L'élément $x(0)$ peut être interprété comme *valeur initiale* de x. Cela étant, *les deux conditions suivantes sont équivalentes*:

(i) *Toute solution x de l'équation $P(D)x = 0$ d'ordre n, satisfaisant aux conditions initiales $x(0) = Dx(0) = D^{n-1}x(0) = 0$, est nulle;*

(ii) *Toute solution $x_1, \ldots, x_n$ du système de n équations (3), satisfaisant aux conditions initiales $x_1(0) = \ldots = x_n(0) = 0$, est nulle.*

La démonstration de cette équivalence est plus élémentaire que pour (I) et (II).

L'inclusion „(II) entraîne (I)" se démontre trivialement.

Pour démontrer l'inclusion „(I) entraîne (II)", multiplions (3) par D^{j-1} $(j = 1, \ldots, n)$

$$D^j x_i = a_{i1} D^{j-1} x_1 + \ldots + a_{in} D^{j-1} x_n \quad (i, j = 1, \ldots, n).$$

De ces égalités il vient, en tenant compte de (10),

$$L^j x_i(0) = a_{i1} D^{j-1} x_1(0) + \ldots + a_{in} D^{j-1} x_n(0),$$

d'où successivement

$$Dx_i(0) = 0, \ \ldots, \ D^{n-1} x_i(0) = 0 \quad (i = 1, \ldots, n).$$

Or, tout élément x_i satisfait à l'équation d'ordre n, $P(D)x = 0$, où $P(D) = |a_{ij} - \delta_{ij} D|$ (cf. [3]). Donc, d'après (i), on a $x_i = 0$ pour $i = 1, \ldots, n$, ce que nous voulions démontrer.

Travaux cités

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[2] J. Mikusiński, *Un théorème d'unicité pour quelques systèmes d'équations différentielles, considérées dans les espaces abstraits*, ibidem 12 (1951), p. 80-83.

[3] — *Sur les solutions linéairement indépendantes des équations différentielles à coefficients constants*, ibidem 16 (1957), p. 41-47.

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Reçu par la Rédaction le 15. 6. 1956

On determinants of Leżański and Ruston

by

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1. Ruston [3], [4] has developed recently a theory of determinants of linear equations

$$x + Tx = x_0 \tag{1}$$

in Banach spaces. His definition of determinants is given only for operators T belonging to the trace class and is based on the notion of the trace of linear[1]) operators. The definition of the trace and of the trace class is as follows.

Let X be a Banach space and Ξ — its conjugate. Let $\mathfrak{K}$ be the class of all linear operators from X into X, and $\mathfrak{K}_0$ — the class of all finitely dimensional operators from X into X, *i. e.*, $\mathfrak{K}_0$ is the class of all operators of the form[2])

$$Kx = \sum_{i=1}^{m} \xi_i x \cdot x_i \tag{2}$$

where $\xi_i \epsilon \Xi$, $x_i \epsilon X$ $(i = 1, 2, \ldots, m)$. The norm of $K \epsilon \mathfrak{K}$ is

$$\|K\| = \sup_{\|x\| \leqslant 1} \|Kx\|.$$

Besides the norm $\| \ \|$ we shall also consider in $\mathfrak{K}_0$ the following norm:

$$\|K\|^* = \inf \sum_{i=1}^{m} \|\xi_i\| \cdot \|x_i\|,$$

where inf is taken over all representations (2) of the operator $K \epsilon \mathfrak{K}_0$. It is easy to see that

$$\|K\| \leqslant \|K\|^* \quad \text{for} \quad K \epsilon \mathfrak{K}_0. \tag{3}$$

[1]) The word "linear" always means "additive and continuous".

[2]) We assume the following notation: the letters x, y (with indices, if necessary) will always denote elements of X, and the letters ξ, η — elements of Ξ. The symbol ξx denotes the value of the functional $\xi \epsilon \Xi$ at the point $x \epsilon X$. The symbol $\xi x \cdot x_0$ denotes the product of the scalar ξx with the element $x_0 \epsilon X$. If K is an operator from X into X, then Kx is the value of K at the point $x \epsilon X$. Consequently, ξKx denotes the value of the functional $\xi \epsilon \Xi$ at the point $Kx \epsilon X$.

Now we complete the space $\mathfrak{K}_0$ with respect to the norm $\| \|^*$. The completed space will be denoted by $\mathfrak{K}^*$, the norm in $\mathfrak{K}^*$ by $\| \|^*$. Clearly, $\mathfrak{K}_0$ is dense in $\mathfrak{K}^*$.

If $T^* \epsilon \mathfrak{K}^*$, $K_n \epsilon \mathfrak{K}_0$ and $\|K_n - T^*\|^* \to 0$, then, by (3), the sequence $\{K_n\}$ satisfies also the Cauchy condition with respect to the norm $\| \|$, *i. e.* there exists an operator $T \epsilon \mathfrak{K}$ such that $\|K_n - T\| \to 0$. The operator T is uniquely determined by the element $T^* \epsilon \mathfrak{K}^*$.

