it_0, \sigma = 1 + 8.4\mathcal{L}^{-1}$, and denote by $\varrho = 1 - \lambda^* + i\gamma$ the zero of $L(s,\chi)$ in R_0, hence $0 \le \lambda \le 1, |\gamma - t_0| \le 5.8\mathcal{L}^{-1}$, and

$$-\Re\frac{1}{s-\varrho} = -\mathcal{L}\frac{8.4+\lambda}{(8.4+\lambda)^2 + ((\gamma - t_0)\mathcal{L})^2} \le -\mathcal{L}\frac{9.4}{9.4^2 + 5.8^2}.$$

By Lemma 1,

$$-\Re\frac{L'}{L}(s,\chi) \le -\sum_{|1+it_0-\varrho|\le\delta} \Re\frac{1}{s-\varrho} + 0.18751\mathcal{L}.$$

If $|1 + it_0 - \varrho| > \delta$ then $\Re\frac{1}{s-\varrho} = O(1)$. So

$$-\Re\frac{L'}{L}(s,\chi) \le \mathcal{L}\left(0.18751 - \frac{9.4n_0}{9.4^2 + 5.8^2}\right).$$

Since $-\frac{\zeta'}{\zeta}(\sigma) \le \frac{1}{\sigma-1} + A$, where A is an absolute constant, we have

$$\frac{9.4n_0}{9.4^2 + 5.8^2} \le \frac{1}{8.4} + 0.18752, \qquad n_0 \le 3.$$

(ii) The rectangles R_1 and R_2 are treated as R_0 in (i) but with $\sigma = 1 + 24\mathcal{L}^{-1}$. Thus $n_1 \le 10$, $n_2 \le 0.2292(\lambda_+ + 42.9)$.

3. The zero density estimate of the Dirichlet L-function near the line $\sigma = 1$. Let $A < q \le Y$ and χ_q be a non-principal character mod q. Write $\alpha = 1 - \lambda/\log D$, and assume

$$\alpha \le \sigma \le 1, \qquad |t| \le D/q. \tag{3.1}$$

Let $S_{jq} = \{\chi_q : L(s, \chi_q)$ has only j zeros in the region (3.1)$\}$. Suppose $A < q_0 \le Y$ and define

$$N_1^*(\alpha, Y) = N_1^*(\lambda, Y) = \sum_{\substack{A<q\le Y\\ [q,q_0]\le D^{\varepsilon}(q,q_0)}} \sum_{j\ge 1} \sideset{}{^*}\sum_{\chi\in S_{jq}} j, \tag{3.2}$$

$$N^*(\alpha, Y) = N^*(\lambda, Y) = \sum_{A<q\le Y} \sum_{j\ge 1} \sideset{}{^*}\sum_{\chi\in S_{jq}} j, \tag{3.3}$$

where $\sum^*$ indicates that the sum is over primitive characters. In this section we will prove the following lemma which improves Lemma 2.1 of [6].

LEMMA 6. *Suppose $A < q_0 \le Y$ and $0 < \lambda \le \varepsilon \log D$. Then*

$$N_1^*(\alpha, Y) = N_1^*(\lambda, Y) \le \begin{cases} 4.356C_1(\lambda)e^{4.064\lambda}, & 0.517 < \lambda \le 0.575,\\ 8.46C_2(\lambda)e^{4.12\lambda}, & 0.575 < \lambda \le 0.618,\\ 14.3C_3(\lambda)e^{4.5\lambda}, & 0.618 < \lambda \le 1,\\ 104.1C_4(\lambda)e^{3.42\lambda}, & 1 < \lambda \le 5,\\ 268.6e^{2.16\lambda}, & 5 < \lambda \le \varepsilon \log D, \end{cases}$$

$$N^*(\alpha, Y) = N^*(\lambda, Y) \le \begin{cases} 3.632C_5(\lambda)e^{5.2\lambda}, & 0.334 < \lambda \le 0.517,\\ 4.338C_6(\lambda)e^{4.82\lambda}, & 0.517 < \lambda \le 0.575,\\ 10.42C_7(\lambda)e^{4.5\lambda}, & 0.575 < \lambda \le 0.618,\\ 14.91C_8(\lambda)e^{5.2\lambda}, & 0.618 < \lambda \le 1,\\ 104.8C_9(\lambda)e^{4.16\lambda}, & 1 < \lambda \le 5,\\ 279.7e^{2.9\lambda}, & 5 < \lambda \le \varepsilon \log D, \end{cases}$$

where

$$C_1(\lambda) = \lambda^{-1}\left(1 - e^{-4.064\lambda}\frac{e^{2.808\lambda} - e^{1.76\lambda}}{1.048\lambda}\right),$$

$$C_2(\lambda) = \lambda^{-1}\left(1 - e^{-4.12\lambda}\frac{e^{2.855\lambda} - e^{1.78\lambda}}{1.075\lambda}\right),$$

$$C_3(\lambda) = \lambda^{-1}\left(1 - e^{-4.5\lambda}\frac{e^{3.198\lambda} - e^{2.013\lambda}}{1.185\lambda}\right),$$

$$C_4(\lambda) = \lambda^{-1}\left(1 - e^{-3.42\lambda}\frac{e^{2.358\lambda} - e^{1.64\lambda}}{0.718\lambda}\right),$$

$$C_5(\lambda) = \lambda^{-1}\left(1 - e^{-5.2\lambda}\frac{e^{3.866\lambda} - e^{2.668\lambda}}{1.198\lambda}\right),$$
$$C_6(\lambda) = \lambda^{-1}\left(1 - e^{-4.82\lambda}\frac{e^{3.565\lambda} - e^{2.51\lambda}}{1.055\lambda}\right),$$
$$C_7(\lambda) = \lambda^{-1}\left(1 - e^{-4.5\lambda}\frac{e^{3.32\lambda} - e^{2.36\lambda}}{0.96\lambda}\right),$$
$$C_8(\lambda) = \lambda^{-1}\left(1 - e^{-5.2\lambda}\frac{e^{3.928\lambda} - e^{2.7312\lambda}}{1.1968\lambda}\right),$$
$$C_9(\lambda) = \lambda^{-1}\left(1 - e^{-4.16\lambda}\frac{e^{3.104\lambda} - e^{2.38\lambda}}{0.724\lambda}\right).$$

