Spaces of continuous functions (I)

(M_0-spaces)

by

Z. SEMADENI and P. ZBIJEWSKI (Poznań)

We know many necessary and sufficient conditions for a Banach lattice to be equivalent to the space of continuous functions on a certain compact Hausdorff space; we believe that the most important are those found by S. Kakutani [2] and M. and S. Krein [4], namely:

(1) $x \wedge y = 0$ implies $\|x+y\| = \|x-y\|$,

(2) $x \geqslant 0$ and $y \geqslant 0$ imply $\|x \vee y\| = \max(\|x\|, \|y\|)$,

(3) the unit exists

(the strong or order unit, *i.e.* the element e such that $\|x\| \leqslant 1$ is equivalent to $x \vee (-x) \leqslant e$).

In this note the procedure of Kakutani is extended to the space of continuous functions on a σ-compact

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n,$$

which, provided with the almost uniform convergence, is a B_0^*-space; we also generalize further results of Kakutani concerning the general form of linear functionals and the topological uniqueness of Ω.

Part 1. Representation

1. Definitions. The functional $\|x\|$ is called an *M-pseudonorm* if it is a B-pseudonorm[1]) and satisfies (1) and (2); it is called an *M-norm* if the condition $\|x\| = 0$ implies $x = 0$.

A normed vector lattice X is called an *M^*-space* if the metric in X is determined by an M-norm. A complete M^*-space is called an *M-space.*

[1]) Notation and auxiliary notions on B_0-spaces are taken from [5].

We shall say that an *M^*-space X has a unit* if there exists in X a strong unit e. A vector lattice X is called an *M_0^*-space* if it is a B_0^*-space under a sequence of M-pseudonorms. A complete M_0^*-space is called an *M_0-space.*

If $\|x\|$ is an M-pseudonorm in the vector lattice X, by the quotient space $X/\|\cdot\|$ we shall understand the space $X/\|\cdot\|$ in the usual linear and metric sense, with an order relation $\geqslant$ defined in the following way: *$x/\|\cdot\| \geqslant y/\|\cdot\|$ if and only if there exist $x_1 \in x/\|\cdot\|$ and $y_1 \in y/\|\cdot\|$ such that $x_1 \geqslant y_1$.*

Since conditions (1) and (2) imply $\|x\| \leqslant \|y\|$ when $|x| \leqslant |y|$, we see that the set $\{x\colon \|x\| = 0\}$ is an l-ideal in the sense of G. Birkhoff and therefore ([1], p. 222, Theorem 9) $X/\|\cdot\|$ is also a lattice. Moreover, we have

LEMMA 1. *If $\|x\|$ is an M-pseudonorm in a vector lattice X, then $X/\|\cdot\|$ is the M^*-space and $(x \vee y)/\|\cdot\| = (x/\|\cdot\|) \vee (y/\|\cdot\|)$.*

This enables us to introduce a definition of unit. An M_0^*-space X will be called an *M^*-space with unit* if there exists an element $e \in X$ and a sequence of M-pseudonorms $\|x\|_1, \|x\|_2, \ldots$ determining the topology of X such that the coset $e/\|\cdot\|_n$ is a unit in $X_n = X/\|\cdot\|_n$ for $n = 1, 2, \ldots$ In this case we shall call e the *unit under the pseudonorms* $\|x\|_1, \|x\|_2, \ldots$ or, shortly, the *unit in X.*

Let X be an arbitrary B_0-space. By X^* we shall denote the space of all linear functionals over X and by X_n^* the set of functionals of order n (*i.e.* continuous with respect to the pseudonorm

$$\max_{i=1,\ldots,n} \|x\|_i).$$

Evidently $X_n^* \subset X_{n+1}^*$ and

$$X_n^* = \bigcup_{n=1}^{\infty} X_n^*$$

(see [6], p. 139, Theorem 2.2.1). Since

$$\max_{i=1,\ldots,n} \|x\|_i = 0$$

and $\xi \in X_n^*$ imply $\xi(x) = 0$, the space X_n^* is equivalent to the conjugate space of $X_n = X/\|\cdot\|_n$.