The set of all linear operators $T \epsilon \mathfrak{K}$ that can be obtained in the way described above is called the *trace class* of X. The canonical transformation which maps the element $T^* \epsilon \mathfrak{K}^*$ onto the operator $T \epsilon \mathfrak{K}$ is linear and, by (3),

$$\|T\| \leqslant \|T^*\|^*. \tag{4}$$

If $T^* \epsilon \mathfrak{K}^*$, $K \epsilon \mathfrak{K}$, $K_n \epsilon \mathfrak{K}$ and $\|K_n - T^*\|^* \to 0$, then the sequences $KK_n \epsilon \mathfrak{K}_0$ and $K_n K \epsilon \mathfrak{K}_0$ satisfy the Cauchy condition with respect to the norm $\| \|^*$. Thus they determine some elements of $\mathfrak{K}^*$ denoted by KT^* and T^*K respectively. Moreover,

$$\|KT^*\|^* = \|T^*K\|^* \leqslant \|K\| \cdot \|T^*\|^*. \tag{5}$$

The *trace* of an operator $K \epsilon \mathfrak{K}_0$ of the form (2) is the number

$$\operatorname{tr}(K) = \sum_{i=1}^{m} \xi_i x_i, \tag{6}$$

which does not depend on the representation of K in the form (2). If $T \epsilon \mathfrak{K}$, and $K \epsilon \mathfrak{K}_0$ is of the form (2), then the superpositions TK, KT belong to $\mathfrak{K}_0$, and

$$\operatorname{tr}(TK) = \operatorname{tr}(KT) = \sum_{i=1}^{m} \xi_i T x_i. \tag{7}$$

Since $|\operatorname{tr}(K)| \leqslant \|K\|^*$ for $K \epsilon \mathfrak{K}_0$, the functional $\operatorname{tr}(K)$ can be extended uniquely to a linear functional on $\mathfrak{K}^*$, denoted also by tr. Moreover,

$$|\operatorname{tr}(T^*)| \leqslant \|T^*\|^* \quad \text{for} \quad T^* \epsilon \mathfrak{K}^*. \tag{8}$$

For every $T^* \epsilon \mathfrak{K}^*$ and $K \epsilon \mathfrak{K}$

$$\operatorname{tr}(T^*K) = \operatorname{tr}(KT^*), \tag{9}$$

$$|\operatorname{tr}(T^*K)| \leqslant \|T^*\|^* \cdot \|K\| \tag{10}$$

by (7), (5) and (8)[3].

[3]) For the detailed examination of the properties of the trace and the trace class, see *e. g.* [3] and [4].

Notice that the space $\mathfrak{K}^*$ defined above in an abstract manner has a simple interpretation. Each finitely dimensional operator K_0 determines uniquely a functional on the Banach space $\mathfrak{K}$ (with the norm $\| \|$), viz. the functional

$$F_{K_0}(K) = \operatorname{tr}(K_0 K) \quad \text{for} \quad K \epsilon \mathfrak{K}.$$

It immediately follows from a general theorem of Schatten ([6], Theorem 1.2) that the norm of the functional F_{K_0} is equal to $\|K_0\|^*$. The linear functionals F_{K_0} ($K_0 \epsilon \mathfrak{K}_0$) will be called *degenerate functionals* on $\mathfrak{K}$. Obviously $\mathfrak{K}^*$ can be interpreted as a subspace of the Banach space of all linear functionals on $\mathfrak{K}$, viz. the closure of the set of all degenerate functionals. According to this interpretation, any element $T^* \epsilon \mathfrak{K}^*$ is identified with the linear functional

$$F(K) = \operatorname{tr}(T^*K) \quad \text{for} \quad K \epsilon \mathfrak{K}.$$

The operator determined by the element $T^* \epsilon \mathfrak{K}^*$ is the operator $T \epsilon \mathfrak{K}$ satisfying the equality

$$F(K) = \xi_0 T x_0$$

for every operator $K \epsilon \mathfrak{K}_0$ of the form

$$Kx = \xi_0 x \cdot x_0 \quad (\xi_0 \epsilon \Xi,\ x_0 \epsilon X).$$

2. Independently of Ruston, Leżański [1], [2] has developed another determinant theory of linear equations (1) in Banach spaces (see also [8]). His theory makes no use of the notion of the trace class.

Let $X, \Xi, \mathfrak{K}, \mathfrak{K}_0$ have the meaning as above. The determinant theory of Leżański can be applied to equation (1) if and only if the operator $T \epsilon \mathfrak{K}$ satisfies the following condition ([8], p. 36):

There exists a linear functional F on the Banach space $\mathfrak{K}$ (with the norm $\| \|$) such that

$$F(K) = \xi_0 T x_0 \tag{11}$$

for every operator $K \epsilon \mathfrak{K}_0$ of the form

$$Kx = \xi_0 x \cdot x_0 \quad (\xi_0 \epsilon \Xi,\ x_0 \epsilon X).$$

Leżański's determinant of (1) is uniquely determined by F.

Condition (11) determines uniquely the values of the functional F on the class $\mathfrak{K}_0$, viz.

$$F(K) = \operatorname{tr}(TK) \quad \text{for} \quad K \epsilon \mathfrak{K}_0. \tag{12}$$

Equation (12) immediately follows from (11), (2) and (7).

