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Algebraic independence of the values of Mahler functions satisfying implicit functional equations

by

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1. Introduction and results. In a sequence of three papers Mahler ([4]–[6]) discussed the transcendence and algebraic independence of values of functions in several variables satisfying a certain type of functional equation. In his survey article [7], 37 years later, he stated three new problems. The third problem (for the first and second problem cf. Loxton and van der Poorten [3]) dealt with implicit functional equations of the type

$$P(z, f(z), f(Tz)) = 0 \tag{1}$$

with $Tz = z^d$, $d \in \mathbb{Z}$, $d \geq 2$ and a polynomial $P(z, y, u)$ with coefficients in $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. Nishioka [8] (cf. Chapter 1.5 in [11]) solved this problem for polynomial transformations T. In [9] she extended her method to functions in several variables and suitable generalizations of the transformation $Tz = z^d$.

Becker [1] generalized the result of Nishioka to algebraic transformations T. Töpfer gave in [15] a quantitative version of Becker's result. In that article Töpfer asked for a proof of the algebraic independence of the values of several functions satisfying implicit functional equations at algebraic points.

In this paper we follow the proof of Töpfer [15] and derive a lower bound for the transcendence degree of the values of functions $f_1, \ldots, f_m$ satisfying a special system of implicit functional equations for the transformation $Tz = z^d$ with an integer $d \geq 2$. It should be easy to generalize the following result to polynomial or even rational or algebraic transformations T (cf. Becker [1] and Töpfer [14, 15]).

For the development of Mahler's method in the last 15 years see the monograph of Nishioka [11] and the overview article of Waldschmidt [16] for further references.

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Throughout the paper let $\mathbb{K}$ denote an algebraic number field and $\mathcal{O}_{\mathbb{K}}$ the ring of integers in $\mathbb{K}$. As usual we denote by $\overline{|\alpha|}$ the *house* of an algebraic number α, which is the maximum of the absolute values of the conjugates of α. A *denominator* of an algebraic number α is a positive integer D such that $D\alpha \in \mathcal{O}_{\mathbb{K}}$. If $P(z, y_1, \ldots, y_m) =: P(z, \underline{y})$ is a polynomial with complex coefficients, $\deg_z P =: d_z P$ denotes the partial degree of P with respect to z, $\deg_{\underline{y}} P =: d_{\underline{y}} P$ denotes the total degree in $\underline{y} := (y_1, \ldots, y_m)$ and analogous notations in other cases. If the coefficients of P are algebraic, the *height* $H(P)$ of P is defined as the maximum of the houses of the coefficients of P, and the *length* $L(P)$ is the sum of the houses of the coefficients. In what follows let $c, c_0, c_1, \ldots$ and $\gamma_0, \gamma_1, \ldots$ denote positive constants which are independent of the parameters M, N, k, k_0, k_1 used. For a vector $\underline{\mu} \in \mathbb{C}^m$ we define $|\underline{\mu}| := |\mu_1| + \ldots + |\mu_m|$ and by $\mathbb{N}$ and $\mathbb{N}_0$ we denote the positive and nonnegative integers.

Theorem 1. *Let $f_1, \ldots, f_m$ be analytic in a neighborhood U of the origin, algebraically independent over $\mathbb{C}(z)$ and suppose that the coefficients of their power series*

$$f_i(z) = \sum_{j=0}^{\infty} f_{i,j} z^j \qquad (i = 1, \ldots, m)$$

belong to a fixed algebraic number field $\mathbb{K}$ and satisfy

$$\overline{|f_{i,j}|} \le \exp(c_0(1 + j^L)) \quad \text{and} \quad D^{[c_0(1+j^L)]} f_{i,j} \in \mathcal{O}_{\mathbb{K}}$$

for $j \in \mathbb{N}_0$ and $i = 1, \ldots, m$ with suitable constants $D \in \mathbb{N}$ and $L \ge 1$. Let $\underline{n} \in \mathbb{N}^m$ and $\beta := n_1 \cdot \ldots \cdot n_m$. Suppose that the functions $f_1, \ldots, f_m$ satisfy the functional equations

$$a(z) f_j(z^d)^{n_j} = \sum_{\nu=0}^{n_j - 1} P_{\nu,j}(z, \underline{f}(z)) f_j(z^d)^{\nu} \tag{2}$$

with polynomials $a \in \overline{\mathbb{Q}}[z] \backslash \{0\}$ and $P_{0,1}, \ldots, P_{n_m-1,m} \in \overline{\mathbb{Q}}[z, \underline{y}]$ and an integer d satisfying $d > \max\{\beta^L, d_{\underline{y}}(\underline{P})\}$, where $d_{\underline{y}}(\underline{P})$ is defined by

$$d_{\underline{y}}(\underline{P}) := \max\{\deg_{\underline{y}}(P_{0,1}), \ldots, \deg_{\underline{y}}(P_{n_m-1,m})\}.$$

Assume $\alpha \in \overline{\mathbb{Q}}^* \cap U$ and $a(\alpha^{d^k}) \ne 0$ for all $k \in \mathbb{N}_0$. Let m_0 be the smallest integer satisfying

$$m_0 \ge \frac{m \log d - L(m+1) \log \beta \big(1 + \frac{\log \beta}{\log d}\big)}{\log \beta + \log d + \big(L(m+1)\big(1 + \frac{\log \beta}{\log d}\big) + m\big)\big(2 \log \beta + \log d_{\underline{y}}(\underline{P})\big)}.$$

Then

$$\operatorname{trdeg}_{\mathbb{Q}} \mathbb{Q}(f_1(\alpha), \ldots, f_m(\alpha)) \ge m_0.$$

As an application of this theorem we obtain easily the following

COROLLARY 2. *Under the assumptions of Theorem* 1, *if* α, $\underline{f}$ *and the parameters* d, β *and* $d_{\underline{y}}(\underline{P})$ *satisfy for* $m>1$ *the inequality*

$$\frac{\log d_{\underline{y}}(\underline{P})}{\log d} < \frac{1-\frac{\log\beta}{\log d}\big(2m^2-m-1+L(m+1)\big(1+\frac{\log\beta}{\log d}\big)(2m-1)\big)}{(m-1)\big(L(m+1)\big(1+\frac{\log\beta}{\log d}\big)+m\big)},$$

then $f_1(\alpha),\ldots,f_m(\alpha)$ *are algebraically independent.*

REMARKS. (i) Nishioka [8] proved the transcendence of $f(\alpha)$ under the condition $d^2>n^2\max\{d,\deg_y(P)\}$, where f satisfies the functional equation (1) and $n=\deg_u(P)$.

Under the hypotheses of Theorem 1 we get the transcendence of $f(\alpha)$ only under the stronger condition $d>\max\{n^{\sqrt{3}+1},\deg_y(P)\}$. The reason for this is that we have to construct a sequence of polynomials $(Q_k)_{k_0\le k\le k_1}$, where the difference k_1-k_0 has to be relatively large (cf. Lemma 8). In the simpler case $m=1$ it suffices to find just one integer k to obtain a contradiction. By an improvement of the method of proof we get the transcendence of $f(\alpha)$ under the condition $d>\max\{n^2,\deg_y(P)\}$, which coincides with the condition of Nishioka in the case $d>\deg_y(P)$. Note that we have to assume $d>d_{\underline{y}}(\underline{P})$ only for technical reasons (cf. formula (24)).

(ii) Töpfer proved in [15] a transcendence measure for $f(\alpha)$ under the condition $d>n\max\{n,\deg_y(P)\}$.

(iii) For $m\ge 1$ and $\beta=1$ we get the result of Nishioka [10]. In [10] one can also find a lot of applications. For other examples in this case, but $d_{\underline{y}}(\underline{P})=1$, see Chirskiĭ [2] and Töpfer [14].

Our next example deals with infinite products of the form

$$f_n(z):=\prod_{j=0}^{\infty}(1-z^{d^j})^{n^j},$$

where d and n are positive integers with $d\ge 2$.

