Since

$$\lim_{n=\infty} (2k_0/\vartheta_n) = 0,$$

the neighbourhood

$$U = \underset{y}{E}\{\|y\| < \varepsilon\}$$

satisfies the condition (**), *i. e.* is bounded.

COROLLARY. *From the proof of Theorem 3 it follows directly that if in an F-space a norm has the property* W_3, *then an equivalent norm has it also.*

Remark 1. The above theorem is false in the case of the F^*-space. An example is provided by the space K of all the sequences $x = (\xi_n)$ almost all elements of which vanish, the norm being

$$\|x\| = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|\xi_k|}{1+|\xi_k|}.$$

It is easily verified that the sequence $\vartheta_n = n$ is a rate of growth for the norm $\|x\|$.

Since K, being a B_0^*-space, is not a B^*-space (see [6]) there are not any bounded neighbourhoods in K.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warszawa 1932.

[2] M. Eidelheit and S. Mazur, *Eine Bemerkung über die Räume vom Typus F*, Studia Math 7 (1938), p. 159-161.

[3] S. Kakutani, *Über die Metrisation der topologischen Gruppen*, Proc. Imp. Acad. Tokyo 12 (1936), p. 159-161.

[4] V. L. Klee, *Boundedness and continuity of linear functionals*, Duke Math. Journal 22 (1955), p. 263-269.

[5] D. Maharam, *An algebraic characterization of measure algebras*, Annals of Math. 48 (1947), p. 154-167.

[6] S. Mazur and W. Orlicz, *Sur les espaces métriques linéaires I*, Studia Math. 10 (1948), p. 184-208; *II*, ibidem 13 (1953), p. 137-179.

[7] S. Rolewicz, *On certain class of linear-metric spaces*, Bull Acad. Pol. Sci., Cl. III, 5 (1957), p. 473-476.

Reçu par la Rédaction le 13. 9. 1956

Spaces of continuous functions (II)
(On multiplicative linear functionals over some Hausdorff classes)

by

Z. SEMADENI (Poznań)

S. Mazur [5] has proved that with every bounded sequence $\{x_n\}$ a real number $\operatorname{Lim}_n x_n$ can be associated in such a way that $\operatorname{Lim} x_n$ is equal to the usual limit of a subsequence of $\{x_n\}$; consequently

$$(1) \qquad \underline{\lim}\, x_n \leqslant \operatorname{Lim} x_n \leqslant \overline{\lim}\, x_n,$$

$$(2) \qquad \operatorname{Lim}(ax_n+by_n) = a\operatorname{Lim} x_n + b\operatorname{Lim} y_n,$$

$$(3) \qquad \operatorname{Lim}(x_n y_n) = \operatorname{Lim} x_n \cdot \operatorname{Lim} y_n.$$

In this note a construction of generalized limits for some classes of functions is given. This construction is non-effective, just as those of Mazur; it is based on the theorem of Kakutani on the representation of abstract (M)-spaces. It is easily seen that this limit can also be derived from the theorem of Tychonoff, but I think that the way which I have chosen leads to more consequences.

The generalization of the theorem of Mazur to the case of real-valued, bounded functions defined on $\langle 0, 1\rangle$ is trivial, *e. g.*, we can put

$$\operatorname*{Limes}_{t\to t_0} x(t) = \operatorname*{Lim}_n x(t_n)$$

where Lim denotes an arbitrary limit of Mazur and $t_n \to t_0$. The functional "Limes" constructed in the Theorems 1, 1a, 1b and 2 satisfies also some additional conditions. It can be considered as a solution of the following problem: given a space of equivalence classes of functions how to assign in a reasonable way the value to every function at every point.

The second part of this paper contains some applications (the existence of certain multiplicative measures and a negative solution of two questions concerning the extension of linear functionals).

1. The generalized limits. Let E be a topological space and $\boldsymbol{R}$ a σ-ideal of boundary sets (*i. e.* $A\epsilon\boldsymbol{R}$ and $B\subset A$ imply $B\epsilon\boldsymbol{R}$, $A_n\epsilon\boldsymbol{R}$ imply for $n=1,2$

$$\bigcup_{n=1}^{\infty} A_n\epsilon\boldsymbol{R},$$

no open non-empty set belongs to $\boldsymbol{R}$). The family $\boldsymbol{H}$ of all sets of form $G\cup A$ (where G is open and $A\epsilon\boldsymbol{R}$) is multiplicative and σ-additive.

