On the Convergence and Summability of Power Series on the Circle of Convergence (I)

by

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§ 1.

1. Let $F(z)$ be a function holomorphic in the circle $|z|<1$, that is

(1.1) $$F(z)=\frac{1}{2}c_0+\sum_{n=1}^{\infty}{}' c_n z^n.$$

The function $F(z)$ is said to belong to the class H^λ, where $\lambda>0$, if the integral

(1.2) $$I_\lambda(\varrho)=I_\lambda(\varrho,F)=\frac{1}{2\pi}\int_0^{2\pi}|F(\varrho e^{i\theta})|^\lambda\, d\theta$$

is bounded for $0\leqslant\varrho<1$. Instead of H^1, we shall write H. It is well known that, if $F(z)$ belongs to H^λ, then the limit

$$F(e^{i\theta})=\lim_{z\to e^{i\theta}}F(z)$$

exists for almost every θ, provided that z tends to $e^{i\theta}$ along any non-tangential path. The function $|F(e^{i\theta})|^\lambda$ is integrable over the interval $0\leqslant\theta\leqslant2\pi$.

Let

$$c_n=a_n-ib_n$$

for $n\geqslant0$, and let

(1.3) $$\frac{1}{2}a_0+\sum_{n=1}^{\infty}{}'(a_n\cos n\theta+b_n\sin n\theta),$$

(1.4) $$-\frac{1}{2}b_0+\sum_{n=1}^{\infty}{}'(a_n\sin n\theta-b_n\cos n\theta)$$

be the real and imaginary parts of the series (1.1) on the circle $z=e^{i\theta}$.

If $\lambda\geqslant1$, the necessary and sufficient condition that the series (1.1) should belong to H^λ, is that both the series (1.3) and (1.4) should be Fourier series of functions of the class L^λ.

Without loss of generality we may suppose that the constant term $\frac{1}{2}c_0$ of the series (1.1) is real, for otherwise we may multiply the series (1.1) by a suitable unit factor $e^{i\alpha}$, which does not influence the value of the integral (1.2). In other words, we may suppose that the constant term of the series (1.4) is equal to zero, and so that that series is of the form

(1.5) $$\sum_{n=1}^{\infty}{}'(a_n\sin n\theta-b_n\cos n\theta).$$

Let $f(\theta)$ be an arbitrary function of period 2π, integrable L over the interval $0\leqslant\theta\leqslant2\pi$. By $s_n(\theta)=s_n(\theta,f)$ and $\bar{s}_n(\theta)=\bar{s}_n(\theta,f)$, we shall denote the partial sums of the series (1.3) and of the conjugate series (1.5) respectively. The partial sums of the series (1.1) will be denoted by $S_n(z)=S_n(z,F)$. Hence

$$s_n(\theta)=\frac{1}{2}a_0+\sum_{\nu=1}^{n}{}'(a_\nu\cos\nu\theta+b_\nu\sin\nu\theta),\qquad \bar{s}_n(\theta)=\sum_{\nu=1}^{n}{}'(a_\nu\sin\nu\theta-b_\nu\cos\nu\theta),$$

$$S_n(z)=\frac{1}{2}c_0+\sum_{\nu=1}^{n}{}' c_\nu z^\nu.$$

We shall also write

(1.6) $$s^*(\theta)=\operatorname*{Max}_n|s_n(\theta)|,\qquad \bar{s}^*(\theta)=\operatorname*{Max}_n|\bar{s}_n(\theta)|,\qquad S^*(z)=\operatorname*{Max}_n|S_n(z)|.$$

The first arithmetic means of the series (1.3), (1.5), and (1.1) will be denoted by $\sigma_n(\theta)=\sigma_n(\theta,f)$, $\bar{\sigma}_n(\theta)=\bar{\sigma}_n(\theta,f)$, $\tau_n(z)=\tau_n(z,F)$ respectively. For example

$$\tau_n(z)=\frac{1}{2}c_0+\sum_{\nu=1}^{n}{}'\left(1-\frac{\nu}{n+1}\right)c_\nu z^\nu.$$

Using a notation analogous to (1.6), we write:

$$\sigma^*(\theta)=\operatorname*{Max}_n|\sigma_n(\theta)|,\qquad \bar{\sigma}^*(\theta)=\operatorname*{Max}_n|\bar{\sigma}_n(\theta)|,\qquad \tau^*(z)=\operatorname*{Max}_n|\tau_n(z)|.$$

By $n_1,n_2,\ldots,n_k,\ldots$ we shall always mean any sequence of positive integers satisfying an inequality

(1.7) $$\frac{n_{k+1}}{n_k}>q>1.$$

We shall also write

$$t^*(\theta)=\operatorname*{Max}_k |s_{n_k}(\theta)|, \quad \bar{t}^*(\theta)=\operatorname*{Max}_k |\bar{s}_{n_k}(\theta)|, \quad T^*(z)=\operatorname*{Max}_k |S_{n_k}(z)|. \tag{1.8}$$

By $A_{\alpha,\beta,\ldots}, B_{\alpha,\beta,\ldots}$, etc. we shall mean positive numbers, not always the same in different contexts, depending only on the parameters $\alpha,\beta,\ldots$ shown explicitly. By A, B, etc. we shall mean positive absolute constants.

2. The Fourier series of functions of L^r, where $r>1$, are known to possess certain interesting properties, some of which fail to hold in the case $r=1$. We shall collect here a number of these properties.

Theorem A [1]). *If f belongs to L^r, where $r>1$, then*

(i) *the sequence $s_{n_k}(\theta)$ converges almost everywhere to $f(\theta)$,*

(ii) *the function $t^*(\theta)$ belongs to L^r, and*

$$\left(\int_0^{2\pi} \{t^*(\theta)\}^r d\theta\right)^{\frac{1}{r}} \leqslant A_{r,\alpha}\left(\int_0^{2\pi} |f(\theta)|^r d\theta\right)^{\frac{1}{r}}. \tag{2.1}$$

Part (i) of this theorem is false for $r=1$. For Kolmogoroff's well known construction of an integrable function whose Fourier series diverges almost everywhere, permits also to obtain an integrable f such that *e. g.*

$$\overline{\lim_{n\to\infty}} |s_{2^n}(\theta, f)| = \infty$$

almost everywhere [2]). Hence also part (ii) of the theorem is false for $r=1$.

Theorem B. *If f belongs to L^r, where $r>1$, the series*

$$\sum_{n=1}^{\infty} \frac{|s_n(\theta)-\sigma_n(\theta)|^2}{n} \tag{2.2}$$

converges almost everywhere [3]). *In particular*

$$\frac{1}{n+1}\sum_{\nu=0}^{n} |s_\nu(\theta)-\sigma_\nu(\theta)|^2 \to 0 \tag{2.3}$$

and so also

$$\frac{1}{n+1}\sum_{\nu=0}^{n} |s_\nu(\theta)-f(\theta)|^2 \to 0 \tag{2.4}$$

for almost every θ.

[1]) Littlewood and Paley [5]. [2]) see Kolmogoroff [4]. [3]) Cf. Zygmund [10].

That (2.4) follows from (2.3) is plain: for almost every θ we have $\sigma_n(\theta)\to f(\theta)$ and so also

$$\frac{1}{n+1}\sum_{\nu=0}^{n} |\sigma_\nu(\theta)-f(\theta)|^2 \to 0. \tag{2.5}$$

In view of the inequality $a^2+b^2 \leqslant 2(a^2+b^2)$, at every point θ where we have (2.3) and (2.5) we also have (2.4).

Schwarz's inequality shows that (2.4) implies

$$\frac{1}{n+1}\sum_{\nu=0}^{n} |s_\nu(\theta)-f(\theta)| \to 0. \tag{2.6}$$

It is known that the exponent 2 in (2.4) may be replaced by any number $l>0$ [4]). The problem whether the relation (2.4), or at least (2.6), holds almost always in the case $r=1$, remains open.

Theorem C [5]). *If f belongs to L^r, where $r>1$, then, for almost every θ, the sequence $1,2,\ldots$ can be broken up into two complementary subsequences $\{\nu_k\}$ and $\{\mu_k\}$, depending in general on θ, and such that*

(a) *$s_{\nu_k}(\theta)$ tends to $f(\theta)$,*

(b) *the series $\sum 1/\mu_k$ converges.*

The result is false for $r=1$ [5]).

3. The main purpose of the paper is to show that part (i) of Theorem A, as well as Theorems B and C, hold for power series of the class H, that is for such Fourier series, whose conjugate series are also Fourier series. More precisely, we have the following propositions.

Theorem 1. *If $F(z)$ belongs to H, and the sequence $\{n_k\}$ satisfies (1.7), then*

(i) *$S_{n_k}(e^{i\theta})$ converges almost everywhere to $F(e^{i\theta})$,*

(ii) $\left(\int_0^{2\pi} \{T^*(e^{i\theta})\}^\mu d\theta\right)^{\frac{1}{\mu}} \leqslant B_{\mu,\alpha}\int_0^{2\pi} |F(e^{i\theta})|\, d\theta$

for every $0<\mu<1$.

[4]) See Hardy and Littlewood [3], Carleman [1], or Zygmund [9], p. 238. The latter book will be quoted, for short, *T. S.*

[5]) Cf. Zygmund [10].

Theorem 2. *Let the integral*

$$\int_0^{2\pi} |F(\varrho e^{i\theta})| \log^+ |F(\varrho e^{i\theta})| \, d\theta \tag{3.1}$$

be bounded for $0 \leqslant \varrho < 1$. Then

$$\int_0^{2\pi} T^*(\theta)\, d\theta \leqslant B \int_0^{2\pi} |F(e^{i\theta})| \log^+ |F(e^{i\theta})| \, d\theta + B.$$

(ii) *Let $m_1 < m_2 < \ldots$ be an arbitrary sequence of positive integers. Let $\varepsilon(u)$ be an arbitrary non-negative function, defined and bounded for $0 \leqslant u < \infty$, and tending to 0 as u tends to infinity. Then there is a function $F(z)$ satisfying the relation*

$$\int_0^{2\pi} |F(\varrho e^{i\theta})| \log^+ |F(\varrho e^{i\theta})| \, \varepsilon(|F(\varrho e^{i\theta})|)\, d\theta = O(1),$$

and such that the function

$$\operatorname*{Max}_k |S_{m_k}(e^{i\theta})|$$

is not integrable over the interval $0 \leqslant \theta \leqslant 2\pi$.

