Les ANNALES POLONICI MATHEMATICI constituent une continuation des ANNALES DE LA SOCIÉTÉ POLONAISE DE MATHÉMATIQUE vol. I-XXV) fondées en 1921 par Stanisław Zaremba.

Les ANNALES POLONICI MATHEMATICI publient, en langues des congrès internationaux, des travaux consacrés à l'Analyse Mathématique, la Géométrie et la Théorie des Nombres. Chaque volume paraît en quelques (2-3) fascicules.

PRINTED IN POLAND

WROCŁAWSKA DRUKARNIA NAUKOWA

ANNALES
POLONICI MATHEMATICI
V. (1958)

Polynomial Hausdorff transformations

I. Mercerian theorems

by Z. POLNIAKOWSKI (Poznań)

1. In 1906 J. Mercer proved the following theorem:

If $m_n = \dfrac{s_0+s_1+\ldots+s_n}{n+1}$, $a>0$ *and* $p_n = as_n+(1-a)m_n$, *then the hypothesis* $\lim\limits_n p_n = s$ *implies* $\lim\limits_n s_n = s$.

G. H. Hardy showed that the assumption $a>0$ may be replaced by $\operatorname{re} a>0$. The theorem remains true under the hypothesis $a>\frac{1}{2}$, if we replace m_n by the transform

$$t_n = \sum_{\nu=0}^{\infty} c_{n\nu} s_\nu,$$

where $(c_{n\nu})$ is a matrix of a regular transformation.

If $a<\frac{1}{2}$, then the conclusion of Mercer's theorem ceases to be true in general. In the case of Euler's transform

$$e_n = 2^{-n}\sum_{\nu=0}^{n}\binom{n}{\nu}s_\nu$$

and $a<\frac{1}{2}$ we may give an example of a divergent sequence $\{s_n^*\}$ such that the corresponding sequence with terms $p_n^* = as_n^*+(1-a)e_n^*$ converges.

In 1938 H. R. Pitt [6] proved the Mercerian theorem, by use of the Mellin transformations, in the case of Hausdorff transforms giving a necessary and sufficient condition of the equivalence of the regular Hausdorff method to the ordinary convergence.

In some cases theorems of this kind may be proved by a more elementary method. We prove (theorem 1A) by the use of Cauchy's theorem for sequences that *the Hausdorff transformation with* $\mu_n = W_1(n)/W(n)$, *where* $W(z)$ *is a polynomial of degree* k *and* $W_1(z)$ *a polynomial of degree* $l\leqslant k$, *is regular if and only if the real parts of all the roots of the equation* $W(z)=0$ *are negative.*

From this we obtain Mercerian theorems for Cesàro and Hölder means of integral order and, more generally, for Hausdorff means with $\mu_n = 1/W(n)$, where $W(x)$ is a polynomial. In the case of the Cesàro mean $c_n^{(k)}$, $k = 1, 2, \ldots$ (*i. e.* for $W(x) = \binom{x+k}{k}$), there exists a constant a_k' such that the assumption $\lim\limits_n p_n = s$, where $p_n = as_n + (1-a)c_n^{(k)}$, implies $\lim\limits_n s_n = s$ if and only if $a > a_k'$. Similarly in the case of the Hölder mean $h_n^{(k)}$ (*i. e.* for $W(x) = (x+1)^k$) there exists a constant a_k'' with the analogous property. The sequences $\{a_k'\}$ and $\{a_k''\}$ are non-negative, increasing for $k \geqslant 2$ and tend to the limit $\frac{1}{2}$. We give the asymptotic representations of these sequences.

Using theorem 1A we prove also the following Tauberian theorem: *If* $\lim\limits_n c_n^{(k)} = s$ *and* $\lim\limits_n n(s_n - s_{n+1}) = 0$ *then* $\lim\limits_n s_n = s$.

In all these theorems we suppose that $|s| < \infty$; the considered sequences are complex.

In section 3 we prove analogous theorems for the difference transform $t(x)$ of the function $f(x)$ defined as

$$t(x) = \sum_{n=0}^{\infty} \mu_n \binom{x}{n} \Delta^n f(0)$$

and the differential transform defined by the formula

$$t(x) = \sum_{n=0}^{\infty} \mu_n \frac{x^n}{n!} f^{(n)}(0).$$

The author is very much obliged to Professor W. Orlicz for his suggestions and remarks in the course of preparing of this paper.

2. We now introduce the following notation: We denote by (a_n) and $[a_n]$ respectively the infinite matrices

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \\ \vdots \end{pmatrix} \quad \text{and} \quad \begin{bmatrix} a_0 & 0 & \ldots & 0 & \ldots \\ 0 & a_1 & & & \\ \vdots & & \ddots & & \\ 0 & & & a_n & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Furthermore let $(c_{n\nu})$ denote any infinite triangular matrix, *i. e.* a matrix satisfying the condition $c_{n\nu} = 0$ for $\nu > n$. We set $\delta = (a_{n\nu})$, where $a_{n\nu} = (-1)^\nu \binom{n}{\nu}$ for $n, \nu = 0, 1, 2, \ldots$ Let us remark that $\delta^2 = I$ (here I denotes the unit matrix of infinite order; the product of two matrices is formed by the multiplication of the rows by the columns). This follows from the fact that for every x we have $(y^n) = \delta(x^n)$ if $y = 1-x$ and $(x^n) = \delta(y^n) = \delta^2(x^n)$, since $x = 1-y$.