LEMMA 2. *The topology induced on X_n^* by the weak topology of X^* (i.e. the topology with the system of neighbourhoods*

$$U(\xi_0; x_1, \ldots, x_n; \varepsilon) = \{\xi \in X^*\colon |\xi(x_i) - \xi_0(x_i)| < \varepsilon\} \quad \textit{for} \quad i = 1, \ldots, n)$$

is equivalent to the weak topology of X_n^* considered as the conjugate space to X_n. Moreover, X_n^* is closed in X^*.

2. Examples[2]). We now give some examples of M_0-spaces.

1° The space $X = C_0(-\infty, \infty)$ of all functions $x(t)$ continuous in an infinite interval, with pseudonorms

$$\|x\|_n = \max_{\langle -n,n\rangle} |x(t)|$$

and the usual order. The function $e(t) = 1$ is the unit.

1ª The same space has no unit under the equivalent pseudonorms

$$\|x\|_n' = n\cdot \max_{\langle -n,n\rangle} |x(t)|.$$

1ᵇ The same space with the pseudonorms

$$\|x\|_n'' = \max_{\langle -n,n\rangle} |x(t)\varrho_n(t)| \quad \text{where} \quad \varrho_n(t) = \begin{cases} 1 & \text{for} \quad t \leqslant n-1, \\ n-t & \text{for} \quad n-1 \leqslant t \leqslant n. \end{cases}$$

Here every quotient space X_n is incomplete and without unit.

2° The space of all real sequences $\{x_n\}$ with $\|x\|_n = |x_n|$. The sequence $\{1, 1, \ldots\}$ is the unit.

3° A countable product

$$X = \mathop{\mathrm{P}}_{n=1}^{\infty} X_n$$

of M-spaces $X_1, X_2, \ldots$ If $x = \{x_1, x_2, \ldots\}$ (where $x_n \epsilon X_n$), then $\|x\|_n = \|x_n\|$. The space X has a unit if and only if every space X_n has a unit e_n. In this case the unit of X is $e = \{e_1, e_2, \ldots\}$.

4° The space X of all functions continuous on $(-\infty, \infty)$ such that

$$\lim_{t\to\infty} x(t) = 0,$$

with pseudonorms

$$\|x\|_n = \sup_{\langle -n,\infty\rangle} |x(t)|.$$

X has no unit.

5° The linear lattice spanned on all polynomials defined in $\langle 0, 1\rangle$ such that $x(0) = 0$ is an M^*-space without unit.

6° The linear lattice spanned on all polynomials in $(-\infty, \infty)$ is an M_0^*-space with a unit.

[2]) Detailed analysis of these examples will be published later.

7° Let Ω be a σ-compact Hausdorff space and let

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$$

where the Ω_n are compact. The space $C_0(\Omega)$ of continuous functions defined on Ω is an M_0^*-space with pseudonorms

$$\|x\|_n = \max_{\Omega_n} |x(t)| \tag{4}$$

and unit $e(t) \equiv 1$[3]).

8° Let Ω be σ-compact, $\{t_a'\}$ and $\{t_a\}$ $(a \epsilon \mathfrak{A})$ two sets of points of Ω and $\{\lambda_a\}$ $(a \epsilon \mathfrak{A})$ a set of real numbers. The space

$$C_0(\Omega;\, t_a,\, t_a',\, \lambda_a,\, a \epsilon \mathfrak{A}) \tag{5}$$

of all functions continuous on Ω such that $x(t_a') = \lambda_a x(t_a)$ for $a \epsilon \mathfrak{A}$ is an M_0^*-space with pseudonorms (4). If, for certain $a \epsilon \mathfrak{A}$, $\lambda_a \neq 1$ and t_a, t_a' are limit points of Ω, then there exists no unit under these pseudonorms.

9° The space X of all functions continuous on $\langle 0, 1\rangle$ and such that

$$\lim_{t\to 0} \frac{x(t)}{t^n} = 0 \quad \text{for} \quad n = 1, 2, \ldots,$$

with pseudonorms

$$\|x\|_n = \sup_{(0,1)} \left|\frac{x(t)}{t^n}\right|,$$

has no unit. Moreover, X cannot be equivalent to a space (5) under any equivalent system of pseudonorms.

