[5]) Die gleichzeitige Verfeinerung des Peano-Jordanschen Verfahrens in den beiden angedeuteten Richtungen ist mit Rücksicht auf die bekannten „Paradoxien der abzählbaren Zerlegungsgleichheit" unmöglich (vgl. die in Anm. 2 zitierte Arbeit von Banach und dem Verfasser).

[6]) Vgl. etwa Hausdorff, op. cit., S. 418 f. (insbesondere S. 419, Z. 20–24 von oben) und W. Sierpiński, dieser Band, S. 96–99.

[7]) Zu den Sätzen 1.2–1.6, 1.9 und 1.10 vgl. Banach-Tarski, op. cit.; Satz 1.14 wurde von D. König, Fund. Math. **8** (1926), S. 114 ff. bewiesen; Satz 1.22 stammt von Hausdorff, op. cit., S. 401 f. Die übrigen Sätze von § 1 (größtenteils in einer viel allgemeineren Formulierung) wurden von Lindenbaum und dem Verfasser gewonnen; vgl. hiezu A. Lindenbaum et A. Tarski, C. R. Soc. Sc. Vars. **19** (1926), Cl. III, S. 316 ff. und 328 f., sowie A. Tarski, Atti Congr. Mat. Bologna 1928, **2**, S. 243 ff.. Genaue Beweise dieser Sätze sind noch nicht veröffentlicht und sollen in besonderen Arbeiten erscheinen.

[8]) Ein kurzer Beweis des Satzes 1.7 (oder genauer: eines analogen Satzes für die Relation der Gleichmächtigkeit) findet sich im Buch von W. Sierpiński, *Zarys teorii mnogości* (*Grundriß der Mengenlehre*, polnisch), 1. Teil, 3. Aufl., Warszawa 1928, S. 90.

[9]) Sätze 1.11–1.13 wurden als Verschärfungen des von D. König stammenden Satzes 1.14 gewonnen; ihr Beweis ist in großen Zügen dem Königschen analog (vgl. Anm. 7).

[10]) Korollar 1.19 kann aus dem in Anm. 4 erwähnten Satz von Banach abgeleitet werden (vgl. Banach-Tarski, op. cit., S. 257 f.); der Beweis von 1.19 auf Grund von 1.15 ist jedoch viel einfacher und stützt sich nicht auf das Auswahlsaxiom.

[11]) Vgl. die Beweise analoger Sätze für ebene Punktmengen in Banach-Tarski, op. cit., S. 258 ff., sowie in meinem oben zitierten Aufsatz aus Przegl. mat.-fiz. **2**, S. 53 ff. Es ist zu bemerken, daß Korollar 1.21 einleuchtend und seine Anwendung in weiteren Überlegungen entbehrlich wäre, wenn wir uns entschließen wollten, lediglich halboffene Strecken (d. h. Strecken mit einem Endpunkt) hier zu betrachten.

[12]) Vgl. hiezu meinen in Anm. 7 zitierten Aufsatz aus Atti Congr. Mat. Bologna 1928 sowie J. v. Neumann, Fund. Math. **13** (1929), S. 81 f.

[13]) Vgl. S. Mazurkiewicz et W. Sierpiński, Comptes Rendus **158** (1914), S. 618 und S. Ruziewicz, Fund. Math. **2** (1921), S. 4 ff.

[14]) Zum Beweis vgl. Banach-Tarski, S. 257 f. (im Gegensatz zu dem Fall der geraden Linie ist kein direkter Beweis dafür bekannt; vgl. Anm. 11).

[15]) Die Kenntnis der Theorie der Zerlegungsgleichheit vorausgesetzt, bietet die Begründung der Ergebnisse aus § 2 keine große Schwierigkeiten; die Schlußweisen sind denen aus der Theorie des Peano-Jordanschen und Lebesgueschen Maßes analog (vgl. Anm. 3).

[16]) Satz 3.13 in Bezug auf das System S der Mengen von I. Kategorie wurde von E. Szpilrajn gewonnen.

On the equivalence of some classes of sets [1]).

By

Edward Szpilrajn (Warszawa).