Let $1\le n_1<\ldots<n_m$ ($m\ge 2$). Then the functions f_{n_i} are analytic for $|z|<1$ and satisfy the functional equations

$$f_{n_i}(z)=(1-z)f_{n_i}(z^d)^{n_i}\qquad (i=1,\ldots,m).$$

Hence we have the following:

COROLLARY 3. *Let* $1\le n_1<\ldots<n_m$ *be integers and* $\beta:=n_1\cdot\ldots\cdot n_m$. *If* α *is algebraic with* $0<|\alpha|<1$ *and* d *is an integer with*

$$\log d>(2m^2-1+\sqrt{4m^4-2m^2+m})\log\beta,$$

then the values

$$\prod_{j=0}^{\infty}(1-\alpha^{d^j})^{n_1^j},\ \ldots,\ \prod_{j=0}^{\infty}(1-\alpha^{d^j})^{n_m^j}$$

are algebraically independent over $\mathbb{Q}$. *Under the corresponding conditions on* α, d *and* n *we get the algebraic independence of*

$$\prod_{j=0}^{\infty}(1-\alpha^{d^j}),\ \prod_{j=0}^{\infty}(1-\alpha^{d^j})^{2^j},\ \ldots,\ \prod_{j=0}^{\infty}(1-\alpha^{d^j})^{n^j}.$$

Remark. Nishioka proved (Theorem 3.4.13 in [11]) the algebraic independence of

$$\prod_{j=0}^{\infty}(1-\alpha^{d^j})\quad (d=2,3,\ldots)$$

for any algebraic number α with $0<|\alpha|<1$.

Proof (of Corollary 3). The algebraic independence of the functions $f_{n_1},\ldots,f_{n_m}$ over $\mathbb{C}(z)$ will be shown in the last section.

By the remark after Lemma 4, $f_{n_1},\ldots,f_{n_m}$ satisfy the conditions for the houses and denominators of the coefficients in Theorem 1 for any $L>1$. Then the assumption of Corollary 3 follows immediately from Theorem 1 and Corollary 2. ∎

2. Preliminaries and auxiliary results. For $\mu\in\mathbb{N}_0$, $\underline{\mu}\in\mathbb{N}_0^m$ and $f_i(z):=\sum_{j=0}^{\infty} f_{i,j}z^j$ $(i=1,\ldots,m)$ we define

$$f_i(z)^\mu := \sum_{j=0}^{\infty} f_{i,j}^{(\mu)} z^j,\qquad f_{i,j}^{(\mu)} := \sum_{\substack{\nu_1,\ldots,\nu_\mu\in\mathbb{N}_0\\ \nu_1+\ldots+\nu_\mu=j}} f_{i,\nu_1}\cdot\ldots\cdot f_{i,\nu_\mu}, \tag{3}$$

$$\begin{gathered} \underline{f}(z)^{\underline{\mu}} := f_1(z)^{\mu_1}\cdot\ldots\cdot f_m(z)^{\mu_m} = \sum_{j=0}^{\infty} f_j^{(\underline{\mu})} z^j,\\ f_j^{(\underline{\mu})} := \sum_{\substack{\nu_1,\ldots,\nu_m\in\mathbb{N}_0\\ \nu_1+\ldots+\nu_m=j}} f_{1,\nu_1}^{(\mu_1)}\cdot\ldots\cdot f_{m,\nu_m}^{(\mu_m)}. \end{gathered} \tag{4}$$

Lemma 4. *If* $\overline{|f_{i,j}|}\le\exp(c_0(1+j^L))$ *and* $D^{[c_0(1+j^L)]}f_{i,j}\in\mathcal{O}_{\mathbb{K}}$ *for* $i=1,\ldots,m$ *and all* $j\in\mathbb{N}_0$ *with* $L\ge 1$ *and* $D\in\mathbb{N}$, *then for all* $\mu\in\mathbb{N}_0$ *and* $\underline{\mu}\in\mathbb{N}_0^m$ *the following assertions hold*:

(i) $\overline{|f_{i,j}^{(\mu)}|}\le\exp(c_1(\mu+j^L))$, $D^{[c_1(\mu+j^L)]}f_{i,j}^{(\mu)}\in\mathcal{O}_{\mathbb{K}}$,

(ii) $\overline{|f_j^{(\underline{\mu})}|}\le\exp(c_2(|\underline{\mu}|+j^L))$, $D^{[c_2(|\underline{\mu}|+j^L)]}f_j^{(\underline{\mu})}\in\mathcal{O}_{\mathbb{K}}$.

Proof. Assertions (i) and (ii) are consequences of the identities (3) and (4) using the fact that the number of $\underline{\nu} \in \mathbb{N}_0^\mu$ with $\nu_1 + \ldots + \nu_\mu = j$ is bounded by $\binom{j+\mu-1}{\mu-1} \le 2^{j+\mu}$. ■

REMARK. If the functions $f_1, \ldots, f_m$ satisfy functional equations of type

$$P_i(z, f_i(z), f_i(z^d)) = 0 \qquad (i = 1, \ldots, m)$$

with polynomials $P_i \in \overline{\mathbb{Q}}[z, y, u]\backslash\{0\}$ and $\deg_u(P_i) \ge 1$, we see that there exist an algebraic number field $\mathbb{K}$, an explicit computable constant $c > 0$ and a positive integer $D \in \mathbb{N}$ such that for $j \in \mathbb{N}_0$ and all $\varepsilon > 0$:

(i) $f_{i,j} \in \mathbb{K}$,
(ii) $\overline{|f_{i,j}|} \le \exp(c(1 + j^{1+\varepsilon}))$,
(iii) $D^{1+j} f_{i,j} \in \mathcal{O}_\mathbb{K}$

hold, i.e. the conditions of Lemma 4 are fulfilled for all $L > 1$. For a proof of this remark see Lemma 1.5.3 of Nishioka [11] and Proposition 1 of Becker [1] for a more general result.

LEMMA 5. *For* $N \in \mathbb{N}$ *there exists a polynomial* $R \in \mathcal{O}_\mathbb{K}[z, \underline{y}]\backslash\{0\}$ *with the following properties*:

(i) $\deg_z R \le N$, $\deg_{\underline{y}} R \le N$,
(ii) $\log H(R) \le c_3 N^{(m+1)L}$,
(iii) $\nu := \operatorname{ord}_0 R(z, \underline{f}(z)) \ge c_4 N^{m+1}$

for suitable constants $c_3, c_4 \in \mathbb{R}_+$.

Proof. Put

$$R(z, \underline{y}) := \sum_{\lambda=0}^{N} \sum_{|\underline{\mu}| \le N} r_{\lambda,\underline{\mu}} z^\lambda \underline{y}^{\underline{\mu}}$$

with $(N+1)\binom{N+m}{m}$ unknown coefficients $r_{\lambda,\underline{\mu}}$. Then

$$R(z, \underline{f}(z)) := \sum_{\lambda=0}^{N} \sum_{|\underline{\mu}| \le N} r_{\lambda,\underline{\mu}} z^\lambda \underline{f}(z)^{\underline{\mu}} = \sum_{h=0}^{\infty} \beta_h z^h \quad \text{(say)}$$

with (cf. the identity (4))

$$\beta_h = \sum_{\lambda=0}^{\min\{h,N\}} \sum_{|\underline{\mu}| \le N} r_{\lambda,\underline{\mu}} f_{h-\lambda}^{(\underline{\mu})}. \tag{5}$$

Assertion (iii) is equivalent to the condition $\beta_h = 0$ for $0 \le h < c_4 N^{m+1}$, and this yields at most $[c_4 N^{m+1}] + 1$ equations in the

$$(N+1)\binom{N+m}{m} \ge \frac{1}{m!} N^{1+m} > 2c_4 N^{m+1} + 1$$

unknowns $r_{\lambda,\underline{\mu}}$ for a suitable constant c_4. After multiplication with a suitable denominator $D^{[c_2 N^{(1+m)L}]}$ according to Lemma 4 the coefficients $f_{h-\lambda}^{(\underline{\mu})}$ are algebraic integers and their houses are bounded by $\exp(c_5(N^{(1+m)L}))$. Siegel's lemma (cf. Hilfssatz 31 in Schneider [12]) yields the assertion. ∎

LEMMA 6. *Let ν be as in Lemma 5 and β_h denote the Taylor coefficients of $R(z, \underline{f}(z))$ as in the proof. Then*

(i) $|\beta_h| \le \exp(c_6(h + N^{(1+m)L})) \le \exp(c_7(h + \nu^L))$.
(ii) $|\beta_\nu| \ge \exp(-c_8 \nu^L)$.
(iii) *Suppose that $k \in \mathbb{N}$ satisfies $d^k \ge c_9 \nu^L$ with ν, N, L as above and a suitable constant $c_9 \in \mathbb{R}_+$ depending only on $\underline{f}$ and α. Then there exist constants $c_{10}, c_{11} \in \mathbb{R}_+$ depending only on $\underline{f}$ and α such that*

$$-c_{10}\nu d^k \le \log|R(T^k(\alpha), \underline{f}(T^k(\alpha)))| \le -c_{11}\nu d^k,$$

where $T^k(\alpha)$ denotes the kth iterate of T at the point α.