3. Representation. In this section we obtain the necessary and sufficient conditions for a given M_0-space X to be equivalent to one of the spaces $C_0(\Omega)$ and $C_0(\Omega;\, t_a,\, t_a',\, \lambda_a,\, a \epsilon \mathfrak{A})$.

LEMMA 3. *The completion $\tilde{Y}$ (in the usual sense) of an M^*-space Y is an M-space.*

Proof. We shall prove only axiom (1), *i. e.* that $y \wedge z = 0$ implies $\|y+z\| = \|y-z\|$ for any $y, z \epsilon \tilde{Y}$. Let

$$y = \lim_{n\to\infty} y_n \quad \text{and} \quad z = \lim_{n\to\infty} z_n$$

where $y_n \epsilon Y$ and $z_n \epsilon Y$. Since

$$[y_n - (y_n \wedge z_n)] \wedge [z_n - (y_n \wedge z_n)] = 0,$$

[3]) $C_0(\Omega)$ is complete (metrically) if Ω is locally compact.

we have

$$\|[y_n-(y_n\wedge z_n)]+[z_n-(y_n\wedge z_n)]\| = \|[y_n-(y_n\wedge z_n)]-[z_n-(y_n\wedge z_n)]\|.$$

Passing to the limit we obtain

$$\|z+y\| = \lim_{n\to\infty}\|y_n-2(y_n\wedge z_n)+z_n\| = \lim_{n\to\infty}\|y_n-z_n\| = \|y-z\|$$

because

$$\lim_{n\to\infty} y_n\wedge z_n = y\wedge z = 0.$$

LEMMA 4. *If $\|x\|_1, \|x\|_2, \ldots$ are M-pseudonorms, then*

$$\|x\|_n^* = \max_{i=1,\ldots,n}\|x\|_i \quad (n = 1, 2, \ldots)$$

forms the equivalent system of M-pseudonorms.

The proof is trivial. By this lemma we may restrict ourselves to the consideration of only M_0^*-spaces with monotone pseudonorms, *i. e.* those satisfying the condition $\|x\|_n \leqslant \|x\|_{n+1}$ for $n = 1, 2, \ldots$ and every $x\in X$.

Now, denote by Φ_n the set of all functionals $\xi\in X_n^*$ satisfying the following conditions:

(6) $\xi \geqslant 0$; $x\wedge y = 0$ implies $\xi(x)\cdot\xi(y) = 0$;

(7) $$\|\xi\|_n = 1.$$

Denote by Ω_n the closure of Φ_n in the weak topology. Put

$$\Phi = \bigcup_{n=1}^{\infty}\Phi_n \quad\text{and}\quad \Omega = \bigcup_{n=1}^{\infty}\Omega_n.$$

Evidently (by lemma 2) Ω is σ-compact in the weak topology induced from X^*. Any $\xi\in\Omega$ of order n is also of order $n+1$, moreover for $\xi\in\Omega$ we have

$$\|\xi\|_n = \sup_{\|x\|_n\leqslant 1}\xi(x) \geqslant \sup_{\|x\|_{n+1}\leqslant 1}\xi(x) = \|\xi\|_{n+1}.$$

If for every $\xi\in\Omega$ the condition $\xi\in X_n^*$ implies $\|\xi\|_n = \|\xi\|_{n+1}$, we say that the pseudonorms $\|x\|_1, \|x\|_2, \ldots$ are *consistent*. In other words: the consistency of pseudonorms is equivalent to the inclusions $\Omega_n\subset\Omega_{n+1}$ $(n = 1, 2, \ldots)$

THEOREM 1. *Every M_0^*-space X with monotone and consistent pseudonorms is equivalent in the linear, metric and lattice sense to a dense subset of a space $Y = C_0(\Omega; t_a, t'_a, \lambda_a, a\in\mathfrak{A})$* (see example 8°).