The k-th difference $\Delta^k a_n$ of the sequence $\{a_n\}$ is defined as usual by

$$\Delta^k a_n = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} a_{n+\nu} \quad \text{for} \quad k = 0, 1, 2, \ldots, \qquad \Delta a_n = \Delta^1 a_n;$$

then $(\Delta^n a_0) = \delta(a_n)$, $(a_n) = \delta(\Delta^n a_0)$ and

$$a_n = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \Delta^\nu a_0. \tag{1}$$

If $t_n = \sum\limits_{\nu=0}^{n} c_{n\nu} s_\nu$, we shall say that the sequence $\{s_n\}$ is transformed into the sequence $\{t_n\}$ by means of the transformation with the matrix $C = (c_{n\nu})$. In the matrix form this may be written as $(t_n) = (c_{n\nu})(s_n)$.

Finally, we denote by $\mathfrak{R}$ the class of polynomials $W(z)$ with complex coefficients, such that the real parts of all the roots of the equation $W(z) = 0$ are negative.

2.1. In the sequel we shall use the following theorem essentially due to Cauchy:

THEOREM A. *Let* $\{s_n\}, \{a_n\}, \{b_n\}$ *be given sequences. The hypothesis* $\lim\limits_n s_n = s$ *implies* $\lim\limits_n (a_n/b_n) = s$ *if*

1a) $|b_n| \to \infty$ *and* $\sum\limits_{\nu=0}^{n-1} |\Delta b_\nu| \leqslant K|b_n|$

or

1b) $a_n \to 0$, $b_n \to 0$, $b_n \neq 0$ *for infinitely many indices* n *and* $\sum\limits_{\nu=n}^{\infty} |\Delta b_\nu| \leqslant K|b_n|$ *(where the constant* K *does not depend on* n*)*,

2) $\Delta a_n = s_n \Delta b_n$.

Let us notice that 1a) or 1b) imply the inequalities $b_n \neq 0$ for $n > N$ and $\Delta b_n \neq 0$ for infinitely many n. The inequalities in 1a) and 1b) are satisfied in the case of monotone sequences.

2.1.1. *Suppose that*

1) *the sequence* $\{b_n\}$ *satisfies the hypothesis* 1a) *or* 1b) *of theorem* A,

2) $b_n \neq 0$ *for* $n = 0, 1, 2, \ldots$,

3) $\lim\limits_n s_n = s$

Then there exists a sequence $\{\bar{x}_n\}$ *which is a solution of the difference equation*

$$\Delta x_{n-1} + \frac{\Delta b_{n-1}}{b_{n-1}}(x_n - s_n) = 0 \quad \text{for} \quad n = 1, 2, \ldots, \tag{2}$$

and tends to the limit s.

In the case 1a) *every solution* $\{x_n\}$ *of* (2) *tends to* s, *in the case* 1b) $x_n = \bar{x}_n + c/b_n$, *where* c *is a constant.*

Proof. We seek the solution of (2) in the form $x_n = c_n/b_n$. Substitution into (2) gives the difference equation for c_n

$$\Delta c_{n-1} = s_n \Delta b_{n-1}, \qquad n \geqslant 1.$$

Suppose first that $\{b_n\}$ satisfies 1a); then

$$c_n - c_0 = -\sum_{\nu=0}^{n-1} \Delta c_\nu = -\sum_{\nu=0}^{n-1} s_{\nu+1} \Delta b_\nu .$$

Hence the general solution of (2) in this case is

$$x_n = \frac{c}{b_n} - \frac{1}{b_n} \sum_{\nu=0}^{n-1} s_{\nu+1} \Delta b_\nu,$$

where c is a constant.

If $\{b_n\}$ satisfies 1b), then the series $\sum \Delta b_\nu$ and $\sum s_{\nu+1} \Delta b_\nu$ are convergent and

$$c_n = \sum_{\nu=n}^{\infty} \Delta c_\nu = \sum_{\nu=n}^{\infty} s_{\nu+1} \Delta b_\nu, \qquad x_n = \frac{\bar{c}}{b_n} + \frac{1}{b_n} \sum_{\nu=n}^{\infty} s_{\nu+1} \Delta b_\nu,$$

because the sequence $\{1/b_n\}$ is a particular solution of (2) for $s_n \doteq 0$.

In both cases it now suffices to apply theorem A; in the case 1b) we obtain the expression for $\bar{x}_n$ taking $\bar{c} = 0$ in the second formula.

2.1.2. *Let*

$$u_n = \prod_{\nu=1}^{n} \left(1 + \frac{z}{\nu}\right), \qquad z = a + bi \neq -1, -2, \ldots;$$

then

$$|u_n| \uparrow \infty \quad \textit{and} \quad \frac{\sum_{\nu=0}^{n-1} |\Delta u_\nu|}{|u_n|} \to \frac{|z|}{a} \quad \textit{for} \quad a > 0,$$

$$|u_n| \to 0 \quad \textit{monotonically for} \quad n > N \quad \textit{and} \quad \frac{\sum_{\nu=n}^{\infty} |\Delta u_\nu|}{|u_n|} \to \frac{|z|}{-a} \quad \textit{for} \quad a < 0.$$