Proof. From (5) we get

$$\beta_h = \sum_{\lambda=0}^{\min\{h,N\}} \sum_{|\underline{\mu}| \le N} r_{\lambda,\underline{\mu}} f_{h-\lambda}^{(\underline{\mu})}.$$

This representation together with Lemma 5 and the inequality $|f_{i,j}| \le \exp(\gamma_0(j+1))$ (notice that the functions $f_1, \ldots, f_m$ are analytic in a neighborhood of 0), hence $|f_h^{(\underline{\mu})}| \le \exp(\gamma_1(|\underline{\mu}| + h))$ with $\gamma_0, \gamma_1 \in \mathbb{R}_+$, implies the first estimate of Lemma 6.

For D, L, c_4 as above and ν as in Lemma 5 we get (recall $\nu \ge c_4 N^{1+m}$)

$$D^{[\gamma_2(N+\nu^L)]}\beta_\nu \in \mathcal{O}_{\mathbb{K}}$$

and

$$\overline{|\beta_\nu|} \le \exp(\gamma_3(N^{(1+m)L} + \nu^L + N)) \le \exp(\gamma_4 \nu^L).$$

By a Liouville estimate we obtain the second part.

We now come to the last part of Lemma 6. By Lemma 5 we write

$$R(T^k(\alpha), \underline{f}(T^k(\alpha))) = \beta_\nu (T^k(\alpha))^\nu \left(1 + \sum_{h=1}^{\infty} \frac{\beta_{h+\nu}}{\beta_\nu}(T^k(\alpha))^h\right)$$

and by the assumption on k and the first two parts of Lemma 6 we get

$$\left|\sum_{h=1}^{\infty} \frac{\beta_{h+\nu}}{\beta_\nu}(T^k(\alpha))^h\right| \le \sum_{h=1}^{\infty} \exp(c_7(\nu^L + h) + c_8\nu^L - \gamma_5 h d^k)$$
$$\le \sum_{h=1}^{\infty} \exp(\gamma_6 \nu^L - \gamma_7 h d^k) < \frac{1}{2}.$$

Now the assertion follows from $|T^k(\alpha)|^\nu = \exp(-\gamma_8 \nu d^k)$ and $\exp(-c_8\nu^L) \le |\beta_\nu| \le \exp(2c_7\nu^L)$. ∎

LEMMA 7. *Let* $S, U_1, \ldots, U_d \in \mathbb{C}$ *satisfy* $S^d + U_1 S^{d-1} + \ldots + U_d = 0$ *and*

$$-X_1 \le \log|S| \le -X_2, \qquad \log|U_i| \le Y \quad (1 \le i \le d)$$

for $X_1, X_2, Y \in \mathbb{R}_+$. *Then there exists* $j \in \{1, \ldots, d\}$ *such that*

$$-dX_1 - Y - \log d \le \log|U_j| \le -X_2 + Y + \log d.$$

Proof. This is Lemma 4.2.3 of Wass [17]. ∎

REMARK. The examples $S^d + U_d = 0$ and $S^d + U_1 S^{d-1} = 0$ show that the bounds for $|U_j|$ cannot be improved.

The proof of Theorem 1 depends on the following result from elimination theory, which can be found in Töpfer [13, Theorem 1] with slight modifications.

LEMMA 8. *Suppose* $\underline{\omega} \in \mathbb{C}^m$ *and* $\mathbb{K}$ *is an algebraic number field. Then there exists a constant* $c_{12} = c_{12}(\omega, \mathbb{K}) \in \mathbb{R}_+$ *with the following property: If there exist increasing functions* $\psi_1, \psi_2, \Lambda : \mathbb{N} \to \mathbb{R}_+$, *real numbers* $\Phi_2 \ge \Phi_1 \ge c_{12}$, *positive integers* $k_0 < k_1$, $m_0 \in \{0, \ldots, m\}$ *and polynomials* $(Q_k)_{k_0 \le k \le k_1} \in \mathcal{O}_{\mathbb{K}}[\underline{y}]$ *such that the following assumptions are satisfied*:

(i) $1 \le \psi_1(k+1)/\psi_2(k) \le \Lambda(k)$ *and* $\psi_2(k) \ge c_{12}(\log H(Q_k) + \deg_{\underline{y}} Q_k)$ *for* $k \in \{k_0, \ldots, k_1\}$,

(ii) *the polynomials* $(Q_k)_{k_0 \le k \le k_1}$ *satisfy*, *for* $k \in \{k_0, \ldots, k_1\}$,

(a) $\deg_{\underline{y}} Q_k \le \Phi_1$,
(b) $\log H(Q_k) \le \Phi_2$,
(c) $-\psi_1(k) \le \log|Q_k(\underline{\omega})| \le -\psi_2(k)$,

(iii) $\psi_2(k_1) \ge c_{12}\Lambda(k_1)^{m_0-1}\Phi_1^{m_0-1}\max\{\psi_1(k_0), \Phi_2\}$,

then $\operatorname{trdeg}_{\mathbb{Q}}\mathbb{Q}(\underline{\omega}) \ge m_0$.

3. Construction of an auxiliary function. Since the case $\beta = 1$ (i.e. $n_1 = \ldots = n_m = 1$) was treated by Nishioka [10] we can assume $\beta > 1$.

The proof is rather long, so we give a short sketch of the main steps. In the first step we show how the powers of $f(\alpha)$ can be reduced by using the functional equations. In the second step we consider $R(T^k(\alpha), \underline{f}(T^k(\alpha)))$ for a polynomial R and construct by induction a polynomial R_k, with degrees and height depending only on the degrees and height of R and on $d, \beta, d_{\underline{y}}(\underline{P})$ and k, such that $|R_k(\alpha, \underline{f}(\alpha))|$ has almost the same analytic bounds as $|R(T^k(\alpha), \underline{f}(T^k(\alpha)))|$. In the last step we use this polynomial R_k to construct a suitable sequence of polynomials $Q_k \in \mathcal{O}_{\mathbb{K}}[\underline{y}]$ satisfying the assumptions of Lemma 8 and prove Theorem 1 by Lemma 8.

For a real number a we define $a_+ := \max\{a, 0\} = \frac{1}{2}(a + |a|)$.

Let $\mathbb{K}$ be an algebraic number field containing α, the coefficients of $f_1, \ldots, f_m$ (cf. the assumption of Theorem 1 and Lemma 4) and the coefficients of the polynomials $a, P_{0,1}, \ldots, P_{n_m-1,m}$. Without loss of generality we can assume $a \in \mathcal{O}_{\mathbb{K}}[z]$ and $P_{0,1}, \ldots, P_{n_m-1,m} \in \mathcal{O}_{\mathbb{K}}[z, \underline{y}]$.

In what follows let $k \in \mathbb{N}$ be fixed. Under the conditions of Theorem 1 on α, d and $\underline{f}$ we put for abbreviation

$$\tau_\kappa := \alpha^{d^\kappa}, \quad \varphi_{i,\kappa} := f_i(\alpha^{d^\kappa}) \quad \text{and} \quad \underline{\varphi}_\kappa := (f_1(\alpha^{d^\kappa}), \ldots, f_m(\alpha^{d^\kappa})).$$

For $j = 1, \ldots, m$ let $P_{n_j,j} := a$ and we define the following notations:

$$\begin{aligned} d_z(\underline{P}) &:= \max\{\deg_z(P_{0,1}), \ldots, \deg_z(P_{n_m,m})\}, \\ d_{\underline{y}}(\underline{P}) &:= \max\{\deg_{\underline{y}}(P_{0,1}), \ldots, \deg_{\underline{y}}(P_{n_m,m})\}, \\ L(\underline{P}) &:= \max\{L(P_{0,1}), \ldots, L(P_{n_m,m})\}. \end{aligned}$$

LEMMA 9. *Suppose that $k \in \mathbb{N}$ and $\lambda \in \mathbb{N}_0$. Then for all $j = 1, \ldots, m$ we have*

$$(a(\tau_{k-1})f_j(\tau_k))^\lambda = \sum_{i=0}^{n_j-1} P_{i,\lambda,j}^{(k)}(\tau_{k-1}, \underline{\varphi}_{k-1})(a(\tau_{k-1})f_j(\tau_k))^i$$

with polynomials $P_{i,\lambda,j}^{(k)} \in \mathcal{O}_{\mathbb{K}}[z, \underline{y}]$ satisfying

$$\begin{aligned} d_z(P_{i,\lambda,j}^{(k)}) &\le (\lambda - i)_+ d_z(\underline{P}), \\ d_{\underline{y}}(P_{i,\lambda,j}^{(k)}) &\le (\lambda - i)_+ d_{\underline{y}}(\underline{P}), \\ L(P_{i,\lambda,j}^{(k)}) &\le 2^{(\lambda-n_j)_+} L(\underline{P})^{(\lambda-i)_+}. \end{aligned}$$

Proof. For $\lambda \in \{0, \ldots, n_j - 1\}$ we choose $P_{i,\lambda,j}^{(k)} = \delta_{i,\lambda}$, where $\delta_{i,k}$ is the Kronecker symbol, and the assertions are obvious.