Proof. For every $\xi_a\in\Omega-\Phi$ there exists a $\xi'_a\in\Phi$ such that $\xi_a = \lambda_a\cdot\xi'_a$ and $\lambda_a < 1$, namely $\lambda_a = \|\xi_a\|$ (see [2], p. 1001). Putting

(8) $$x(\xi) = \xi(x) \quad\text{for}\quad \xi\in\Omega,$$

we define a one-to-one mapping φ of X into Y; it is clear that the function $x(\xi)$ is continuous on Ω, and that

$$\varphi(x+y) = \varphi(x)+\varphi(y),$$

$$\|x\|_k = \max_{\xi\in\Omega_k}|x(\xi)| = \|\varphi(x)\|_k,$$

$$x\geqslant y \text{ implies } x(\xi)\geqslant y(\xi) \text{ for all } \xi\in\Omega, \text{ i. e. } \varphi(x)\geqslant\varphi(y).$$

By Lemma 1, Lemma 3 and the theorem of Kakutani ([2], p. 998, Theorem 1) we see that if $x(\xi)\geqslant y(\xi)$ for all ξ, then $x/\|\cdot\|_n \geqslant y/\|\cdot\|_n$ for arbitrary n. But $x/\|\cdot\|_n\geqslant y/\|\cdot\|_n$ means that there exist x_n and y_n such that $\|x-x_n\|_n = \|y-y_n\|_n = 0$ and $x_n\geqslant y_n$. Since $\|x\|_n\leqslant\|x\|_m$ for $n\leqslant m$, we see that $\|x-x_m\|_n = 0$ for $m > n$. Hence

$$x = \lim_{n\to\infty} x_n$$

and similarly

$$y = \lim_{n\to\infty} y_n.$$

From the continuity of the order relation we obtain $x\geqslant y$. It remains to prove that for every $x_0\in Y$ and $\varepsilon > 0$ there is a $y\in\varphi(X)$ such that $\|x-y\| < \varepsilon$. Choose n such that $1/2^n < \varepsilon/2$; then the function $x(t)$ can be approximated on Ω_n by a certain $y_n = \varphi(x_n)$ so that $\|x-y_n\| < \varepsilon/2$, whence

$$\|y_n-x\| = \sum_{k=1}^{n}\frac{1}{2^k}\frac{\|y_n-x\|_k}{1+\|y_n-x\|_k}+\sum_{k=n+1}^{\infty}\frac{1}{2^k}\frac{\|y_n-x\|_k}{1+\|y_n-x\|_k} < \sum_{k=1}^{n}\frac{\varepsilon}{2^{k+1}}+\frac{1}{2^n} < \varepsilon.$$

This completes the proof.

COROLLARY. *If the pseudonorms are consistent and X is complete, then any quotient space $X_n = X/\|\cdot\|_n$ is complete.*

LEMMA 5. *Let $\|x\|$ be an M-pseudonorm. If $e\in X$ is such that $e/\|\cdot\|$ is a unit in $X/\|\cdot\|$ and $\|x\|\leqslant 1$, then $\|e-(e\vee x)\| = 0$.*

Proof. If $\|x\|\leqslant 1$, then $x/\|\cdot\|\leqslant e/\|\cdot\|$, whence $e/\|\cdot\| = x/\|\cdot\|\vee e/\|\cdot\| = x\vee e/\|\cdot\|$.

LEMMA 6. *If e is a unit under the pseudonorms $\|x\|_1, \|x\|_2, \ldots$, then e is a unit under the pseudonorms*

$$\|x\|_n^* = \max_{i=1,\ldots,n}\|x\|_i.$$

Proof. If $\|x\|_n^*\leqslant 1$, then $\|x\|_i\leqslant 1$ for $i = 1, 2, \ldots, n$ and, by Lemma 5, $\|e-(e\vee x)\|_i = 0$ for $i = 1, 2, \ldots, n$, whence $\|e-(e\vee x)\|_n^* = 0$.

THEOREM 2. *For any M_0-space with a unit there exists a σ-compact Hausdorff space*

$$\boldsymbol{\Omega} = \bigcup_{n=1}^{\infty} \boldsymbol{\Omega}_n$$

such that X is linear and lattice-isomorphic with the space $C_0(\boldsymbol{\Omega})$ of continuous functions on $\boldsymbol{\Omega}$ and if $x \to x(t)$ in this isomorphism, then

$$\max_{i=1,\ldots,n} \|x\|_i = \max_{t \in \Omega_n} |x(t)|.$$

Proof. If the unit exists, then condition (7) can be replaced by $\xi(e) = 1$. It follows that the existence of the unit implies the consistency of pseudonorms. Moreover, then all Φ_n are closed (*i. e.* $\Phi_n = \boldsymbol{\Omega}_n$ and $\Phi = \boldsymbol{\Omega}$), and the set $\mathfrak{A}$ of indices is empty. Therefore by Theorem 1, (8) is the mapping from X onto $C_0(\boldsymbol{\Omega})$.