Proof. If $z \neq -1, -2, \ldots$, then from the inequality

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|1 + \frac{z}{n+1}\right| \begin{cases} > 1 & \text{for} \quad a > 0, \\ < 1 & \text{for} \quad a < 0, \; n > N \end{cases}$$

it follows that the sequence $|u_n|$ is monotone for $a > 0$, $n = 0, 1, 2, \ldots$, and for $a < 0$, $n > N$. We remark that

$$|u_n| \sim \frac{n^a}{|\Gamma(z+1)|} \to \begin{cases} \infty & \text{for} \quad a > 0, \\ 0 & \text{for} \quad a < 0. \end{cases}$$

Hence using Cauchy's theorem we have in the case $a > 0$

$$\lim_n \frac{\sum_{\nu=0}^{n-1} |\Delta u_\nu|}{|u_n|} = \lim_n \frac{|\Delta u_{n-1}|}{|u_n| - |u_{n-1}|} = \lim_n \frac{|z|}{n(|1+z/n|-1)} = \frac{|z|}{a}.$$

In the case $a < 0$ we have

$$|\Delta u_{n-1}| = \frac{|z|}{n} |u_{n-1}| \sim \frac{|z|}{|\Gamma(z+1)|} n^{a-1}$$

and the series $\sum |\Delta u_n|$ converges; the proof now follows as in the case $a < 0$.

2.2. Transformations with the matrix $H = (c_{n\nu})$ of the form $H = \delta\mu\delta$, $\mu = [\mu_n]$ are said to be of *Hausdorff type* (Hausdorff [2], Hardy [1], p. 247).

The following properties of the Hausdorff transformations are well known:

If $(t_n) = \delta\mu\delta(s_n)$ then $\Delta^n t_0 = \mu_n \Delta^n s_0$ for $n = 0, 1, 2, \ldots$

If $H_1 = \delta\mu^{(1)}\delta$, $H_2 = \delta\mu^{(2)}\delta$ and λ_1, λ_2 are complex numbers, then

$$\lambda_1 H_1 + \lambda_2 H_2 = \delta(\lambda_1 \mu^{(1)} + \lambda_2 \mu^{(2)})\delta, \qquad H_1 H_2 = \delta\mu^{(1)}\mu^{(2)}\delta.$$

From the latter equation it follows that if $H = H_1 H_2 = \delta\mu\delta$, $\mu_n \neq 0$, $\mu^{(1)} = [\mu_n^{(1)}]$, $\mu^{(2)} = [\mu_n^{(2)}]$, then $\mu_n = \mu_n^{(1)} \mu_n^{(2)}$ and $H^{-1} = \delta[1/\mu_n]\delta$.

It may also be shown that the class of matrices H is identical with the class of matrices commutable with the matrix of the method of the first arithmetic mean (Hardy [1], p. 249).

A sequence $\{s_n\}$ is said to be (H, μ) *summable* if the sequence $\{t_n\}$ defined by the formula $(t_n) = \delta\mu\delta(s_n)$ is convergent.

If $H = (c_{n\nu}) = \delta\mu\delta$, then $c_{n\nu} = \binom{n}{\nu} \Delta^{n-\nu} \mu_\nu$. Using (1) with $a_\nu = \mu_{n-\nu}$ we have $\sum_{\nu=0}^{n} c_{n\nu} = \mu_0$.

Thus the conditions of regularity of the transformation with a matrix H are

$$1^\circ \quad \sum_{\nu=0}^{n} |c_{n\nu}| = \sum_{\nu=0}^{n} \binom{n}{\nu} |\Delta^{n-\nu} \mu_\nu| < K, \qquad \text{with some } K < \infty,$$

$$(3) \qquad 2^\circ \quad \sum_{\nu=0}^{n} c_{n\nu} = \mu_0 = 1,$$

$$3^\circ \quad \lim_n c_{n\nu} = \lim_n \binom{n}{\nu} \Delta^{n-\nu} \mu_\nu = 0 \qquad \text{for} \quad \nu = 0, 1, 2, \ldots$$

It may be shown that the condition 3° for $\nu > 0$ follows form 1° (see Hardy [1], p. 255, for real case).

If the methods $(H, \mu^{(\nu)})$, $\nu = 1, 2, \ldots, k$, are regular, $\sum_{\nu=1}^{k} \lambda_\nu = 1$, $\mu_n = \sum_{\nu=1}^{n} \lambda_\nu \mu_n^{(\nu)}$, then the method (H, μ) is also regular.

In the sequel let $(t_n) = \delta[\mu_n]\delta(s_n)$.

2.2.1. If $\mu_n = (-1)^k \binom{n}{k}$ then

$$t_n = \begin{cases} 0 & \text{for} \quad n < k, \\ \binom{n}{k} \Delta^k s_{n-k} & \text{for} \quad n \geqslant k. \end{cases} \tag{4}$$

We prove that the terms of the sequence $\{t_n\}$ defined above satisfy the relation $\Delta^n t_0 = (-1)^k \binom{n}{k} \Delta^n s_0$. Namely $\Delta^n t_0 = 0$ for $n < k$ and

$$\Delta^n t_0 = \sum_{\nu=k}^{n} (-1)^\nu \binom{n}{\nu}\binom{\nu}{k} \Delta^k s_{\nu-k} = \binom{n}{k} \sum_{\nu=k}^{n} (-1)^\nu \binom{n-k}{\nu-k} \Delta^k s_{\nu-k}$$

$$= (-1)^k \binom{n}{k} \sum_{\nu=0}^{n-k} (-1)^\nu \binom{n-k}{\nu} \Delta^k s_\nu$$

$$= (-1)^k \binom{n}{k} \Delta^{n-k}(\Delta^k s_0) = (-1)^k \binom{n}{k} \Delta^n (s_0)$$

for $n \geqslant k$.