Proof. We use the method of Section 3 of [7]. For $1 \le j \le 4$, let h_j denote positive constants which satisfy

$$h_1 < h_2 < h_3, \quad h_2 + h_4 + 3/8 < h_3, \quad 2h_4 + 3/8 < h_1 \tag{3.4}$$

when we consider $N_1^*(\alpha, Y)$, and

$$h_1 < h_2 < h_3, \quad h_2 + h_4 + 3/8 < h_3, \quad 2h_4 + 3/4 < h_1 \tag{3.5}$$

when we consider $N^*(\alpha, Y)$.

Let

$$z_j := D^{h_j}, \quad \alpha := 1 - \lambda\mathcal{L}^{-1}, \quad \lambda \le \varepsilon\mathcal{L}. \tag{3.6}$$

For positive δ_1, δ_3, let

$$\begin{aligned}\kappa(s) := s^{-2}\{&(e^{-(1-\delta_1)(\log z_1)s} - e^{-(\log z_1)s})\delta_3(\log z_3) \\ &- (e^{-(\log z_3)s} - e^{-(1+\delta_3)(\log z_3)s})\delta_1(\log z_1)\}.\end{aligned} \tag{3.7}$$

For a zero $\varrho_0 \in D$, let

$$M(\varrho_0) := \sum_{\varrho(\chi)} |\kappa(\varrho(\chi) + \overline{\varrho}_0 - 2\alpha)|, \tag{3.8}$$

where the sum is over the zeros of $L(s,\chi)$ in (3.1). Then if $2h_4 + 3/8 < (1-\delta_1)h_1$, then as in (3.17) of [7] we have

$$\begin{aligned}N_1^*(\alpha, Y) &\le \frac{(1+\delta)\max_{\varrho_0} M(\varrho_0)}{2(1-\alpha)(h_2-h_1)\delta_1\delta_3 h_1 h_3 h_4 \mathcal{L}^4} \\ &\quad \times \left(D^{2h_3(1-\alpha)} - \frac{(2\alpha-1)(D^{2h_2(1-\alpha)} - D^{2h_1(1-\alpha)})}{2(1-\alpha)(h_2-h_1)\mathcal{L}}\right) \\ &\le \frac{(1+\delta)\max_{\varrho_0} M(\varrho_0)}{2\lambda(h_2-h_1)\delta_1\delta_3 h_1 h_3 h_4 \mathcal{L}^3}\left(e^{2h_3\lambda} - \frac{e^{2h_2\lambda} - e^{2h_1\lambda}}{2\lambda(h_2-h_1)}\right).\end{aligned} \tag{3.9}$$

If $2h_4+3/4<(1-\delta_1)h_1$, then as in (3.17) of [7] we have

$$(3.10)\quad N^*(\alpha,Y)\le \frac{(1+\delta)\max_{\varrho_0}M(\varrho_0)}{2(1-\alpha)(h_2-h_1)\delta_1\delta_3h_1h_3h_4\mathcal{L}^4} \times\left(D^{2h_3(1-\alpha)}-\frac{(2\alpha-1)(D^{2h_2(1-\alpha)}-D^{2h_1(1-\alpha)})}{2(1-\alpha)(h_2-h_1)\mathcal{L}}\right)$$
$$\le \frac{(1+\delta)\max_{\varrho_0}M(\varrho_0)}{2\lambda(h_2-h_1)\delta_1\delta_3h_1h_3h_4\mathcal{L}^3}\left(e^{2h_3\lambda}-\frac{e^{2h_2\lambda}-e^{2h_1\lambda}}{2\lambda(h_2-h_1)}\right).$$

(i) If $5<\lambda\le\varepsilon\mathcal{L}$, let $\Delta=23.4\mathcal{L}^{-1}$. As in [7], by Lemma 5 we have

$$M(\varrho_0)\le 0.2292(\lambda+42.9)\mathcal{L}^3(1/2) \times\{(\delta_1h_1(2\delta_3+\delta_3^2)h_3^2-\delta_3h_3(2\delta_1-\delta_1^2)h_1^2) + (\pi/23.4)^2(\delta_1h_1+\delta_3h_3)\}.$$

Choose $h_1=0.58$, $h_2=0.669$, $h_3=1.08$, $h_4=0.0353$, $\delta_1h_1=\delta_3h_3=\pi/23.4$. By (3.9) we have

$$N_1^*(\alpha,Y)\le 268.6e^{2.16\lambda}.$$

Choose $h_1=0.95$, $h_2=1.042$, $h_3=1.45$, $h_4=0.0328$, $\delta_1h_1=\delta_3h_3=\pi/23.4$. By (3.10) we have

$$N^*(\alpha,Y)\le 279.7e^{2.9\lambda}.$$

(ii) If $1<\lambda\le 5$, then as in [7], by Lemma 5 ($n_1\le 10$) we have

$$M(\varrho_0)\le (10/2)\mathcal{L}^3 \times\{(\delta_1h_1(2\delta_3+\delta_3^2)h_3^2-\delta_3h_3(2\delta_1-\delta_1^2)h_1^2) + (\pi/23.4)^2(\delta_1h_1+\delta_3h_3)\}.$$