COROLLARY. *If the unit exists and X is complete, then all X_n are also complete.*

Part II. Linear functionals over $C_0(\Omega)$

1. General form of the linear functional over $C_0(\Omega)$. By the theorem of Kakutani ([2], p. 1012, Theorem 10) about the general form of linear functionals in M-spaces and a theorem of Mazur and Orlicz ([6], p. 139, th. 2.21) we get

THEOREM 3. *Every linear functional on the M_0-space $X = C_0(\Omega)$ can be represented by the integral*

$$\xi(x) = \int_{\Omega} x(t)\, d\mu, \tag{9}$$

where μ is a σ-additive set function of bounded variation defined for Borel subsets of Ω and vanishing outside a certain Ω_n. Moreover, $\xi \in X_n^*$ and

$$\|\xi\|_m = \operatorname*{Var}_{\Omega_n} \mu \quad \textit{for} \quad m \geqslant n.$$

2. Weak convergence in $C_0(\Omega)$. We prove

THEOREM 4. *The sequence $\{x_m\} \subset X = C_0(\Omega)$ is weakly convergent to x_0 (i. e. $\xi_0(x) = \lim\limits_{m\to\infty} \xi(x_m)$ for each $\xi \in X^*$) if and only if*

$$\sup_{m=1,2,\ldots} \|x_m\|_n < \infty \quad (n = 1, 2, \ldots), \tag{10}$$

$$x_m(t) \to x_0(t) \quad \textit{for every} \quad t \in \Omega\,[4]). \tag{11}$$

[4]) A generalization of a theorem of Banach (see S. Banach, *Théorie des opérations linéaire*, Warszawa-Lwów 1932, p. 224).

Proof. The necessity of (10) follows from the Banach-Steinhaus theorem, (11) is trivial. The sufficiency is an immediate result of the integral representation (9) and the Lebesgue theorem on term by term integration.

THEOREM 5 (The generalized theorem of Dini). *If $x_m \in X$, $x_m \geqslant x_{m+1}$ for $m = 1, 2, \ldots$ and x_m are weakly convergent to x_0, then x_m are strongly convergent to x_0, i. e.*

$$\lim_{m\to\infty} \|x_m - x_0\|_n = 0 \quad \textit{for} \quad n = 1, 2, \ldots$$

Proof. Given integer n and $\varepsilon > 0$, write

$$\overline{F_m^{(n)}} = \{t \in \Omega_n : |x_m(t) - x_0(t)| \geqslant \varepsilon\}.$$

Since $F_m^{(n)} = \overline{F_m^{(n)}} \supset F_{m+1}^{(n)}$ and

$$\bigcap_{m=1}^{\infty} F_m^{(n)} = 0,$$

from the compactness of Ω_n follows the existence of an m such that $F_m^{(n)} = 0$, whence $\|x_m - x_0\|_n < \varepsilon$.

Part III. Topological properties of Ω

In this section we establish the topological invarience of σ-compact Ω under some tranformations of $C_0(\Omega)$ and some connections between the properties of Ω and $C_0(\Omega)$. The method used is a generalization of that of Kaplansky (see [3], p. 617-620, and also [1], p. 175-176) who used it in the case of compact Ω.

1. Topological uniqueness of Ω. We remark first that *if Ω is completely regular and σ-compact, then the M_0^*-space $C_0(\Omega)$ determines topologically Ω.*

In fact, if $C(\Omega_1) \equiv C(\Omega_2)$, then $\Omega_1 \underset{\text{top}}{=} \boldsymbol{\Omega}_1 \underset{\text{top}}{=} \boldsymbol{\Omega}_2 \underset{\text{top}}{=} \Omega_2$.