2.2.2. *If $W(z)$ is a polynomial of degree k, $\mu_n = W(n)$, then*

$$t_n = L(s_n), \quad n = 0, 1, 2, \ldots, \tag{5}$$

where $L(s_n) = \sum_{\nu=0}^{k} \lambda_\nu \binom{n}{\nu} \Delta^\nu s_{n-\nu}$ *(for* $n < \nu$ *we set* $\binom{n}{\nu} \Delta^\nu s_{n-\nu} = 0$*) and* $\lambda_\nu = (-1)^\nu \Delta^\nu W(0) = \sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} W(j)$.

Since $W(n) = \sum_{\nu=0}^{k} \binom{n}{\nu} \Delta^\nu W(0)$, the proof follows immediately from 2.2.1.

2.2.3. If μ_n is defined as in 2.2.2 and $s_n = \dfrac{n!}{\Gamma(n-r+1)}$, then

$$t_n = W(r) \frac{n!}{\Gamma(n-r+1)} \quad \text{for } n \geqslant k.$$

Using 2.2.2 we obtain

$$t_n = L(s_n) = \sum_{\nu=0}^{k} (-1)^\nu \binom{n}{\nu} \Delta^\nu W(0) \Delta^\nu \frac{(n-\nu)!}{\Gamma(n-r-\nu+1)}$$

$$= \frac{n!}{\Gamma(n-r+1)} \sum_{\nu=0}^{k} \binom{r}{\nu} \Delta^\nu W(0) = \frac{n!}{\Gamma(n-r+1)} W(r),$$

since

$$(-1)^\nu \binom{r}{\nu} \Delta^\nu \frac{(n-\nu)!}{\Gamma(n-r-\nu+1)}$$

$$= \binom{n}{\nu} \frac{(n-\nu)!}{\Gamma(n-r+1)} r(r-1)\ldots(r-\nu+1) = \binom{r}{\nu} \frac{n!}{\Gamma(n-r+1)}.$$

We set here $n!/\Gamma(n-r+1) = 0$ if r is an integer and $r > n$.

Compare also Rogosinski [8] I, L. 1.

2.2.4. *Let p be a positive integer. If* $\operatorname{re} a > 0$ *then the Hausdorff transformation with* $\mu_n = 1/(an+1)^p$ *is regular.*

We may write the relation $(t_n) = \delta[\mu_n]\delta(s_n)$ in the form of a system of equations

$$\delta[an+1]\delta(x_n^{(\nu)}) = (x_n^{(\nu+1)}), \quad \nu = 1, 2, \ldots, p,$$

where $x_n^{(1)} = t_n$, $x_n^{(p+1)} = s_n$, which by 2.2.1 is equivalent to the system of difference equations

$$-an\Delta x_{n-1}^{(\nu)} + x_n^{(\nu)} = x_n^{(\nu+1)}. \tag{6}$$

Assuming $\lim_n s_n = s$ and using 2.1.1 succesively for $\nu = p, p-1, \ldots, 1$ and $b_n = \prod_{\nu=1}^{n} (1+1/a\nu)$, we find from (6) by means of 2.1.2 that $\lim_n x_n = s$ for every sequence $\{x_n\}$ satisfying any of the difference equations (6). Then the sequence $\{t_n\} = \{x_n^{(1)}\}$ converges to the limit s and the considered transformation is regular. Compare Perron [5].

THEOREM 1A. *Let $W(z)$ and $W_1(z)$ be polynomials of degrees k and $l \leqslant k$ respectively, without common zeros, $W(n) \neq 0$ for $n = 0, 1, 2, \ldots$ and $W(0) = W_1(0)$.*

The Hausdorff transformation with a matrix $H = \delta[\mu_n]\delta$, where $\mu_n = W_1(n)/W(n)$, is regular if and only if $W(z) \epsilon \mathfrak{R}$ (see p. 3).

Proof. We may evidently assume that $k \geqslant 1$. Let r_ν be the roots of the equation $W(z) = 0$, $\operatorname{re} r_\nu < 0$ for $\nu = 1, 2, \ldots, k$ and

$$\frac{W_1(z)}{W(z)} = l_0 + \sum_{\nu=1}^{k} \frac{l_\nu}{(1-z/r_\nu)^{p_\nu}}, \quad \text{where} \quad \sum_{\nu=0}^{k} l_\nu = 1 \quad \text{and} \quad z \neq r_\nu.$$

If the multiplicity of the root r_ν is k_ν, it appears in this sum k_ν times with $p_\nu = 1, 2, \ldots, k_\nu$. Next we set

$$t_n^{(0)} = s_n,$$

$$(t_n^{(\nu)}) = \delta\left[\frac{1}{(1-n/r_\nu)^{p_\nu}}\right]\delta(s_n) \quad \text{for} \quad \nu = 1, 2, \ldots, k,$$

$$t_n = \sum_{\nu=0}^{k} l_\nu t_n^{(\nu)}.$$

Then

$$(t_n) = \left(\sum_{\nu=0}^{k} l_\nu t_n^{(\nu)}\right) = \delta\left[l_0 + \sum_{\nu=1}^{k} \frac{l_\nu}{(1-n/r_\nu)^{p_\nu}}\right]\delta(s_n) = \delta\left[\frac{W_1(n)}{W(n)}\right]\delta(s_n).$$

Moreover $\lim\limits_n t_n = s$, since $\lim\limits_n t_n^{(\nu)} = s$ for $\nu = 1, 2, \ldots, k$ by 2.2.4. This proves that the transformation with $\mu_n = W_1(n)/W(n)$ is regular in the considered case.