Theorem 3. *If $F(z)$ belongs to H, the series*

$$\sum_{n=1}^{\infty} \frac{|S_n(e^{i\theta}) - \tau_n(e^{i\theta})|^2}{n} \tag{3.2}$$

converges for almost every θ. In particular,

$$\frac{1}{n+1} \sum_{\nu=0}^{n} |S_\nu(e^{i\theta}) - F(e^{i\theta})|^2 \to 0 \tag{3.3}$$

for almost every θ.

Theorem 4. *If $F(z)$ is of the class H, then, for almost every θ, the sequence $1,2,3,\ldots$ can be broken up into two complementary subsequences $\{\nu_k\}$ and $\{\mu_k\}$, such that*

(a) *$S_{\nu_k}(e^{i\theta})$ tends to $F(e^{i\theta})$,*

(b) *the series $\sum 1/\mu_k$ converges.*

4. Before we pass on to the proof of these results, we shall make a few eplanatory remarks.

α) Part (ii) of Theorem 2 is stated here for the sake of completeness only. Its proof is given elsewhere, and we shall not reproduce it here [6]).

β) Part (i) of Theorem A follows from Theorem 1. Part (ii) of Theorem A, however, is not a consequence of Theorem 1. Since the inequality (2.1) is interesting, and its proof, as given by Littlewood and Paley, rather difficult, we shall give here another proof of this inequality. The new proof uses the main ideas of Littlewood and Paley, but at certain points it seems to be simpler. The simplifications can also be applied in other cases.

γ) It has been shown elsewhere that Theorem 4 is a simple consequence of the convergence of the series (3.2) [7]). We shall not repeat the argument here.

Theorems 1, 2, 3, and 4 may also be enunciated for Fourier series. For this purpose it is sufficient to observe that if the function

$$|f(\theta)| \log^+ |f(\theta)| \tag{3.4}$$

is integrable, and (1.3) is the Fourier series of f, the conjugate series (1.5) is the Fourier series of the function

$$\bar{f}(\theta) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta+t) \frac{1}{2} \operatorname{ctg} \frac{1}{2} t \, dt \quad {}^{8}). \tag{3.5}$$

(The integral (3.5) is known to exist, in the principal-value sense, for almost every θ, if f is integrable. The function $\bar{f}$ satisfies the following two inequalities

$$\int_0^{2\pi} |\bar{f}(\theta)| \, d\theta \leqslant A \int_0^{2\pi} |f(\theta)| \log^+ |f(\theta)| \, d\theta + A \quad {}^{9}) \tag{3.6}$$

$$\int_0^{2\pi} |\bar{f}(\theta)| \log^+ |\bar{f}(\theta)| \, d\theta \leqslant B \int_0^{2\pi} |f(\theta)| (\log^+ |f(\theta)|)^2 \, d\theta + B \quad {}^{10}). \tag{3.7}$$

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[6]) Cf. Zygmund [8].
[7]) Cf. Zygmund [10].
[8]) *T. S.*, p. 150.
[9]) *T. S.*, p. 150.
[10]) *T. S.*, p. 165 (ex. 7).

From (3.6) and from Theorems 1 and 3, we see that if the function (3.4) is integrable, then

$$s_{n_k}(\theta)\to f(\theta),\qquad \frac{1}{n+1}\sum_{\nu=0}^{n}|s_\nu(\theta)-f(\theta)|^2\to 0$$

for almost every θ. The reader will have no difficulty in stating the other results in terms of Fourier series.

§ 2.

4. We now pass on to the proofs of the results stated above.

Lemma 1. *Let $\{\varphi_n(t)\}$, where $n=1,2,\ldots$, be the sequence of Rademachers functions, that is $\varphi_n(t)=\operatorname{sign}\sin 2^n\pi t$. Let*

$$s(t)=a_1\varphi_1(t)+a_2\varphi_2(t)+\ldots+a_n\varphi_n(t),\qquad S=\Big(\sum_{\nu=1}^{n}|a_\nu|^2\Big)^{\frac{1}{2}}$$

where the a's are constants, real or complex. Then

$$A_r' S\leqslant\Big(\int_0^1|s(t)|^r\,dt\Big)^{\frac{1}{r}}\leqslant A_r S \tag{4.1}$$

for every $r>0$, and

$$K'S\log^+S-L\leqslant\int_0^1|s(t)|\log^+|s(t)|\,dt\leqslant KS\log^+S+L. \tag{4.2}$$

The inequality (4.1) is well known [11]), and we restrict ourselves to proving (4.2). Let $\varphi(u)=\log(u+e)$ for $u\geqslant 0$. The function $\varphi^2(u)$ is concav for $u\geqslant 0$. Applying Schwarz's and Jensen's inequalities we obtain

$$\int_0^1|s|\log^+|s|\,dt\leqslant\int_0^1|s|\,\varphi(|s|)\,dt\leqslant\Big(\int_0^1|s|^2\,dt\Big)^{\frac{1}{2}}\Big(\int_0^1\varphi^2(|s|)\,dt\Big)^{\frac{1}{2}}$$
$$\leqslant\Big(\int_0^1|s|^2\,dt\Big)^{\frac{1}{2}}\Big\{\varphi^2\Big(\int_0^1|s|\,dt\Big)\Big\}^{\frac{1}{2}}\leqslant A_2S\varphi(A_1S)=S\varphi(S),$$

in view of (4.1) and of the fact that we may take $A_1=A_2=1$. It is easy to verify that $\varphi(u)\leqslant\log^+u+2$. Hence, considering separately the cases $S\geqslant e$ and $S<e$, we may write

$$\int_0^1|s|\log^+|s|\,dt\leqslant S\log^+S+2S\leqslant 3S\log^+S+3e.$$

The second inequality (4.2) is thus established.

[11]) see e.g. *T. S.*, p. 129 (ex. 8).

In order to establish the remaining inequality, we write $\psi(u)=u\log^+u$, $\chi(u)=\psi(u/A_1')$ for $u\geqslant 0$. The functions $\psi(u)$ and $\chi(u)$ are convex and non-decreasing. In view of the first inequality (4.1) and of Jensen's inequality,

$$S\log^+S=\psi(S)\leqslant\psi\Big(\frac{1}{A_1'}\int_0^1|s|\,dt\Big)=\chi\Big(\int_0^1|s|\,dt\Big)\leqslant$$
$$\leqslant\frac{1}{A_1'}\int_0^1|s|\log^+\Big|\frac{s}{A_1'}\Big|\,dt\leqslant\frac{1}{A_1'}\int_0^1|s|\log^+|s|\,dt+\frac{\log 1/A_1'}{A_1'}\int_0^1|s|\,dt. \tag{4.3}$$

Integrating the obvious inequality $|s|\leqslant|s|\log^+|s|+e$ over the interval $0\leqslant t\leqslant 1$, and substituting the result into the right-hand side of (4.3), we obtain the first inequality in (4.2). This completes the proof of the lemma.

The inequalities analogous to (4.2) hold if the function $u\log^+u$ is replaced by more general functions. For example, if $\mu(u)$ is non-negative, increasing, and such that $\mu^2(u)$ is concave, then

$$\int_0^1|s|^r\mu(|s|)\,dt\leqslant A_{2r}^rS^r\mu(S).$$

(If, instead of Schwarz's, we apply Hölder's inequality, we may suppose that $\mu^{1+\varepsilon}(u)$ is concave, where $\varepsilon>0$). A similar argument may be used to obtain an analogue of the first inequality (4.2), but we may then also argue as follows. It is known [11a]) that

$$|s(t)|\geqslant\frac{1}{2}S$$

in a set of t contained in (0,1) and of measure $\geqslant 1/16$. Hence, if $\omega(u)$ denotes an arbitrary non-negative and non-decreasing function, then

$$\int_0^1\omega(|s|)\,dt\geqslant\frac{1}{16}\,\omega\Big(\frac{1}{2}S\Big).$$

Lemma 2. *Let $f_1(\theta),f_2(\theta),\ldots,f_N(\theta)$ be functions of period 2π, integrable L, and let $\bar f_\nu(\theta)$ be the function conjugate to $f_\nu(\theta)$ (that is the function derived from f_ν by means of the formula (3.5)). Then*

$$\int_0^{2\pi}\Big(\sum_\nu|\bar f_\nu|^2\Big)^{\frac{1}{2}r}d\theta\leqslant P_r^r\int_0^{2\pi}\Big(\sum_\nu|f_\nu|^2\Big)^{\frac{1}{2}r}d\theta\ \ ^{12)}\qquad (r>1) \tag{4.4}$$

[11a]) Paley and Zygmund [6].
[12]) Cf. Littlewood and Paley [5], for the case $r=2,4,6,\ldots$

$$(4.5)\qquad \int_0^{2\pi}\Big(\sum_\nu |\bar f_\nu|^2\Big)^{\frac12} d\theta \leqslant Q\int_0^{2\pi}\Big(\sum_\nu|f_\nu|^2\Big)^{\frac12}\log^+\Big(\sum_\nu|f_\nu|^2\Big)\,d\theta+Q,$$

$$(4.6)\qquad \int_0^{2\pi}\Big(\sum_\nu|\bar f_\nu|^2\Big)^{\frac12\mu}d\theta\leqslant R_\mu^\mu\Big(\int_0^{2\pi}\Big(\sum_\nu|f_\nu|^2\Big)^{\frac12}d\theta\Big)^\mu \qquad (0<\mu<1).$$

These inequalities can be deduced respectively from the well known inequalities [13])

$$(4.7)\qquad \int_0^{2\pi}|\bar f|^r\, d\theta \leqslant P_r^{*r}\int_0^{2\pi}|f|^r\, d\theta \qquad (r>1),$$

$$(4.8)\qquad \int_0^{2\pi}|\bar f|\, d\theta\leqslant Q^*\int_0^{2\pi}|f|\log^+|f|\,d\theta+Q^*,$$

$$(4.9)\qquad \int_0^{2\pi}|\bar f|^\mu\, d\theta\leqslant R_\mu^{*\mu}\Big(\int_0^{2\pi}|f|\,d\theta\Big)^\mu \qquad (0<\mu<1),$$

and it is plain that the latter inequalities are contained in the former ones.