Let now $\lambda = n_j + l$ for $l \in \mathbb{N}_0$. We show the assertion by induction on l. This is obvious for $l = 0$ because of (2) and

$$(a(\tau_{k-1})f_j(\tau_k))^{n_j} = \sum_{i=0}^{n_j-1} P_{i,j}(\tau_{k-1}, \underline{\varphi}_{k-1}) a(\tau_{k-1})^{n_j-1-i}(a(\tau_{k-1})f_j(\tau_k))^i,$$

with $P_{i,n_j,j}^{(k)}(z, \underline{y}) := P_{i,j}(z, \underline{y}) a(z)^{n_j-1-i}$.

In the induction step the assertion follows from

$$\begin{aligned} (a(\tau_{k-1})f_j(\tau_k))^{n_j+l+1} &= (a(\tau_{k-1})f_j(\tau_k))^{n_j+l}(a(\tau_{k-1})f_j(\tau_k)) \\ &= \sum_{i=0}^{n_j-1} P_{i,n_j+l,j}^{(k)}(\tau_{k-1}, \underline{\varphi}_{k-1})(a(\tau_{k-1})f_j(\tau_k))^{i+1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{n_j-2} P^{(k)}_{i,n_j+l,j}(\tau_{k-1},\underline{\varphi}_{k-1})(a(\tau_{k-1})f_j(\tau_k))^{i+1} \\
&\quad + P^{(k)}_{n_j-1,n_j+l,j}(\tau_{k-1},\underline{\varphi}_{k-1})(a(\tau_{k-1})f_j(\tau_k))^{n_j} \\
&= \sum_{i=0}^{n_j-2} P^{(k)}_{i,n_j+l,j}(\tau_{k-1},\underline{\varphi}_{k-1})(a(\tau_{k-1})f_j(\tau_k))^{i+1} \\
&\quad + P^{(k)}_{n_j-1,n_j+l,j}(\tau_{k-1},\underline{\varphi}_{k-1}) \\
&\quad \times \sum_{i=0}^{n_j-1} P^{(k)}_{i,n_j,j}(\tau_{k-1},\underline{\varphi}_{k-1})(a(\tau_{k-1})f_j(\tau_k))^{i} \\
&= \sum_{i=0}^{n_j-1} P^{(k)}_{i,n_j+l+1,j}(\tau_{k-1},\underline{\varphi}_{k-1})(a(\tau_{k-1})f_j(\tau_k))^{i}.
\end{aligned}
$$

So we get

$$
P^{(k)}_{i,n_j+l+1,j}(z,\underline{y}\,) := P^{(k)}_{i-1,n_j+l,j}(z,\underline{y}\,) + P^{(k)}_{n_j-1,n_j+l,j}(z,\underline{y}\,)P^{(k)}_{i,n_j,j}(z,\underline{y}\,),
$$

where $P^{(k)}_{-1,n_j+l,j}(z,\underline{y}\,) := 0$.

By induction it follows that $P^{(k)}_{i,n_j+l+1,j} \in \mathcal{O}_{\mathbb{K}}[z,\underline{y}\,]$ and

$$
\begin{aligned}
d_z(P^{(k)}_{i,n_j+l+1,j}) &\le (n_j+l+1-i)d_z(\underline{P}), \\
d_{\underline{y}}(P^{(k)}_{i,n_j+l+1,j}) &\le (n_j+l+1-i)d_{\underline{y}}(\underline{P}), \\
L(P^{(k)}_{i,n_j+l+1,j}) &\le 2^{l+1}L(\underline{P})^{n_j+l+1-i}. \ \blacksquare
\end{aligned}
$$

In the reduction step we replace $R(\tau_k,\underline{\varphi}_k) =: R_0(\tau_k,\underline{\varphi}_k)$ for an arbitrary polynomial $R \in \mathcal{O}_{\mathbb{K}}[z,\underline{y}\,]$ inductively by $R_l(\tau_{k-l},\underline{\varphi}_{k-l})$ and finally get a polynomial R_k with almost the same bounds for $|R_k(\alpha,\underline{f}(\alpha))|$, the degrees and the height of R_k as R_0.

Lemma 10. *Suppose $k \in \mathbb{N}$ and $R \in \mathcal{O}_{\mathbb{K}}[z,\underline{y}\,]$. Then there exists a polynomial*

$$
R^*(z,\underline{u},\underline{y}\,) := \sum_{\underline{\mu}\in M} R^*_{\underline{\mu}}(z,\underline{u})\underline{y}^{\underline{\mu}} \in \mathcal{O}_{\mathbb{K}}[z,\underline{u},\underline{y}\,]
$$

with $M := \{0,1,\ldots,n_1-1\}\times\ldots\times\{0,1,\ldots,n_k-1\}$ and

$$
\begin{aligned}
d_{y_j}(R^*) &\le n_j-1 \qquad (j=1,\ldots,m), \\
d_z(R^*_{\underline{\mu}}) &\le d\,d_z(R) + d_z(\underline{P})d_{\underline{y}}(R), \\
d_{\underline{u}}(R^*_{\underline{\mu}}) &\le d_{\underline{y}}(\underline{P})d_{\underline{y}}(R), \\
L(R^*_{\underline{\mu}}) &\le L(R)L(\underline{P})^{d_{\underline{y}}(R)}2^{d_{\underline{y}}(R)}
\end{aligned}
$$

such that

$$a(\tau_{k-1})^{d_{\underline{y}}(R)} R(\tau_k, \underline{\varphi}_k) = R^*(\tau_{k-1}, \underline{\varphi}_{k-1}, a(\tau_{k-1})\underline{\varphi}_k).$$

Proof. From the representation

$$R(z, \underline{y}\,) := \sum_{i=0}^{d_z(R)} \sum_{|\underline{j}| \le d_{\underline{y}}(R)} R_{i,\underline{j}} z^i \underline{y}^{\underline{j}}$$

we get, by Lemma 9,

$$\begin{aligned} a(\tau_{k-1})^{d_{\underline{y}}(R)} R(\tau_k, \underline{\varphi}_k) &= \sum_{i=0}^{d_z(R)} \sum_{|\underline{j}| \le d_{\underline{y}}(R)} R_{i,\underline{j}} \tau_k^i a(\tau_{k-1})^{d_{\underline{y}}(R) - |\underline{j}|} (a(\tau_{k-1})\underline{\varphi}_k)^{\underline{j}} \\ &= \sum_{\underline{\mu} \in M} R^*_{\underline{\mu}}(\tau_{k-1}, \underline{\varphi}_{k-1})(a(\tau_{k-1})\underline{\varphi}_k)^{\underline{\mu}}, \end{aligned}$$

where

$$\begin{aligned} &R^*_{\underline{\mu}}(z, \underline{u}) \\ &\quad := \sum_{i=0}^{d_z(R)} \sum_{|\underline{j}| \le d_{\underline{y}}(R)} R_{i,\underline{j}} z^{d^i} a(z)^{d_{\underline{y}}(R) - |\underline{j}|} P^{(k)}_{\mu_1, j_1, 1}(z, \underline{u}) \cdot \ldots \cdot P^{(k)}_{\mu_m, j_m, m}(z, \underline{u}). \end{aligned}$$

Now the bounds for the partial degrees d_{y_j} are obvious. From Lemma 9 we get

$$\begin{aligned} d_z(R^*_{\underline{\mu}}) &\le d d_z(R) + d_z(\underline{P}) d_{\underline{y}}(R) \\ &\qquad + d_z(\underline{P}) \max\Big\{\sum_{i=1}^{m} (j_i - \mu_i)_+ - j_i : |\underline{j}| \le d_{\underline{y}}(R)\Big\} \\ &\le d d_z(R) + d_z(\underline{P}) d_{\underline{y}}(R) \end{aligned}$$

and similarly we derive the upper bound for $d_{\underline{u}}$. The length can be bounded in an analogous way by

$$\begin{aligned} L(R^*_{\underline{\mu}}) &\le L(R) 2^{\max\{\sum_{i=1}^{m} (j_i - n_i)_+ : |\underline{j}| \le d_{\underline{y}}(R)\}} L(\underline{P})^{d_{\underline{y}}(R)} \\ &\le L(R) L(\underline{P})^{d_{\underline{y}}(R)} 2^{d_{\underline{y}}(R)}. \ \blacksquare \end{aligned}$$

