On Mikusiński's algebraical theory of differential equations

by

R. SIKORSKI (Warszawa)

Let F be a linear space over an algebraical field C with the characteristic zero. The letters x, y, z (with indices, if necessary) will always denote elements of F, the letters a, b — elements of C, and the letters P, Q — polynomials of a variable ξ with coefficients belonging to C.

Let D be an abstract derivation in F, *i. e.* an endomorphism[1]) of F such that

(I) if

(1) $$P(\xi) = a_0\xi^p + a_1\xi^{p-1} + \ldots + a_p$$

is a polynomial of degree p, then the equation[2]) $P(D)x = 0$, *i. e.* the equation

(2) $$a_0D^px + a_1D^{p-1}x + \ldots + a_{p-1}Dx + a_px = 0$$

has at most p linearly independent solutions $x_1, \ldots, x_p$;

(II) if the equations $P_1(D)x = 0$ and $P_2(D)x = 0$ (where P_1 and P_2 are polynomials) have exactly p_1 and p_2 linearly independent solutions respectively, then the equation $P_1(D)P_2(D)x = 0$ has exactly $p_1 + p_2$ linearly independent solutions.

Suppose also that

(III) each element $x \epsilon F$ is a solution of an equation $P(D)x = 0$ for a non-zero polynomial P.

The equation (2) is an abstract homogeneous differential equation with constant coefficients. By (III), F is a space of solutions of equations (2). In the theory of ordinary differential equations (2) with real (or complex) coefficients, the space F' of all real (or complex) solutions of all

[1]) U is said to be an *endomorphism* of a linear space F (over a field C) if U is a mapping of F into itself and $U(ax+by) = aUx + bUy$.

[2]) If P is a polynomial of the form (1), then $P(D)$ obviously denotes the endomorphism $a_0D^p + a_1D^{p-1} + \ldots + a_pD^0$, where D^0 is the identity endomorphism.

equations (2) is such that if a function $x(t)$ is in F', then the function $tx(t)$ also belongs to F', *i. e.*, the transformation T defined by the equation

(3) $$Tx(t) = tx(t)$$

is a linear endomorphism of F'. Moreover, $DTx = TDx + x$ for $x \epsilon F'$ (D denotes here the usual derivation).

Mikusiński[3]) has proved that, in the case of an arbitrary linear space F with an abstract derivation D, there exists an endomorphism T which is an abstract analogue of the endomorphism (3). More exactly, he has proved the following theorem:

(M) *Under the hypotheses* (I), (II), (III), *there is an endomorphism T of F such that $DTx = TDx + x$ for every $x \epsilon F$.*

The purpose of this paper is to give another proof of Theorem (M). Since the knowledge of Mikusiński's paper is not assumed here, we start with the proof of some simple lemmas.

(i) *If x_0 is a solution of the equation $P(D)x = 0$, and P_1 is any polynomial, then the element $P_1(D)x_0$ is also a solution of this equation.*

In fact, $P(D)P_1(D)x_0 = P_1(D)P(D)x_0 = 0$.

The letter Q will always denote a fixed polynomial, irreducible in C, $Q(\xi) = \xi^q + b_1\xi^{q-1} + \ldots + b_q$, such that the equation $Q(D)x = 0$ has a solution $x \neq 0$.

Let E_n ($n = 0, 1, 2, \ldots$) denote the linear space of all $x \epsilon F$ such that $Q(D)^n x = 0$, and let $F_Q = E_0 + E_1 + E_2 + \ldots$ be the linear space of all solutions x of all the equations $Q(D)^n x = 0$, $n = 0, 1, 2, \ldots$ Of course, we assume that $Q(D)^0$ is the identity endomorphism of F, therefore E_0 contains only the zero element 0 of F. Observe that $E_0 \subset E_1 \subset E_2 \subset \ldots$

It follows from (II) that the set $E_{n+1} - E_n$ is not empty ($n = 0, 1, 2, \ldots$).

(ii) *If $Q^{n+1}(D)y = 0$ but $Q(D)^n y \neq 0$ (i. e., if $y \epsilon E_{n+1} - E_n$), then $P(D)y = 0$ if an only if the polynomial P is divisible by Q^{n+1}.*

Consequently $P(D)y \neq 0$ for every non-zero polynomial P of degree $< (n+1)q$.

If $P = P_1Q^{n+1}$, then $P(D)y = P_1(D)Q(D)^{n+1}y = 0$.