For further applications, it is important to observe that the functions f are not supposed to be real. If $f(\theta)=u(\theta)+iv(\theta)$, where u and v are real, then by $\bar f(\theta)$ we mean the function $\bar u(\theta)+i\bar v(\theta)$. (The inequalities (4.7)-(4.9) are usually stated in the case when the function f is real, but it is not difficult to see that they hold for f complex).

We restrict ourselves to proving only one of the inequalities (4.4)-(4.6), *e. g.* to proving (4.6), the argument in the remaining cases being essentially the same.

Let

$$(4.10)\qquad g_t(\theta)=\sum_{\nu=1}^N f_\nu(\theta)\,\varphi_\nu(t), \qquad \bar g_t(\theta)=\sum_{\nu=1}^N \bar f_\nu(\theta)\,\varphi_\nu(t).$$

The function $\bar g_t(\theta)$, where t is a parameter, is conjugate to $g_t(\theta)$. Hence, in view of (4.9),

$$\int_0^{2\pi}|\bar g_t(\theta)|^\mu\, d\theta\leqslant R_\mu^{*\mu}\Big(\int_0^{2\pi}|g_t(\theta)|\,d\theta\Big)^\mu.$$

[13]) Cf. *T. S.*, pp. 147, 150.

Integrating both sides of this inequality with respect to t over the interval $0\leqslant t\leqslant 1$, and making use of Hölder's inequality, we obtain

$$\int_0^1 dt\int_0^1|\bar g_t(\theta)|^\mu\, d\theta\leqslant R_\mu^{*\mu}\Big(\int_0^1 dt\int_0^{2\pi}|g_t(\theta)|\,d\theta\Big)^\mu.$$

If we now invert the order of integration with respect to θ and t, and take account of Lemma 1, we obtain (4.6), with $R_\mu=R_\mu^* A_1/A'_\mu=R_\mu^*/A'_\mu$.

This completes the proof. The lemma holds, of course, also in the case $N=\infty$.

5. *Lemma 3.* *Let $f_1,f_2,\dots,f_N$, be the functions of Lemma 2, and let $s_{n,\nu}$ denotes the ν-th partial sum of the Fourier series of f_n. Let $k=k_n$ be an arbitrary function of n. Then,*

$$(5.1)\qquad \int_0^{2\pi}\Big(\sum_{n=1}^N|s_{n,k}|^2\Big)^{\frac12 r}d\theta\leqslant A_r^r\int_0^{2\pi}\Big(\sum_{n=1}^N|f_n|^2\Big)^{\frac12 r}d\theta \qquad (r>1),$$

$$(5.2)\qquad \int_0^{2\pi}\Big(\sum_{n=1}^N|s_{n,k}|^2\Big)^{\frac12}d\theta\leqslant B\int_0^{2\pi}\Big(\sum_{n=1}^N|f_n|^2\Big)^{\frac12}\log^+\Big(\sum_{n=1}^N|f_n|^2\Big)\,d\theta+B,$$

$$(5.3)\qquad \int_0^{2\pi}\Big(\sum_{n=1}^N|s_{n,k}|^2\Big)^{\frac12\mu}d\theta\leqslant C_\mu^\mu\Big(\int_0^{2\pi}\Big(\sum_{n=1}^\infty|f_n|^2\Big)^{\frac12}d\theta\Big)^\mu \qquad (0<\mu<1).$$

The inequalities hold if we replace $s_{n,\nu}$ by $\bar s_{n,\nu}$, that is by the ν-th partial sum of the series conjugate to the Fourier series of f_n.

It is again sufficient to restrict ourselves to one of the inequalities, *e. g.* to (5.3). The argument uses a familiar device. We write

$$(5.4)\qquad \overline{f_n\cos k\theta}=g_n(\theta), \qquad \overline{f_n\sin k\theta}=h_n(\theta).$$

Using (3.5), the formula for $s_{n,k}$ may be written

$$(5.5)\qquad s_{n,k}=g_n(\theta)\sin k\theta-h_n(\theta)\cos k\theta+a_n(\theta),$$

where

$$(5.6)\qquad a_n(\theta)=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cos k_n(t-\theta)\,dt.$$

Let I denote the left-hand side of (5.3). In virtue of (5.5), we may write

(5.7)
$$I \leqslant 3^{\frac{1}{2}\mu} \int_0^{2\pi} \Big(\sum_n |g_n|^2 + \sum_n |h_n|^2 + \sum_n |a_n|^2\Big)^{\frac{1}{2}\mu} d\theta$$
$$\leqslant 3^{\frac{1}{2}\mu} \int_0^{2\pi} \Big\{\Big(\sum_n |g_n|^2\Big)^{\frac{1}{2}\mu} + \Big(\sum_n |h_n|^2\Big)^{\frac{1}{2}\mu} + \Big(\sum_n |a_n|^2\Big)^{\frac{1}{2}\mu}\Big\} d\theta.$$

From (5.6) we obtain

$$\int_0^{2\pi} \Big(\sum_n |a_n|^2\Big)^{\frac{1}{2}\mu} d\theta \leqslant (2\pi)^{1-\mu} \Big\{\int_0^{2\pi} \Big(\sum_n |a_n|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu}$$
$$\leqslant (2\pi)^{1-\mu} \Big\{\int_0^{2\pi} \Big(\sum_n \Big(\int_0^{2\pi} |f_n(t)|\, dt\Big)^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu} = 2\pi \Big\{\sum_n \Big(\int_0^{2\pi} |f_n|\, dt\Big)^2\Big\}^{\frac{1}{2}\mu}.$$

Hence, by Minkowskis inequality,

(5.8)
$$\int_0^{2\pi} \Big(\sum_n |a_n|^2\Big)^{\frac{1}{2}\mu} d\theta \leqslant 2\pi \Big\{\int_0^{2\pi} \Big(\sum_n |f_n|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu}.$$

On the other hand, from (5.4) and (4.6) we deduce

(5.9)
$$\int_0^{2\pi} \Big(\sum_n |g_n|^2\Big)^{\frac{1}{2}\mu} d\theta \leqslant R_\mu^\mu \Big\{\int_0^{2\pi} \Big(\sum_n |f_n|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu},$$
$$\int_0^{2\pi} \Big(\sum_n |h_n|^2\Big)^{\frac{1}{2}\mu} d\theta \leqslant R_\mu^\mu \Big\{\int_0^{2\pi} \Big(\sum_n |f_n|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu}.$$

From (5.7), (5.8), and (5.9) we obtain

$$I \leqslant (2R_\mu^\mu + 2\pi)\, 3^{\frac{1}{2}\mu} \Big\{\int_0^{2\pi} \Big(\sum_n |f_n|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu},$$

which gives (5.3).

6. If (1.3) is the Fourier series of a function f, we write

(6.1)
$$s_\nu(\varrho,\theta) = \frac{1}{2} a_0 + \sum_{m=1}^{\nu} (a_m \cos m\theta + b_m \sin m\theta)\varrho^m,$$
$$f(\varrho,\theta) = \frac{1}{2} a_0 + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta)\varrho^m.$$

Lemma 4. *Let $f_1, f_2, \ldots, f_N$ be a set of functions integrable L and of period 2π, and let $s_{n,\nu}(\varrho,\theta)$ denote the sum analogous to (6.1) but derived from the Fourier series of the function f_n. Let $k=k_n$ and $\varrho=\varrho_n$ (where $0 \leqslant \varrho_n \leqslant 1$) be two arbitrary functions of n. Then*

(6.2)
$$\int_0^{2\pi} \Big(\sum_{n=1}^N |s_{n,k}(\varrho_n,\theta)|^2\Big)^{\frac{1}{2}r} d\theta \leqslant K_r^r \int_0^{2\pi} \Big(\sum_{n=1}^N |f_n(\theta)|^2\Big)^{\frac{1}{2}r} d\theta \qquad (r>1)$$

(6.3)
$$\int_0^{2\pi} \Big(\sum_{n=1}^N |s_{n,k}(\varrho_n,\theta)|^2\Big)^{\frac{1}{2}} d\theta \leqslant L \int_0^{2\pi} \Big(\sum_{n=1}^N |f_n(\theta)|^2\Big)^{\frac{1}{2}} \log^+ \Big(\sum_{n=1}^N |f_n(\theta)|^2\Big) d\theta + L$$

(6.4)
$$\int_0^{2\pi} \Big(\sum_{n=1}^N |s_{n,k}(\varrho_n,\theta)|^2\Big)^{\frac{1}{2}\mu} d\theta \leqslant M_\mu^\mu \Big\{\int_0^{2\pi} \Big(\sum_{n=1}^N |f_n(\theta)|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu} \qquad (0<\mu<1).$$

In the left-hand sides we may replace s_{n,k_n} by $\bar{s}_{n,k_n}$, where $\bar{s}_{n,k_n}$ is the polynomial conjugate to s_{n,k_n}.