Denote by H the class of all real-valued functions $x(t)$ on E such that the sets $\{t\colon a<x(t)<b\}$ belong to $\boldsymbol{H}$ for every a and b. Hausdorff ([3], p. 235) has established that H is closed with respect to addition, multiplication, supremum and infimum of two elements. Next, denote by

$$\sup_{E}{}_{\boldsymbol{R}}\, x(t)$$

the least upper bound of the totality of numbers a such that the se $\{t\colon x(t)>a\}$ belongs to $\boldsymbol{R}$. In particular, $\sup_L x(t)$ denotes the usua essential supremum and $\sup_B x(t)$ denotes the essential supremum with respect to the sets of Baire's first category. We introduce also the *R-essential limit* in t_0 in the following manner:

$$\overline{\lim_{t\to t_0}}{}_{\boldsymbol{R}}\, x(t)=\inf_{A\epsilon\boldsymbol{R}}\Big[\overline{\lim_{\substack{t\to t_0\\ t\epsilon E-A}}}\, x(t)\Big],\qquad \underline{\lim_{t\to t_0}}{}_{\boldsymbol{R}}\, x(t)=-\overline{\lim_{t\to t_0}}{}_{\boldsymbol{R}}[-x(t)].$$

LEMMA 1. *For any $x\epsilon\boldsymbol{H}$ the sets*

$$A=\{t\colon \underline{\lim_{\tau\to t}}{}_{\boldsymbol{R}}\, x(\tau)\neq\overline{\lim_{\tau\to t}}{}_{\boldsymbol{R}}\, x(\tau)\}\quad \textit{and}\quad B=\{t\colon x(t)\neq\lim_{\tau\to t}{}_{\boldsymbol{R}}\, x(\tau)\}$$

belong to $\boldsymbol{R}$.

Proof. According to Alexiewicz ([1], p. 64) a function $x(t)$ belongs to H if and only if the set D of its points of discontinuity belongs to $\boldsymbol{R}$. It follows that $A\epsilon\boldsymbol{R}$ and $B\epsilon\boldsymbol{R}$, because $A\cup B\subset D$.

In the class X_0 of bounded functions of H we introduce the reflexive, symmetric and transitive relation $x_1\sim x_2$ when $\{t\colon x_1(t)\neq x_2(t)\}\epsilon\boldsymbol{R}$, and we identify *R-equivalent functions*. Denote by $x,y,\ldots$ the classes of equivalence under the relation $\sim$, corresponding to elements $x_0,y_0,\ldots$ Evidently the space $X=X_0/\sim$ is a Banach space with the norm

$$\|x\|=\sup_{E}{}_{\boldsymbol{R}}|x_0(t)|.$$

In X we introduce a partial ordering by the relation: $x\leqslant y$ if and only if the set $\{t\colon x_0(t)>y_0(t)\}$ belongs to $\boldsymbol{R}$ for $x_0\epsilon x$ and $y_0\epsilon y$. If $x\wedge y=0$ then the set $\{t\colon |x_0(t)+y_0(t)|\neq|x_0(t)-y_0(t)|\}$ belongs to $\boldsymbol{R}$; it follows that $x\wedge y=0$ implies $\|x+y\|=\|x-y\|$. Similarly $x\geqslant 0$ and $y\geqslant 0$ imply

$\|x\vee y\|=\|x\|\vee\|y\|$, therefore X is an (M)-space with a unit element (see [3]). Denote it by $\mathcal{H}(E,\boldsymbol{R})$ and by $\boldsymbol{\Omega}$ the set of all linear functionals defined on X which satisfy the conditions

(4) $\quad \|\xi\|=1,\ \xi(x)\geqslant 0$ for $x\geqslant 0$, $x\wedge y=0$ implies $\xi(x)\cdot\xi(y)=0$.

By the theorem of Kakutani X can be linearly, isometrically and isotonically mapped on the space $C(\boldsymbol{\Omega})$ of continuous functions defined on $\boldsymbol{\Omega}$ (which is compact in a weak topology).