Let us remark that by the hypothesis of regularity of the transformations $(H, \mu^{(\nu)})$ with $\mu_n^{(\nu)} = \dfrac{1}{(1-n/r_\nu)^{p_\nu}}$, the regularity of the transformation (H, μ) with $\mu_n = l_0 + \sum\limits_{\nu=1}^{k} l_\nu \mu_n^{(\nu)}$ also (as mentioned above) follows immediately from (3).

Now let $\mathrm{re}\, r_1 \geqslant 0$. We define the sequences $\{s_n^*\}$ and $\{t_n^*\}$ by

$$(s_n^*) = \delta[W(n)]\delta\left(\frac{n!}{\Gamma(n-r_1+1)}\right), \qquad (t_n^*) = \delta[W_1(n)]\delta\left(\frac{n!}{\Gamma(n-r_1+1)}\right).$$

Whence $(t_n^*) = \delta[W_1(n)/W(n)]\delta(s_n^*)$. By means of 2.2.3 it follows for $n \geqslant k$ that

$$s_n^* = W(r_1)\frac{n!}{\Gamma(n-r_1+1)} = 0, \qquad t_n^* = W_1(r_1)\frac{n!}{\Gamma(n-r_1+1)} \neq 0.$$

Thus $\lim\limits_n s_n^* = 0$, whereas the sequence $\{t_n^*\}$ does not converge and the transformation with $\mu_n = W_1(n)/W(n)$ is not regular.

2.3. *For every $k \geqslant 1$ there exists a_k with the following properties: If $a > a_k$, then the real parts of all the roots of the equation*

(7) $$\prod_{\nu=1}^{k}\left(1+\frac{z}{x_\nu}\right) = 1-\frac{1}{a}, \qquad x_\nu > 0,$$

are negative. If $a \leqslant a_k$, $a \neq 0$, then there exists a root of the equation (7) with a non-negative real part. In particular $a_1 = a_2 = 0$; if $k \geqslant 3$, then $a_k \in (0, \frac{1}{2})$ and

(8) $$\frac{1}{a_k} - 1 = \sqrt{\prod_{\nu=1}^{k}\left(1+\frac{r_k^2}{x_\nu^2}\right)},$$

where $r = r_k$ is the unique root of the equation

(9) $$\sum_{\nu=1}^{k} \operatorname{arctg}(r/x_\nu) = \pi.$$

Proof. Let r and $a^{(r)}$ satisfy with some integer n ($|n| < \frac{1}{2}k$) the equations

(10) $$\sum_{\nu=1}^{k} \operatorname{arctg}(r/x_\nu) = n\pi, \qquad \prod_{\nu=1}^{k}(1+r^2/x_\nu^2) = (1/a^{(r)}-1)^2 \quad \text{for} \quad k \geqslant 3.$$

If

$$1+\frac{r}{x_\nu}i = \sqrt{1+\frac{r^2}{x_\nu^2}}\, e^{i\varphi_\nu}, \quad \text{where} \quad \varphi_\nu = \operatorname{arctg}\frac{r}{x_\nu}, \quad 0 < |\varphi_\nu| < \frac{\pi}{2},$$

then

$$\prod_{\nu=1}^{k}\left(1+\frac{r}{x_\nu}i\right) = e^{i\Sigma\varphi_\nu}\sqrt{\prod_{\nu=1}^{k}\left(1+\frac{r^2}{x_\nu^2}\right)} = e^{in\pi}\left|\frac{1}{a^{(r)}}-1\right| = (-1)^n\left|\frac{1}{a^{(r)}}-1\right|.$$

It follows that the value $z = ri$ satisfies the equation (7) for $a = a^{(r)}$ if and only if r and $a^{(r)}$ satisfy (10) with a certain n ($|n| < \frac{1}{2}k$), which is odd in the case $0 < a < 1$ and even in the cases $a > 1$ and $a < 0$.

We denote by r_k the root of the first of the equations (10) with $n = 1$ and by a_k the positive root $a^{(r)}$ of the second for $r = r_k$. The constant a_k is the largest value of the parameter a for which the equation (7) has a purely imaginary root. From (10) follows the inequality $a^{(r)} \leqslant \frac{1}{2}$, whence $a_k \leqslant \frac{1}{2}$.

We remark that if $z \neq 0$, $\mathrm{re}\, z \geqslant 0$, then

$$\left|\prod_{\nu=1}^{k}\left(1+\frac{z}{x_\nu}\right)\right| > 1.$$

Since $|1/a-1| \leqslant 1$ for $a \geqslant \frac{1}{2}$, we see that the real parts of all the roots of (7), for those a, are negative. They are continuous functions of the parameter a, whence it follows that they are negative for $a > a_k$.