It is sufficient to prove (6.4). We shall write $s_{n,k}(\theta)$ instead of $s_{n,k}(1,\theta)$. Abel's transformation and Schwarz's inequality give

$$|s_{n,k}(\varrho_n,\theta)|^2 = \Big|(1-\varrho_n)\sum_{\nu=0}^{k-1} s_{n,\nu}(\theta)\varrho_n^\nu + s_{n,k}(\theta)\varrho_n^k\Big|^2 \leqslant$$
$$\leqslant 2\Big((1-\varrho_n)\sum_{\nu=0}^{k-1} |s_{n,\nu}(\theta)|^2 \varrho_n^\nu + |s_{n,k}(\theta)|^2 \varrho_n^{2k}\Big).$$

Let I denote the left-hand side of (6.4). We may write

$$I \leqslant 2^{\frac{1}{2}\mu} \int_0^{2\pi} \Big(\sum_{n=1}^N \sum_{\nu=0}^{k_n} (1-\varrho_n)\, |s_{n,\nu}(\theta)|^2 \varrho_n^\nu + \sum_{n=1}^N |s_{n,k_n}(\theta)|^2\Big)^{\frac{1}{2}\mu} d\theta.$$

In view of the inequality (5.3) of Lemma 3, the last expression does not exceed

$$2^{\frac{1}{2}\mu} C_\mu^\mu \Big\{\int_0^{2\pi} \Big(\sum_{n=1}^N \sum_{\nu=0}^{k_n} (1-\varrho_n)\, |f_n(\theta)|^2 \varrho_n^\nu + \sum_{n=1}^N |f_n(\theta)|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu} \leqslant$$
$$\leqslant 2^\mu C_\mu^\mu \Big\{\int_0^{2\pi} \Big(\sum_{n=1}^N |f_n|^2\Big)^{\frac{1}{2}} d\theta\Big\}^{\mu},$$

and the inequality (6.4) is established.

If we take $k_1 = k_2 = \ldots = k_N = \infty$, we obtain inequalities for harmonic functions $f_1(\varrho,\theta), f_2(\varrho,\theta), \ldots, f_N(\varrho,\theta)$.

Lemma 5. *Let $0\leqslant\varrho_n<1$ for $n=1,2,\ldots,N$, and let Δ_n denote an arbitrary interval situated in $(\varrho_n,1)$ as well as the length of this interval. Then, under the hypotheses of Lemma 4,*

$$(6.5)\quad \int_0^{2\pi}\Big(\sum_{n=1}^{N}|s_{n,k}(\varrho_n,\theta)|^2\Big)^{\frac{1}{2}r}d\theta\leqslant K_r^r\int_0^{2\pi}\Big(\sum_{n=1}^{N}\frac{1}{\Delta_n}\int_{\Delta_n}|f_n(\varrho,\theta)|^2d\varrho\Big)^{\frac{1}{2}r}d\theta\quad(r>1),$$

$$(6.6)\quad \begin{aligned}&\int_0^{2\pi}\Big(\sum_{n=1}^{N}|s_{n,k}(\varrho_n,\theta)|^2\Big)^{\frac{1}{2}}d\theta\leqslant\\ &\leqslant L\int_0^{2\pi}\Big(\sum_{n=1}^{N}\frac{1}{\Delta_n}\int_{\Delta_n}|f_n(\varrho,\theta)|^2d\varrho\Big)^{\frac{1}{2}}\log^+\Big(\sum_{n=1}^{N}\frac{1}{\Delta_n}\int_{\Delta_n}|f_n(\varrho,\theta)|^2d\varrho\Big)d\theta+L,\end{aligned}$$

$$(6.7)\quad \begin{aligned}&\int_0^{2\pi}\Big(\sum_{n=1}^{N}|s_{n,k}(\varrho_n,\theta)|^2\Big)^{\frac{1}{2}\mu}d\theta\leqslant\\ &\leqslant M_\mu^\mu\Big\{\int_0^{2\pi}\Big(\sum_{n=1}^{N}\frac{1}{\Delta_n}\int_{\Delta_n}|f_n(\varrho,\theta)|^2d\varrho\Big)^{\frac{1}{2}}d\theta\Big\}^\mu\qquad(0<\mu<1),\end{aligned}$$

where the coefficients K_r, L, M_μ are the same as in Lemma 4. The sums $s_{n,k}(\theta)$ may be replaced by $\bar{s}_{n,k}(\theta)$.

We restrict ourselves to proving (6.7). We may suppose without loss of generality that no interval Δ_n contains the point $\varrho=1$. If ϱ_n' is any number between ϱ_n and 1, the inequality (6.4) gives

$$(6.8)\qquad \int_0^{2\pi}\Big(\sum_{n=1}^{N}|s_{n,k}(\varrho_n,\theta)|^2\Big)^{\frac{1}{2}\mu}d\theta\leqslant M_\mu^\mu\Big\{\int_0^{2\pi}\Big(\sum_{n=1}^{N}|f(\varrho_n',\theta)|^2\Big)^{\frac{1}{2}}d\theta\Big\}^\mu.$$

Now let m be a positive integer, and $\varrho_n^{(i)}$, where $i=1,2,\ldots,m$, the left-hand ends of m equal intervals into which we divide Δ_n. We replace every term $|s_{n,k}(\varrho_n,\theta)|^2$ on the left of (6.8) by m terms, each equal to $m^{-1}|s_{n,k}(\varrho_n,\theta)|^2$. Similarly we replace the term $|f_n(\varrho_n',\theta)|^2$ on the right by the sum $\sum_{i=1}^{m}m^{-1}|f_n(\varrho_n^{(i)},\theta)|^2$. If in this new inequality we make m tend to ∞, and note that $f_n(\varrho,\theta)$ is continuous for ϱ of Δ_n and arbitrary θ, we obtain (6.7).

7. Lemma 6. *Let $F(z)$ be a function of the class H^λ, where $\lambda>0$, and let*

$$(7.1)\qquad g(\theta)=\Big(\int_0^1(1-\varrho)|F'(\varrho e^{i\theta})|^2d\varrho\Big)^{\frac{1}{2}}.$$

Then

$$(7.2)\qquad \Big(\int_0^{2\pi}g^\lambda(\theta)\,d\theta\Big)^{\frac{1}{\lambda}}\leqslant A_\lambda\Big(\int_0^{2\pi}|F(e^{i\theta})|^\lambda d\theta\Big)^{\frac{1}{\lambda}}.$$

This result is due to Littlewood and Paley [14]). They state it for $\lambda>1$, but the proof holds, without any change, for any $\lambda>0$. We shall require the lemma in the case $\lambda\geqslant1$ only. The limiting case $\lambda=1$ is the only one which is required for the proof of Theorems 1-4.

Littlewood and Paley state the inequality (7.2) in a slightly different form, *viz.* they replace the right hand side of (7.2) by

$$A_\lambda\Big(\int_0^{2\pi}|f(\theta)|^\lambda d\theta\Big)^{1/\lambda},$$

where $f(\theta)=\Re F(e^{i\theta})$. In this form the inequality cannot be extended to the values $0<\lambda\leqslant1$.

In stating their inequality (7.2), Paley and Littlewood assume, for simplicity, that $F(0)=0$. This hypothesis is not necessary. It is easy to see that if $\lambda\geqslant1$, and if (7.2) is valid for the functions $F(z)$ vanishing at the origin, then that inequality is also valid for any function of the class H^λ. The same result holds in the case $0<\lambda<1$, as the usual argument, based on F. Riesz's decomposition theorem, shows.

8. The following is the main lemma of the paper.

Lemma 7. *Let $S_n(z)$ and $\tau_n(z)$ denote respectively the partial sums and the first arithmetic means of the series (1.1), representing a function $F(z)$ of the class H. Then*

$$(8.1)\qquad \int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{|S_n(e^{i\theta})-\tau_n(e^{i\theta})|^2}{n}\Big)^{\frac{1}{2}r}d\theta\leqslant P_r^r\int_0^{2\pi}|F(e^{i\theta})|^r d\theta\qquad(r>1),$$

$$(8.2)\qquad \int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{|S_n(e^{i\theta})-\tau_n(e^{i\theta})|^2}{n}\Big)^{\frac{1}{2}}d\theta\leqslant Q\int_0^{2\pi}|F(e^{i\theta})|\log^+|F(e^{i\theta})|\,d\theta+Q,$$

$$(8.3)\qquad \int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{|S_n(e^{i\theta})-\tau_n(e^{i\theta})|^2}{n}\Big)^{\frac{1}{2}\mu}d\theta\leqslant\Big(R_\mu\int_0^{2\pi}|F(e^{i\theta})|\,d\theta\Big)^\mu\qquad(0<\mu<1).$$

[14]) loc. cit.

Similarly, if $\{n_k\}$ *denotes any sequence of positive integers satisfying the condition* (1.7), *then*

$$(8.4)\qquad \int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|S_{n_k}(e^{i\theta})-\tau_{n_k}(e^{i\theta})|^2\Big)^{\frac{1}{2}r}d\theta\leqslant P_{r,\alpha}^r\int_0^{2\pi}|F(e^{i\theta})|^r\,d\theta \qquad (r>1),$$

$$(8.5)\qquad \int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|S_{n_k}(e^{i\theta})-\tau_{n_k}(e^{i\theta})|^2\Big)^{\frac{1}{2}}d\theta\leqslant Q_\alpha\int_0^{2\pi}|F(e^{i\theta})|\log^+|F(e^{i\theta})|\,d\theta+Q_\alpha,$$

$$(8.6)\qquad \int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|S_{n_k}(e^{i\theta})-\tau_{n_k}(e^{i\theta})|^2\Big)^{\frac{1}{2}\mu}d\theta\leqslant R_{\mu,\alpha}^{\mu}\Big(\int_0^{2\pi}|F(e^{i\theta})|\,d\theta\Big)^{\mu}.$$

The most important for our purposes are the inequalities (8.3) and (8.6), and we shall confine our attention to these inequalities only (the proofs in the remaining cases are similar).