Let X be a lattice $C_0(\Omega)$ of real-valued continuous functions defined on σ-compact Ω. Denote by I the family of sets $I \subset X$ satisfying the following conditions:

(12) $x \in I$ and $y \leqslant x$ imply $y \in I$,

(13) if $x_n \in I$ and $\bigvee\limits_{n=1}^{\infty} x_n$ exists, then $\bigvee\limits_{n=1}^{\infty} x_n \in I$,

(14) if $y_n \in X - I$ and $\bigwedge\limits_{n=1}^{\infty} y_n$ exists, then $\bigwedge\limits_{n=1}^{\infty} y_n \in X - I$.

In other words, J is the family of σ-prime σ-ideals.

We shall say that *an ideal $I \epsilon J$ is associated with a point $t \epsilon \Omega$* when the conditions $x \epsilon I$ and $y(t) < x(t)$ imply $y \epsilon I$.

LEMMA 7. *If Ω is locally compact, then every ideal $I \epsilon J$ is associated with a certain point $t \epsilon \Omega$.*

Proof. Suppose that I is not associated with any point $t \epsilon \Omega$. In particular, for any $t \epsilon \Omega_n$ there exist functions $x_t(\tau)$ and $y_t(\tau)$ such that $x_t \epsilon I$, $y_t \epsilon X-I$ and $x_t(t) > y_t(t)$. The sets $G(x_t, y_t) = \{\tau: x_t(\tau) > y_t(\tau)\}$ consist of an open covering of the compact set Ω_n, whence there exist a finite number of points $t_1, \ldots, t_m$ and corresponding pairs of functions $x_1, y_1, \ldots, x_m, y_m$ such that $x_i \epsilon I$, $y_i \epsilon X-I$ $(i = 1, \ldots, n)$ and

$$\Omega_n \subset \bigcup_{i=1}^{m} G(x_i, y_i) \subset G(\bigvee_{i=1}^{m} x_i, \bigwedge_{i=1}^{m} y_i).$$

By (13) the function

$$a_n = \bigvee_{i=1}^{m} x_i$$

belongs to I and by (14)

$$b_n = \bigwedge_{i=1}^{m} y_i$$

belongs to $X-I$ $(n = 1, 2, \ldots)$, moreover $a_n(t) \geqslant b_n(t)$ on Ω_n. We put

$$g_n = \bigvee_{i=1}^{n} a_i, \quad h_n = \bigwedge_{i=1}^{n} b_i, \quad u_1 = g_1 \quad \text{and} \quad v_1 = h_1;$$

then $g_n \epsilon I$ and $h_n \epsilon X-I$.

Suppose that $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are already defined so that $u_n \epsilon I$, $v_n \epsilon X-I$, $g_n \geqslant u_n$, $h_n \leqslant v_n$, $u_n(t) = u_{n-1}(t)$ for $t \epsilon \Omega_{n-1}$, $v_n(t) = v_{n-1}(t)$ for $t \epsilon \Omega_{n-1}$ and $u_n(t) \geqslant v_n(t)$ for $t \epsilon \Omega_n$. We put

$$u_{n+1} = g_{n+1} \wedge (v_n \vee v_n) \quad \text{and} \quad v_{n+1} = h_{n+1} \vee (u_n \wedge v_n).$$

Evidently $u_{n+1} \epsilon I$, $v_{n+1} \epsilon X-I$ (from (12)), $g_{n+1} \geqslant u_{n+1} \geqslant u_n$ and $h_{n+1} \leqslant v_{n+1} \leqslant v_n$; moreover $u_{n+1}(t) = u_n(t)$ and $v_{n+1}(t) = v_n(t)$ for $t \epsilon \Omega_n$, $u_{n+1}(t) \geqslant v_{n+1}(t)$ for $t \epsilon \Omega_{n+1}$.

The sequences $\{u_n\}$ and $\{v_n\}$ are convergent with respect to the norm

$$\|x\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\max_{\Omega_n} |x(t)|}{1 + \max_{\Omega_n} |x(t)|}. \tag{15}$$

Let $u_0 = \lim u_n$ and $v_0 = \lim v_n$. Since $\{u_n\}$ and $\{v_n\}$ are monotone, we have

$$u_0 = \bigvee_{n=1}^{\infty} u_n \quad \text{and} \quad v_0 = \bigwedge_{n=1}^{\infty} v_n.$$

Next, by (13) $u_0 \epsilon I$ and by (14) $v_0 \epsilon X-I$. This contradicts (12), for $u_0 \geqslant v_0$.