Let $z = a+bi$ be a root of equation (7) with $a \neq 0$ given; then for some n ($|n| < \frac{1}{2}k$)

(11) $$\sum_{\nu=1}^{k} \operatorname{arctg}\frac{b}{a+x_\nu} = n\pi, \qquad \prod_{\nu=1}^{k}\left[\left(1+\frac{a}{x_\nu}\right)^2 + \frac{b^2}{x_\nu^2}\right] = \left(\frac{1}{a}-1\right)^2.$$

In virtue of the first equation of (11) we may consider the parameter a as a function of b. In the case $n = 1$ we see that

1° the value $z = z_1 = a_1 + b_1 i$ satisfies (7) and (11) if $a = a_k$, $b_1 = r_k$, $a_1 = a_1(r_k) = 0$,

2° the function $a_1(b_1)$ is continuous and increasing for $b_1 > r_k - \varepsilon$, with some $\varepsilon > 0$.

By the second equation of (11) it follows that $b_1 > r_k$ and $a_1 > 0$ in the case $0 < a < a_k$.

In the case $a<0$ it is easy to prove that equation (7) has a positive root.

The values a_1 and a_2 we compute immediately.

2.4. *If $0<x_\nu\leqslant x_{\nu+1}$ for $\nu=1,2,\ldots$, then the sequence $\{a_k\}$ defined in 2.3 increases for $k\geqslant 2$.*

Proof. Let

$$f(x)=\sum_{\nu=1}^{k}\ln(1+x^2/x_\nu^2),\qquad g(x)=\sum_{\nu=1}^{k}\operatorname{arctg}(x/x_\nu).$$

By means of Cauchy's mean value theorem we obtain for some $\xi_k\in(r_{k+1},r_k)$, where r_k are defined in 2.3 and $k\geqslant 3$:

$$f(r_k)-f(r_{k+1})=[g(r_k)-g(r_{k+1})]\frac{f(r_k)-f(r_{k+1})}{g(r_k)-g(r_{k+1})}$$

$$=\operatorname{arctg}\frac{r_{k+1}}{x_{k+1}}\frac{\sum_{\nu=1}^{k}2\xi_k/(x_\nu^2+\xi_k^2)}{\sum_{\nu=1}^{k}x_\nu/(x_\nu^2+\xi_k^2)}>\frac{2r_{k+1}}{x_{k+1}}\operatorname{arctg}\frac{r_{k+1}}{x_{k+1}},$$

because $r_k>r_{k+1}$ by (9). Hence

$$\ln p_k-\ln p_{k+1}=f(r_k)-f(r_{k+1})-\ln\left(1+\frac{r_{k+1}^2}{x_{k+1}^2}\right)$$

$$>2a\operatorname{arctg}a-\ln(1+a^2)=2\int_0^a\operatorname{arctg}x\,dx>0,$$

where $p_k=\prod_{\nu=1}^{k}(1+r_k^2/x_\nu^2)$, $a=r_{k+1}/x_{k+1}$.

Thus $p_k=(1/a_k-1)^2$ is for $k\geqslant 3$ a decreasing and a_k an increasing sequence. Let us notice that from (8) and (9) follows $a_3>0$.

2.5. *If $\sum_{\nu=1}^{\infty}1/x_\nu=\infty$ and $x_\nu\geqslant\varepsilon>0$ then the sequences $\{r_k\}$ and $\{a_k\}$ defined in 2.3 have the following asymptotic representations:*

$$r_k\sim\pi/s_k,\qquad a_k\sim\tfrac{1}{2}-\tfrac{1}{8}r_k^2S_k\sim\tfrac{1}{2}-\tfrac{1}{8}\pi^2(S_k/s_k^2),$$

where $s_k=\sum_{\nu=1}^{k}1/x_\nu$, $S_k=\sum_{\nu=1}^{k}1/x_\nu^2$. It follows in particular that $r_k\downarrow 0$, $a_k\to\frac{1}{2}$.

If $\sum_{\nu=1}^{\infty}1/x_\nu<\infty$ then $r_k\downarrow r_\infty>0$ and $a_k\to a_\infty<\frac{1}{2}$.

Proof. Let $\sum 1/x_\nu=\infty$. From (9) it follows that $r_k\downarrow 0$; by hypothesis $r_k\downarrow r_\infty>0$ we obtain namely $\sum\operatorname{arctg}(r_\infty/x_\nu)=\infty$ in contradiction to (9). Using the inequality $x-\frac{1}{3}x^3<\operatorname{arctg}x<x$, true for $x>0$, we have

$$r_ks_k-\tfrac{1}{3}r_k^3\sigma_k<\sum_{\nu=1}^{k}\operatorname{arctg}(r_k/x_\nu)<r_ks_k,\qquad\text{where}\qquad\sigma_k=\sum_{\nu=1}^{k}1/x_\nu^3;$$

hence by (9)

$$0<r_ks_k-\pi<\tfrac{1}{3}r_k^3\sigma_k,\qquad 0<1-\pi/r_ks_k<\tfrac{1}{3}r_k^2(\sigma_k/s_k).$$

It follows $\lim_k r_ks_k=\pi$, because $\overline{\lim_k}\frac{\sigma_k}{s_k}\leqslant\overline{\lim_k}\frac{1/x_k^3}{1/x_k}\leqslant\frac{1}{\varepsilon^2}$.