We first observe that

$$(8.7)\qquad S_n(e^{i\theta})-\tau_n(e^{i\theta})=-i\frac{S_n'(e^{i\theta})}{n+1}.$$

Here, and everywhere in the proof of Lemma 7, a dash denotes differentiation with respect to θ. For $0\leqslant\varrho<1$, Abel's transformation gives

$$S_n'(e^{i\theta})=\varrho^{-n}S_n'(\varrho e^{i\theta})-(1-\varrho)\sum_{\nu=0}^{n-1}\varrho^{-\nu-1}S_\nu'(\varrho e^{i\theta}).$$

Hence

$$(8.8)\qquad |S_n'(e^{i\theta})|^2\leqslant 2\Big\{\varrho^{-2n}|S_n'(\varrho e^{i\theta})|^2+(1-\varrho)^2\Big(\sum_{\nu=0}^{n-1}\varrho^{-\nu-1}|S_\nu'(\varrho e^{i\theta})|\Big)^2\Big\}$$

$$\leqslant 2\Big\{\varrho^{-2n}|S_n'(\varrho e^{i\theta})|^2+\frac{1-\varrho}{\varrho^n}\sum_{\nu=0}^{n-1}\varrho^{-\nu-1}|S_\nu'(\varrho e^{i\theta})|^2\Big\}.$$

We write $\varrho=\varrho_n=1-\dfrac{1}{n+1}$. Let $\Delta_n=(\varrho_n,\varrho_{n+1})$. From (8.7), (8.8), and Lemma 5 we obtain that the left-hand side I of (8.3) satisfies the inequality

$$I\leqslant 2^{\frac{1}{2}\mu}\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{|S_n'(\varrho_n e^{i\theta})|^2}{n^3\varrho_n^{2n}}+\sum_{n=1}^{\infty}\frac{1-\varrho_n}{n^3\varrho_n^n}\sum_{\nu=0}^{n-1}\varrho_n^{-\nu-1}|S_\nu'(\varrho_n e^{i\theta})|^2\Big)^{\frac{1}{2}\mu}d\theta$$

$$\leqslant 2^{\frac{1}{2}\mu}e^\mu M_\mu^\mu\Big\{\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{1}{n^3\Delta_n}\int_{\Delta_n}|F'(\varrho e^{i\theta})|^2\,d\varrho+\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{\nu=0}^{n-1}\frac{1}{\Delta_\nu}\int_{\Delta_\nu}|F'(\varrho e^{i\theta})|^2d\varrho\Big)^{\frac{1}{2}}d\theta\Big\}^\mu$$

$$\leqslant (2eM_\mu)^\mu\Big\{\int_0^{2\pi}\Big(\sum_{n=1}^{\infty}\frac{(n+1)(n+2)^2}{n^3}\int_{\Delta_n}(1-\varrho)|F'(\varrho e^{i\theta})|^2d\varrho\Big)^{\frac{1}{2}}d\theta\Big\}^\mu.$$

Thence we deduce that

$$(8.9)\qquad I\leqslant(10\,eM_\mu)^\mu\Big(\int_0^{2\pi}g(\theta)\,d\theta\Big)^\mu,$$

since $\Big|\dfrac{d}{d\theta}F(\varrho e^{i\theta})\Big|\leqslant\Big|\dfrac{d}{dz}F(z)\Big|_{z=\varrho e^{i\theta}}$. The inequality (8.3) is a consequence of (8.9) and of Lemma 6.

The proof of (8.6) is analogous to that of (8.3). From the condition (1.7) it follows that there is a number $\alpha'=\alpha'(\alpha)>1$, such that $(n_{m+1}+1)/(n_m+1)>\alpha'$ for $m=1,2,\dots$ Let

$$\varrho_m=1-\frac{1}{n_m+1},\qquad \Delta_m=\Big(\varrho_m,1-\frac{1}{\alpha'(n_m+1)}\Big),$$

so that no two Δ_m have points in common. Let I' denote the left-hand side of (8.6). If we replace n by n_m and ϱ by ϱ_m in (8.8), we may write

$$I'\leqslant(2e^2)^{\frac{1}{2}\mu}\int_0^{2\pi}\Big\{\sum_{m=1}^{\infty}\Big(n_m^{-2}|S_{n_m}'(\varrho_m e^{i\theta})|^2+n_m^{-3}\sum_{\nu=0}^{n_m-1}|S_\nu'(\varrho_m e^{i\theta})|^2\Big)\Big\}^{\frac{1}{2}\mu}d\theta$$

$$\leqslant(2e^2)^{\frac{1}{2}\mu}M_\mu^\mu\Big\{\int_0^{2\pi}\Big(2\sum_{m=1}^{\infty}n_m^{-2}\frac{1}{\Delta_m}\int_{\Delta_m}|F'(\varrho e^{i\theta})|^2d\varrho\Big)^{\frac{1}{2}}d\theta\Big\}^\mu$$

$$\leqslant(2eM_\mu)^\mu\Big\{\int_0^{2\pi}\Big(\frac{\alpha'^2}{\alpha'-1}\sum_{m=1}^{\infty}\Big(\frac{n_m+1}{n_m}\Big)^2\int_{\Delta_m}|F'(\varrho e^{i\theta})|^2(1-\varrho)\,d\varrho\Big)^{\frac{1}{2}}d\theta\Big\}^\mu.$$

Hence

$$(8.10)\qquad I'\leqslant\Big(\frac{4\alpha' eM_\mu}{(\alpha'-1)^{\frac{1}{2}}}\Big)^\mu\Big(\int_0^{2\pi}g(\theta)\,d\theta\Big)^\mu.$$

The inequality (8.4) follows from (8.10) and (7.2).

9. It is now a simple matter to prove Theorems 1-4.

Part (i) of Theorem 1 is immediate. If $F(z)$ belongs to H, then, in view of (8.6), $\sum|S_{n_k}-\tau_{n_k}|^2<\infty$, and so $S_{n_k}-\tau_{n_k}\to 0$, for almost every θ. Since $\tau_{n_k}(e^{i\theta})\to F(e^{i\theta})$ for almost every θ, the result follows.

In order to prove part (ii) of Theorem 1, we note that $|S_{n_k}|\leqslant|\tau_{n_k}|+|S_{n_k}-\tau_{n_k}|$. Hence

$$(9.1)\qquad T^*(\theta)\leqslant\sigma^*(\theta)+\Big(\sum_{k=1}^{\infty}|S_{n_k}-\tau_{n_k}|^2\Big)^{\frac{1}{2}}.$$

The inequality (ii) follows from (9.1), (8.6), and from the well known inequality

$$(9.2)\qquad \Big(\int_0^{2\pi}\{\sigma^*(\theta)\}^{\mu}\,d\theta\Big)^{\frac{1}{\mu}}\leqslant C_{\mu}\int_0^{2\pi}|F(e^{i\theta})|\,d\theta$$ [15].

Part (i) of Theorem 2 is established similarly. Instead of (8.6) and (9.2) we apply respectively (8.5) and the inequality

$$\int_0^{2\pi}\sigma^*(\theta)\,d\theta\leqslant A\int_0^{2\pi}|F(e^{i\theta})|\log^+|F(e^{i\theta})|\,d\theta+A$$ [15].

Theorem 3 is a consequence of the inequality (8.3).

Theorem 4 follows, as has already been observed, from the convergence of the series (3.2).

§ 3.

10. In this paragraph we shall prove some minor results.

Theorem 5. *Let the sequence $\{n_k\}$ satisfy* (1.7), *and let the functions $t^*(\theta)$ and $\bar{t}^*(\theta)$ be defined by* (1.8). *If $|f(\theta)|\leqslant 1$, then there exist two positive absolute constants λ and μ such that*

$$(10.1)\qquad \int_0^{2\pi}\exp\lambda t^*(\theta)\,d\theta\leqslant\mu,\qquad \int_0^{2\pi}\exp\lambda\bar{t}^*(\theta)\,d\theta\leqslant\mu.$$

If f is continuous, the integrals in (10.1) *are finite for every positive λ.*

We restrict ourselves to the first inequality (10.1).

Lemma 8. *If the sequence $\{n_k\}$ satisfies the condition* (1.7), *and if f belongs to the class L^q, where $q\geqslant 2$, then*

$$(10.2)\qquad \int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|s_{n_k}(\theta)-\sigma_{n_k}(\theta)|^q\Big)\,d\theta\leqslant A_{q,\alpha}^q\int_0^{2\pi}|f(\theta)|^q\,d\theta$$ [16].

The constant A depends on q and α only. If α is fixed, then

$$(10.3)\qquad A_{q,\alpha}=A_q\leqslant Aq.$$

[15]) See Hardy and Littlewood [2] or *T. S.*, p. 248.

[16]) This inequality is not new. It was communicated to me a few years ago by Prof. Littlewood. The proof given here seems to be slightly simpler than the original proof.