LEMMA 8. *Any ideal $I \epsilon J$ is associated with only one point of Ω.*

LEMMA 9. *Two ideals $I_1 \epsilon J$ and $I_2 \epsilon J$ are associated with the same point $t \epsilon \Omega$ if and only if $I_1 \cap I_2$ contains an ideal $I_3 \epsilon J$.*

LEMMA 10. *Let x_0 be a fixed function from X, and S any subset of Ω. Then a point t belongs to $\bar{S}$ if and only if a certain ideal $I_0 \epsilon J$ associated with t contains the intersection $J(S, x_0)$ of the totality of all ideals $I \epsilon J$ containing x_0 which are associated with points of S.*

THEOREM 6 (The generalized theorem of Kaplansky). *If a space $X = C_0(\Omega)$ of continuous functions on a locally compact and σ-compact Hausdorff space Ω can be mapped on a space $Y = C_0(\Phi)$ in a one-to-one and order preserving manner, then $\Omega \underset{\text{top}}{=} \Phi$.*

The proofs of Lemmas 8-10 and of Theorem 6 do not differ from the proofs given by Kaplansky [3] (or by Birkhoff [1], p. 175); they are true in completely regular spaces.

2. Other cases of the invariance of locally compact and σ-compact Ω. In the previous section it has been proved that the lattice $C_0(\Omega)$ determines Ω (to within the homeomorphism); we now give other results of the same kind.

THEOREM 7 (The generalized theorem of Gelfand and Kolmogorov). *The ring $C_0(\Omega)$ of continuous functions determines Ω topologically*[5]).

Proof. We put $x \geqslant y$ if and only if there exists a function $z \epsilon X$ such that $x - y = z^2$. Thus the points of Ω and its neighbourhoods can be constructed in terms of addition and multiplication only.

THEOREM 8 (The generalized Banach-Eilenberg theorem). *If $X = C_0(\Omega)$ can be mapped onto $Y = C_0(\Phi)$ in a one-to-one manner so that all pseudodistances are preserved (i. e. if $x_1 \to y_1$ and $x_2 \to y_2$, then $\max_{t \epsilon \Omega_n} |x_1(t) - x_2(t)| = \max_{t \epsilon \Omega_n} |y_1(t) - y_2(t)|$ for any n), then*

$$\Omega \underset{\text{top}}{=} \Phi.$$

Proof. Let $y = f(x)$ be a transformation of X into Y such that $\|f(x_1) - f(x_2)\|_n = \|x_1 - x_2\|_n$ for $n = 1, 2, \ldots$ We can suppose that $f(0) = 0$. In particular $\|f(x_1) - f(x_2)\|_n = 0$ if and only if $\|x_1 - x_2\|_n = 0$. This means

[5]) This theorem is due to E. Hewitt (Trans. Amer. Math. Soc. 64(1948), p. 88-89).

that the mapping $\tilde{y} = f_n(\tilde{x})$ (where $x \in X/\|\cdot\|_n = C(\Omega_n)$ and $\tilde{y} \in Y/\|\cdot\|_n = C(\Phi_n)$), defined as $\tilde{y} = f_n(\tilde{x})$ when $y = f(x)$ for $x \in \tilde{x}$ and $y \in \tilde{y}$, is an isometry between $C(\Omega_n)$ and $C(\Phi_n)$, hence a linear isometry. Let e be the unit in X, *i. e.* the function $e(t) = 1$ on Ω. The partial function $e_n(t) = e|\Omega_n$ is an extreme point of the unit sphere in $C(\Omega_n)$, whence $f_n(e_n)$ is an extreme one in $C(\Phi_n)$. It follows that $f(e)$ is a function taking only the values $+1$ on Φ' and -1 on Φ'', where Φ' and Φ'' are open-and-closed subsets of Φ. Hence if we define $x \geqslant 0$ when

$$\|x\|_n = 0 \quad \text{or} \quad \left\| \frac{X}{\|x\|_n} - e \right\|_n \leqslant 1 \quad \text{for} \quad n = 1, 2, \ldots$$

we preserve the order on Φ' and invert it on Φ''. Thus X and Y are lattice isomorphic and, by theorem 6, $\Omega \underset{\text{top}}{=} \Phi$.