Choose $h_1=0.82$, $h_2=1.179$, $h_3=1.71$, $h_4=0.155$, $\delta_1h_1=\delta_3h_3=\pi/23.4$. By (3.9) we have

$$N_1^*(\alpha,Y)\le 104.1C_4(\lambda)e^{3.42\lambda}.$$

Choose $h_1=1.19$, $h_2=1.552$, $h_3=2.08$, $h_4=0.1528$, $\delta_1h_1=\delta_3h_3=\pi/23.4$. By (3.10) we have

$$N^*(\alpha,Y)\le 104.8C_9(\lambda)e^{4.16\lambda}.$$

(iii) If $0.618<\lambda\le 1$, then as in [7], by Lemma 5 we have

$$\left(\frac{1}{a}-\frac{1}{a+1}-\frac{2(a+1)}{(a+1)^2+5.8^2}+0.1876\right) \times\max\left\{\frac{a+1}{5.8^2}+\frac{1}{a+1},\frac{a+0.618}{5.8^2}+\frac{1}{a+0.618}\right\}\le 0.014621.$$

For $a = 6.3$,

$$M(\varrho_0) \le \{1.5(\delta_1 h_1(2\delta_3 + \delta_3^2)h_3^2 - \delta_3 h_3(2\delta_1 - \delta_1^2)h_1^2) + 2 \cdot 0.014621 \cdot (\delta_1 h_1 + \delta_3 h_3)\}\mathcal{L}^3.$$

Choose $h_1 = 1.0065$, $h_2 = 1.599$, $h_3 = 2.25$, $h_4 = 0.2759$, $\delta_1 = 0.079$, $\delta_3 = 0.094$. By (3.9) we have

$$N_1^*(\alpha, Y) \le 14.3 C_3(\lambda) e^{4.5\lambda}.$$

Choose $h_1 = 1.3656$, $h_2 = 1.964$, $h_3 = 2.6$, $h_4 = 0.26$, $\delta_1 = 0.07$, $\delta_3 = 0.094$. By (3.10) we have

$$N^*(\alpha, Y) \le 14.91 C_8(\lambda) e^{5.2\lambda}.$$

(iv) If $0.575 < \lambda \le 0.618$, then by Lemma 4 there are at most two zeros satisfying $\varrho = 1 - \beta/\mathcal{L} - i\gamma/\mathcal{L}$, $\beta < 0.618$. Then, as in (v) of [7], when (3.4) holds we have

$$\text{(3.11)}\quad N_1^*(\alpha, Y) \le \frac{(1+\delta)\widetilde{M}}{2(1-\alpha)(h_2 - h_1)h_4\mathcal{L}^2} \times \left(D^{2h_3(1-\alpha)} - \frac{(2\alpha - 1)(D^{2h_2(1-\alpha)} - D^{2h_1(1-\alpha)})}{2(1-\alpha)(h_2 - h_1)\mathcal{L}}\right) \le \frac{(1+\delta)\widetilde{M}}{2\lambda(h_2 - h_1)h_4\mathcal{L}}\left(e^{2h_3\lambda} - \frac{e^{2h_2\lambda} - e^{2h_1\lambda}}{2\lambda(h_2 - h_1)}\right).$$

Similarly, when (3.5) holds we have

$$\text{(3.12)}\quad N^*(\alpha, Y) \le \frac{(1+\delta)\widetilde{M}}{2\lambda(h_2 - h_1)h_4\mathcal{L}}\left(e^{2h_3\lambda} - \frac{e^{2h_2\lambda} - e^{2h_1\lambda}}{2\lambda(h_2 - h_1)}\right)$$

where

$$\widetilde{M} := \max_{\substack{\chi \bmod q \\ q \le Y}} \max_{1 \le j \le 2} \frac{1}{j} \int_{\log z_1}^{\log z_3} \Big| \sum_{l=1}^{j} e^{-(\varrho(l,\chi) - \alpha)x} \Big|^2 \, dx.$$

We have

$$\int_{\log z_1}^{\log z_3} |e^{-(\varrho(\chi) - \alpha)x}|^2 \, dx \le (h_3 - h_1)\mathcal{L},$$

$$\frac{1}{2} \int_{\log z_1}^{\log z_3} \Big| \sum_{l=1}^{2} e^{-(\varrho(l,\chi) - \alpha)x} \Big|^2 \, dx \le 2(h_3 - h_1)\mathcal{L}.$$

Choose $h_1 = 0.89$, $h_2 = 1.4275$, $h_3 = 2.06$, $h_4 = 0.2574$. By (3.11) we have

$$N_1^*(\alpha, Y) \le 8.46 C_2(\lambda) e^{4.12\lambda}.$$

Choose $h_1 = 1.18$, $h_2 = 1.66$, $h_3 = 2.25$, $h_4 = 0.214$. By (3.12) we have

$$N^*(\alpha, Y) \le 10.42 C_7(\lambda) e^{4.5\lambda}.$$

(v) If $0.517 < \lambda \le 0.575$, then by Lemma 4 there is at most one zero satisfying $\varrho = 1 - \beta/\mathcal{L} - i\gamma/\mathcal{L}$, $\beta < 0.575$. Then, as in (v) of [7], when (3.4) holds we have

$$\begin{aligned}(3.13)\qquad N_1^*(\alpha, Y) &\le \frac{(1+\delta)(h_3-h_1)}{2(1-\alpha)(h_2-h_1)h_4\mathcal{L}} \\ &\quad\times \left(D^{2h_3(1-\alpha)} - \frac{(2\alpha-1)(D^{2h_2(1-\alpha)} - D^{2h_1(1-\alpha)})}{2(1-\alpha)(h_2-h_1)\mathcal{L}}\right) \\ &\le \frac{(1+\delta)(h_3-h_1)}{2\lambda(h_2-h_1)h_4}\left(e^{2h_3\lambda} - \frac{e^{2h_2\lambda} - e^{2h_1\lambda}}{2\lambda(h_2-h_1)}\right).\end{aligned}$$

