A PROOF OF AN AUERBACH - BANACH - MAZUR - ULAM THEOREM ON CONVEX BODIES

BY

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THEOREM. *Given any family $\mathfrak{V} = \{V_i\}$ of k-dimensional convex bodies with diameters not exceeding D and such that $S = \sum_i |V_i| < \infty$* [1]*, there exists a k-dimensional cube $C(D, S)$ in which all the family can be embedded in such a manner that no two of its members intersect.*

This theorem has been discovered (and called "theorem on a sack of potatoes") by H. Auerbach, S. Banach, S. Mazur and S. Ulam, but a proof has never been published. The aim of this note is to give a simple proof of the Theorem [2].

By a parallelepided we shall always mean a *rectangular* parallelepiped.

LEMMA 1. *Let V be a k-dimensional convex body. There exists a parallelepiped P containing V and such that $|P|/|V| \leqslant k!$.*

Proof. The lemma is trivially true for $k = 1$. Suppose it is true also for $k < n$ and let V be an n-dimensional convex body in the Euclidean n-dimensional space. Without loss of generality we may assume that the segment $\langle 0, D\rangle$ of the x_n-axis is the diameter of V. Let V' be the orthogonal projection of V into the hyperplane $x_n = 0$. If $x = (x_1, \ldots, x_{n-1})$ is a point of V', then the intersection of the straight line through x parallel to the x_n-axis with V is a segment with ends $(x_1, \ldots, x_{n-1}, x_n^1)$ and $(x_1, \ldots, x_{n-1}, x_n^2)$ where the numeration is so chosen that $x_n^1 \leqslant x_n^2$. Let $f_i(x) = x_n^i$, $i = 1, 2$.

Now, observe that $f_2(0, \ldots, 0) - f_1(0, \ldots, 0) = D$ and that the difference $f_2(x) - f_1(x)$ is a concave function. Hence

$$(1) \qquad |V| = \int_{V'} \big(f_2(x) - f_1(x)\big)\, dx \geqslant |C| = \frac{1}{n} |V'| \cdot D,$$

where C denotes the cone of base V' and vertex $(0, \ldots, 0, D)$.

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[1]) $|V|$ denotes the k-dimensional measure of V.

[2]) I am indebted to Prof. Drobot for calling my attention to the Theorem. Prof. Drobot's own proof (communicated at the meeting of the Polish Math. Soc., see Colloquium Mathematicum 1(1948), p. 341) also has not been published. I am also indebted to K. Radziszewski for a remark simplifying my previous proof of lemma 1.

The inductive assumption guarantees that in the hyperplane $x_n = 0$ there exists a parallelepiped P' containing V' and such that

$$(2) \qquad |P'|/|V'| \leqslant (n-1)!$$

Let P be the parallelepiped with base P' and height D. Then P contains V and $|P| = |P'| \cdot D$. It follows by (1) and (2) that

$$|P|/|V| \leqslant n|P'|D/|V'|D \leqslant n(n-1)! = n!$$

Now, let the family $\mathfrak{V}$ be as in the Theorem. Let P_i be the parallelepiped containing V_i and such that $|P_i| \leqslant k!\,|V_i|$. Since the series $\sum_i |V_i|$ is convergent, so is also the series $\sum_i |P_i|$. Thus in order to prove the Theorem it suffices to establish the following

LEMMA 2. *Let $\mathfrak{V} = \{V_i\}$ be a family of k-dimensional parallelepipeds, such that $\sum_i a_i^1 \cdot \ldots \cdot a_i^k = V < \infty$ and $a_i^j \leqslant D$ for $i = 1, 2, \ldots$ and $j = 1, 2, \ldots, k$, where a_i^j are the edges of V_i.*

Then the family $\mathfrak{V}$ can be embedded in a k-dimensional parallelepiped with edges $3D, 3D, \ldots, 3D, (V + D^k)/D^{k-1}$ in such a manner that no two of its members intersect.

Proof. The lemma is trivially true for $k = 1$. Suppose it is true for $k = n-1$. We shall prove that it is also true for $k = n$.