Remark. If for every n there exists a mapping f_n from X on Y preserving the pseudodistance $\|x_1 - x_2\|$, then it may happen that there exists no mapping f preserving all pseudodistances simultaneously. This is connected with the well known fact that the conditions $A_1 \subset B_1$, $A_2 \subset B_2$, $A_1 \underset{\text{top}}{=} A_2$ and $B_1 \underset{\text{top}}{=} B_2$ do not necessarily imply the existence of a homeomorphism between B_1 and B_2 transforming A_1 on A_2. Moreover, the conditions $\Omega_n \underset{\text{top}}{=} \Phi_n$ $(n = 1, 2, \ldots)$ do not imply $\Omega \underset{\text{top}}{=} \Phi$, even if Ω and Φ are locally compact (*e. g.* the set Ω of all integers and the set Φ of all numbers $1/n$ with 0).

3. Metrizability of Ω. We have

LEMMA 11 (The generalized Stone-Weierstrass theorem). *Let X_0 be the smallest ring spanned upon a set $A \subset X = C_0(\Omega)$; X_0 is dense in X (in the metric (4)) if and only if A separates Ω (i. e. for any $t_1, t_2 \in \Omega$ $(t_1 \neq t_2)$ there exists a function $z \in A$ such that $z(t_1) \neq z(t_2)$).*

THEOREM 9. *Ω being locally compact and σ-compact, the space $C_0(\Omega)$ is separable if and only if Ω is metrizable (or, which is equivalent, if Ω satisfies the second countability axiom).*

Proof. Necessity. If X is separable, so is $X/\|.\|_n = C(\Omega_n)$ too; let $\{x_m\}$ be a sequence dense in $C(\Omega_n)$ and U_0 a neighbourhood of a point $t_0 \in \Omega_n$. There exists a function $y_0(t)$ continuous on Ω_n equal to 1 at t_0 and 0 outside U_0. If $\|y_0 - x_m\|_n < 1/3$, then the set $G_m = \{t: x_m(t) > 1/2\}$ is a neighbourhood of t_0 contained in U_0. It follows that the family $\{G_m\}$, $m = 1, 2, \ldots$, forms the base in Ω_n, whence Ω_n is metrizable.

Sufficiency. Let $\{R_m\}$ be the base in Ω. For each pair R_n, R_m such that $\bar{R}_n \cap \bar{R}_m = 0$ there exists a function $y_{nm}(t)$ equal to 0 on R_n and to 1 on R_m. The family $\{y_{nm}\}$ separates Ω, therefore the set of all polynomials over $\{y_{nm}\}$ with rational coefficients is countable and dense in X.

4. Direct product. We have

THEOREM 10. *If $X = C_0(\Omega)$ is a direct product of lattices X_1 and X_2 (see [1], p. 13), then Ω is the sum of two open-and-closed subsets Ω_1 and Ω_2 such that $X_1 = C_0(\Omega_1)$ and $X_2 = C_0(\Omega_2)$.*

The proof is not different from that of Kaplansky ([3], p. 620).

THEOREM 11. *If $X_1 = C_0(\Omega_1)$, $X_2 = C_0(\Omega_2)$ and if X is the direct product of X_1 and X_2 in the B_0-sense, then X is a space $C_0(\Omega)$ and Ω is homeomorphic to $\Omega_1 \cup \Omega_2$ (Ω_1 and Ω_2 are regarded as disjoint).*

Proof. This immediately follows from Theorem 8.

References

[1] G. Birkhoff, *Lattice Theory*, New York 1948.

[2] S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. of Math. 42 (1941), p. 994-1024.

[3] I. Kaplansky, *Lattices of continuous functions*, Bull. Amer. Math. Soc. 53 (1947), p. 617-622.

[4] M. Krein et S. Krein, *L'espace des fonctions continues sur un bicompact de Hausdorff et leur sous-espaces semiordonnés*, Mat. Sbornik 12 (1943), p. 1-38.

[5] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires I*, Studia Math. 10 (1948), p. 184-208.

[6] — *Sur les espaces métriques linéaires II*, ibidem 13 (1953), p. 137-179.

Reçu par la Rédaction le 23. 7. 1956