2.5.1. We show that if $a_n>0$, $\lambda_n>0$, $\lambda_nA_n\to 0$, $p_n=\prod_{\nu=1}^{n}(1+\lambda_na_\nu)$, where $A_n=\sum_{\nu=1}^{n}a_\nu$, then $p_n\sim 1+\lambda_nA_n$. Using the inequality $x/(1+x)<\ln(1+x)<x$, true for $x>-1$, we obtain

$$\frac{\exp\left(\lambda_n\sum_{\nu=1}^{n}\frac{a_\nu}{1+\lambda_na_\nu}\right)-1}{\lambda_n\sum_{\nu=1}^{n}\frac{a_\nu}{1+\lambda_na_\nu}}\cdot\frac{\sum_{\nu=1}^{n}\frac{a_\nu}{1+\lambda_na_\nu}}{A_n}<\frac{p_n-1}{\lambda_nA_n}<\frac{\exp(\lambda_nA_n)-1}{\lambda_nA_n}.$$

We have $0<\lambda_n\sum_{\nu=1}^{n}\frac{a_\nu}{1+\lambda_na_\nu}<\lambda_nA_n\to 0$; moreover, if $\lim_n A_n=\infty$, we obtain by Cauchy's theorem

$$\lim_n\frac{\sum_{\nu=1}^{n}a_\nu/(1+\lambda_na_\nu)}{A_n}=\lim_n\frac{a_n/(1+\lambda_na_n)}{a_n}=1;$$

if $\lim_n A_n=s<\infty$, it is easy to state that $\lim_n\sum_{\nu=1}^{n}\frac{a_\nu}{1+\lambda_na_\nu}=s$, because $\lambda_n\to 0$. Hence $\lim_n\frac{p_n-1}{\lambda_nA_n}=1$.

If $\lambda_n=r_n^2$, $a_n=1/x_n^2$, then

$$0\leqslant\underline{\lim_n}\lambda_nA_n=\underline{\lim_n}r_n^2\sum_{\nu=1}^{n}\frac{1}{x_\nu^2}\leqslant\lim_n r_n\lim_n r_ns_n\overline{\lim_n}\frac{S_n}{s_n}\leqslant\pi\lim_n r_n\overline{\lim_n}\frac{1}{x_n}=0.$$

Thus $\lim_n\lambda_nA_n=0$ and by 2.5.1

$$p_k=\prod_{\nu=1}^{k}\left(1+\frac{r_k^2}{x_\nu^2}\right)\sim 1+r_k^2S_k.$$

Therefore $\lim_k p_k=1$, $\lim_k a_k=\frac{1}{2}$ by (8) and

$$\lim_k\frac{1}{r_k^2S_k}\left(\frac{1}{2}-a_k\right)=\lim_k\frac{a_k}{2r_k^2S_k}\left(\frac{1}{a_k}-2\right)=\frac{1}{4}\lim_k\frac{1}{r_k^2S_k}(\sqrt{p_k}-1)=\frac{1}{8};$$

hence $a_k\sim\frac{1}{2}-\frac{1}{8}r_k^2S_k$.

Under the hypothesis $\sum_{\nu=1}^{\infty} 1/x_\nu < \infty$ the series $\sum_{\nu=1}^{\infty} \operatorname{arctg}(r/x_\nu)$ converges for every real r. Denoting its sum by $F(r)$, we find from (9) that $r_\infty = \lim_k r_k$ satisfies the equation $F(r) = \pi$. Hence

$$r_\infty > 0, \quad \frac{1}{a_\infty} = 1+\sqrt{\prod_{\nu=1}^{\infty}\left(1+\frac{r_\infty^2}{x_\nu^2}\right)} > 2 \quad \text{and} \quad 0 < a_\infty < \frac{1}{2},$$

where $a_\infty = \lim_k a_k$.

2.6. We now return to theorem 1A. Let us remark that a simple modification of its proof enables us to state the following variant of this theorem:

Under the hypothesis $W_1(n) \neq 0$ instead of $W(n) \neq 0$, $n = 0, 1, 2, \ldots$, the Hausdorff method of summability corresponding to $\mu_n = W(n)/W_1(n)$ is not stronger than the ordinary convergence if and only if $W(z) \in \mathfrak{R}$.

From this we obtain

THEOREM 2A. *Let $W(z)$ be a polynomial, $W(0) = 1$, $W(n) \neq 0$ for $n = 1, 2, \ldots$, $(t_n) = \delta[1/W(n)]\delta(s_n)$ and $p_n = as_n+(1-a)t_n$ with $a \neq 0$. In order that for every sequence $\{s_n\}$ the hypothesis $\lim_n p_n = s$ should imply $\lim_n s_n = s$ it is necessary and sufficient that $W(z)-1+1/a \in \mathfrak{R}$.*

In the case fo the Cesàro and Hölder transforms $c_n^{(k)}$ and $h_n^{(k)}$ of the sequence $\{s_n\}$ (corresponding to $\mu_n = 1\Big/\binom{n+k}{n}$ and $\mu_n = 1/(n+1)^k$ respectively) we obtain for positive integers k

THEOREM 3A. *Suppose a real. For every $k \geqslant 1$ there exist constants a_k' and a_k'' with the following properties: the hypothesis $\lim_n p_n = s$, where $p_n = as_n+(1-a)c_n^{(k)}$, implies $\lim_n s_n = s$ if and only if $a > a_k'$; the hypothesis $\lim_n q_n = s$, where $q_n = as_n+(1-a)h_n^{(k)}$, implies $\lim_n s_n = s$ if and only if $a > a_k''$.*