When (3.5) holds we have

$$(3.14)\qquad N^*(\alpha, Y) \le \frac{(1+\delta)(h_3-h_1)}{2\lambda(h_2-h_1)h_4}\left(e^{2h_3\lambda} - \frac{e^{2h_2\lambda} - e^{2h_1\lambda}}{2\lambda(h_2-h_1)}\right).$$

Choose $h_1 = 0.88$, $h_2 = 1.404$, $h_3 = 2.032$, $h_4 = 0.2524$. By (3.13) we have

$$N_1^*(\alpha, Y) \le 4.356 C_1(\lambda) e^{4.064\lambda}.$$

Choose $h_1 = 1.255$, $h_2 = 1.7825$, $h_3 = 2.41$, $h_4 = 0.2524$. By (3.14) we have

$$N^*(\alpha, Y) \le 4.338 C_6(\lambda) e^{4.82\lambda}.$$

(vi) If $0.334 < \lambda \le 0.517$, then as above, when (3.5) holds we have

$$(3.15)\qquad N^*(\alpha, Y) \le \frac{(1+\delta)(h_3-h_1)}{2\lambda(h_2-h_1)h_4}\left(e^{2h_3\lambda} - \frac{e^{2h_2\lambda} - e^{2h_1\lambda}}{2\lambda(h_2-h_1)}\right).$$

Choose $h_1 = 1.334$, $h_2 = 1.933$, $h_3 = 2.6$, $h_4 = 0.291$. By (3.15) we have

$$N^*(\alpha, Y) \le 3.632 C_5(\lambda) e^{5.2\lambda}.$$

If $q_1, q_2 \le Y$, we consider the zeros of $L(s, \chi_{q_1})$ and $L(s, \chi_{q_2})$ for non-principal characters χ_{q_1} and χ_{q_2}. If $\varrho_1 = \beta_1 + i\gamma_1 = 1 - \lambda_1/\log Y + i\gamma_1$ is a zero of $L(s, \chi_{q_1})$ satisfying $q_1(|\gamma_1| + 1) \le Y^{1+\varepsilon}$ and $\varrho_2 = \beta_2 + i\gamma_2 = 1 - \lambda_2/\log Y + i\gamma_2$ is a zero of $L(s, \chi_{q_2})$ satisfying $q_2(|\gamma_2| + 1) \le Y^{1+\varepsilon}$, then we have the lower bounds for λ_2 given in Table 1. If $[q_1, q_2] \le Y^{\varepsilon}(q_1, q_2)$, then we have the lower bounds for λ_2 given in Table 2.

Table 1. The lower bounds for λ_2

λ_1	λ_2
0.24	0.444
0.26	0.418
0.28	0.393
0.30	0.37
0.32	0.349
0.334	0.334

Table 2. The lower bounds for λ_2

λ_1	λ_2	λ_1	λ_2
0.22	1.189	0.38	0.745
0.24	1.116	0.40	0.706
0.26	1.050	0.42	0.669
0.28	0.989	0.44	0.634
0.30	0.933	0.46	0.601
0.32	0.881	0.48	0.570
0.34	0.832	0.50	0.541
0.36	0.787	0.517	0.517

In each table, following the convention of [6], the entries indicate that if λ_1 does not exceed the first entry, then λ_2 is no smaller than the second entry.

4. The circle method. Suppose x is a sufficiently large positive number, and $Y = x^{\lambda}$ where $\lambda = 0.0862$. Let

$$S(\alpha) = \sum_{Y<p\le x} \log p \, e(\alpha p), \quad D(n) = D(n; x, Y) = \sum_{\substack{n=p_1+p_2\\ Y<p_1,p_2\le x}} \log p_1 \log p_2.$$

Then

$$D(n) = \int_0^1 S^2(\alpha) e(-\alpha n)\, d\alpha. \tag{4.1}$$

Trivially, $D(n) = 0$ if $n \le 2Y$ or $n > 2x$, and n is a Goldbach number if $D(n) > 0$.

Let $Q = x^{1-\lambda}$, $\tau = Q^{-1}$ and

$$E_1 = \bigcup_{1\le q\le Y} \bigcup_{\substack{1\le a\le q\\ (a,q)=1}} I(a,q), \qquad E_2 = (-\tau, 1-\tau] \setminus E_1$$

where

$$I(a,q) = \left[\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}\right].$$

Then

$$\begin{aligned} D(n) &= \int_{-\tau}^{1-\tau} S^2(\alpha) e(-\alpha n)\, d\alpha \qquad (4.2)\\ &= \int_{E_1} S^2(\alpha) e(-\alpha n)\, d\alpha + \int_{E_2} S^2(\alpha) e(-\alpha n)\, d\alpha \\ &= D_1(n) + D_2(n). \end{aligned}$$

LEMMA 7. *Let $M(x)$ denote the number of integers $n \in [(1-\varepsilon)x, x]$ for which*

$$|D_2(n)| > 0.5 x^{1-10^{-5}\lambda}.$$

Then

$$M(x) \ll x^{1-(1-10^{-4})\lambda}.$$

Proof. Apply Lemma 8 of [3].