We may write

$$|s_n(\theta)-\sigma_n(\theta)|=\frac{1}{n+1}|\bar{s}_n'(\theta)|\leqslant\frac{1}{n+1}\Big|(1-\varrho)\sum_{\nu=1}^{n-1}\bar{s}_\nu'(\varrho,\theta)\,\varrho^{-\nu-1}\Big|+\frac{1}{n+1}|\bar{s}_n'(\varrho,\theta)|\,\varrho^{-n}.$$

Hence if $n>0$,

$$(10.4)\qquad |s_n(\theta)-\varrho_n(\theta)|^q\leqslant\frac{2^{q-1}}{n^q}(1-\varrho)^q\Big|\sum_{\mu=1}^{n-1}\bar{s}_\nu'(\varrho,\theta)\,\varrho^{-\nu-1}\Big|^q+\frac{2^{q-1}}{n^q}|\bar{s}_n'(\varrho,\theta)|^q\varrho^{-nq}$$

$$\leqslant\frac{2^{q-1}}{n^q}(1-\varrho)^q\Big(\sum_{\nu=1}^{n-1}|\bar{s}_\nu'(\varrho,\theta)|^q\Big)\Big(\sum_{\nu=1}^{n-1}\varrho^{-(\nu+1)q'}\Big)^{q-1}+\frac{2^{q-1}}{n^q}|\bar{s}_n'(\varrho,\theta)|^q\varrho^{-nq}$$

$$\leqslant\frac{2^{q-1}}{n^q}\varrho^{-nq}(1-\varrho)\sum_{\nu=1}^{n-1}|\bar{s}_\nu'(\varrho,\theta)|^q+\frac{2^{q-1}}{n^q}\varrho^{-nq}|\bar{s}_n'(\varrho,\theta)|\,.$$

By M. Riesz's theorem,

$$(10.5)\qquad \int_0^{2\pi}|\bar{s}_\nu'(\varrho,\theta)|^q\,d\theta\leqslant R_q^q\int_0^{2\pi}|f'(\varrho,\theta)|^q\,d\theta,$$

where a dash denotes differentiation with respect to θ. The constant R_q satisfies an inequality

$$(10.6)\qquad R_q\leqslant Kq.$$

If we write $\varrho=\varrho_n=1-1/n$ in (10.4), an application of (10.5) and (10.6) gives

$$\int_0^{2\pi}|s_n(\theta)-\sigma_n(\theta)|^q\,d\theta\leqslant\frac{C^qq^q}{n^q}\int_0^{2\pi}|f'(\varrho_n,\theta)|^q\,d\theta.$$

We now observe that the integral $\int_0^{2\pi}|f'(\varrho,\theta)|^q\,d\theta$ is an increasing function of ϱ. Hence, if $\varrho_{n_k}=1-1/n_k$, $\Delta_{n_k}=(1-1/n_k,\ 1-1/\alpha n_k)$, we have

$$\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|s_{n_k}(\theta)-\sigma_{n_k}(\theta)|^q\Big)\,d\theta\leqslant C^q q^q\sum_{k=1}^{\infty}\frac{1}{n_k^q}\frac{1}{\Delta_{n_k}}\int_0^{2\pi}d\theta\int_{\Delta_{n_k}}|f'(\varrho,\theta)|^q\,d\varrho\leqslant$$

$$C^q\frac{q^q\alpha^q}{\alpha-1}\sum_{k=1}^{\infty}\int_0^{2\pi}d\theta\int_{\Delta_{n_k}}|f'(\varrho,\theta)|^q(1-\varrho)^{q-1}\,d\varrho\leqslant C^q\frac{q^q\alpha^q}{\alpha-1}\int_0^{2\pi}d\theta\int_0^1|f'(\varrho,\theta)|^q(1-\varrho)^{q-1}\,d\varrho.$$

It is now sufficient to apply the following known result due to Littlewood and Paley [17]):

[17]) loc. cit.

Lemma 9. *If $f(\theta)$ belongs to the class L^q, where $q\geqslant 2$, then*

$$\int_0^{2\pi} d\theta\left(\int_0^1 |f'(\varrho,\theta)|^q(1-\varrho)^{q-1}\,d\varrho\right)\leqslant B^q\int_0^{2\pi}|f(\theta)|^q\,d\theta.$$

This completes the proof of Lemma 7.

The proof of Theorem 5 is now easy. Let $\varphi(u)=e^u-u-1=$ $=u^2/2!+u^3/3!+\dots$ From (10.2), (10.3), and from the fact that $|f|\leqslant 1$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{k=1}^{\infty}\varphi(\lambda|s_{n_k}-\sigma_{n_k}|)\,d\theta\leqslant\sum_{q=2}^{\infty}\frac{(A\lambda q)^q}{q!}. \tag{10.7}$$

Using Stirling's formula we see that the sum γ_λ of the series on the right is finite provided that $A\lambda e<1$. Let us fix any positive λ satysfying this inequality. Since $|s_{n_k}|\leqslant|\sigma_{n_k}|+|s_{n_k}-\sigma_{n_k}|\leqslant 1+|s_{n_k}-\sigma_{n_k}|$, we have $\varphi(\lambda t^*)\leqslant\varphi(\lambda)+\sum\varphi(\lambda|s_{n_k}-\sigma_{n_k}|)$, and so

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(\lambda t^*)\,d\theta\leqslant e^\lambda+\gamma_\lambda.$$

In view of the inequality $u\leqslant\varphi(u)+2$ $(u\geqslant 0)$, this gives

$$\int_0^{2\pi}e^{\lambda t^*}d\theta=\int_0^{2\pi}(\varphi(\lambda t^*)+\lambda t^*+1)\,d\theta\leqslant\int_0^{2\pi}(2\varphi(\lambda t^*)+3)\,d\theta\leqslant 4\pi e^\lambda+4\pi\gamma_\lambda+6\pi,$$

and the proof of the first inequality (10.1) is complete.

To establish the second part of the theorem, concerning continuous functions, we make the decomposition $f=f_1+f_2$, where f_1 is a trigonometrical polynomial and the upper bound of $|f_2|$ is as small as we please.

11. Let $f(\theta)$ be a function of the class L^2, and let $p_1<p_2<p_3<\dots$ be any sequence of positive integers satisfying the condition

$$p_n^2\sum_{\nu=n}^{\infty}\frac{1}{\nu p_\nu^2}=O(1).$$

(which is certainly satisfied if p_ν/ν is an increasing sequence). Then

$$\frac{1}{n}\sum_{\nu=1}^{n}|s_{p_\nu}(\theta)-f(\theta)|^2\to 0$$

almost everywhere. In particular, the sequence $\{s_{p_\nu}(\theta)\}$ is summable $(C,1)$ for almost every θ [18]).

A similar result holds for the functions of the class H.

[18]) See Zalcwasser [7].

Theorem 6. *Let $F(z)$ belong to H, and let $\{p_n\}$ be any increasing sequence of positive integers such that*

$$p_n/n=O(p_{n+1}-p_n). \tag{11.1}$$

(This condition is satisfied if $p_{\nu+1}-p_\nu$ is an increasing sequence). Then

$$\frac{1}{n}\sum_{\nu=1}^{n}|S_{p_\nu}(e^{i\theta})-F(e^{i\theta})|^2\to 0$$

for almost every θ.

It is sufficient to establish the convergence almost everywhere of the series $\sum|S_{p_\nu}-\tau_{p_\nu}|^2/\nu$, which is a consequence of the inequality

$$\int_0^{2\pi}\left(\sum_{\nu=1}^{\infty}\frac{|S_{p_\nu}-\tau_{p_\nu}|^2}{\nu}\right)^{\frac{1}{2}\mu}d\theta\leqslant R_\mu^\mu\left(\int_0^{2\pi}|F(e^{i\theta})|\,d\theta\right)^\mu.$$

We shall not give here the proof of this inequality, for this would be a repetition of the proof of the inequality (8.3). We only observe that in defining Δ_{p_n} we distinguish two cases, and we write

$$\Delta_{p_n}=(1-1/p_n,\ 1-1/p_{n+1}),\qquad\text{if}\qquad p_{n+1}\leqslant 2p_n;$$
$$\Delta_{p_n}=(1-1/p_n,\ 1-1/2p_n)\qquad\text{if}\qquad p_{n+1}\geqslant 2p_n.$$

It is not excluded that the hypothesis (11.1) may be relaxed. We have not investigated this problem.

12. Let $s_n(\theta)$ denote the n-th partial sum of the Fourier series of a function f. It has been shown by Littlewood and Paley (loc. cit.) that, if f belongs to L^p, where $1<p\leqslant 2$, then the function

$$m(\theta)=\operatorname*{Max}_n\{|s_n(\theta)|/\log^{1/p}(n+2)\}$$

belongs to L^p [19]). This theorem is false for $p=1$. For example, the series

$$\sum_{n=3}^{\infty}\frac{\cos nx}{\log\log n}$$

is a Fourier series [20]), but the sequence $|s_n(\theta)/\log(n+2)|$ can not in this case be majorised by any integrable function.

[19]) Littlewood and Paley [5].

[20]) Cf. e. g. *T. S.* p. 110.

Theorem 7. *Let $F(z)$ be a function of the class H, and let*

$$M(\theta)=\operatorname*{Max}_{n}\{|S_n(e^{i\theta})|/\log(n+2)\}.$$

Then $M(\theta)$ is also integrable and

$$\int_0^{2\pi} M(\theta)\,d\theta\leqslant C\int_0^{2\pi}|F(e^{i\theta})|\,d\theta.$$

It is sufficient to consider the case when F does not vanish for $|z|<1$. Then $F=G^2$, where G belongs to H^2. Let

$$G(z)=\sum_{n=0}^{\infty}d_n z^n \tag{12.1}$$

$$A_n^{\alpha}=\binom{n+\alpha}{n}\sim\frac{n^{\alpha}}{\Gamma(\alpha+1)},\quad S_n^{\alpha}(z)=S_n^{\alpha}(z,G)=\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha}\,d_\nu z^{\nu},\quad \tau_n^{\alpha}(z)=S^{\alpha}(z)/A_n^{\alpha}.$$

In other words, S_n^{α} and τ_n^{α} denote respectively the α-th Cesàro sums and the α-th Cesàro means of the series (12.1).