Thus we infer that the determinant theory of Leżański can be applied to equation (1) if and only if the operator $T \epsilon \mathfrak{R}$ satisfies the following condition:

(∗) The functional

$$\operatorname{tr}(TK) \tag{13}$$

defined for $K \epsilon \mathfrak{R}_0$ by (7) is continuous on the space $\mathfrak{R}_0$ with respect to the norm $\| \ \|$.

In fact, if there is a linear functional F satisfying (11), then it is an extension of (13), and (13) is continuous. Conversely, if (13) is continuous, it can be extended to a linear functional F on $\mathfrak{R}$ satisfying (11).

3. The purpose of this paper is to explain the relation between the determinant theories of Ruston and Leżański.

As I remarked in [8], p. 48, Leżański's theory is not weaker than that of Ruston, namely:

THEOREM I. *If an operator T belongs to the trace class of X, then it satisfies the condition* (∗).

In fact, T is then determined by an element $T^* \epsilon \mathfrak{R}^*$. Thus there exists a sequence $K_n \epsilon \mathfrak{R}_0$ such that $\|K_n - T^*\|^* \to 0$ and $\|K_n - T\| \to 0$. Hence, for every operator $K \epsilon \mathfrak{R}_0$ of the form (2),

$$\operatorname{tr}(T^* K) = \lim_{n\to\infty} \operatorname{tr}(K_n K) = \lim_{n\to\infty} \sum_{i=1}^m \xi_i K_n x_i = \sum_{i=1}^m \xi_i T x_i = \operatorname{tr}(TK).$$

Consequently, by (10), $|\operatorname{tr}(TK)| \leqslant \|T^*\|^* \|K\|$ for $K \epsilon \mathfrak{R}_0$, *i. e.* (13) is continuous.

Setting $F(K) = \operatorname{tr}(T^* K)$ for $K \epsilon \mathfrak{R}$ in Leżański's theory (see *e. g.* [8]), we obtain the same formulae as in Ruston's paper [3]. There is a certain difference between the definitions of Leżański's subdeterminants (see [8]) and Ruston's [4] subdeterminants.

Now we shall prove that Leżański's theory is essentially more general than Ruston's theory, *i. e.* we shall give examples of operators T which satisfy the condition (∗), but do not belong to the trace class (see §§ 5 and 6).

These examples will be given in the case where X is the space $\boldsymbol{L}$ of all integrable functions on the unit interval. The examples will be preceded by several simple lemmas.

4. In §§ 4, 5, 6 we assume $X = \boldsymbol{L}$. Obviously Ξ is the space $\boldsymbol{M}$ of all measurable essentially bounded functions in the unit interval $I = (0, 1)$, and

$$\|x\| = \|x\|_{\boldsymbol{L}} = \int_0^1 |x(t)|\,dt \quad \text{for} \quad x \epsilon \boldsymbol{L},$$

$$\|\xi\| = \|\xi\|_{\boldsymbol{M}} = \sup_t \operatorname{ess} |\xi(t)| \quad \text{for} \quad \xi \epsilon \boldsymbol{M}.$$

$\boldsymbol{M}_0$ will denote the set of all functions $\xi \epsilon \boldsymbol{M}$ which assume only a finite number of values.

Let $K(s, t)$ be a measurable function defined on $I \times I$ such that

$$\int_0^1 \int_0^1 |\xi(s) K(s, t) x(t)|\,ds\,dt < \infty \quad \text{for every} \quad \xi \epsilon \boldsymbol{M} \text{ and } x \epsilon \boldsymbol{L}. \tag{14}$$

Then, as is well known, the function $K(s, t)$ determines a linear operator K from $\boldsymbol{L}$ into $\boldsymbol{L}$ which transforms any element $x \epsilon \boldsymbol{L}$ into the element $y = Kx \epsilon \boldsymbol{L}$ defined by the formula

$$y(s) = \int_0^1 K(s, t) x(t)\,dt.$$

Any operator K of the above form is called an *integral operator*. The function $K(s, t)$ is called the *kernel* of K. The norm of the integral operator K with the kernel $K(s, t)$ is the number

$$\|K\| = \sup_t \operatorname{ess} \int_0^1 |K(s, t)|\,ds. \tag{15}$$

The condition (14) is satisfied if and only if the number on the right-hand side of (15) is finite.

The kernel $K(s, t)$ is said to be *degenerate* if it is of the form

$$K(s, t) = \sum_{i=1}^m x_i(s)\, \xi_i(t) \tag{16}$$

where $x_i \epsilon \boldsymbol{L}$ and $\xi_i \epsilon \boldsymbol{M}$ for $i = 1, 2, \ldots, m$.

The class $\mathfrak{R}_0$ of all finitely dimensional operators (2) from $\boldsymbol{L}$ into $\boldsymbol{L}$ is the class of all integral operators with degenerate kernels (16).

Let $\mathfrak{R}_1$ denote the class of all operators K with degenerate kernels $K(s, t)$ of the form (16) where $x_i \epsilon \boldsymbol{M}_0$ and $\xi_i \epsilon \boldsymbol{M}_0$ $(i = 1, 2, \ldots, m)$. Obviously $\mathfrak{R}_1 \subset \mathfrak{R}_0$.