Now we consider the integral on the major arcs. For $\alpha \in I(a,q) \subset E_1$, we write $\alpha = a/q + \theta$, $(a,q) = 1$, $q \le Y$, $|\theta| \le 1/(qQ)$. Moreover, suppose that $\widetilde{q}$, $\widetilde{\chi}$ and $\widetilde{\beta}$ are the possible modulus, primitive character and zero respectively, with $\widetilde{q} \le Y$. Let

$$T(\theta) = \sum_{Y<m\le x} e(m\theta), \tag{4.3}$$

$$\widetilde{T}(\theta) = -\sum_{Y<m\le x} m^{\widetilde{\beta}-1} e(m\theta), \tag{4.4}$$

$$\widehat{S}(\theta,\chi) = \sum_{Y<p\le x} \chi(p)\log p\, e(p\theta), \tag{4.5}$$

χ being a character modulo q, $q \le Y$, and

$$\begin{cases} \widehat{S}(\theta,\chi_q^0) = T(\theta) + W(\theta,\chi_q^0), & \\ \widehat{S}(\theta,\chi_q^0\widetilde{\chi}) = \widetilde{T}(\theta) + W(\theta,\chi_q^0\widetilde{\chi}) & \text{if } \widetilde{q}\mid q, \\ \widehat{S}(\theta,\chi_q) = W(\theta,\chi_q) & \text{otherwise.} \end{cases} \tag{4.6}$$

Then if the exceptional character exists we have

$$D_1(n) = \sum_{q\le Y} \sum_{\substack{a\le q\\(a,q)=1}} \int_{a/q-1/(qQ)}^{a/q+1/(qQ)} S^2(\alpha)e(-\alpha n)\,d\alpha = \sum_{j=1}^{6} D_{1j}(n). \tag{4.7}$$

Otherwise we have

$$D_1(n) = \sum_{j=1}^{3} D_{1j}(n). \tag{4.8}$$

For the definitions of $D_{1j}(n)$, see [3]. By the method of [8] one has

$$D_{11}(n) = nC(n) + O(x^{1+\varepsilon}Y^{-1}), \tag{4.9}$$

$$D_{14}(n) = \widetilde{C}(n)\widetilde{I}(n) + O((n,\widetilde{q})x^{1+\varepsilon}Y^{-1}), \tag{4.10}$$

where

$$C(n) = \sum_{q=1}^{\infty} \frac{\mu^2(q)}{\phi^2(q)} C_q(-n) = \frac{n}{\phi(n)} \prod_{p\nmid n}\left(1 - \frac{1}{(p-1)^2}\right), \tag{4.11}$$

$$\widetilde{C}(n) = \sum_{\substack{q=1\\ \widetilde{q}\mid q}}^{\infty} \frac{\tau^2(\chi_q^0\widetilde{\chi})}{\phi^2(q)} C_q(-n) = \widetilde{\chi}(-1)\mu\left(\frac{\widetilde{q}}{(\widetilde{q},n)}\right) \prod_{\substack{p\mid\widetilde{q}\\ p\nmid n}} \left(\frac{1}{p-2}\right) C(n), \tag{4.12}$$

$$\widetilde{I}(n) = \sum_{Y<m\le n-Y} (m(n-m))^{\widetilde{\beta}-1} \le x^{(1-\varepsilon)(\widetilde{\beta}-1)} n^{\widetilde{\beta}}, \tag{4.13}$$

with

$$\tau(\chi) = \sum_{h=1}^{q} \chi(h) e\left(\frac{h}{q}\right), \qquad C_q(m) = \sum_{\substack{h\le q\\(h,q)=1}} e\left(\frac{mh}{q}\right).$$

Let

$$W(\chi_d)=\Big(\int\limits_{-1/(dQ)}^{1/(dQ)}|W(\theta,\chi_d)|^2\,d\theta\Big)^{1/2}. \tag{4.14}$$

Then by (20) of [3] one has

$$D_{12}(n)\le\frac{n}{\phi(n)}\bigg\{8x^{1/2}W(\log^{10}x)+O\bigg(\frac{x^{1/2}W(Y)}{\log^6 x}\bigg)\bigg\}, \tag{4.15}$$

where

$$W(Y)=\sum_{d\le Y}\sum_{\chi_d}{}^{*}W(\chi_d), \tag{4.16}$$

the $*$ denoting that the sum is over primitive characters χ_d. We have

$$D_{15}(n)\ll\widetilde{\chi}^2(n)\frac{\widetilde{q}}{\phi^2(\widetilde{q})}\cdot\frac{n}{\phi(n)}x. \tag{4.17}$$

From $\prod_{p\ge5}(1+1/(p-1)^2)\le1.132$, by the method of [1], we have

$$D_{16}(n)\le4.1594\frac{n}{\phi(n)}x^{1/2}W(Y,\widetilde{q})+\frac{n}{\phi(n)}W(Y)x^{(1-\varepsilon)/2}, \tag{4.18}$$

$$D_{13}(n)\le2.0797\frac{n}{\phi(n)}W(Y)W'(Y), \tag{4.19}$$

where

$$W(Y,\widetilde{q})=\sum_{\substack{d\le Y\\ [d,\widetilde{q}]\le x^{\varepsilon}(d,\widetilde{q})}}\ \sum_{\chi_d}{}^{*}W(\chi_d), \tag{4.20}$$

$$W'(Y)=\max\sum_{\substack{d\le Y\\ [d_1,d]\le x^{\varepsilon}(d_1,d)}}\ \sum_{\chi_d}{}^{*}W(\chi_d). \tag{4.21}$$

Here the max is over $A<d_1\le Y$.

5. The estimation of $W'(Y), W(Y)$ and $W(Y,\widetilde{q})$. By Section III of [2] we have

$$\begin{aligned}W(\chi_d)\le{}&(1+2\cdot10^{-5})x^{1/2}\sum_{\substack{\beta\ge1/4\\ |\gamma_{\chi_d}|\le Y^{1+\varepsilon}d^{-1}}}{}^{\prime}x^{(1-\varepsilon)(\beta-1)}\\ &+O\Big(x^{1/2-\varepsilon}\sum_{\substack{\beta\ge1/4\\ |\gamma_{\chi_d}|\le Y^{1.01}d^{-1}}}{}^{\prime}x^{\beta-1}\Big)\\ &+O\Big(x^{1/2-0.01\lambda}\sum_{\substack{\beta\ge1/4\\ |\gamma_{\chi_d}|\le Y^{2.01}}}{}^{\prime}x^{\beta-1}\Big)+O(x^{1/2-1.01\lambda+\varepsilon}d^{-1}),\end{aligned} \tag{5.1}$$

where $\sum'$ indicates that the sum does not contain the exceptional zero $\widetilde{\beta}$.