Lemma 10. *If $G(z)$ belongs to H^2, then*

$$\int_0^{2\pi}\sum_{n=0}^{\infty}\frac{|\tau_n^{-\frac12}(e^{i\theta})-\tau_n^{\frac12}(e^{i\theta})|^2}{(n+1)\log(n+2)}\,d\theta\leqslant K\int_0^{2\pi}|G(e^{i\theta})|^2\,d\theta. \tag{12.2}$$

We first observe that

$$\tau_n^{\alpha-1}(z)-\tau_n^{\alpha}(z)=\frac{1}{\alpha A_n^{\alpha}}\sum_{\nu=0}^{n}\nu A_{n-\nu}^{\alpha-1}\,d_\nu z^{\nu},$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi}|\tau_n^{-\frac12}(e^{i\theta})-\tau_n^{\frac12}(e^{i\theta})|^2\,d\theta=\frac{4}{(A_n^{\frac12})^2}\sum_{\nu=0}^{n}\nu^2(A_{n-\nu}^{-\frac12})^2|d_\nu|^2.$$

Writing, for brevity, $l_n=1/\log(n+2)$, we see therefore that the left-hand side of (12.2) does not exceed

$$K_1\sum_{n=0}^{\infty}\frac{l_n}{(n+1)^2}\sum_{\nu=0}^{n}|d_\nu|^2\nu^2(n-\nu+1)^{-1}=K_1\sum_{\nu=0}^{\infty}|d_\nu|^2\nu^2\sum_{n=\nu}^{\infty}\frac{l_n}{(n+1)^2(n-\nu+1)}=$$

$$K_1\sum_{\nu=0}^{\infty}|d_\nu|^2\nu^2\sum_{n=\nu}^{2\nu}\frac{l_n}{(n+1)^2(n-\nu+1)}+K_1\sum_{\nu=0}^{\infty}|d_\nu|^2\nu^2\sum_{n=2\nu+1}^{\infty}\frac{l_n}{(n+1)^2(n-\nu+1)}=P+Q,$$

say, where

$$P\leqslant K_1\sum_{\nu=1}^{\infty}|d_\nu|^2\nu^2 l_\nu(\nu+1)^{-2}\sum_{n=\nu}^{2\nu}\frac{1}{n-\nu+1}\leqslant K_2\sum_{\nu=1}^{\infty}|d_\nu|^2,$$

$$Q\leqslant K_3\sum_{\nu=1}^{\infty}|d_\nu|^2\nu^2\sum_{n=2\nu+1}^{\infty}\frac{l_n}{n^3}\leqslant K_3\sum_{\nu=1}^{\infty}|d_\nu|^2,$$

since $l_n<1$ for $n>1$. This completes the proof of (12.2).

The proof of Theorem 7 is based on the formula

$$S_n(e^{i\theta},F)=\sum_{\nu=0}^{n}S_{n-\nu}^{-\frac12}(e^{i\theta},G)\,S_\nu^{-\frac12}(e^{i\theta},G),$$

which is a consequence of the equation $F=G^2$. By Schwarz's inequality,

$$|S_n(e^{i\theta},F)|\leqslant\sum_{\nu=0}^{n}|S_\nu^{-\frac12}(e^{i\theta},G)|^2=\sum_{\nu=0}^{n}|\tau_\nu^{-\frac12}(e^{i\theta},G)A_\nu^{-\frac12}|^2 \tag{12.3}$$

$$\leqslant K_4\sum_{\nu=0}^{n}\frac{|\tau_\nu^{-\frac12}-\tau_\nu^{\frac12}|^2}{\nu+1}+K_4\sum_{\nu=0}^{n+1}\frac{|\tau_\nu^{\frac12}|^2}{\nu+1}=U_n(\theta)+V_n(\theta),$$

say. Let $\psi(\theta)=\operatorname*{Max}_{n}|\tau_n^{\frac12}(e^{i\theta},G)|$. Then

$$V_n(\theta)\leqslant K_5\,\psi^2(\theta)\log(n+2),\quad\text{where}\quad\int_0^{2\pi}\psi^2(\theta)\,d\theta\leqslant K_6\int_0^{2\pi}|G(e^{i\theta})|^2\,d\theta. \tag{12.4}$$

Let $\varphi^2(\theta)$ denote the integrand on the left-hand side of (12.2). From (12.3) we see that

$$U_n(\theta)\leqslant K_4\,\varphi^2(\theta)\log(n+2),\quad\text{where}\quad\int_0^{2\pi}\varphi^2(\theta)\,d\theta\leqslant K_7\int_0^{2\pi}|G(e^{i\theta})|^2\,d\theta. \tag{12.5}$$

Theorem 7 is a consequence of the inequalities (12.3), (12.4), (12.5), and of the equation $F=G^2$.

Remark. From Theorem 7 we may deduce that, if $F(z)=\sum c_n z^n$ belongs to H, and if $N(\theta)=\operatorname*{Max}_{n}\left|\sum_{\nu=0}^{n}c_\nu e^{i\nu\theta}/\log(\nu+2)\right|$, then $N(\theta)$ is integrable and

$$\int_0^{2\pi}N(\theta)\,d\theta\leqslant K\int_0^{2\pi}|F(e^{i\theta})|\,d\theta.$$

In order to prove this, we apply Abel's transformation four times to the sum $\sum_{\nu=0}^{n}c_\nu e^{i\nu\theta}/\log(\nu+2)$, so as to introduce the third arithmetic means of the series $\sum_{\nu=0}^{\infty} c_\nu e^{i\nu\theta}$. It is then sufficient to observe that the expression

$$\mu(\theta)=\operatorname{Max} |\tau_n^3(e^{i\theta},F)|$$

is integrable and that

$$\int_0^{2\pi} \mu(\theta)\,d\theta \leqslant C\int_0^{2\pi} |F(e^{i\theta})|\,d\theta.$$

The latter inequality follows from the fact that, if $F=G^2$, then

$$|S_n^3(e^{i\theta},F)|=\Big|\sum_{\nu=0}^{n} S_{n-\nu}^1(e^{i\theta},G)\,S_\nu^1(e^{i\theta},G)\Big|\leqslant\sum_{\nu=0}^{n}|S_\nu^1(e^{i\theta},G)|^2=\sum_{\nu=0}^{n}(\nu+1)^2|\tau_\nu^1(e^{i\theta},G)|^2$$

and that, if $\nu(\theta)=\operatorname{Max}_n |\tau_n^1(e^{i\theta},G)|$, then

$$\int_0^{2\pi}\nu^2(\theta)\,d\theta\leqslant A\int_0^{2\pi}|G(e^{i\theta})|^2\,d\theta=A\int_0^{2\pi}|F(e^{i\theta})|\,d\theta.$$

13. One of the most interesting results of the Littlewood-Paley [21]) paper is the following

Theorem D. *Let $n_0=0<n_1<n_2<\dots$ be any sequence of integers satisfying the condition*

$$1<\alpha<n_{k+1}/n_k<\beta \qquad (k=1,2,\dots). \tag{13.1}$$

If $F(z)=\sum c_\nu z^\nu$ belongs to H^r, where $r>1$, and if

$$\Delta_0=c_0,\qquad \Delta_k=\sum_{\nu=n_{k-1}+1}^{n_k} c_\nu e^{i\nu\theta}\qquad (k=1,2,\dots),$$

then

$$L_{r,\alpha,\beta}\Big(\int_0^{2\pi}\Big(\sum_{k=0}^{\infty}|\Delta_k|^2\Big)^{\frac{r}{2}}d\theta\Big)^{\frac1r}\leqslant\Big(\int_0^{2\pi}|F(e^{i\theta})|^r d\theta\Big)^{\frac1r}\leqslant K_{r,\alpha,\beta}\Big(\int_0^{2\pi}\Big(\sum_{k=0}^{\infty}|\Delta_k|^2\Big)^{\frac{r}{2}}d\theta\Big)^{\frac1r}. \tag{13.2}$$

It is not difficult to see that the first of these inequalities may be obtained exactly by the same argument by means of which we have obtained the inequalities (8.1) and (8.4). The argument gives even slightly more, viz. the following

Theorem 8. *If $F(z)$ belongs to H, then*

$$\Big(\int_0^{2\pi}\Big(\sum_{k=0}^{\infty}|\Delta_k|^2\Big)^{\frac12\mu}d\theta\Big)^{\frac1\mu}\leqslant A_{r,\alpha,\beta}\int_0^{2\pi}|F(e^{i\theta})|\,d\theta \qquad (0<\mu<1).$$

If the integral (3.1) is bounded, then

$$\int_0^{2\pi}\Big(\sum_{k=0}^{\infty}|\Delta_k|^2\Big)^{\frac12}d\theta\leqslant A_{\alpha,\beta}\int_0^{2\pi}|F(e^{i\theta})|\log^+|F(e^{i\theta})|\,d\theta+A_{\alpha,\beta}.$$

21) loc. cit.

The second inequality (13.2) may be obtained by an analogous argument, but instead of Lemma 6 we must then apply Lemma 12 (we give this argument below). It must, however, be observed that the original proof of the second inequality (13.2) is more elementary.

14. In this section we shall show that the inequalities opposite to (8.1) and (8.4) are also true. Although the new inequalities seem to have no interesting applications, we shall prove them because they are not difficult consequences of known results.

Theorem 9. *Let the function $F(z)$ belong to H^r, where $F(0)=0$, $r>1$, and let the sequence $\{n_k\}$ satisfy the last inequality (13.1). Then*

$$\int_0^{2\pi}|F(e^{i\theta})|^r d\theta\leqslant B_r^r\int_0^{2\pi}\Big(\sum_{\nu=1}^{\infty}\frac{|S_\nu(e^{i\theta})-\tau_\nu(e^{i\theta})|^2}{\nu}\Big)^{\frac12 r}d\theta \tag{14.1}$$ [22])

$$\int_0^{2\pi}|F(e^{i\theta})|^r d\theta\leqslant B_{r,\beta}^r\int_0^{2\pi}\Big(\sum_{k=1}^{\infty}|S_{n_k}(e^{i\theta})-\tau_{n_k}(e^{i\theta})|^2\,d\theta\Big)^{\frac12 r}d\theta. \tag{14.2}$$