(i) *If $K_0 \epsilon \mathfrak{R}_0$ and $\varepsilon > 0$, then there exists an operator $K \epsilon \mathfrak{R}_1$ such that $\|K_0 - K\|^* < \varepsilon$.*

The operator K_0 is determined by a degenerate kernel

$$K_0(s, t) = \sum_{i=1}^m y_i(s)\, \eta_i(t)$$

where $y_i \epsilon \boldsymbol{L}$ and $\eta_i \epsilon \boldsymbol{M}$. Let $\xi_i \epsilon \boldsymbol{M}_0$ be such a function that

$$\|\eta_i - \xi_i\|_{\boldsymbol{M}} < \frac{\varepsilon}{2m\|y_i\|_{\boldsymbol{L}}}$$

and let $x_i \epsilon \boldsymbol{M}_0$ be such a function that

$$\|x_i - y_i\|_{\boldsymbol{L}} < \frac{\varepsilon}{2m\|\xi_i\|_{\boldsymbol{M}}}.$$

The operator K determined by the kernel (16) satisfies the conditions of lemma (i). In fact, $K_0 - K$ is determined by the kernel

$$\sum_{i=1}^{m} y_i(s)\big(\eta_i(t) - \xi_i(t)\big) + \sum_{i=1}^{m}\big(y_i(s) - x_i(s)\big)\xi_i(t).$$

Consequently,

$$\|K_0 - K\|^* \leqslant \sum_{i=1}^{m}\|y_i\|\cdot\|\eta_i - \xi_i\| + \sum_{i=1}^{m}\|y_i - x_i\|\cdot\|\xi_i\| < \varepsilon.$$

Let $\mathfrak{K}_*$ be the set of all integral operators K (from $\boldsymbol{L}$ into $\boldsymbol{L}$) with kernels $K(s,t)$ such that the number

$$\|K\|_* = \int_0^1 \operatorname*{sup\,ess}_t |K(s,t)|\,ds$$

is finite. Obviously $\mathfrak{K}_0 \subset \mathfrak{K}_*$ and $\|K\| \leqslant \|K\|_*$ for $K \epsilon \mathfrak{K}_*$.

(ii) *$\boldsymbol{K}_*$ is a Banach space with the norm* $\|\;\|_*$.

Only the completeness of $\boldsymbol{K}_*$ should be proved. The proof is a modification of the usual proof of the completeness of the space $\boldsymbol{L}$.

Suppose $\|K_n - K_m\|_* \to 0$ for $n, m \to \infty$. There exist an increasing sequence $\{m_n\}$ of positive integers and a sequence $\{A_n\}$ of measurable sets such that $|A_n| < 1/2^n$ and

$$\operatorname*{sup\,ess}_t |K_{m_{n+1}}(s,t) - K_{m_n}(s,t)| < 1/2^n \quad \text{for} \quad s \epsilon I - A_n.$$

Let $B_n = A_n + A_{n+1} + \ldots$ and $B = B_1 B_2 \ldots$ The series

$$\sum_{n=1}^{\infty} \operatorname*{sup\,ess}_t |K_{m_{n+1}}(s,t) - K_{m_n}(s,t)|.$$

converges for $s \epsilon I - B$, and $|B| = 0$. Consequently, there exists a measurable function $K(s,t)$ such that

$$\operatorname*{sup\,ess}_t |K_{m_n}(s,t) - K(s,t)| \to 0 \quad \text{for} \quad s \epsilon I - B.$$

Since

$$\int_0^1 \operatorname*{sup\,ess}_t |K_n(s,t) - K_{m_p}(s,t)|\,ds \leqslant \varepsilon \quad \text{for} \quad n, p \geqslant n_0,$$

thus, by the Fatou lemma,

$$\int_0^1 \operatorname*{sup\,ess}_t |K_n(s,t) - K(s,t)|\,ds \leqslant \varepsilon \quad \text{for} \quad n \geqslant n_0.$$

This implies that $K \epsilon \boldsymbol{K}_*$ and $\|K_n - K\|_* \to 0$, q. e. d.

(iii) *For every* $K \epsilon \mathfrak{K}_0$

$$\|K\|_* = \|K\|^*.$$

Supposed K is determined by the kernel (16). We have

$$\begin{aligned}\sum_{i=1}^{m}\|\xi_i\|_{\boldsymbol{M}}\|x_i\|_{\boldsymbol{L}} &= \sum_{i=1}^{m}\operatorname*{sup\,ess}_t|\xi_i(t)|\int_0^1|x_i(s)|\,ds\\ &= \int_0^1\sum_{i=1}^{m}\operatorname*{sup\,ess}_t|x_i(s)\,\xi_i(t)|\,ds\\ &\geqslant \int_0^1\operatorname*{sup\,ess}_t\sum_{i=1}^{m}|x_i(s)\,\xi_i(t)|\,ds\\ &\geqslant \int_0^1\operatorname*{sup\,ess}_t|K(s,t)|\,ds = \|K\|_*.\end{aligned}$$