LEMMA 11. *Suppose that* $R^* \in \mathcal{O}_{\mathbb{K}}[z, \underline{u}, \underline{y}\,]$ *is the polynomial in Lemma* 10. *Then there exist polynomials* $U_1, \ldots, U_\beta \in \mathcal{O}_{\mathbb{K}}[z, \underline{u}]$ *such that*

$$R^{*\beta} + U_1 R^{*\beta - 1} + \ldots + U_\beta = 0$$

at the point $(z_0, \underline{u}_0, \underline{y}_0) := (\tau_{k-1}, \underline{\varphi}_{k-1}, a(\tau_{k-1})\underline{\varphi}_k)$ *and*

$$\begin{aligned} d_z(U_l) &\le \beta d d_z(R) + \beta d_z(\underline{P})(d_{\underline{y}}(R) + |\underline{n}|), \\ d_{\underline{u}}(U_l) &\le \beta d_{\underline{y}}(\underline{P})(d_{\underline{y}}(R) + |\underline{n}|), \\ L(U_l) &\le \exp(c_{13}(d_z(R) + d_{\underline{y}}(R))) H(R)^\beta. \end{aligned}$$

Proof. With $R^*(z,\underline{u},\underline{y}\,) := \sum_{\underline{\mu}\in M} R^*_{\underline{\mu}}(z,\underline{u})\underline{y}^{\underline{\mu}}$ as in Lemma 10 we put for $\underline{\nu}\in M$,

$$\begin{aligned} R^*(\tau_{k-1},\underline{\varphi}_{k-1},a(\tau_{k-1})\underline{\varphi}_k)(a(\tau_{k-1})\underline{\varphi}_k)^{\underline{\nu}} \\ &= \sum_{\underline{\mu}\in M} R^*_{\underline{\mu}}(\tau_{k-1},\underline{\varphi}_{k-1})(a(\tau_{k-1})\underline{\varphi}_k)^{\underline{\mu}+\underline{\nu}} \\ &= \sum_{\underline{\lambda}\in M} R_{\underline{\lambda},\underline{\nu}}(\tau_{k-1},\underline{\varphi}_{k-1})(a(\tau_{k-1})\underline{\varphi}_k)^{\underline{\lambda}}, \end{aligned}$$

with (cf. Lemma 9)

$$R_{\underline{\lambda},\underline{\nu}}(z,\underline{u}) := \sum_{\underline{\mu}\in M} R^*_{\underline{\mu}}(z,\underline{u})P^{(k)}_{\lambda_1,\mu_1+\nu_1,1}(z,\underline{u})\cdot\ldots\cdot P^{(k)}_{\lambda_m,\mu_m+\nu_m,m}(z,\underline{u}).$$

The degrees and length of $R_{\underline{\lambda},\underline{\nu}}$ can be bounded by Lemmas 9 and 10:

$$\begin{aligned} d_z(R_{\underline{\lambda},\underline{\nu}}) &\le \max_{\underline{\mu}\in M}\Big\{d_z(R^*_{\underline{\mu}}) + \sum_{j=1}^m d_z(P^{(k)}_{\lambda_j,\mu_j+\nu_j,j})\Big\} \\ &\le dd_z(R) + d_z(\underline{P})d_{\underline{y}}(R) + d_z(\underline{P})\max_{\underline{\mu}\in M}\Big\{\sum_{j=1}^m(\mu_j+\nu_j-\lambda_j)_+\Big\} \\ &\le dd_z(R) + d_z(\underline{P})(d_{\underline{y}}(R) + |\underline{n}| + |\underline{\nu}| - |\underline{\lambda}|). \end{aligned}$$

Similarly

$$\begin{aligned} d_{\underline{u}}(R_{\underline{\lambda},\underline{\nu}}) &\le d_{\underline{y}}(\underline{P})(d_{\underline{y}}(R) + |\underline{n}| + |\underline{\nu}| - |\underline{\lambda}|), \\ L(R_{\underline{\lambda},\underline{\nu}}) &\le L(R)L(\underline{P})^{d_{\underline{y}}(R)+|\underline{n}|+|\underline{\nu}|-|\underline{\lambda}|}2^{d_{\underline{y}}(R)+|\underline{\nu}|} \le \gamma_1 L(R)\gamma_2^{d_{\underline{y}}(R)}, \end{aligned}$$

where the constants $\gamma_1,\gamma_2\in\mathbb{R}_+$ depend only on $\underline{P}$ and $\underline{n}$.

Thus the system of β linear equations with β unknowns,

$$\sum_{\underline{\lambda}\in M}\{R_{\underline{\lambda},\underline{\nu}}(\tau_{k-1},\underline{\varphi}_{k-1}) - \delta_{\underline{\lambda},\underline{\nu}}R^*(\tau_{k-1},\underline{\varphi}_{k-1},a(\tau_{k-1})\underline{\varphi}_k)\}\underline{\omega}_{\underline{\lambda}} = 0,$$

where

$$\delta_{\underline{\lambda},\underline{\nu}} := \begin{cases} 1 & \text{if } \underline{\lambda}=\underline{\nu}, \\ 0 & \text{else} \end{cases}$$

is the generalized Kronecker symbol, has for $\underline{\omega} := (\underline{\omega}_{\underline{\lambda}})_{\underline{\lambda}\in M}$ a nontrivial solution

$$\underline{\omega}_{\underline{\lambda}} := (a(\tau_{k-1})\underline{\varphi}_k)^{\underline{\lambda}}.$$

Hence the determinant of the matrix of coefficients must vanish at the point $(z_0,\underline{u}_0,\underline{y}_0) := (\tau_{k-1},\underline{\varphi}_{k-1},a(\tau_{k-1})\underline{\varphi}_k)$, and the expansion of the determinant with respect to the powers of $R^*(\tau_{k-1},\underline{\varphi}_{k-1},a(\tau_{k-1})\underline{\varphi}_k)$ implies

$$0 = \det(R_{\underline{\lambda},\underline{\nu}} - \delta_{\underline{\lambda},\underline{\nu}} R^*)_{\underline{\lambda},\underline{\nu}\in M} = \pm(R^{*\beta} + U_1 R^{*\beta-1} + \ldots + U_\beta)$$

with polynomials $U_l \in \mathcal{O}_{\mathbb{K}}[z,\underline{u}]$.

Since the polynomials U_l are sums of products of the form

$$R_{\underline{\lambda}_1,\underline{\sigma}(\underline{\lambda}_1)} \cdot \ldots \cdot R_{\underline{\lambda}_s,\underline{\sigma}(\underline{\lambda}_s)},$$

where $\underline{\lambda}_1,\ldots,\underline{\lambda}_s \in M$ are pairwise distinct and $\underline{\sigma} := (\sigma_1,\ldots,\sigma_m)$ is a permutation of $\{0,\ldots,n_1-1\} \times \ldots \times \{0,\ldots,n_m-1\}$, for $l \in \{1,\ldots,\beta\}$ we get

$$d_{\underline{u}}(U_l) \le \max_{\underline{\sigma}} \Big\{ \sum_{\underline{\lambda}\in M} d_{\underline{u}}(R_{\underline{\lambda},\underline{\sigma}(\underline{\lambda})}) \Big\} \le \beta d_{\underline{y}}(\underline{P})(d_{\underline{y}}(R) + |\underline{n}|)$$

because

$$\sum_{\underline{\lambda}\in M} (|\underline{\lambda}| - |\underline{\sigma}(\underline{\lambda})|) = 0.$$

By analogy we obtain

$$d_z(U_l) \le \max_{\underline{\sigma}} \Big\{ \sum_{\underline{\lambda}\in M} d_z(R_{\underline{\lambda},\underline{\sigma}(\underline{\lambda})}) \Big\} \le \beta d d_z(R) + \beta d_z(\underline{P})(d_{\underline{y}}(R) + |\underline{n}|).$$

The length of U_l can be bounded by

$$L(U_l) \le \beta! \max\{L(R_{\underline{\lambda},\underline{\nu}}) : \underline{\lambda},\underline{\nu} \in M\}^\beta,$$

with

$$L(R_{\underline{\lambda},\underline{\nu}}) \le \exp(c_{13}(d_z(R) + d_{\underline{y}}(R)))H(R).$$