Let $g(\theta)$ be defined by (7.1). We may write

$$|F'(\varrho e^{i\theta})|=\Big|\sum_{\nu=1}^{\infty}\nu c_\nu\varrho^{\nu-1}e^{i\nu\theta}\Big|=(1-\varrho)\Big|\sum_{\nu=1}^{\infty}S'_\nu\varrho^{\nu-1}\Big|,$$

where $S'_\nu=S'_\nu(e^{i\theta})$, and where the dash in S' (but *not* in F') denotes differentiation with respect to θ. If $\varrho_n=1-1/n$, then

$$g^2(\theta)=\sum_{n=1}^{\infty}\int_{\varrho_n}^{\varrho_{n+1}}(1-\varrho)|F'(\varrho e^{i\theta})|^2 d\varrho\leqslant\sum_{n=1}^{\infty}\frac{1}{n^3}\Big((1-\varrho_n)\sum_{\nu=1}^{\infty}|S'_\nu|\varrho_{n+1}^{\nu-1}\Big)^2$$

$$\leqslant 2\sum_{n=1}^{\infty}\frac{1}{n^5}\Big(\sum_{\nu=1}^{n}|S'_\nu|\varrho_{n+1}^{\nu-1}\Big)^2+2\sum_{n=1}^{\infty}\frac{1}{n^5}\Big(\sum_{\nu=n+1}^{\infty}|S'_\nu|\varrho_{n+1}^{\nu-1}\Big)^2=P+Q,$$

say. Now

$$P\leqslant 2\sum_{n=1}^{\infty}\frac{1}{n^5}\Big(\sum_{\nu=1}^{n}|S'_\nu|^2\Big)\Big(\sum_{\nu=1}^{n}\varrho_{n+1}^{2(\nu-1)}\Big)\leqslant 2\sum_{n=1}^{\infty}\frac{1}{n^4}\sum_{\nu=1}^{n}|S'_\nu|^2\leqslant A\sum_{\nu=1}^{\infty}\frac{|S'_\nu|^2}{\nu^3},$$

$$Q\leqslant 2\sum_{n=1}^{\infty}\frac{1}{n^5}\Big(\sum_{\nu=n+1}^{\infty}\frac{|S'_\nu|^2}{\nu^4}\Big)\Big(\sum_{\nu=n+1}^{\infty}\nu^4\varrho_{n+1}^{2(\nu-1)}\Big)\leqslant B\sum_{n=1}^{\infty}\sum_{\nu=n+1}^{\infty}\frac{|S'_\nu|^2}{\nu^4}\leqslant B\sum_{\nu=1}^{\infty}\frac{|S'_\nu|^2}{\nu^3}.$$

22) The inequality (14.1) seems to have been known to Paley, but no proof of it has ever been published.

Observing that $|S'_\nu|/(\nu+1)=|S_\nu-\tau_\nu|$, we may state the result in the form of

Lemma 11. *For every* $0\leqslant\theta\leqslant 2\pi$,

$$g^2(\theta)\leqslant A^2\sum_{\nu=1}^{\infty}\frac{|S_\nu(e^{i\theta})-\tau_\nu(e^{i\theta})|^2}{\nu}.$$

In order to prove (14.1), it is sufficient to combine this lemma with the following known result:

Lemma 12[23]). *If $F(z)$ belongs to H^r, where $r>1$, and $F(0)=0$, then*

$$\left(\int_0^{2\pi}|F(e^{i\theta})|^r d\theta\right)^{\frac1r}\leqslant K_r\left(\int_0^{2\pi}g^r(\theta)\,d\theta\right)^{\frac1r}.$$

The inequality (14.2) can be deduced from (14.1). For, in view of Lemma 3,

$$\int_0^{2\pi}\left(\sum_{\nu=1}^{\infty}\frac{|S'_\nu|^2}{\nu^3}\right)^{\frac12 r}d\theta=\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\sum_{\nu=n_{k-1}}^{n_k-1}\frac{|S'_\nu|^2}{\nu^3}\right)^{\frac12 r}d\theta\leqslant \tag{14.3}$$

$$\leqslant A_r^r\int_0^{2\pi}\left(\sum_{k=1}^{\infty}|S'_{n_k}|^2\sum_{n_{k-1}}^{n_k-1}\frac{1}{\nu^3}\right)^{\frac12 r}d\theta\leqslant A_{r,\beta}^r\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{|S'_{n_k}|^2}{n_k^2}\right)^{\frac12 r}d\theta,$$

which completes the proof.

We shall now show that the second inequality (13.2) may be deduced from (14.2). Let $U^r(\theta)$ denote the integrand on the right hand-side of (14.3). Since $|S'_n|=\left|\sum_{\nu=0}^{n}(S_n-S_\nu)\right|$, we may write

$$U^2(\theta)=\sum_{k=1}^{\infty}\frac{1}{n_k^2}\left(\sum_{\nu=0}^{n_k}|S_{n_k}-S_\nu|\right)^2\leqslant\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{\nu=0}^{n_k}|S_{n_k}-S_\nu|^2$$

$$\leqslant\sum_{k=1}^{\infty}\frac{1}{n_k}|S_{n_k}|^2+\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{i=1}^{k}\sum_{\nu=n_{i-1}+1}^{n_i}|S_{n_k}-S_\nu|^2,$$

since $n_0=0$. If we note that Lemma 3 applies not only to partial sums of Fourier series, but to any connected block of terms of Fourier series (for such blocks are differences of two partial sums), we obtain

[23]) See Littlewood and Paley [5].

$$\int_0^{2\pi}U^r(\theta)\,d\theta\leqslant \tag{14.4}$$

$$\leqslant 2^{\frac r2}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{n_k}|S_{n_k}|^2\right)^{\frac12 r}d\theta+2^{\frac r2}A_r^r\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{i=1}^{k}\sum_{\nu=n_{i-1}+1}^{n_i}|S_{n_k}-S_{n_{i-1}}|^2\right)^{\frac12 r}d\theta$$

$$\leqslant 2^{\frac r2}\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{n_k}|S_{n_k}|^2\right)^{\frac12 r}d\theta+2^{\frac r2}A_r^r\int_0^{2\pi}\left(\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{i=1}^{k}n_i|S_{n_k}-S_{n_{i-1}}|^2\right)^{\frac12 r}d\theta.$$

Now $|S_{n_k}-S_{n_{i-1}}|\leqslant|\Delta_i|+|\Delta_{i+1}|+\ldots+|\Delta_k|$. Hence, writing $\Delta_j=\Delta_j n_j^{\frac14}\cdot n_j^{-\frac14}$ and applying Schwarz's inequality, we obtain

$$\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{i=1}^{k}n_i|S_{n_k}-S_{n_{i-1}}|^2\leqslant$$

$$\leqslant\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{i=1}^{k}n_i\left(\sum_{j=i}^{k}|\Delta_j|^2n_j^{\frac12}\right)\left(\sum_{j=i}^{\infty}n_j^{-\frac12}\right)$$

$$\leqslant A_\alpha\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{i=1}^{k}n_i^{\frac12}\sum_{j=i}^{k}|\Delta_j|^2n_j^{\frac12}=A_\alpha\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{j=1}^{k}|\Delta_j|^2n_j^{\frac12}\sum_{i=1}^{j}n_i^{\frac12}$$

$$\leqslant A_\alpha\sum_{k=1}^{\infty}\frac{1}{n_k}\sum_{j=1}^{k}|\Delta_j|^2n_j=A_\alpha\sum_{j=1}^{\infty}|\Delta_j|^2n_j\sum_{k=j}^{\infty}\frac{1}{n_k}\leqslant A_\alpha\sum_{j=1}^{\infty}|\Delta_j|^2.$$

A similar inequality may be obtained for the integrand of the remaining integral on the right of (14.4) (the lower limit of summation being now 0). Hence

$$\int_0^{2\pi}U^r\,d\theta\leqslant A_\alpha\int_0^{2\pi}\left(\sum_{j=0}^{\infty}|\Delta_j|^2\right)^{\frac12 r}d\theta.$$

This, in connection, with (14.3) and (14.2) gives the required inequality.

A similar argument permits to obtain a limiting case of the second inequality (13.2), when $r=1$. We shall return to this on another occasion.

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Eine äquivalente Formulierung des Auswahlaxioms.

Von

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Es sind heute mehrere Sätze bekannt, die auf Grund des Zermelo-Fraenkelschen Axiomensystems dem Auswahlaxiom äquivalent sind, so z. B. der Wohlordnungssatz, der Vergleichbarkeitssatz (d. i. der Satz der Trichotomie) sowie verschiedene Theoreme von speziellerem Charakter aus des Theorie der Gleichmächtigkeit und der Arithmetik der Kardinalzahlen [1]). Im vorliegenden Aufsatz möchte ich einen neuen Satz dieser Art formulieren, der seinem Inhalt nach sowohl dem Auswahlaxiom selbst als auch allen oben erwähnten Sätzen ziemlich ferne liegt [2]).

Dieser Satz lautet folgendermaßen

Satz $\mathfrak{S}$. Zu jeder Menge N gibt es eine Menge M, die folgender Bedingung genügt:

X ist dann und nur dann ein Element von M, wenn X eine Teilmenge von M ist und wenn dabei N mit keiner Teilmenge Y von X gleichmächtig ist.

Es soll nun gezeigt werden, daß der Satz $\mathfrak{S}$ dem Auswahlaxiom tatsächlich äquivalent ist [3]).

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[1]) Vgl. hiezu F. Hartogs, *Über das Problem der Wohlordnung*, Math. Ann. **76** (1915), S. 438 ff., ferner meinen Aufsatz *Sur quelques théorèmes qui équivalent à l'axiome du choix*, Fund. Math. **5** (1924), S. 147 ff., sowie die gemeinsame Mitteilung von A. Lindenbaum und Verfasser, *Communication sur les recherches de la théorie des ensembles*, C. R. Soc. Sc. Vars. **19** (1926), S. 311 f.

[2]) Über das in diese Artikel vorgebrachte Ergebnis hat der Verfasser am 12. XI. 1937 in der Warschauer Sektion der Polnischen Mathematischen Gesellschaft berichtet.

[3]) Satz $\mathfrak{S}$ hängt mit einem Satz zusammen, der in meiner Arbeit *Über unerreichbare Zahlen*, dieser Band, S. 84, formuliert und dort als Axiom der unerreichbaren Mengen bezeichnet wurde. Mit Rücksicht hierauf sind die beiden Teile des hier gebrachten Beweises mit gewissen dortigen Überlegungen eng verknüpft, und zwar der I. Teil mit dem Beweis des Hilfssatzes 18 (S. 77 ff.) und der II. Teil mit der Ableitung des Auswahlaxioms aus dem Axiom der unerreichbaren Mengen (S. 85 ff.).