The representation (16) of the operator K being arbitrary, we obtain

(17) $$\|K\|^* \geqslant \|K\|_* \quad \text{for} \quad K \epsilon \mathfrak{K}_0.$$

Let $K \epsilon \mathfrak{K}_1$. The kernel $K(s,t)$ of K can be represented in the form (16) where x_i are characteristic functions of some pairwise disjoint measurable subsets of I. Hence we have

$$\|K\|_* = \sum_{i=1}^{m}\|x_i\|_{\boldsymbol{L}}\|\xi_i\|_{\boldsymbol{M}} \geqslant \|K\|^*.$$

This proves that

(18) $$\|K\|_* = \|K\|^* \quad \text{for} \quad K \epsilon \mathfrak{K}_1.$$

It follows from (i) that, for any operator $K \epsilon \mathfrak{K}_0$, there exists a sequence $K_n \epsilon \mathfrak{K}_1$ such that $\|K - K_n\|^* \to 0$. Since $\|K - K_n\|_* \to 0$ by (17) and $\|K_n\|_* = \|K_n\|^*$ by (18), we obtain

$$\|K\|^* = \lim_{n\to\infty}\|K_n\|^* = \lim_{n\to\infty}\|K_n\|_* = \|K\|_*,$$

which completes the proof of (iii).

It immediately follows from (i), (ii), (iii) and the definition of the trace class that

(iv) *A linear operator T from $\boldsymbol{L}$ into $\boldsymbol{L}$ belongs to the trace class of $\boldsymbol{L}$ if and only if there exists a sequence of operators $K_n \epsilon \mathfrak{R}_1$ such that $\|K_n - T\|_* \to 0$.*

The space $\mathfrak{R}^*$ can be interpreted as the subspace of the Banach space $\mathfrak{R}_*$ (with the norm $\| \|_*$), viz. the closure of $\mathfrak{R}_0$ in $\mathfrak{R}^*$. The norms $\| \|^*$, $\| \|_*$ are equal[4]).

Leżański [1], p. 267, has proved that

(v) *If $T \epsilon \mathfrak{R}_*$, then*

(19) $$|\mathrm{tr}(TK)| \leqslant \|T\|_* \|K\| \quad \textit{for} \quad K \epsilon \mathfrak{R}_0.$$

Consequently, T satisfies the condition (*).

5. There exist operators $T \epsilon \mathfrak{R}_*$ which are not completely continuous[5]). An example of such operators has been given by J. von Neumann (see [9], p. 322). We quote here another, simpler example due to C. Ryll-Nardzewski.

Let $\{r_n(s)\}$ be the Rademacher orthogonal system, and let T be the integral operator with the kernel

(20) $$T(s,t) = \sum_{n=1}^{\infty} r_n(s)\chi_n(t)$$

where $\chi_n(t)$ is the characteristic function of the interval $(1/(n+1), 1/n)$. In other words,

$$T(s,t) = r_n(s) \quad \text{for} \quad \frac{1}{n+1} < t < \frac{1}{n}.$$

Clearly, $T \epsilon \mathfrak{R}_*$ since the kernel $T(s,t)$ is bounded.

Let $x_n(t) = n(n+1)\chi_n(t)$. We have $\|x_n\|_{\boldsymbol{L}} = 1$ and $Tx_n = r_n$.

The set of Rademacher functions r_n is not compact in $\boldsymbol{L}$ since $\|r_n - r_m\|_{\boldsymbol{L}} = 1$ for $n \neq m$. This proves that the operation T is not completely continuous.

Each operator belonging to the trace class is completely continuous since it is the limit (in the norm) of a sequence of finitely dimensional operators (for the case $\boldsymbol{X} = \boldsymbol{L}$ — see (iv)). Hence we infer that the integral operator T with the kernel (20) satisfies the condition (*) (see (v)) but does not belong to the trace class of $\boldsymbol{L}$. Consequently, the Leżański theory can be applied to the operator T, but the Ruston theory cannot.

[4]) During the print of this paper C. Ryll-Nardzewski has proved that $\mathfrak{R}^*$ is identical with $\mathfrak{R}_*$.

[5]) On the other hand, if $T \epsilon \mathfrak{R}_*$, then the superposition TT is completely continuous. See [9], p. 323.

Incidentally we have proved

THEOREM II. *There exist linear operators which satisfy the hypothesis* (*) *of the determinant theory of Leżański but are not completely continuous.*

6. The examples given in § 5 are rather complicated. Now we shall prove that there exist very simple and natural operators which satisfy the Leżański theory but do not satisfy the Ruston theory. As an example of this type we quote here the integration operator T:

$$Tx = y \quad \text{where} \quad y(s) = \int_0^s x(t)\,dt,$$

i. e. the integral operator T (from $\boldsymbol{L}$ into $\boldsymbol{L}$) determined by the kernel

$$T(s,t) = \begin{cases} 1 & \text{if} \quad 0 \leqslant t \leqslant s \leqslant 1, \\ 0 & \text{if} \quad 0 \leqslant s < t \leqslant 1. \end{cases}$$

The proof will be preceded by the following lemmas.