By the same methods as in [1] we have

$$\sum_{\substack{d\le Y\\ [d_1,d]\le Y^{\varepsilon}(d_1,d)}}\ \sideset{}{^*}\sum_{\chi_d}\ \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le Y^{2.01}}} x^{\beta-1}\ll x^{0.7\varepsilon},$$
$$\sum_{d\le Y}\ \sideset{}{^*}\sum_{\chi_d}\ \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le Y^{2.01}}} x^{\beta-1}\ll x^{0.7\varepsilon}. \tag{5.2}$$

Let

$$I_1=\sum_{\substack{d\le Y\\ [d_1,d]\le Y^{\varepsilon}(d_1,d)}}\ \sideset{}{^*}\sum_{\chi_d}\ \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le Y^{1+\varepsilon}d^{-1}}} x^{(1-\varepsilon)(\beta-1)},$$
$$I_2=\sum_{d\le Y}\ \sideset{}{^*}\sum_{\chi_d}\ \sideset{}{'}\sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le Y^{1+\varepsilon}d^{-1}}} x^{(1-\varepsilon)(\beta-1)}. \tag{5.3}$$

Suppose $\varrho_{\chi_d}=\beta_{\chi_d}+i\gamma_{\chi_d}, |\gamma_{\chi_d}|\le Y^{1+\varepsilon}d^{-1}$, is a zero of $L(s,\chi_d)$. Let $\mathcal{L}=(1+\varepsilon)\log Y$.

1) If $1-0.24/\mathcal{L}\le\beta_{\chi_d}\le 1-0.239/\mathcal{L}$, then by Lemma 6 and Tables 1 and 2 we have

$$I_1\le 2e^{-0.239/(\lambda+\varepsilon)}+\frac{1}{\lambda+\varepsilon}\int_{1.116}^{\infty}e^{-(1-\varepsilon)t/(\lambda+\varepsilon)}N_1^*(t,Y)\,dt\le 0.136,$$
$$I_2\le 2e^{-0.239/(\lambda+\varepsilon)}+\frac{1}{\lambda+\varepsilon}\int_{0.444}^{\infty}e^{-(1-\varepsilon)t/(\lambda+\varepsilon)}N^*(t,Y)\,dt\le 1.009.$$

2) If $1-0.26/\mathcal{L}\le\beta_{\chi_d}\le 1-0.24/\mathcal{L}$, we have $I_1\le 0.143$, $I_2\le 1.098$.
3) If $1-0.28/\mathcal{L}\le\beta_{\chi_d}\le 1-0.26/\mathcal{L}$, we have $I_1\le 0.129$, $I_2\le 1.177$.
4) If $1-0.30/\mathcal{L}\le\beta_{\chi_d}\le 1-0.28/\mathcal{L}$, we have $I_1\le 0.118$, $I_2\le 1.271$.
5) If $1-0.32/\mathcal{L}\le\beta_{\chi_d}\le 1-0.30/\mathcal{L}$, we have $I_1\le 0.114$, $I_2\le 1.377$.
6) If $1-0.34/\mathcal{L}\le\beta_{\chi_d}\le 1-0.32/\mathcal{L}$, we have $I_1\le 0.118$, $I_2\le 1.464$.
7) If $1-0.36/\mathcal{L}\le\beta_{\chi_d}\le 1-0.34/\mathcal{L}$, we have $I_1\le 0.131$, $I_2\le 1.374$.
8) If $1-0.38/\mathcal{L}\le\beta_{\chi_d}\le 1-0.36/\mathcal{L}$, we have $I_1\le 0.153$, $I_2\le 1.249$.
9) If $1-0.40/\mathcal{L}\le\beta_{\chi_d}\le 1-0.38/\mathcal{L}$, we have $I_1\le 0.185$, $I_2\le 1.141$.
10) If $1-0.42/\mathcal{L}\le\beta_{\chi_d}\le 1-0.40/\mathcal{L}$, we have $I_1\le 0.229$, $I_2\le 1.047$.
11) If $1-0.44/\mathcal{L}\le\beta_{\chi_d}\le 1-0.42/\mathcal{L}$, we have $I_1\le 0.287$, $I_2\le 0.967$.
12) If $1-0.46/\mathcal{L}\le\beta_{\chi_d}\le 1-0.44/\mathcal{L}$, we have $I_1\le 0.337$, $I_2\le 0.897$.
13) If $1-0.48/\mathcal{L}\le\beta_{\chi_d}\le 1-0.46/\mathcal{L}$, we have $I_1\le 0.372$, $I_2\le 0.835$.
14) If $1-0.50/\mathcal{L}\le\beta_{\chi_d}\le 1-0.48/\mathcal{L}$, we have $I_1\le 0.395$, $I_2\le 0.784$.
15) If $1-0.517/\mathcal{L}\le\beta_{\chi_d}\le 1-0.50/\mathcal{L}$, we have $I_1\le 0.420$, $I_2\le 0.738$.
16) If $1-0.517/\mathcal{L}\ge\beta_{\chi_d}$, we have $I_1\le 0.414$, $I_2\le 0.704$.