Lemma 11 is proved. ∎

Now the necessary tools for the reduction step from R_0 to R_k are complete, and we prove for $j = 0,\ldots,k$ the existence of polynomials $R_j \in \mathcal{O}_{\mathbb{K}}[z,\underline{y}]$ such that for $j = 0$,

$$\begin{gathered} d_z(R_0) := d_{1,0}, \quad d_{\underline{y}}(R_0) := d_{2,0}, \quad \log H(R_0) := H_0, \\ \exp(-\psi_1(0)) \le |R_0(\tau_k,\underline{\varphi}_k)| \le \exp(-\psi_2(0)), \end{gathered} \tag{6}$$

and for $j \ge 1$:

$$d_{\underline{y}}(R_j) =: d_{2,j} \le \beta d_{\underline{y}}(\underline{P})(d_{2,j-1} + |\underline{n}|), \tag{7}$$

$$d_z(R_j) =: d_{1,j} \le \beta d d_{1,j-1} + \beta d_z(\underline{P})(d_{2,j-1} + |\underline{n}|), \tag{8}$$

$$\log H(R_j) =: H_j \le \beta H_{j-1} + c_{14}(d_{1,j-1} + d_{2,j-1}). \tag{9}$$

Here the constant $c_{14} > 0$ depends only on $\underline{f}$ and α and

$$\exp(-\psi_1(j)) \le |R_j(\tau_{k-j},\underline{\varphi}_{k-j})| \le \exp(-\psi_2(j)). \tag{10}$$

The functions ψ_1, ψ_2 satisfy for $j \ge 1$ the following recurrence equalities:

$$\psi_1(j) := \beta\psi_1(j-1) + \beta H_{j-1} + c_{15}(d_{1,j-1} + d^{k-j} d_{2,j-1}) + \log\beta, \tag{11}$$

$$\psi_2(j) := \psi_2(j-1) - \beta H_{j-1} - c_{16}(d_{1,j-1} + d_{2,j-1}) - \log\beta \tag{12}$$

provided that

$$\psi_2(0) \geq c_{17}\beta^k(H_0 + d^k(d_{1,0} + d_{2,0})), \tag{13}$$

where $c_{15}, c_{16}, c_{17} \in \mathbb{R}_+$ are suitable constants depending only on $\underline{f}$ and α.

The existence of the polynomials will be proved in the next section. First we will derive upper bounds for $d_{1,j}$, $d_{2,j}$, H_j and $\psi_1(j)$ and a lower bound for $\psi_2(j)$.

Obviously (7) implies

$$d_{2,j} \leq \gamma_0(\beta d_{\underline{y}}(\underline{P}))^j(d_{2,0} + |\underline{n}|) \leq c_{18}(\beta d_{\underline{y}}(\underline{P}))^j d_{2,0},$$

and for $d_{1,j}$ we get inductively (note that $d > d_{\underline{y}}(\underline{P})$ by the condition of Theorem 1)

$$d_{1,j} \leq (\beta d)^j d_{1,0} + \beta d_z(\underline{P}) \sum_{i=0}^{j-1} (\beta d)^i (d_{2,j-i-1} + |\underline{n}|) \leq c_{19}(\beta d)^j(d_{1,0} + d_{2,0}).$$

For H_j, the logarithm of the height of R_j, we get in a similar way

$$H_j \leq \beta^j H_0 + \gamma_1 \sum_{i=0}^{j-1} \beta^i (d_{1,j-i-1} + d_{2,j-i-1}) \leq \beta^j H_0 + c_{20}(d_{1,0} + d_{2,0})(\beta d)^j.$$

Now we can easily deduce from (11) and the above estimates that

$$\begin{aligned} \psi_1(k) &= \beta^k \psi_1(0) \\ &\quad + \sum_{i=0}^{k-1} \beta^i \{\beta H_{k-i-1} + c_{15}(d_{1,k-i-1} + d^i d_{2,k-i-1}) + \log \beta\} \\ &\leq \beta^k \psi_1(0) + k\beta^k H_0 + c_{21}(\beta d)^k (d_{1,0} + d_{2,0}). \end{aligned} \tag{14}$$

In a similar way (cf. (13)) we can derive a lower bound for $\psi_2(k)$:

$$\begin{aligned} \psi_2(k) &= \psi_2(0) - \sum_{i=0}^{k-1} \{\beta H_{k-i-1} + c_{16}(d_{1,k-i-1} + d_{2,k-i-1}) + \log \beta\} \\ &\geq \psi_2(0) - c_{22}\beta^k(H_0 + d^k(d_{1,0} + d_{2,0})). \end{aligned} \tag{15}$$

Now we prove by induction on $j = 0, \ldots, k$ the existence of a sequence of polynomials $R_j \in \mathcal{O}_{\mathbb{K}}[z, \underline{y}]$ satisfying the conditions (6)–(10). For $j = 0$, this is a consequence of Lemmas 5 and 6 with $R_0 := R$ and

$$\begin{aligned} &d_{1,0}, d_{2,0} \leq N, \quad H_0 \leq c_3 N^{(m+1)L}, \\ &\psi_1(0) := c_{10}\nu d^k, \quad \psi_2(0) := c_{11}\nu d^k \end{aligned} \tag{16}$$

provided that $d^k \geq c_9 \nu^L$ for a suitable constant $c_9 > 0$. Now suppose that the assertions are true for $j - 1$ ($j \in \{1, \ldots, k\}$). We apply Lemmas 10 and 11 with R replaced by R_{j-1}. This yields the existence of polynomials

$U_1,\dots,U_\beta \in \mathcal{O}_{\mathbb{K}}[z,\underline{u}]$ with

$$\begin{aligned} d_z(U_l) &\le \beta d d_{1,j-1} + \beta d_z(\underline{P})(d_{2,j-1}+|\underline{n}|),\\ d_{\underline{u}}(U_l) &\le \beta d_{\underline{y}}(\underline{P})(d_{2,j-1}+|\underline{n}|),\\ \log H(U_l) &\le \gamma_1(d_{1,j-1}+d_{2,j-1}) + \beta H_{j-1} \end{aligned}$$

for $l=1,\dots,\beta$ such that

$$R_{j-1}^{*\beta} + U_1 R_{j-1}^{*\beta-1} + \dots + U_\beta = 0$$

for $(z_0,\underline{u}_0,\underline{y}_0) := (\tau_{k-j},\underline{\varphi}_{k-j}, a(\tau_{k-j})\underline{\varphi}_{k-(j-1)})$. Here $R_{j-1}^* \in \mathcal{O}_{\mathbb{K}}[z,\underline{u},\underline{y}]$ is defined analogously to Lemma 10 by

$$\begin{aligned} &a(\tau_{k-j})^{d_{2,j-1}} R_{j-1}(\tau_{k-(j-1)},\underline{\varphi}_{k-(j-1)})\\ &\qquad\qquad = R_{j-1}^*(\tau_{k-j},\underline{\varphi}_{k-j}, a(\tau_{k-j})\underline{\varphi}_{k-(j-1)}). \end{aligned}$$

The induction hypothesis together with the fact that $-\gamma_2 d^k \le \log|a(\tau_k)| \le \gamma_3$ for $k\in\mathbb{N}_0$, implies

$$\begin{aligned} -\psi_1(j-1) - \gamma_4 d^{k-j} d_{2,j-1} \le\ & \log|R_{j-1}^*(\tau_{k-j},\underline{\varphi}_{k-j}, a(\tau_{k-j})\underline{\varphi}_{k-(j-1)})|\\ \le\ & -\psi_2(j-1) + \gamma_5 d_{2,j-1}. \end{aligned}$$

For $l=1,\dots,\beta$ we obtain by a standard estimate together with Lemma 11,

$$\begin{aligned} |U_l(\tau_{k-j},\underline{\varphi}_{k-j})| &\le L(U_l)\max\{1,|\tau_{k-j}|,|\varphi_{1,k-j}|,\dots,|\varphi_{m,k-j}|\}^{d_z(U_l)+d_{\underline{u}}(U_l)}\\ &\le \exp(\beta H_{j-1} + \gamma_6(d_{1,j-1}+d_{2,j-1})), \end{aligned}$$

where the constant $\gamma_6\in\mathbb{R}_+$ depends only on $\underline{f}$ and α.