(vi) *Let $I = A_1 + \ldots + A_r$ be a decomposition of the unit interval I into disjoint measurable sets. For every $s \epsilon I$ except a finite number*[6]), *there exists a positive integer $i = i(s) \leqslant r$ such that*

$$|A_{i(s)} \cdot (0,s)| > 0 \quad \textit{and} \quad |A_{i(s)}(s,1)| > 0.$$

For suppose that it is not so. Then there exists an infinite set $B \subset I$ and a set C of positive integers $i \leqslant r$ such that, for every $s \epsilon B$,

$$|A_i \cdot (0,s)| = |A_i| \quad \text{and} \quad |A_i \cdot (s,1)| = 0 \quad \text{for} \quad i \epsilon C,$$

$$|A_i \cdot (0,s)| = 0 \quad \text{and} \quad |A_i \cdot (s,1)| = |A_i| \quad \text{for} \quad i \bar{\epsilon} C.$$

Hence it follows that the set

$$\sum_{i \epsilon C} A_i$$

differs from the interval $(0,s)$ only by a set of measure zero. Consequently,

$$s = |(0,s)| = \sum_{i \epsilon C} |A_i| \quad \text{for} \quad s \epsilon B,$$

which is impossible since the right-hand side is constant and the left-hand side is not.

(vii) *For every operator $K \epsilon \boldsymbol{K}_0$ with the kernel* (16), *where $x_i \epsilon \boldsymbol{L}$ and $\xi_i \epsilon \boldsymbol{M}_0$, we have $\|T - K\|_* \geqslant \frac{1}{2}$.*

[6]) More exactly: except a number $\leqslant 2^r$.

Let $I = A_1 + \ldots + A_r$ be a decomposition of I into disjoint measurable sets such that each function ξ_j is constant on each set A_i ($j = 1, 2, \ldots, m$; $i = 1, 2, \ldots, r$). Let $i(s)$ have the same meaning as in Lemma (vi). The function $K(s, t)$, considered as a function of one variable t, is constant on $A_{i(s)}$. On the other hand,

$$T(s, t) = \begin{cases} 1 & \text{for} \quad t \in A_{i(s)}(0, s), \\ 0 & \text{for} \quad t \in A_{i(s)}(s, 1). \end{cases}$$

Since these sets have a positive measure, we infer that

$$\sup_t \operatorname{ess} |T(s, t) - K(s, t)| \geqslant \tfrac{1}{2},$$

which implies (vii).

THEOREM III. *The integration operator has the property* (*) *but does not belong to the trace class of* $\boldsymbol{L}$[7]).

The first remark immediately follows from (v) since $\|T\|_* < \infty$.

The second remark follows from (vii) and the first part of (iv).

By a similar method we can prove that if f is a bounded function continuous in the interval $\langle -1, 1 \rangle$, except a point s_0 $(-1 < s_0 < 1)$ where

$$\lim_{s \to s_0 -} f(s) \neq \lim_{s \to s_0 +} f(s),$$

then the integral operator T with the kernel $T(s, t) = f(s - t)$ belongs to $\mathfrak{K}_*$ and, consequently, satisfies the condition (*) but does not belong to the trace class of $\boldsymbol{L}$. The hypothesis about f can be weakened.

In particular, if g is a continuous function on $\langle 0, 1 \rangle$ and $g(0) \neq 0$, then the integral operator $y = Tx$ defined by the convolution formula

$$y(s) = \int_0^s g(s - t) x(t)\, dt \qquad (x \in \boldsymbol{L})$$

belongs to $\mathfrak{K}_*$, but does not belong to the trace class of $\boldsymbol{L}$. This example shows that the trace class of $\boldsymbol{L}$ is rather small.

7. Consider now the case of the space $\boldsymbol{l}$ of all absolutely convergent series. Its conjugate is the space $\boldsymbol{m}$ of all bounded sequences.

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[7]) During the print of this paper I observed that Theorem III is an immediate consequence of a more general theorem of A. Grothendieck (cf. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoires of Amer. Math. Soc. 1955, p. 59).

Each linear operator K from $\boldsymbol{l}$ into $\boldsymbol{l}$ is uniquely determined by an infinite square matrix $\{\varkappa_{ij}\}$, *i. e.*, it is of the form

$$Kx = \Big\{\sum_{j=1}^{\infty} \varkappa_{ij} a_j\Big\} \quad \text{for} \quad x = \{a_j\} \in \boldsymbol{l}.$$

The necessary and sufficient condition for a matrix $\{\varkappa_{ij}\}$ to determine a linear operator K from $\boldsymbol{l}$ into $\boldsymbol{l}$ is that

$$\sup_j \sum_{i=1}^{\infty} |\varkappa_{ij}| < \infty. \tag{21}$$

Number (21) is then the norm $\|K\|$ of K.

Let

$$\|K\|_* = \sum_{i=1}^{\infty} \sup_j |\varkappa_{ij}|.$$

Obviously $\|K\| \leqslant \|K\|_*$.