If P is not divisible by Q^{n+1}, then Q^n is a multiple of the largest common divisor of P and Q^{n+1}. Therefore there exist polynomials P_1 and P_2 such that $P_1P + P_2Q^{n+1} = Q^n$. This implies that $P_1(D)P(D)y = Q^n(D)y \neq 0$. Hence $P(D)y \neq 0$.

[3]) J. Mikusiński, *Sur l'espace linéaire avec dérivation*, ce fascicule, p. 113-123.

(iii) *Let $y \in E_{n+1} - E_n$. An element $x \in F$ belongs to E_{n+1} if and only if it is of the form*

$$x = P(D)y, \tag{4}$$

where P is a polynomial of degree $< (n+1)q$. The representation of x in the form (4) is unique.

The element x of the form (4) belongs to E_n if and only if P is divisible by Q.

By (i), every element x of the form (4) belongs to E_{n+1}.

In particular, the elements

$$y, Dy, D^2y, \ldots, D^{(n+1)q-1}y \tag{5}$$

belong to E_{n+1}. By the second part of (ii), they are linearly independent. Let $x \in E_{n+1}$. By (I), the element x and the elements (5) are linearly dependent, *i. e.*

$$x = a_0 y + a_1 Dy + \ldots + a_{(n+1)q-1} D^{(n+1)q-1} y = P(D)y,$$

where $P(\xi) = a_0 + a_1\xi + \ldots + a_{(n+1)q-1}\xi^{(n+1)q-1}$. Since the elements (5) are linearly independent, the coefficients a_j are uniquely determined by x, *i. e.*, the representation of x in the form (4) is unique.

If $x = P(D)y \in E_n$, then $Q(D)^n P(D)y = 0$. By (ii), the polynomial $Q^n P$ is divisible by Q^{n+1}, which implies that P is divisible by Q. On the other hand, if $x = P(D)y$ and P is divisible by Q, *i. e.*, $P = P_1 Q$, then $Q(D)^n x = Q(D)^n P(D)y = P_1(D)Q(D)^{n+1}y = 0$, *i. e.*, $x \in E_n$.

(iv) *If $y \in E_{n+1} - E_n$, then every element $x \in E_{n+1}$ can be uniquely represented in the form*

$$x = P(D)y + x_1, \tag{6}$$

where P is a polynomial of degree $< q$ and $x_1 \in E_n$, i. e. in the form

$$x = a_0 y + a_1 Dy + \ldots + a_{q-1} D^{q-1} y + x_1 \quad (x_1 \in E_n). \tag{7}$$

The element x belongs to E_n if and only if $P \equiv 0$.

By (iii), we have $x = P_1(D)y$, where P_1 is a polynomial of degree $< (n+1)q$. There exist polynomials P and P_2 such that $P_1 = QP_2 + P$ and the degree of P is $< q$. We have $x = P(D)y + x_1$, where $x^1 = Q(D)P_2(D)y \in E_n$ on account of the second part of (iii).

On the other hand, if equation (6) holds, then x_1 is of the form $x_1 = Q(D)P_2(D)y$ by the second part of (iii). Consequently $x = P_1(D)y$, where $P_1 = QP_2 + P$. Since P_1 is uniquely determined by x on account of (iii), so are P and P_2. This proves the uniqueness of the decomposition (6).

(v) *If $x \in E_{n+1}$, then there exists an element $z \in E_{n+2}$ such that*

$$Q(D)z = x. \tag{8}$$

If $x \in E_{n+1} - E_n$, then $z \in E_{n+2} - E_{n+1}$.

Let $y \in E_{n+2} - E_{n+1}$. By (iii) (where n should be replaced by $n+1$), there exists a polynomial $P_1(D)$ of degree $< (n+1)q$, such that $x = Q(D)P_1(D)y$. The element $z = P_1(y)$ fulfils equation (8).

If $z \in E_{n+1}$, then $Q(D)^n x = Q(D)^{n+1} z = 0$ by (ii), *i. e.* $x \in E_n$.

(vi) *There exist an endomorphism T of F_Q such that $DTx = TDx + x$ for $x \in F_Q$.*

Let $x_0 \neq 0$ be an element such that $Q(D)x_0 = 0$, *i. e.*,

$$x_0 \in E_1 - E_0. \tag{9}$$

We define, by induction, the linear transformation T on the subspaces E_n ($n = 0, 1, 2, \ldots$) of the space F_Q in such a way that

$$DTx = TDx + x \quad \text{for} \quad x \in E_n, \tag{10}$$

$$T(E_j - E_{j-1}) \subset E_{j+1} - E_j \quad \text{for} \quad j = 1, 2, \ldots, n. \tag{11}$$

On the subspace E_0 the transformation T is obviously defined by the formula $T(0) = 0$.