We say that two mathematical objects A and B (sets, classes of sets, sequences of sets, functions, etc.) defined respectively in two spaces X and Y are *equivalent in the sense of the General Theory of Sets*, or briefly *equivalent*, if there exists a one-one transformation of the space X into the space Y which transforms A into B [2]).

This paper contains some simple theorems on the equivalence, concerning the category, the property of Baire and measures.

Terminology and notation. X being any metrical space and E a subset of X, we shall denote by

$\widetilde{X}$ the smallest complete space including X;

$\mathrm{D}(E)$ the set of all the points $x \in X$ at which E is of the second category in X [3]);

$\mathrm{Int}(E)$ the interior of E;

$\mathrm{Fr}(E)$ the boundary of E;

$\mathbf{B}(X)$ the class of all the Borel subsets of X;

$\mathbf{K}(X)$ the class of all the sets of the first category in X;

$\mathbf{R}(X)$ the class of all the sets possessing the Baire property in X i. e. the sets of the form $G - K_1 + K_2$, where G is open in X and $K_1, K_2 \in \mathbf{K}(X)$.

Next, we shall denote by

ϑ the closed interval $\langle 0, 1\rangle$;

$\mathbf{N}(\vartheta)$ the class of all the sets of measure zero contained in ϑ;

$\mathbf{M}(\vartheta)$ the class of all the sets measurable (L) contained in ϑ.

A metrical space X will be said *Borel space* if $X \in \mathbf{B}(\widetilde{X})$. A one-one transformation will be called a *generalized homeomorphism* if both the transformation itself and its inversion are measurable (B) [4]).

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[1]) Presented to the Polish Mathematical Society, Warsaw section, on February 25, 1938.

[2]) For the equivalence of sequences of sets see Szpilrajn [2].

[3]) Cf. Kuratowski [1], p. 45.

[4]) See Kuratowski [2] and [1], p. 221.

1. Auxiliary theorems.

I. *Let X be a separable metrical space, dense in itself, possessing the property of Baire in $\tilde{X}$, of the second category in itself. Then there exists a decomposition into two sets without common points: $X = N + K$, where N is homeomorphic with the set of all the irrational numbers, $N \subset D(X)$ and K is a non enumberable set of the first category (in X).*

Remarks 1. It is easy to see that if we omitt the relation $N \subset D(X)$, we may at the same time replace the condition concerning the second category by that of the non enumerability of the space X.

2. The space $\tilde{X}$ cannot be replaced by an arbitrary complete space containing X.

Proof. Putting $H_1 = D(X)$ we may evidently write:

(1) $$X = H_1 + K_1, \qquad H_1 K_1 = 0,$$

where $K_1 \epsilon \mathbf{K}(X)$. Since the set H_1 is closed in X and $X \epsilon \mathbf{R}(\tilde{X})$ by hypothesis, we have $H_1 \epsilon \mathbf{R}(\tilde{X})$. Consequently,

(2) $$H_1 = H_2 + K_2, \qquad H_2 K_2 = 0,$$

where H_2 is a G_δ in $\tilde{X}$ and $K_2 \epsilon \mathbf{K}(\tilde{X})$. Hence the set H_2 is an absolute G_δ and, the space X being dense in $\tilde{X}$, $K_2 \epsilon \mathbf{K}(X)$.

H_2, as a non enumerable absolute G_δ, contains a perfect set K_3 non-dense in H_2 and therefore we may write

(3) $$H_2 = H_3 + K_3, \qquad H_3 K_3 = 0,$$

where H_3 is an absolute G_δ and K_3 a non enumerable set non-dense in X.

According to a known theorem we may decompose the set H_3:

(4) $$H_3 = H_4 + K_4, \qquad H_4 K_4 = 0,$$

where H_4 is a G_δ in H_3 of dimension zero and K_4 a set of the first category in H_3 [1]).

We have finally by Mazurkiewicz's known theorem [2]):

(5) $$H_4 = N + K_5, \qquad N K_5 = 0,$$

where N is homeomorphic with the set of all the irrational numbers and K_5 is at most enumerable.

[1]) M denoting a metrical separable space and $\{V_n\}$ its basis, $\mathrm{Fr}(V_1) + \mathrm{Fr}(V_2) + \ldots$ is a set F_σ of the first category (in M), the complement of which is of dimension 0. See Hurewicz [1], p. 755, Satz XXIII.