By (13) and (16) we see that

$$\psi_2(j-1) - (\beta H_{j-1} + \gamma_7(d_{1,j-1}+d_{2,j-1}) + \log\beta) > 0$$

and by Lemma 7 we get the existence of $l_0\in\{1,\dots,\beta\}$ such that

$$\begin{aligned} \log|U_{l_0}(\tau_{k-j},\underline{\varphi}_{k-j})| \le\ & -\psi_2(j-1) + \gamma_8 d_{2,j-1} + \beta H_{j-1}\\ & + \gamma_9(d_{1,j-1}+d_{2,j-1}) + \log\beta\\ \le\ & -\psi_2(j-1) + \beta H_{j-1} + c_{16}(d_{1,j-1}+d_{2,j-1}) + \log\beta\\ =\ & -\psi_2(j) \end{aligned}$$

and

$$\begin{aligned} &\log|U_{l_0}(\tau_{k-j},\underline{\varphi}_{k-j})|\\ &\quad\ge -\beta\psi_1(j-1) - \gamma_{10}\beta d^{k-j} d_{2,j-1} - \beta H_{j-1}\\ &\qquad - \gamma_{11}(d_{1,j-1}+d_{2,j-1}) - \log\beta\\ &\quad\ge -\beta\psi_1(j-1) - \beta H_{j-1} - c_{15}(d_{1,j-1} + \beta d^{k-j} d_{2,j-1}) - \log\beta\\ &\quad= -\psi_1(j). \end{aligned}$$

Thus we put $R_j(z,\underline{y}) := U_{l_0}(z,\underline{y}) \in \mathcal{O}_{\mathbb{K}}[z,\underline{y}]$ and see that (6)–(10) are proved for the polynomial R_j.

4. Proof of Theorem 1. Now the necessary tools for the proof of Theorem 1 are complete. From the preceding section with $j = k$ we know that for $k, N \in \mathbb{N}$ sufficiently large with

$$d^k \geq c_9 \nu^L, \tag{17}$$

$$\nu d^k \geq c_{23} \beta^k (N^{(1+m)L} + d^k N) \tag{18}$$

for sufficiently large constants $c_9, c_{23} > 0$, there exist polynomials $R_k \in \mathcal{O}_{\mathbb{K}}[z,\underline{y}]$ with

$$d_z(R_k) \leq c_{24}(\beta d)^k N, \tag{19}$$

$$d_{\underline{y}}(R_k) \leq c_{18}(\beta d_{\underline{y}}(\underline{P}))^k N, \tag{20}$$

$$\log H(R_k) \leq c_{25}(\beta d)^k N, \tag{21}$$

$$-c_{26}(\beta d)^k \nu \leq \log |R(\alpha, \underline{f}(\alpha))| \leq -c_{27} d^k \nu. \tag{22}$$

The estimates for the degrees (19) and (20) are obvious from (16) and the above estimates. The upper bound for the height (21) of R_k and a lower bound for the right-hand side of (22) could be derived from (18) and (15).

With (14) and (16) it follows from (18) that

$$\psi_1(k) \leq \gamma_1 \beta^k d^k \nu + \gamma_2 k \beta^k (N^{(1+m)L} + d^k N) \leq \gamma_1 \beta^k d^k \nu + \gamma_3 k d^k \nu$$

and this gives the left-hand inequality of (22); note that $\beta \geq 2$.

In order to use Lemma 8 we define the polynomials $(Q_k)_{k_0 \leq k \leq k_1} \in \mathcal{O}_{\mathbb{K}}[\underline{y}]$ by

$$Q_k(\underline{y}) := D^{d_z(R_k)} R_k(\alpha, \underline{y}),$$

where $D \in \mathbb{N}$ is a denominator of α.

Because of (18) and (19) and the condition $d_{\underline{y}}(\underline{P}) < d$ we obtain, for $k \in \mathbb{N}$,

$$\begin{aligned}
d_{\underline{y}}(Q_k) &\leq c_{18}(\beta d_{\underline{y}}(\underline{P}))^k N,\\
\log H(Q_k) &\leq c_{28}(\beta d)^k N,\\
\log |Q_k(\underline{f}(\alpha))| &\leq -c_{29} d^k \nu + c_{30}(\beta d)^k N \leq -c_{31} \nu d^k,\\
\log |Q_k(\underline{f}(\alpha))| &\geq -c_{32} \nu (\beta d)^k.
\end{aligned}$$

Now for $N \in \mathbb{N}$ we define a number $M \geq N$ by $\nu := c_4 M^{m+1}$ and for positive integers $k_0 \leq k \leq k_1$, where $k_0 < k_1$ will be specified later, we define the following functions:

$$
\begin{aligned}
&\Phi_1 := c_{18}(\beta d_{\underline{y}}(\underline{P}))^{k_1}M, \quad \Phi_2 := c_{28}(\beta d)^{k_1}M,\\
&\psi_1(k) := c_{32}\nu(\beta d)^k, \quad \psi_2(k) := c_{31}\nu d^k,\\
&\Lambda(k) := \frac{\psi_1(k+1)}{\psi_2(k)} = \frac{c_{32}d\beta}{c_{31}}\beta^k.
\end{aligned}
\tag{23}
$$

With a sufficiently large constant $\gamma_4 \in \mathbb{R}_+$ we define, for $\nu = c_4M^{1+m}$,

$$
k_0 := \left[\frac{(1+m)L\log M}{\log d} + \gamma_4\right].
$$

Then (17) and condition (i) of Lemma 8 are obviously fulfilled for all $k \ge k_0$.

For $M \ge N$ large enough we have to find a positive integer $k_1 = k_1(M) > k_0$ such that the inequalities (ii) and (iii) of Lemma 8 are satisfied, where the condition (iii) is equivalent to the following two inequalities:

$$
\left(\frac{d}{\beta^{2(m_0-1)}d_{\underline{y}}(\underline{P})^{m_0-1}}\right)^{k_1} \ge c_{33}M^{m_0-1}(d\beta)^{k_0}, \tag{24}
$$

$$
M^{m+1-m_0} \ge c_{34}(\beta^{2(m_0-1)+1}d_{\underline{y}}(\underline{P})^{m_0-1})^{k_1} \tag{25}
$$

with ineffective constants $c_{33}, c_{34} \in \mathbb{R}_+$.

Remark. In the inequality (24) we see that the condition $d > d_{\underline{y}}(\underline{P})$ is necessary to obtain nontrivial results.

Since

$$
m_0 < \frac{m - \sigma L(m+1)(1+\sigma)}{\sigma + 1 + (L(m+1)(1+\sigma)+m)(2\sigma + \log d_{\underline{y}}(\underline{P})/\log d)} + 1
$$

with $\sigma := \log\beta/\log d$, the inequality

$$
\begin{aligned}
((m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P})) + \log\beta)((m_0-1) + L(m+1)(1+\sigma))&\\
< (m+1-m_0)(\log d - (m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P})))&
\end{aligned}
$$

holds. So we can find $\gamma \in \mathbb{R}_+$ satisfying

$$
\begin{aligned}
m+1-m_0 &> \gamma((m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P})) + \log\beta),\\
(m_0-1) + L(m+1)(1+\sigma) &< \gamma(\log d - (m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P}))).
\end{aligned}
$$

Now we choose $N \in \mathbb{N}$ and thereby M large enough, define k_1 by $k_1 := [\gamma\log M]$ and show that the conditions $k_0 < k_1$ and (18) are fulfilled.

Without loss of generality, we may assume that $m_0 \ge 1$ and see that

$$
\gamma > \frac{(m_0-1) + L(m+1)(1+\sigma)}{\log d - (m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P}))} \ge \frac{L(m+1)}{\log d},
$$

which shows $k_0 < k_1$.

To see that (18) is fulfilled, we show that $\nu d^k \geq \gamma_5 \beta^k d^k M$ and $\nu d^k \geq \gamma_6 \beta^k M^{L(m+1)}$ is valid for $k_0 \leq k \leq k_1$.

As $m_0 \geq 1$ we get

$$\gamma < \frac{m+1-m_0}{(m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P})) + \log\beta} \leq \frac{m}{\log\beta},$$

and the inequality $\nu d^k \geq \gamma_5 \beta^k d^k M$ is obvious.

A similar argument leads to $\nu d^k \geq \gamma_6 \beta^k M^{L(m+1)} \geq 1$. From the condition $d > \beta^L$ we obtain, for $k \geq k_0$,

$$k(\log d - \log\beta) > \frac{(1+m)L}{\log d} \log M(\log d - \log\beta) > (L-1)(m+1)\log M,$$

hence

$$\left(\frac{d}{\beta}\right)^k \geq \left(\frac{d}{\beta}\right)^{k_0} \geq \gamma_7 M^{(L-1)(m+1)}.$$

Now we can finish the proof of Theorem 1. We have shown that the conditions (17) and (18) are satisfied with this choice of parameters, if $N \in \mathbb{N}$ is large enough with respect to a constant depending only on α and $\underline{f}$. We get

$$k_1(\log d - (m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P}))) \geq ((m_0-1) + L(m+1)(1+\sigma))\log M + c,$$

$$(m+1-m_0)\log M \geq k_1((m_0-1)\log(\beta^2 d_{\underline{y}}(\underline{P})) + \log\beta) + c,$$

for a suitable constant $c > 0$. This implies the inequalities (24) and (25), hence the condition (iii) of Lemma 8 and thereby the assertion of Theorem 1. ∎

5. Proof of the algebraic independence of the functions considered in Corollary 3. Let

$$f_n(z) := \prod_{j=0}^{\infty} (1 - z^{d^j})^{n^j}.$$

By induction on k we prove that for $\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$, where $i_1, \ldots, i_k$ are pairwise distinct, the functions $f_{n_{i_1}}, \ldots, f_{n_{i_k}}$ are algebraically independent over $\mathbb{C}(z)$. We follow the proof of Proposition 6 in [10].