Suppose that T is defined on E_n so that (10) and (11) hold. We extend T to E_{n+1} as follows.

Let $y = T^n x_0$. Condition (11) implies that $T^n(E_1 - E_0) \subset E_{n+1} - E_n$. In particular

$$y = T^n x_0 \in E_{n+1} - E_n. \tag{12}$$

Consequently (see (iii))

$$D^i y \in E_{n+1} - E_n \quad \text{for} \quad i = 0, 1, \ldots, q-1. \tag{13}$$

It follows also from (iii) that $Q(D)y \in E_n$, which implies, by (11), $TQ(D)y \in E_{n+1}$.

Let Q' be the ordinary derivative of the polynomial Q, *i. e.*

$$Q'(\xi) = q\xi^{q-1} + (q-1)b_1\xi^{q-1} + \ldots + b_{q-1}.$$

Since $Q'(D)y \in E_{n+1}$ by (13), we infer that $TQ(D)y + Q'(D)y \in E_{n+1}$. By (v) there exists an element $z \in E_{n+2}$ such that

$$Q(D)z = TQ(D)y + Q'(D)y. \tag{14}$$

We define the values of the transformation T at elements (13) by the formulae

$$(15)\quad \begin{aligned} Ty &= z,\\ TDy &= Dz-y,\\ TD^2y &= D^2z-2Dy,\\ &\dots\dots\dots\dots\\ TD^iy &= D^iz-iD^{i-1}y\\ &\dots\dots\dots\dots\dots\\ TD^{q-1}y &= D^{q-1}z-(q-1)D^{q-2}y.\end{aligned}$$

For an arbitrary element $x \epsilon E_{n+1}$ of the form (7) we define Tx by the formula $Tx = a_0Ty+a_1TDy+\ldots+a_{q-1}TD^{q-1}y+Tx_1$.

Of course, T is a linear transformation on E_{n+1}. It immediately follows from (15) that, for every polynomial P of degree $<q$,

$$(16)\qquad P(D)z = TP(D)y+P'(D)y.$$

In order to show that

$$(17)\qquad DTx = TDx+x \quad \text{for} \quad x \epsilon E_{n+1}$$

it suffices to prove equality (17) for elements (13) only. Multiplying the i-th equation (15) by D and substracting it from the $(i+1)$-th equation (15), we infer that (17) is true for elements $y, Dy, \ldots, D^{q-2}y$, *i. e.*

$$DTD^iy = TD^{i+1}y+D^iy \quad (i = 0, 1, \ldots, q-2).$$

Replacing $P(D)$ by $D^q-Q(D)$ in equality (16), we obtain

$$\big(D^q-Q(D)\big)z = T\big(D^q-Q(D)\big)y+\big(qD^{q-1}-Q'(D)\big)y.$$

Consequently, by (14),

$$(18)\qquad D^qz = TD^qy+qD^{q-1}y.$$

Multiplying the last of the equalities (15) by D and substracting from (18), we obtain $DTD^{q-1}y = TD^qy+D^{q-1}$, which completes the proof of (17).

Now we shall prove that the transformation T extended to E_{n+1} satisfies also the second inductive hypothesis, *i. e.* that

$$(19)\qquad T(E_{n+1}-E_n) \subset E_{n+2}-E_{n+1}.$$

On account of (11) we have

$$(20)\qquad T^i(E_1-E_0) \subset E_{i+1}-E_i \quad \text{for} \quad i = 1, \ldots, n.$$

It follows from (17) that, for every polynomial P,

$$P(D)T^nx_0 = \binom{n}{0}T^nP(D)x_0+\binom{n}{1}T^{n-1}P'(D)x_0+\binom{n}{0}P''(D)x_0+\ldots,$$

where $P', P'', \ldots$ are the ordinary derivatives of the polynomial P. Hence, by (9) and (20),

$$\begin{aligned} Q(D)z &= TQ(D)y+Q'(D)y = TQ(D)T^nx_0+Q'(D)T^nx_0\\ &= \binom{n+1}{1}T^nQ'(D)x_0+\binom{n+1}{2}T^{n-1}Q''(D)x_0+\\ &\quad +\binom{n+1}{3}T^{n-2}Q'''(D)x_0+\ldots \epsilon E_{n+1}-E_n\end{aligned}$$