THEOREM 1. *Suppose that E satisfies the first axiom of countability at a certain point $t_0\epsilon E$, and suppose that there exists a base $\{U_n\}$ of neighbourhoods of t_0 and a sequence of continuous functions φ_n from E into U_n such that no $\varphi_n(E)$ belongs to $\boldsymbol{R}$ $(n=1,2,\ldots)$. Then to every $x\epsilon X=\mathcal{H}(E,\boldsymbol{R})$ corresponds a generalized limit*

$$\operatorname*{Lim}_{t\to t_0} x(t)=\xi_{t_0}(x)$$

such that

(5) $$\underline{\lim_{t\to t_0}}{}_{\boldsymbol{R}}\, x(t)\leqslant\xi_{t_0}(x)\leqslant\overline{\lim_{t\to t_0}}{}_{\boldsymbol{R}}\, x(t),$$

(6) $$\xi_{t_0}(ax+by)=a\cdot\xi_{t_0}(x)+b\cdot\xi_{t_0}(y),$$

(7) $$\xi_{t_0}(x\cdot y)=\xi_{t_0}(x)\cdot\xi_{t_0}(y),$$

(8) $$\xi_{t_0}(x\vee y)=\max[\xi_{t_0}(x),\ \xi_{t_0}(y)].$$

If this limit exists for any $t_0\epsilon E$, then the function $u(t)=\xi_t(x)$ is R-equivalent to x, i. e.,

(9) $$\{t\colon \xi_t(x)\neq x_0(t)\}\epsilon\boldsymbol{R}\quad \textit{for}\quad x_0\epsilon x.$$

Proof. Choose an arbitrary fixed $\xi\epsilon\boldsymbol{\Omega}$. We put $x_n(t)=x_0(\varphi_n(t))$ for $t\epsilon E$ and $\xi_n(x)=\xi(x_n)$. Evidently $\xi_n\epsilon\boldsymbol{\Omega}$. Every limit point ξ_{t_0} of the set $\{\xi_n\}$ satisfies (6), (7) and (8); (5) follows from the identity

$$\overline{\lim_{t\to t_0}}{}_{\boldsymbol{R}}\, x(t)=\lim_{n\to\infty}\sup_{U_n}{}_{\boldsymbol{R}}\, x(t),$$

and (9) results by lemma 1.

Now, we specialize the space E and the family $\boldsymbol{R}$ to obtain some applications of theorem 1.

a) Let E be the interval $\langle 0,1\rangle$ and $\boldsymbol{R}$ the family $\boldsymbol{L}$ of sets of Lebesgue's measure zero. Then

Theorem 1a. *To every function $x(t)$ Riemann-integrable in $\langle 0,1\rangle$ corresponds a measurable function $u(t)=\xi_t(x)$ satisfying* (6), (7), (8),

$$\underline{\lim}_{\tau\to t}{}_L\, x(\tau)\leqslant \xi_t(x)\leqslant \overline{\lim}_{\tau\to t}{}_L\, x(\tau), \tag{10}$$

and such that $x(t)$ and $u(t)$ are equal almost everywhere.

b) Let E be a complete metric space and $\boldsymbol{R}$ the family $\boldsymbol{B}$ of sets of the first category. Then

Theorem 1b. *To every bounded function $x(t)$, point-wise discontinuous in E, corresponds a function $u(t)=\xi_t(x)$ satisfying* (6), (7), (8),

$$\underline{\lim}_{\tau\to t}{}_B\, x(\tau)\leqslant \xi_t(x)\leqslant \overline{\lim}_{\tau\to t}{}_B\, x(\tau) \tag{11}$$

and

(12) *$x(t)$ and $u(t)$ are equal except a set of the first category.*

Lemma 2. *Let X denote the space of all bounded functions satisfying the condition of Baire (in a wide sense), with the norm*

$$\|x\|=\sup_E{}_B|x(t)|.$$

For arbitrary $x\in X$ the set

$$A=\{t:\underline{\lim}_{\tau\to t}{}_B\, x(\tau)\neq \overline{\lim}_{\tau\to t}{}_B\, x(\tau)\}$$

is of the first category[1]).

Proof. Let us notice first that it is sufficient to prove this lemma for the functions taking only the values 0 and 1, because the simple functions (*i. e.*, the functions with a finite set of values) form a dense set in X, and from the inequality

$$|\overline{\lim}_{\tau\to t}{}_B\, x(\tau)-\overline{\lim}_{\tau\to t}{}_B\, y(\tau)|\leqslant\|x-y\|$$

it follows that the operations $\overline{\lim}_{\tau\to t}{}_B x(\tau)$ and $\underline{\lim}_{\tau\to t}{}_B x(\tau)$ are continuous in X.