For abbreviation we put for $j = 1, \ldots, k$ and a positive integer $\nu \in \mathbb{N}$

$$f_{n_{i_j}}(z) := \varphi_j \quad \text{and} \quad f_{n_{i_j}}(z^{d^\nu}) := \varphi_j^{(\nu)}.$$

Assume that φ_1 is algebraic over $\mathbb{C}(z)$. Then by Theorem 1.3 of [11] it is a rational function. Let $\varphi_1 = a(z)/b(z)$, where $a(z)$ and $b(z)$ are relatively prime polynomials. By the functional equation we obtain

$$a(z)b(z^d)^{n_{i_1}} = (1-z)a(z^d)^{n_{i_1}}b(z).$$

Since a and b are relatively prime polynomials, we get $a(z^d)^{n_{i_1}} \mid a(z)$, hence $a \in \mathbb{C}^*$ and

$$(1-z)a^{n_{i_1}-1}b(z) = b(z^d)^{n_{i_1}}.$$

If $dn_{i_1} > 2$ or $\deg b \geq 2$, we get a contradiction by comparing the degrees. In the remaining case it is enough to assume $b(z) = \alpha z + \beta$; then by considering the equation $(1-z)b(z) = b(z^2)$ we see $\alpha = \beta = 0$ and again we obtain a contradiction.

Assume now that the assertion is true for $k-1$, but $\{f_{n_{i_1}}(z), \ldots, f_{n_{i_k}}(z)\}$ $=: \{\varphi_1, \ldots, \varphi_k\}$ are algebraically dependent over $\mathbb{C}(z)$. By $D^{(\nu)}$ and $D^{(\nu)}_\kappa$ we denote the degrees of the following field extensions:

$$D^{(\nu)} := [\mathbb{C}(z)(\varphi_1^{(\nu)}, \ldots, \varphi_k^{(\nu)}) : \mathbb{C}(z)(\varphi_1, \ldots, \varphi_k)],$$
$$D^{(\nu)}_\kappa := [\mathbb{C}(z)(\varphi_1^{(\nu)}, \ldots, \widehat{\varphi_\kappa^{(\nu)}}, \ldots, \varphi_k^{(\nu)}) : \mathbb{C}(z)(\varphi_1, \ldots, \widehat{\varphi}_\kappa, \ldots, \varphi_k)],$$

where $(\varphi_1, \ldots, \widehat{\varphi}_\kappa, \ldots, \varphi_k) := (\varphi_1, \ldots, \varphi_{\kappa-1}, \varphi_{\kappa+1}, \ldots, \varphi_k)$.

In a first step we show that for arbitrary positive integers n and ν,

$$[\mathbb{C}(z)(f_n(z^{d^\nu})) : \mathbb{C}(z)(f_n(z))] = n^\nu;$$

but this is trivial by induction, since the polynomial $P(y) := (1-z)y^n - f_n(z) \in \mathbb{C}(z, f_n(z))[y]$ is irreducible. (Note that $f_n(z)$ is not an algebraic function.)

Now we are able to prove

$$D^{(\nu)}_\kappa = \Big(\prod_{\lambda=1,\, \lambda\neq\kappa}^{k} n_{i_\lambda}\Big)^\nu = \Big(\prod_{\lambda=1}^{k} n_{i_\lambda}\Big)^\nu n_{i_\kappa}^{-\nu}.$$

We prove this formula for simplicity just for $k = \kappa = 3$, but the general case follows similarly.

Since by assumption φ_1 and φ_2 are algebraically independent, we see by the functional equation that $\varphi_1^{(\nu)}$ and $\varphi_2^{(\nu)}$ are also algebraically independent. Hence $\mathbb{C}(z)(\varphi_1^{(\nu)})$ and $\mathbb{C}(z)(\varphi_2^{(\nu)})$ are regular field extensions (cf. Weil [18]), which are linearly disjoint by [18, Theorem I.6]. The assumption now follows from [18, Proposition I.14].

Let d_κ be the degree of φ_κ over $\mathbb{C}(z)(\varphi_1, \ldots, \widehat{\varphi}_\kappa, \ldots, \varphi_k)$, then we get $d^{(\nu)}_\kappa \leq d_\kappa$, where $d^{(\nu)}_\kappa$ denotes the degree of $\varphi^{(\nu)}_\kappa$ over $\mathbb{C}(z)(\varphi_1^{(\nu)}, \ldots, \widehat{\varphi_\kappa^{(\nu)}}, \ldots$ $\ldots, \varphi_k^{(\nu)})$. Finally we obtain by a standard formula

$$D^{(\nu)} d_\kappa = d^{(\nu)}_\kappa D^{(\nu)}_\kappa.$$

Let $\mu, \kappa \in \{1, \ldots, k\}$ and $n_{i_\mu} < n_{i_\kappa}$. By the above formulas we get

$$\left(\frac{n_{i_\kappa}}{n_{i_\mu}}\right)^{\nu} = \frac{D_\mu^{(\nu)}}{D_\kappa^{(\nu)}} = \frac{d_\mu}{d_\mu^{(\nu)}} \cdot \frac{d_\kappa^{(\nu)}}{d_\kappa} \le d_\mu.$$

Since $n_{i_\mu} < n_{i_\kappa}$, this is a contradiction as ν tends to infinity. Thus the algebraic independence of the functions $f_{n_1}, \ldots, f_{n_m}$ is proved.

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References

[1] P.-G. Becker, *Transcendence of the values of functions satisfying generalized Mahler type functional equations*, J. Reine Angew. Math. 440 (1993), 111–128.

[2] V. G. Chirskiĭ, *On the algebraic independence of the values of functions satisfying systems of functional equations*, Proc. Steklov Inst. Math. 218 (1997), 433–438.

[3] J. H. Loxton and A. J. van der Poorten, *Transcendence and algebraic independence by a method of Mahler*, in: Transcendence Theory—Advances and Applications, A. Baker and D. W. Masser (eds.), Academic Press, New York, 1977, 211–226.

[4] K. Mahler, *Arithmetische Eigenschaften der Lösungen einer Klasse von Funktionalgleichungen*, Math. Ann. 101 (1929), 342–366.

[5] —, *Arithmetische Eigenschaften einer Klasse transzendental-transzendenter Funktionen*, Math. Z. 32 (1930), 545–585.

[6] —, *Über das Verschwinden von Potenzreihen mehrerer Veränderlichen in speziellen Punktfolgen*, Math. Ann. 103 (1930), 573–587.

[7] —, *Remarks on a paper by W. Schwarz*, J. Number Theory 1 (1969), 512–521.

[8] K. Nishioka, *On a problem of Mahler for transcendency of function values*, J. Austral. Math. Soc. Ser. A 33 (1982), 386–393.

[9] —, *On a problem of Mahler for transcendency of function values II*, Tsukuba J. Math. 7 (1983), 265–279.

[10] —, *New approach in Mahler's method*, J. Reine Angew. Math. 407 (1990), 202–219.

[11] —, *Mahler Functions and Transcendence*, Lecture Notes in Math. 1631, Springer, 1996.

[12] T. Schneider, *Einführung in die transzendenten Zahlen*, Springer, Berlin, 1957.

[13] T. Töpfer, *An axiomatization of Nesterenko's method and applications on Mahler functions*, J. Number Theory 49 (1994), 1–26.

[14] —, *Algebraic independence of the values of generalized Mahler functions*, Acta Arith. 70 (1995), 161–181.

[15] —, *Simultaneous approximation measures for functions satisfying generalized functional equations of Mahler type*, Abh. Math. Sem. Univ. Hamburg 66 (1996), 177–201.

[16] M. Waldschmidt, *Algebraic independence of transcendental numbers: a survey*, in: Proceedings of the Fifth Conference of the Canadian Number Theory Association; Monograph on Number Theory, Indian National Science Academy, R. P. Bambah *et al.* (eds.), to appear; http://www.math.jussieu.fr/˜miw, 1998.
[17] N. C. Wass, *Algebraic independence of the values at algebraic points of a class of functions considered by Mahler*, Dissertationes Math. 303 (1990).
[18] A. Weil, *Foundations of Algebraic Geometry*, Colloq. Publ. 29, Amer. Math. Soc., 1962.

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