since $Q'(D)x_0 \neq 0$ (see (ii)), *i. e.* $Q'(D)x_0 \epsilon E_1-E_0$. Hence we infer on account of the second part of (v) that

$$(21)\qquad z = Ty = T^{n+1}x_0 \epsilon E_{n+2}-E_{n+1}.$$

If $x \epsilon E_{n+1}-E_n$, then, by (iv), $x = P(D)y+x_1$, where P is a non-zero polynomial of degree $<q$, and $x_1 \epsilon E_n$. Hence, by (16),

$$Tx = TP(D)y+Tx_1 = P(D)z-P'(D)y+Tx_1.$$

Since $P(D)z \epsilon E_{n+2}-E_{n+1}$ by (21) and (iii), $P'(D)y \epsilon E_{n+1}$ by (12) and (i), and $Tx_1 \epsilon E_{n+1}$ by (11), we infer that $Tx \epsilon E_{n+2}-E_{n+1}$, which completes the proof of (19).

(vii) *The space F is the direct sum of all subspaces F_Q where Q is any irreducible polynomial such that the equation $Q(D)x = 0$ has a solution $x \neq 0$.*

It suffices to prove that every element $x \epsilon F$ can be represented in the form

$$(22)\qquad x = x_1+\ldots+x_r,$$

where $x_i \epsilon F_{Q_i}$ $(i = 1, \ldots, r)$ and Q_i are different irreducible polynomials, and that the decomposition (22) is unique (up to summands equal to zero).

By (III) we have

$$(23)\qquad P(D)x = 0$$

for a non-zero polynomial P. We have then $P = Q_1^{n_1}Q_2^{n_2}\cdot\ldots\cdot Q_s^{n_s}$, where Q_j $(1 \leqslant j \leqslant s)$ are different irreducible polynomials of degree q_j respectively, and $a \neq 0$. Suppose that if $1 \leqslant j \leqslant r$, the equation $Q_j(D)x = 0$ has a non-zero solution, and if $r < j \leqslant s$, it has only the zero solution.

If $1 \leqslant j \leqslant r$, let y_j be an element such that $Q_j^{n_j}(D)y_j = 0$ but $Q_j^{n_j-1}(D)y_j \neq 0$.

The elements

$$y_j,\ Dy_j,\ D^2y_j, \ldots, D^{n_jq_j-1}y_j \quad (j = 1, \ldots, r) \tag{24}$$

are solutions of (23). Suppose that a linear combination of (24) is equal to zero, *i. e.*

$$P_1(D)y_1 + \ldots + P_r(D)y_r = 0, \tag{25}$$

where P_j is a polynomial of degree $< n_jq_j$. Let

$$H_j = Q_1^{n_1} \cdot \ldots \cdot Q_{j-1}^{n_j-1} Q_{j+1}^{n_j+1} \cdot \ldots \cdot Q_r^{n_r}. \tag{26}$$

Multiplying (25) by $H_j(D)$ we infer that $H_j(D)P_j(D)y_j = 0$. Hence, by (ii), $P_j \equiv 0$ $(j = 1, \ldots, r)$. This proves that if a liner combination of (24) is equal to zero, then all the coefficients are also equal to zero. Therefore elements (24) are linearly independent.

It follows from (II) and (iii) that equation (22) has exactly $n = n_1q_1 + \ldots + n_rq_r$ linearly independent solutions. Since the number of elements (24) is equal to n, they constitute a basis for the space of all solutions of (23). Therefore, if x satisfies (23), we have

$$x = P_1(D)y_1 + \ldots + P_r(D)y_r,$$

for some polynomials $P_1, \ldots, P_r$, *i. e.* the decomposition (22), where $x_j = P_j(D)y_j$.

To prove the uniqueness of (22) it suffices to show that if the decomposition (22) holds and $x = 0$, then $x_j = 0$ for $j = 1, \ldots, r$.

In fact, let n_j be the least non-negative integer such that $Q_j^{n_j}(D)x_j = 0$. Multiplying (22) by $H_j(D)$, where H_j is defined by (26), we infer that $H_j(D)x_j = 0$. By (ii), H_j is divisible by Q^{n_j}, which implies $n_j = 0$. Therefore $x_j = 0$.

Theorem (M) is an immediate consequence of (vi) and (vii). In fact, first we define the transformation T on each of the subspaces F_Q separately. If $x \in F$ is of the form (22), we assume $Tx = Tx_1 + \ldots + Tx_r$.

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