Let H be an arbitrary subset of E satisfying the condition of Baire, *i. e.*, $H=(G\cup P)-Q$ where G is open and P,Q of the first category. Its characteristic function $\chi_H(t)$ is B-equivalent to the characteristic function of G. Since the set

$$\{t:\underline{\lim}_{\tau\to t}\chi_G(\tau)\neq\overline{\lim}_{\tau\to t}\chi_G(\tau)\}$$

is contained in the non-dense boundary of G, A is of the first category.

[1]) This lemma is an immediate corollary from a theorem due to C. Kuratowski (see. C. Kuratowski, *Topologie I*, Warszawa 1948, p. 306).

Theorem 2. *For every bounded function satisfying the condition of Baire in a complete metric space E there exists a function $u(t)=\xi_t(x)$ such that* (6), (7), (8), (11) *and* (12) *hold.*

The proof is analogous to the proof of Theorem 1.

The method presented here may be applied to other functional spaces, but the second part of Theorem 1 is not true in all cases.

2. Multiplicative measures. Let us consider the space X of bounded sequences $x=\{x_1,x_2,\ldots\}$ with the norm

$$\|x\|=\sup_n|x_n|,$$

the ordering $x\leqslant y$ *if* $x_n\leqslant y_n$ *for* $n=1,2,\ldots$ and the unit $e=\{1,1,\ldots\}$. By the above mentioned theorem of Kakutani X is strongly equivalent to the space $C(\boldsymbol{\Omega})$ (where $\boldsymbol{\Omega}$ is given by (4)). Every functional

$$\xi_n(x)=x_n \tag{13}$$

obviously belongs to $\boldsymbol{\Omega}$. Any limit point ξ_0 of the sequence $\{\xi_n\}$ satisfies (1), (2) and (3), whence it follows that ξ_0 is a limit of Mazur. Conversely, each functional which satisfies (1), (2) and (3) is a limit point of $\{\xi_n\}$ because it belongs to $\boldsymbol{\Omega}$ and by (1) it is none of the functionals (13). In other words: the Stone-Čech compactification $\beta(N)$ of the countable isolate set N consists of the functionals (13) and of the limits of Mazur.

Let S denote a subset of N and χ_S its characteristic function. Given $\xi\in\boldsymbol{\Omega}$, we put $m(s)=\xi(\chi_S)$.

It is easily seen that (a) $m(S)\geqslant 0$, (b) $m(S_1\cup S_2)=m(S_1)+m(S_2)$ if $S_1\cap S_2=0$, (c) $m(S_1\cap S_2)=m(S_1)\cdot m(S_2)$, (d) $m(N)=1$, (e) if S is finite and ξ is no of (13), then $m(S)=0$.

Thus $m(S)$ is a finitely-additive and multiplicative set function defined on all subsets of N. The condition (c) can be interpreted as a stochastical independence. Conversely, to every measure of such kind there corresponds a multiplicative functional

$$\xi(x)=\int_N x\,dm.$$

In other words *the functional ξ is multiplicative if and only if it is multiplicative on nought-or-one sequences.*

This procedure may be generalized. Let X be the space of Baire-functions in E (see Theorem 2). For an arbitrary Baire-set $A\subset E$ and $\xi\in\boldsymbol{\Omega}$ we establish that $m_\xi(A)=\xi(\chi_A)$ is a finitely-additive and multiplicative measure vanishing on sets of the first category. The general form of linear functional over X is the integral

$$f(x) = \int_E x(t)\, dm_f$$

and $\|f\| = \operatorname{Var}_E m_f$. More generally, we can consider Hausdorff classes corresponding to arbitrary Boolean algebras.

THEOREM 3. *The following conditions are equivalent for linear functionals over the space $C(\Omega)$ of continuous functions defined on a 0-dimensional compact space Ω:*

1° $\xi \in \boldsymbol{\Omega}$ *(i. e. $\xi(x) = x(t_0)$ for fixed $t_0 \in \Omega$);*

2° *ξ is multiplicative and $\xi \neq 0$;*

3° *the measure m_ξ does not vanish everywhere and for $A, B \subset \Omega$ open-and-closed we have $m_\xi(A \cap B) = m_\xi(A) \cdot m_\xi(B)$;*

4° *$m_\xi(\Omega) = 1$, $m_\xi \geqslant 0$, and $A \cap B = 0$ implies $m_\xi(A) \cdot m_\xi(B) = 0$.*