Without loss of generality we may assume that $a_i^j \geqslant a_i^1$ for $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots$ Thus we can rearrange the family $\mathfrak{V}$ so that

$$(3) \qquad a_i^1 \geqslant a_{i+1}^1.$$

Consider the sequence V_i' of $(n-1)$-dimensional parallelepipeds, where V_i' has the edges $a_i^2, \ldots, a_i^n$. Put $i_1 = 1$ and let i_{k+1} be such a number that

$$(4) \qquad D^{n-1} < \sum_{j=i_k}^{i_{k+1}-1} a_j^2 \ldots a_j^n \leqslant 2D^{n-1}.$$

Inductive assumption yields an $(n-1)$-dimensional parallelepiped P_k' with edges $3D, \ldots, 3D$, $(2D^{n-1} + D^{n-1})/D^{n-2} = 3D$ in which all the parallelepipeds $V_{i_k}', \ldots, V_{i_{k+1}-1}'$ can be embedded in the desired manner. (All P_k' are then congruent $(n-1)$-dimensional cubes.) It follows from (3) that the parallelepipeds $V_{i_k}, \ldots, V_{i_{k+1}-1}$ can be embedded in the same manner in the parallelepiped P_k with base P_k' and height $a_{i_k}^1$.

Now, it follows from (3) and (4) that if the sequence $\{i_k\}$ is infinite, then

$$V = \sum_{k=1}^{\infty} \sum_{j=i_k}^{i_{k+1}-1} a_j^1 \ldots a_j^n \geqslant \sum_{k=1}^{\infty} a_{i_{k+1}}^1 \sum_{j=i_k}^{i_{k+1}-1} a_j^2 \ldots a_j^n > D^{n-1} \sum_{k=1}^{\infty} a_{i_{k+1}}^1;$$

and if this sequence contains k_0 terms, then

$$V = \sum_{k=1}^{k_0-1} \sum_{j=i_k}^{i_{k+1}-1} a_j^1 \dots a_j^n + \sum_{j=i_{k_0}}^{\infty} a_j^1 \dots a_j^n > \sum_{k=1}^{k_0-1} a_{i_{k+1}}^1 \sum_{j=i_k}^{i_{k+1}-1} a_j^2 \dots a_j^n > D^{n-1} \sum_{k=1}^{k_0-1} a_{i_{k+1}}^1 .$$

Thus in any case

$$\sum_k a_{i_k}^1 < \frac{V}{D^{n-1}} + a_1^1 \leqslant \frac{V}{D^{n-1}} + D. \tag{5}$$

Assign to each i_k a positive number ε_k such that

$$\sum_k \varepsilon_k = \frac{V}{D^{n-1}} + D - \sum_k a_{i_k}^1$$

(such a sequence exists by (5)) and let Q_k be the parallelepiped with base P_k' and height ε_k. Set the parallelepipeds P_k, Q_k on top of each other in the following order: $P_1, Q_1, P_2, Q_2, \dots$ Then we shall obtain an n-dimensional parallelepiped with edges $3D, 3D, \dots, 3D, \sum_k a_{i_k}^1 + \sum_k \varepsilon_k = (V+D^n)/D^{n-1}$, and all the familly $\mathfrak{V}$ will be embedded in P in the desired manner.

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SUR LES PARTAGES DU TRIANGLE

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Supposons que dans un triangle il y ait des lignes divisant son intérieur en quelques parties; dans ce cas nous disons que nous avons un *partage* de ce triangle. Nous désignerons le partage par un nombre entre parenthèses ou par une lettre grecque.

Nous dirons que deux partages du triangle sont *équivalents*, si l'on peut transformer l'un d'eux en l'autre à l'aide d'une transformation homéomorphe qui laisse invariants les sommets du triangle divisé. Dorénavant nous ne distinguerons pas les partages équivalents, et nous désignerons l'équivalence des deux partages α et β en écrivant $\alpha = \beta$.

Nous appellerons *noeuds* du partage: 1° les sommets du triangle divisé, en les notant toujours par les lettres A, B, C; 2° les points où se rencontrent au moins trois lignes de partage, en tenant aussi compte des segments des côtés du triangle ABC.

La notion de noeud est une notion invariante des transformations homéomorphes.

Étant donné un partage du triangle, on peut en déduire un nouveau partage en ajoutant de nouvelles lignes de partage. Nous appelons ce nouveau partage — *condensation* du partage précédent.

On peut distinguer parmi tous les partages du triangle une classe spéciale. A cette classe appartiennent les partages qui divisent le triangle ABC en un nombre fini de triangles qui ne possèdent pas de noeuds sur leurs côtés (à l'exception de leurs sommets). Nous appellerons ces partages — *simpliciaux*. Étant donné que nous ne distinguons pas les partages équivalents, nous appelons ici partage simplicial non seulement le partage en triangles — décrit plus haut — mais aussi chaque partage qui lui est équivalent. Par conséquent la notion de partage simplicial a ici un sens plus large que dans la topologie combinatoire. Par exemple, dans la fig. 2, les partages (a) et (b) sont tous les deux simpliciaux alors que, d'après la définition classique, le partage (b) n'est pas simplicial.

Un partage simplicial est donc un partage du triangle ABC en un nombre fini de domaines sur les frontières desquels il y a exactement trois