Hence in all cases we have

$$I_1 I_2 \le 0.311. \tag{5.4}$$

LEMMA 8. *Let χ_1 be a real non-principal character mod q, $\beta_1 = 1 - \delta_1$ a real zero of $L(s, \chi_1)$, χ a character mod q, and $\varrho = \beta + i\gamma = 1 - \delta + i\gamma$ a zero of $L(s, \chi)$ with $\delta < 1/6$, $\beta \le \beta_1$. Suppose that $D = q(|\gamma| + 1)$ is sufficiently large, that is, $D \ge D_0(\varepsilon)$. Then*

$$\delta_1 \ge (2/3 - \varepsilon)(1 - 6\delta) D^{-(3/2+\varepsilon)\delta/(1-6\delta)} / \log D.$$

This is Theorem 2 of [9].

LEMMA 9. *If the exceptional primitive real character $\widetilde{\chi}$ (mod $\widetilde{q}$) exists, and the unique exceptional zero $\widetilde{\beta}$ of $L(s, \widetilde{\chi})$ satisfies $\widetilde{\delta}(\lambda + \varepsilon) \log x \le 0.239$ where $\widetilde{\delta} = 1 - \widetilde{\beta}$, let χ_q be a primitive character mod q, and $\varrho = \beta + i\gamma = 1 - \delta + i\gamma$ a zero of $L(s, \chi_q)$ with $0 < \delta < \varepsilon$. Suppose that $D_1 = [q, \widetilde{q}](|\gamma| + 1)$ is sufficiently large, that is, $D_1 \ge D_1(\varepsilon)$. Then*

$$\widetilde{\delta} \ge (2/3 - \varepsilon)(1 - 6\delta) D_1^{-(3/2+\varepsilon)\delta/(1-6\delta)} / \log D_1.$$

Proof. This follows by Lemma 8 and the method of Lemma 15 of [1].

By (26) of [1] we have

$$W((\log x)^{10}) \le 10^{-10} x^{1/2}. \tag{5.5}$$

By (5.1)–(5.4) and definitions of $W(Y)$ and $W'(Y)$ we have

$$W(Y) W'(Y) \le 0.311x. \tag{5.6}$$

Now we suppose that the exceptional primitive real character $\widetilde{\chi}$ (mod $\widetilde{q}$) exists, and the unique exceptional real zero $\widetilde{\beta}$ of $L(s, \widetilde{\chi})$ satisfies $\widetilde{\delta}(\lambda + \varepsilon) \log x \le 0.239$ where $\widetilde{\delta} = 1 - \widetilde{\beta}$. In this case, as above we have

$$W(Y, \widetilde{q}) \le W'(Y) \le 0.0107 x^{1/2}, \qquad W(Y) \le 0.884 x^{1/2}. \tag{5.7}$$

Hence we have

$$W(Y) W'(Y) \le 0.0095x. \tag{5.8}$$

We suppose, as we may, that $\widetilde{q} \le Y$, $q \le Y$, $[q, \widetilde{q}] \le x^{\varepsilon}(q, \widetilde{q})$ and $|\gamma| \le Y^{1+\varepsilon} q^{-1}$, and then we may take $D_1 = x^{\lambda + 2\varepsilon}$ in Lemma 9. Therefore if $\widetilde{\delta}(\lambda + \varepsilon) \log x \le 0.005$ and $\delta \le \varepsilon$, then we have

$$\delta \ge \frac{3.26}{\lambda \log x}. \tag{5.9}$$

If $\widetilde{\delta}(\lambda + \varepsilon) \log x \ge 0.005$, $\widetilde{\delta} \ge (2/3 - \varepsilon)(D_1^{1.501\varepsilon} \log D_1)^{-1}$, $\delta \le \varepsilon$, then as above, by Lemma 9 one has

$$\delta \ge -\frac{\log(1.501 \widetilde{\delta} \log D_1)}{1.501 \log D_1}. \tag{5.10}$$

By Lemma 6 we have

$$\sum_{\substack{d\le Y\\ [d_1,d]\le Y^{\varepsilon}(d_1,d)}}\ \sum_{\chi_d}^{*}\ \sum_{\substack{\beta\ge 1/4\\ |\gamma_{\chi_d}|\le Y^{1.01}d^{-1}}}' x^{(1-\varepsilon)(\beta-1)} \tag{5.11}$$
$$\le \frac{1}{\lambda+\varepsilon}\int\limits_{-(\log(1.501\widetilde{\delta}\log D_1))/1.501}^{\infty} e^{-(1-\varepsilon)t/(\lambda+\varepsilon)}N_1^*(t,Y)\,dt+O(x^{-\varepsilon})$$
$$\le 10^{-8}(\widetilde{\delta}\log x)+O(x^{-\varepsilon}).$$

Hence

$$W'(Y)\le 10^{-8}(\widetilde{\delta}\log x)x^{1/2}+O(x^{1/2-\varepsilon}). \tag{5.12}$$

Similarly we have

$$W((\log x)^{10}),W(Y,\widetilde{q})\le 10^{-8}(\widetilde{\delta}\log x)x^{1/2}+O(x^{1/2-\varepsilon}). \tag{5.13}$$

If $x^{-\lambda/10^5}\le\widetilde{\delta}\le(2/3-\varepsilon)(D_1^{1.501\varepsilon}\log D_1)^{-1}$, then as above, by Lemma 9 one has

$$W((\log x)^{10}),W(Y,\widetilde{q}),W'(Y)\le\varepsilon(\widetilde{\delta}\log x)x^{1/2}+O(x^{1/2-0.01}). \tag{5.14}$$

6. Proof of the Theorem. First of all, we suppose that there is no exceptional character. When $(1-\varepsilon)x\le n\le x$, by (4.8), (4.9), (4.15) and (4.19) we have

$$D_1(n)\ge nC(n)-\frac{n}{\phi(n)}\biggl\{8x^{1/2}W((\log x)^{10})$$
$$+2.0797W(Y)W'(Y)+O\biggl(\frac{x^{1/2}W(Y)}{(\log x)^6}\biggr)\biggr\}+O(x^{1-\lambda+\varepsilon}).$$

Since $\lambda=0.0862$, $\prod_{p\ge3}(1-1/(p-1)^2)\ge 0.6601$, by (5.5) and (5.6) it follows that

$$D_1(n)\ge\frac{nx}{\phi(n)}\biggl\{\prod_{p\ge3}\biggl(1-\frac{1}{(p-1)^2}\biggr)-2.0797\cdot 0.311-10^{-9}\biggr\}\ge 0.01x,$$

which proves the assertion.