[2]) See e.g. Kuratowski [1], p. 227, I.

Now we put

(6) $$K = K_1 + K_2 + K_3 + K_4 + K_5.$$

Accordingly K is a set of the first category in X, non enumerable as containing K_3. Furthermore, the formulae (1)-(6) imply:

$$X = N + K, \qquad NK = 0,$$
$$N \subset H_4 \subset H_3 \subset H_2 \subset H_1 = D(X).$$

This completes the proof.

II. *Let X be a metrical space and Z a subset of $D(X)$ such that $X - Z$ is of the first category in X. Then, in order that a set $K \subset X$ be of the first category in X it is necessary and sufficient that the set KZ be of the first category in Z.*

Proof. The sufficiency of this condition is obvious, so that we have only to prove it necessary.

Let us then suppose that $K \epsilon \mathbf{K}(X)$. Evidently we may write:

(7) $$K \cdot D(X) = K \cdot \mathrm{Int} D(X) + K \cdot \mathrm{Fr} D(X).$$

The set $K \cdot \mathrm{Int} D(X)$ is of the first category in $\mathrm{Int} D(X)$ and consequently it is so in $D(X)$. Since the set $D(X)$ equals the closure of $\mathrm{Int} D(X)$ [1]) the set $\mathrm{Fr} D(X)$ is non-dense in $D(X)$.

Hence, it follows from (7) that $K \cdot D(X) \epsilon \mathbf{K}[D(X)]$ and a fortiori $KZ \epsilon \mathbf{K}[D(X)]$. On the other hand, since $X - Z \epsilon \mathbf{K}(X)$, we have $D(X) = D(Z) \subset \bar{Z}$ and therefore the set Z is dense in $D(X)$. It follows from this that $KZ \epsilon \mathbf{K}(Z)$.

2. Definitions of equivalence and of *B*-equivalence.

Two classes $\mathbf{K}$ and $\mathbf{M}$ of subsets of spaces X and Y are termed *equivalent* if there exists a one-one transformation φ of the space X into the space Y such that $K \epsilon \mathbf{K}$ if and only if $\varphi(K) \epsilon \mathbf{M}$. Then the transformation φ is said *to realize* the equivalence of $\mathbf{K}$ and $\mathbf{M}$.

If X and Y are metrical Borel spaces and if there exists a generalized homeomorphism realizing the equivalence of $\mathbf{K}$ and $\mathbf{M}$, then $\mathbf{K}$ and $\mathbf{M}$ are called *B-equivalent*.

Analogically two functions $f(B)$ and $g(B)$ of a Borel set on spaces X and Y are termed *B-equivalent* if there exists a generalized homeomorphism φ between X and Y such that $f(B) = g[\varphi(B)]$ for each Borel subset B of X (and it is φ which *realizes* this equivalence).

[1]) See Kuratowski [1], p. 47, 11).

3. The category and the property of Baire.

Theorem 1. *Suppose that two metrical Borel spaces X_1 and X_2 are separable, dense in themselves, and of the second category in themselves. Then the classes $\mathbf{K}(X_1)$ and $\mathbf{K}(X_2)$ of the sets of the first category in X_1 and in X_2 respectively (and analogically the classes $\mathbf{R}(X_1)$ and $\mathbf{R}(X_2)$ of the sets possessing the property of Baire) are B-equivalent* [1]).

Proof. Let

$$X_i = N_i + K_i, \qquad N_i K_i = 0 \qquad \text{for} \quad i = 1, 2$$

be a decomposition according to auxiliary theorem I.

Denote by h a homeomorphism transforming N_1 into N_2 and by g a generalized homeomorphism transforming K_1 into K_2 [2]) Putting

$$\varphi(x) = \begin{cases} h(x) & \text{for} \quad x \in N_1 \\ g(x) & \text{for} \quad x \in K_1, \end{cases}$$

we obtain a generalized homeomorphism which transforms X_1 into X_2. Since φ establishes a homeomorphism of sets N_1 and N_2, it transforms the class $\mathbf{K}(N_1)$ into the class $\mathbf{K}(N_2)$. Consequently it follows from the auxiliary theorem II (applied for $X = X_i$, $Z = N_i$; $i = 1, 2$) that φ transforms the class $\mathbf{K}(X_1)$ into the class $\mathbf{K}(X_2)$.