Just as in §§ 4-6, let $\mathfrak{K}_*$ be the space of all operators K such that $\|K\|_* < \infty$. Let $\mathfrak{K}_0$ be the set of all finitely dimensional operators, and let $\mathfrak{K}_1$ be the set of all operators K of the form

$$Kx = \sum_{i=1}^{m} \xi_i x \cdot e_i,$$

where $\xi_i \in \boldsymbol{m}$ and e_i is the i-th unit vector of $\boldsymbol{l}$, *i. e.*,

$$e_1 = \{1, 0, 0, 0, \ldots\},$$
$$e_2 = \{0, 1, 0, 0, \ldots\},$$
$$\ldots\ldots\ldots\ldots$$

Obviously, $\mathfrak{K}_1 \subset \mathfrak{K}_0 \subset \mathfrak{K}_*$.

(viii) *For every* $\varepsilon > 0$ *and* $K_0 \in \mathfrak{K}_0$ *there is an operator* $K \in \mathfrak{K}_1$ *such that* $\|K_0 - K\|^* < \varepsilon$.

(ix) *The set* $\mathfrak{K}_*$ *is a Banach space with respect to the norm* $\|\ \|_*$.

(x) *For every* $K \in \mathfrak{K}_0$

$$\|K\|^* = \|K\|_*.$$

(xi) *If* $T \in \mathfrak{K}_*$, *then*

$$|\mathrm{tr}(TK)| \leqslant \|T\|_* \|K\| \quad \textit{for} \quad K \in \mathfrak{K}_0.$$

The proof of (viii), (ix), (x), (xi) is similar to that of (i), (ii), (iii), (v) respectively.

(xii) *The set K_1 is dense in $\mathfrak{K}_*$ (with respect to the norm $\| \|_*$).*

In fact, if $K \in \mathfrak{K}_*$ is given by a matrix $\{\varkappa_{ij}\}$ and K_n is given by the matrix

$$\begin{matrix} \varkappa_{1,1}, & \varkappa_{1,2}, & \varkappa_{1,3}, & \ldots \\ \cdots & \cdots & \cdots & \cdots \\ \varkappa_{n,1}, & \varkappa_{n,2}, & \varkappa_{n,3}, & \ldots \\ 0, & 0, & 0, & \ldots \\ 0, & 0, & 0, & \ldots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

then $K_n \in \mathfrak{K}_1$ and $\|K_n - K\|_* \to 0$.

In the case of the space $\boldsymbol{l}$ the determinant theories of Ruston and Leżański are equivalent. More exactly,

THEOREM IV. *The following conditions are equivalent for any linear operator T from $\boldsymbol{l}$ into $\boldsymbol{l}$:*

(a) *T belongs to the trace class of $\boldsymbol{l}$;*

(b) *$\|T\|_* < \infty$ (i. e. $T \in \mathfrak{K}_*$);*

(c) *T satisfies the condition $(*)$.*

The space $\mathfrak{K}^*$ is identical with the space $\mathfrak{K}_*$ and $\| \|^* = \| \|_*$. If $T \in \mathfrak{K}_*$ is determined by the matrix $\{\tau_{ij}\}$, then

$$\mathrm{tr}(T) = \sum_{i=1}^{\infty} \tau_{ii}.$$

Similarly to (iv), we can prove on account of (viii), (ix), (x) that the space $\mathfrak{K}^*$ can be identified with the subspace of $\mathfrak{K}_*$ which is the closure of $\mathfrak{K}_0$ in $\mathfrak{K}_*$. By (xii), this subspace is identical with the whole space $\mathfrak{K}_*$. Thus proves also that conditions (a) and (b) are equivalent.

By (xi), (b) implies (c). To prove the converse implication suppose that the operator T determined by a matrix $\{\tau_{ij}\}$ satisfies the condition $(*)$. Let $\|F\|$ be the norm of the linear functional $F(K) = \mathrm{tr}(TK)$ for $K \in \mathfrak{K}_0$ (see (13)).

Let $\xi_i = \{\tau_{in}\}$ for $i = 1, 2, \ldots$ Clearly, $\xi_i \in \boldsymbol{m}$. For every $\varepsilon > 0$ there exists a sequence $x_i = \{a_{ni}\} \in \boldsymbol{l}$ such that $\|x_i\|_{\boldsymbol{l}} = 1$ and $\xi_i x_i \geqslant \|\xi_i\|_{\boldsymbol{m}} - \varepsilon$. The operator K (from $\boldsymbol{l}$ into $\boldsymbol{l}$) determined by the matrix

$$\begin{matrix} a_{1,1}, & a_{1,2}, & \ldots, & a_{1,n}, & 0, 0, 0, \ldots \\ a_{2,1}, & a_{2,2}, & \ldots, & a_{2,m}, & 0, 0, 0, \ldots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

belongs to $\mathfrak{K}_0$ since, for every $x \in \boldsymbol{l}$, Kx is a linear combination of elements $x_1, \ldots, x_m$. Moreover

$$\|K\| = \sup_i \sum_{n=1}^{\infty} |a_{ni}| = \sup_i \|x_i\|_{\boldsymbol{l}} = 1$$

and

$$F(K) = \mathrm{tr}(TK) = \sum_{i=1}^{m} \sum_{j=1}^{\infty} \tau_{ij} a_{ji} = \sum_{i=1}^{m} \xi_i x_i$$

$$\geqslant \sum_{i=1}^{m} \|\eta_i\|_{\boldsymbol{m}} - m\varepsilon = \sum_{i=1}^{m} \sup_j |\tau_{ij}| - m\varepsilon.$$

The number $\varepsilon > 0$ being arbitrary, we obtain

$$\|F\| \geqslant \sum_{i=1}^{m} \sup_j |\tau_{ij}| \quad \text{for} \quad m = 1, 2, \ldots,$$

which implies $\|T\|_* \leqslant \|F\| < \infty$.