Now we suppose the exceptional character occurs, and $(1-\varepsilon)x\le n\le x$. By Section 4 we have

$$D_1(n)\ge nC(n)+\widetilde{I}(n)\widetilde{C}(n) \tag{6.1}$$
$$-\frac{n}{\phi(n)}\{8x^{1/2}W((\log x)^{10})+2.0797W(Y)W'(Y)$$
$$+4.1594W(Y,\widetilde{q})+W(Y)x^{(1-\varepsilon)/2}\}$$
$$+O\biggl(\frac{x^{1/2}W(Y)}{(\log x)^6}\biggr)+O\biggl(\widetilde{\chi}^2(n)\frac{\widetilde{q}}{\phi^2(\widetilde{q})}\cdot\frac{n}{\phi(n)}x\biggr)$$
$$+O(x^{1-\lambda+\varepsilon}(n,\widetilde{q})).$$

1) When $(n,\widetilde{q})=1$ or $(n,\widetilde{q})\le x^{(1-10^{-4})\lambda}$ and $\prod_{p\mid\widetilde{q},\,p\nmid n}(p-2)\ge 1/\varepsilon$ we follow the argument of [1]. Thus by (5.7) and (5.8) we have

$$D_1(n)\ge\frac{n}{\phi(n)}\biggl\{x\prod_{p\ge3}\biggl(1-\frac{1}{(p-1)^2}\biggr)-2.0797W(Y)W'(Y)-10^{-8}x \\ -4.1594W(Y,\widetilde{q})x^{1/2}\biggr\}\ge 0.59x. \tag{6.2}$$

2) When $(n,\widetilde{q})>x^{(1-10^{-4})\lambda}$ we have

$$\sum_{\substack{n\le x\\ (n,\widetilde{q})>x^{(1-10^{-4})\lambda}}}1\le x^{1-(1-10^{-4})\lambda+\varepsilon}. \tag{6.3}$$

3) When $1<(n,\widetilde{q})\le x^{(1-10^{-4})\lambda}$ and $\prod_{p\mid\widetilde{q},\,p\nmid n}(p-2)\le 1/\varepsilon$, we notice that $\widetilde{\chi}(n)=0$, and from Lemma 5.1 of [8] we have $\mu(\widetilde{q}/(4,\widetilde{q}))=0$ hence $16\nmid\widetilde{q}$, $p^2\nmid\widetilde{q}$ $(p\ge3)$. Since $\prod_{p\mid\widetilde{q},\,p\nmid n}(p-2)\le 1/\varepsilon$, there exists $\widetilde{q}\le 16(n,\widetilde{q})/\varepsilon^2\le x^{(1-10^{-4})\lambda+\varepsilon}$. By (4.12) and (4.13) we have

$$nC(n)-|\widetilde{I}(n)\widetilde{C}(n)|\ge(n-x^{(1-\varepsilon)(\widetilde{\beta}-1)}n^{\widetilde{\beta}})C(n). \tag{6.4}$$

When $1-\frac{0.239}{(\lambda+\varepsilon)\log x}\le\widetilde{\beta}\le1-\frac{0.005}{(\lambda+\varepsilon)\log x}$, we have

$$x^{(1-\varepsilon)(\widetilde{\beta}-1)}n^{\widetilde{\beta}}\le 0.8905n.$$

By (5.7) and (5.8) we have

$$D_1(n)\ge\frac{nx}{\phi(n)}\biggl\{0.1095\prod_{p\ge3}\biggl(1-\frac{1}{(p-1)^2}\biggr)-2.0797\cdot0.0095 \\ -10^{-8}-4.1594\cdot0.0107\biggr\}\ge0.007x. \tag{6.5}$$

When $1-\frac{0.005}{(\lambda+\varepsilon)\log x}\le\widetilde{\beta}\le1-\bigl(\frac{2}{3}-\varepsilon\bigr)\frac{x^{-1.501\varepsilon\lambda}}{\lambda\log x}$, as in (48) of [1] we have

$$nC(n)-|\widetilde{I}(n)\widetilde{C}(n)|\ge0.62\frac{\widetilde{\delta}nx\log n}{\phi(n)}.$$

By (5.12) and (5.13) we have

$$D_1(n)\ge\frac{\widetilde{\delta}nx\log n}{\phi(n)}\{0.62-2.0797\cdot10^{-7}-(8+4.1594)\cdot10^{-8}\} \\ \ge0.6x^{1-\varepsilon}. \tag{6.6}$$

When $\widetilde{\beta}\ge1-\bigl(\frac{2}{3}-\varepsilon\bigr)\frac{x^{-1.501\varepsilon\lambda}}{\lambda\log x}$, by $\widetilde{q}\le x^{\lambda}$ and Lemma 2 we have

$$x^{-10^{-5}\lambda}\le\widetilde{\delta}\le\biggl(\frac{2}{3}-\varepsilon\biggr)\frac{x^{-1.501\varepsilon\lambda}}{\lambda\log x},$$

and by (5.14) we have

$$D_1(n) \geq \frac{\widetilde{\delta} n x \log n}{\phi(n)}\{0.62 - 20\varepsilon\} \geq 0.6x^{1-10^{-5}\lambda}. \tag{6.7}$$

By (6.1)–(6.7) and Lemma 7 the assertion follows.

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