Now, the class $\mathbf{R}(X_i)$ equals the class of all the sets of the form $B + K$ where $B \in \mathbf{B}(X_i)$ and $K \in \mathbf{K}(X_i)$; besides the class $\mathbf{B}(X_1)$ is transformed by φ into the class $\mathbf{B}(X_2)$, therefore φ transforms $\mathbf{R}(X_1)$ into $\mathbf{R}(X_2)$.

The theorem is thus proved.

It is obvious that the hypothesis concerning the category and the density are essential for the above theorem. We shall prove that the hypothesis that X_i are Borel spaces cannot be omitted either:

Theorem 2. *If the hypothesis of the continuum is true, then there exist two separable metrical spaces of the second category in themselves, dense in themselves and for which the classes of all the sets of the first category are not equivalent.*

[1]) This theorem contains the answer to a question raised by Dr A. Tarski.

[2]) There exists for each two non-enumerable separable Borel spaces A and B a generalized homeomorphism which transforms A into B. See Kuratowski [2], p. 212 and 215.

In fact these conditions are fulfilled by the interval ϑ and the Lusin set (without isolated points) $L \subset \vartheta$ i. e. a non enumerable set each subset of which is either of the second category or at most enumerable. (The existence of such a set follows from the hypothesis of the continuum [1])). The class $\mathbf{K}(L)$ consists only of the sets which are at most enumerable and hence $\mathbf{K}(L)$ and $\mathbf{K}(\vartheta)$ are not equivalent.

4. The category and the measure.

Sierpiński has proved that the hypothesis of the continuum implies the following theorem:

(a) The class of all the sets (contained in ϑ) of measure zero and that of all the sets of the first category (in ϑ) are equivalent [2]).

Observe that conversely *Theorem* (a) *together with the theorem on the existence of a Lusin set of the power* $\mathfrak{c}$ *imply the hypothesis of the continuum.*

In order to show this, suppose that a transformation φ realizes the equivalence of $\mathbf{K}(\vartheta)$ and $\mathbf{N}(\vartheta)$. It is easy to see that φ realizes also the equivalence of the class of Lusin sets and the class of Sierpiński sets (i. e. the non enumerable sets each subset of which is either of positive outer measure or at most enumerable). Hence the existence of a Lusin set of the power $\mathfrak{c}$ implies the existence of a Sierpiński set of the same power. Since the existence of such two sets implies the hypothesis of the continuum (according to a recent result of Rothberger [3])), our remark is proved.

On the other hand, we have

(b) The class of sets measurable (L) (contained in ϑ) and that of sets possessing the property of Baire (in ϑ) are not equivalent [4]).

Hence it will be seen that the equivalence considered in (a) cannot be realized by a generalized homeomorphism:

Theorem 3. *The class of sets of measure zero (contained in ϑ) and that of sets of the first category (in ϑ) are not B-equivalent.*

Proof. If the theorem is not true, then there exists a generalized homeomorphism h realizing the equivalence of $\mathbf{N}(\vartheta)$ and $\mathbf{K}(\vartheta)$. Since h transforms $\mathbf{B}(\vartheta)$ into itself, it transforms also all the sets M of the form:

$$M = B + N, \qquad B \in \mathbf{B}(\vartheta), \qquad N \in \mathbf{N}(\vartheta),$$

[1]) See e. g. Sierpiński [1], p. 37.

[2]) Sierpiński [2] and [1], p. 77.

[3]) Rothberger [1].

[4]) Szpilrajn [1], p. 306.

into all the sets of the form:

$$R = B + K, \qquad B \in \mathbf{B}(\mathcal{S}), \qquad K \in \mathbf{K}(\mathcal{S}),$$

or, in other words, the class $\mathbf{R}(\mathcal{S})$ into the class $\mathbf{M}(\mathcal{S})$. This contradicts Theorem (b) and our theorem is thus proved.