Proof. The implications $1° \to 2°$, $2° \to 3°$, $3° \to 4°$ are trivial, we shall prove only $4° \to 1°$. From the obvious $\xi \geqslant 0$ we have $\|\xi\| = \xi(e) = m_\xi(E) = 1$. Suppose that $x_1, y_1 \in X$ and $x \wedge y = 0$. For any n there exist simple functions $x_n \in X$ and $y_n \in X$ such that $\|x - x_n\| < 1/n$, $\|y - y_n\| < 1/n$ and $x_n \wedge y_n = 0$, moreover, there exist sets $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ such that $A_k \cap B_j = 0$ for $j, k = 1, 2, \ldots, m$,

$$x_n = \sum_{k=1}^{m} a_k \chi_{A_k} \quad \text{and} \quad y_n = \sum_{k=1}^{m} b_k \chi_{B_k}.$$

Then

$$\xi(x_n) \cdot \xi(y_n) = \sum_{k=1}^{m} a_k m_\xi(A_k) \cdot \sum_{j=1}^{m} b_j m_\xi(B_j) = \sum_{k,j=1}^{m} a_k b_j m_\xi(A_k \cap B_j) = 0.$$

Passing to the limit we obtain $\xi(x) \cdot \xi(y) = 0$, whence $\xi \in \boldsymbol{\Omega}$.

THEOREM 4. *A compact space Ω is 0-dimensional if and only if the condition $\xi \in \boldsymbol{\Omega}$ is equivalent to the following: if $x \in C(\Omega)$ and $x(t)$ take only 0 and 1 as the values, then either $\xi(x) = 0$, or $\xi(x) = 1$.*

Proof. Necessity. Since every positive function $x \in X$ can be approximated by non-negative simple functions, we have $\xi \geqslant 0$ and $\|\xi\| = 1$ (if $\xi \neq 0$). Suppose that $E_1 \subset \Omega$, $E_2 \subset \Omega$ and $E_1 \cap E_2 = 0$. Then every number $m_\xi(E_1)$, $m_\xi(E_2)$ and $m_\xi[\Omega - (E_1 \cup E_2)]$ is equal to 0 or 1, whence from $m_\xi(E_1) + m_\xi(E_2) + m_\xi[\Omega - (E_1 \cup E_2)]$ there follows $m_\xi(E_1) \cdot m_\xi(E_2) = 0$. By theorem 3 (proposition 4°) $\xi \in \boldsymbol{\Omega}$.

Sufficiency. Suppose that there exists a connected set $E \subset \Omega$ containing two different points t_1 and t_2. Then the functional

$$\eta(x) = \frac{x(t_1) + x(t_2)}{2}$$

does not belong to $\boldsymbol{\Omega}$, and $\eta(x) = 0$ or $\eta(x) = 1$ holds for any nought-or-one continuous function $x \in X$.

3. Extension of linear functionals. Similarly to Theorem 1 we can prove the existence of a generalized left-hand limit, ξ_t, and a right-hand one, η_t, in the space X of Riemann-integrable functions, at any point $t \in \langle 0, 1 \rangle$. Let X_0 denote the subspace of X of continuous functions on $\langle 0, 1 \rangle$. The functionals ξ_t and η_t are equal on x_0. There follow two propositions:

1° *A norm-preserving extension of linear functional from an M-subspace X_0 on M-space X is not necessarily unique (even if X_0 and X have the same unit).*

However, by a theorem of M. Krein and S. Krein ([4], p. 7) it follows that if Ω_0 is an open-and-closed subset of Ω, then every linear functional has a unique norm-preserving extension from $X_0 = C(\Omega_0)$ to $X = C(\Omega)$; this is also easily deducible from the integral representation of functionals.

2° *If $X_0 = C(\Omega_0)$ is an M-subspace $X = C(\Omega)$, then $\Omega_0 \underset{\text{top}}{\subset} \Omega$ is not necessarily satisfied even, if every functional $\xi \in \boldsymbol{\Omega}_0$ has an extension $\bar{\xi} \in \boldsymbol{\Omega}$.*

References

[1] A. Alexiewicz, *On Hausdorff classes*, Fund. Math. 34 (1947), p. 61-65.

[2] F. Hausdorff, *Mengenlehre*, Göttingen 1927.

[3] S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. of Math. 42 (1941), p. 994-1024.

[4] M. Krein et S. Krein, *Sur l'espace des fonctions continues définies sur un bicompact de Hausdorff et ses sous-espace semiordonnés*, Mat. Sbornik 13 (1943), p. 1-38.

[5] S. Mazur, *On the generalized limit of bounded sequences*, Coll. Math. 2 (1951), p. 173-175.

Reçu par la Rédaction le 21. 9. 1956