Notice that this inequality together with (xi) proves that $\|F\| = \|T\|_*$.

It is easy to see that

$$G(T) = \sum_{i=1}^{\infty} \tau_{ii} \quad (T \in \mathfrak{K}_*)$$

is a linear functional on the space $\mathfrak{K}_*$ (with the norm $\| \|_*$), and that $G(T) = \mathrm{tr}(T)$ for $T \in \mathfrak{K}_1$. Hence, by (xii), $G(T) = \mathrm{tr}(T)$ for every $T \in \mathfrak{K}_*$.

8. In the case of the Hilbert space H the determinant theories of Leżański and of Ruston are also equivalent.

THEOREM V. *The following conditions are equivalent for every linear operator T from the Hilbert space H into H:*

(a) *T belongs to the trace class of H;*

(b) *T satisfies the condition $(*)$;*

(c) *T is the superposition of two operators, T_1 and T_2, belonging to the Schmidt class*[8]*).*

Theorem V is only another formulation of some theorems of von Neumann and Schatten (see [7], Theorems 2.1, 2.2 and 2.3; see also [5], Theorems 5.11, 5.12, 5.13).

Notice that the norm of functional (13) on the space $\mathfrak{K}_0$ of all finitely dimensional operators from H into H is equal to $\|(\overline{T}T)^{1/2}\|$ where $\overline{T}$ is the operator associated with T (see [7], Theorem 2.1).

[8]) An operator T from H into H belongs to the Schmidt class if

$$\sum_{i=1}^{\infty} |(Tx_i, x_i)|^2 < \infty$$

for every orthogonal normed sequence $\{x_i\}$.

The integration operator T in the space $\boldsymbol{L}^2$ of all functions integrable in the second power on $(0, 1)$ does not satisfy the hypotheses of the determinant theories of Leżański and of Ruston. In fact, let us consider the operator $K \epsilon \mathfrak{K}_0$ (from $\boldsymbol{L}^2$ into $\boldsymbol{L}^2$) of the form (2) where $\xi_j(t) = \sin 2\pi jt$ and $x_j(s) = \cos 2\pi js$.

We have $\|K\| = 1$ (see [6], Lemma 3.3) and

$$\operatorname{tr}(TK) = \sum_{j=1}^{m} \int_0^1 \frac{\sin^2 2\pi jt}{2\pi j}\, dt = \frac{1}{4\pi} \sum_{j=1}^{m} \frac{1}{j}.$$

The positive integer m being arbitrary, we infer that

$$\sup_{\substack{K \leqslant 1 \\ K \epsilon \mathfrak{K}_0}} |\operatorname{tr}(TK)| = \infty,$$

i. e. the condition (*) is not satisfied.

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Reçu par la Rédaction le 15. 6. 1956

Sur l'espace linéaire avec dérivation

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1. Soit $\mathcal{F}$ un espace linéaire sur un corps commutatif $\mathcal{C}$ de caractéristique 0. Soit D un endomorphisme, assujetti à la condition suivante:

(I) *Quel que soit n naturel, toute équation $P(D)x = 0$ d'ordre n $(P(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_0)$ a au plus n solutions linéairement indépendantes.*

Dans un travail antérieur [1] j'ai démontré que s'il existe un autre endomorphisme T tel que

$$DTx = TDx + x, \tag{1}$$

la proposition suivante a lieu:

(II) *Si une équation $P_1(D)x = 0$ a exactement p_1 solutions linéairement indépendantes et une autre équation $P_2(D)x = 0$ en a exactement p_2, l'équation $P_1(D)P_2(D)x = 0$ a exactement $p_1 + p_2$ solutions linéairement indépendantes.*

Donc, l'existence d'un endomorphisme T satisfaisant à la condition (1) est une condition suffisante pour que la proposition (II) ait lieu.

Le but principal de cet article est de démontrer que cette condition est aussi nécessaire, pourvu que tout élément de $\mathcal{F}$ soit une solution d'une équation $P(D)x = 0$.

La condition (II) exprime que, si l'on multiplie deux équations, les nombres de leurs solutions linéairement indépendantes s'ajoutent. Lorsque l'endomorphisme D est interprété comme une *dérivation*, l'endomorphisme T joue formellement le rôle d'une multiplication par un argument par rapport auquel on effectue la dérivation.

2. Soit $Q(\xi) = \xi^q + b_{n-1}\xi^{q-1} + \ldots + b_0$ un polynôme irréductible dans $\mathcal{C}$. Posons

$$B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 \\ -b_0 & -b_1 & -b_2 & \ldots & -b_{q-1} \end{bmatrix}.$$