We call a *measure* in a space X every finite non negative and absolutely additive function of a Borel subset of X, vanishing for the sets that consist of a single point. I have proved that

(c) If μ_1 and μ_2 are measures in two non enumerable, separable Borel spaces X_1 and X_2 such that $\mu(X_1)=\mu(X_2)$, then μ_1 and μ_2 are B-equivalent [1]).

Now, we may join this to Theorem 1 as follows:

Theorem 4. *Let X_1 and X_2 be two metrical Borel spaces which are separable, dense in themselves, and of the second category in themselves. Next, let μ_1 and μ_2 be two measures in X_1 and X_2 respectively such that $\mu(X_1)=\mu(X_2)$. Then there exists a generalized homeomorphism realizing simultaneously the equivalence of μ_1 and μ_2 and that of the classes of the first category respectively in X_1 and X_2* [2]).

Proof. There exists a decomposition of X_i into two Borel sets:

$$X_i = K_i + N_i$$

where

$$K_i \in \mathbf{K}(X), \qquad \mu_i(N_i)=0 \qquad \text{for} \quad i=1,2 \text{ [3])}.$$

Denote by Q_i the set consisting of all the isolated points of N_i and of all the points belonging to $N_i - \mathrm{D}(X_i)$, and put

$$K_i^* = K_i + Q_i, \qquad N_i^* = N_i - Q_i.$$

Evidently we may write for $i=1,2$:

$$X_i = K_i^* + N_i^*,$$

$$K_i^* \in \mathbf{K}(X), \qquad \mu_i(N_i^*)=0, \qquad N_i^* \subset \mathrm{D}(X_i),$$

and moreover the set N_i^* is dense in itself.

Theorem (c) implies the existence of a generalized homeomorphism h transforming K_1^* into K_2^* in such a way that

$$\mu_1(B) = \mu_2[h(B)] \quad \text{for each} \quad B \in \mathbf{B}(X_1), \ B \subset K_1^*.$$

[1]) Szpilrajn [3], p. 57, th. 4.1 (ii).
[2]) This theorem contains the answer to a question of Miss S. Braun.
[3]) Szpilrajn [1], p. 304.

Similarly, Theorem 1 implies the existence of another generalized homeomorphism g transforming the class $\mathbf{K}(N_1^*)$ into the class $\mathbf{K}(N_2^*)$.

Define a generalized homeomorphism as follows:

$$\varphi(x) = \begin{cases} h(x) & \text{for each} \quad x \in K_1^* \\ g(x) & \text{for each} \quad x \in N_1^*. \end{cases}$$

It follows from the auxiliary theorem II that φ realizes the equivalence of the classes $\mathbf{K}(X_1)$ and $\mathbf{K}(X_2)$ and on the other hand it is obvious that φ realizes also the equivalence of the measures μ_1 and μ_2. The theorem is thus proved.

References.

Hurewicz, W. [1] *Normalbereiche und Dimensionstheorie.* Math. Ann. **96** (1927), p. 737-764.

Kuratowski, C. [1] *Topologie I.* Monografie Matematyczne **3**. Warszawa-Lwów 1933.

— [2] *Sur une généralisation de la notion d'homéomorphie.* Fund. Math. **22** (1934), p. 206-220.

Rothberger, F. [1] *Eine Äquivalenz zwischen der Kontinuumhypothese und der Existenz der Lusinschen und Sierpińskischen Menge.* Fund. Math. **30** (1938), p. 215-217.

Sierpiński, W. [1] *Hypothèse du continu.* Monografie Matematyczne **4**. Warszawa-Lwów 1934.

— [2] *Sur la dualité entre la première catégorie et la mesure nulle.* Fund. Math. **22** (1934,), p. 276-280.

Szpilrajn, E. [1] *Remarques sur les fonctions complètement additives d'ensembles et sur les ensembles jouissant de la propriété de Baire.* Fund. Math. **22** (1934), p. 303-311.

— [2] *Sur l'équivalence des suites d'ensembles et l'équivalence des fonctions,* Fund. Math. **26** (1936), p. 302-326; *Correction*, Fund. Math. 27 (1936), p. 294.

— [3] *O zbiorach i funkcjach bezwzględnie mierzalnych* [*On absolutely measurable sets and functions*, in Polish]. Comptes Rendus de la Soc. des Sciences et de Lettres de Varsovie **30** (1937), p. 39-68.