In particular $a_1' = a_2' = a_1'' = a_2'' = 0$, $a_3' = \frac{1}{11}$, $a_4' = \frac{4}{25}$. For $k \geqslant 3$ it is $0 < a_k', a_k'' < \frac{1}{2}$; the sequences $\{a_k'\}$ and $\{a_k''\}$ are increasing and tend to the limit $\frac{1}{2}$. The following representations are true:

$$a_k' \sim \frac{1}{2}-\frac{\pi^4}{48\ln^2 k}, \qquad a_k'' = \frac{\cos^k(\pi/k)}{1+\cos^k(\pi/k)} \sim \frac{1}{2}-\frac{\pi^2}{8k}.$$

For $a \neq 0$ the proof follows from theorem 2A by means of 2.3, 2.4 and 2.5 for $x_\nu = \nu$ and $x_\nu = 1$ respectively. The case $a = 0$ is known.

THEOREM 4. *Let $p_n = \lambda_0 s_n+\sum_{\nu=1}^{k}\lambda_\nu c_n^{(\nu)}$, $\sum_{\nu=0}^{k}\lambda_\nu = 1$ with $k \geqslant 1$. The hypothesis $\lim_n p_n = s$ implies the equality $\lim_n s_n = s$ if and only if*

$$\sum_{\nu=0}^{k}\frac{\lambda_{k-\nu}}{\binom{k}{\nu}}\binom{z+k}{\nu} \in \mathfrak{R}.$$

For the proof we observe that the equality

$$(c_n^{(\nu)}) = \delta\left[\frac{1}{\binom{n+\nu}{\nu}}\right]\delta(s_n)$$

implies

$$(p_n) = \delta[\mu_n]\delta(s_n) \quad \text{and} \quad (s_n) = \delta[1/\mu_n]\delta(p_n),$$

where

$$\mu_n = \sum_{\nu=0}^{k}\lambda_\nu\frac{1}{\binom{n+\nu}{\nu}} = \frac{1}{\binom{n+k}{k}}\sum_{\nu=0}^{k}\frac{\lambda_{k-\nu}}{\binom{k}{\nu}}\binom{n+k}{\nu}.$$

We now use theorem 1A in the modified form given at the beginning of 2.6.

THEOREM 5A. *Let $W(z)$, $W_1(z)$ be the polynomials defined in theorem 1A and let $p_n = \sum_{\nu=0}^{l}\lambda_\nu\binom{n}{\nu}\Delta^\nu s_{n-\nu}+t_n$, $n \geqslant k$, where $(t_n) = \delta[W_1(n)/W(n)]\delta(s_n)$. The hypothesis $\lim_n p_n = s$ implies $\lim_n s_n = s$ if and only if*

$$W(z)\sum_{\nu=0}^{l}(-1)^\nu\lambda_\nu\binom{z}{\nu}+W_1(z) \in \mathfrak{R}.$$

By means of (4) the proof follows in the same way as that of theorem 4.

Let us remark that from theorem 5A the following Tauberian theorem results: *if $\lim_n c_n^{(k)} = s$, $\lim_n n\cdot a_n = 0$, then $\lim_n s_n = s$ with $s_n = \sum_{\nu=0}^{n} a_\nu$.* Indeed, if $p_n = -An\Delta s_{n-1}+c_n^{(k)}$, we take $A > 0$ and so large that the equation $\binom{z+k}{k}z+\frac{1}{A} = 0$ has $k+1$ negative roots, and we apply theorem 5A.

3. We now give the analogous theorems in the case of difference and differential Hausdorff transforms of functions.

3.1. We suppose that the functions considered in this section are complex-valued and defined for all real values $x \geqslant a$.

The k-th difference $\Delta^k f(x)$ of the function $f(x)$ is defined as usual by

$$\Delta^k f(x) = \sum_{\nu=0}^{k} (-1)^{k-\nu} \binom{k}{\nu} f(x+\nu) \quad \text{for} \quad k = 0, 1, 2, \ldots, \quad \Delta f(x) = \Delta^1 f(x).$$

The functional analogue of theorem A (see 2.1) is the following theorem:

THEOREM B. *Let $s(x)$, $f(x)$, $g(x)$ be given functions. The hypothesis $\lim\limits_{x\to\infty} s(x) = s$ implies $\lim\limits_{x\to\infty}\big(f(x)/g(x)\big) = s$ if*

1a) *$|g(x)| \to \infty$ as $x \to \infty$ and $\sum\limits_{\nu=1}^{[x]} |\Delta g(x-\nu)| \leqslant K|g(x)|$ or*

1b) *$f(x) \to 0$, $g(x) \to 0$ as $x \to \infty$, $g(x) \neq 0$ for large x and $\sum\limits_{\nu=0}^{\infty} |\Delta g(x+\nu)| \leqslant K|g(x)|$,*

2) *$\Delta f(x) = s(x) \Delta g(x)$.*

3.1.1. *Suppose that*

1) *$g(x)$ satisfies the conditions 1a) or 1b) of theorem B,*

2) *$g(x) \neq 0$ for $x \geqslant a$,*

3) *there exists $X \geqslant a$ such that $g(x)$ is bounded in every interval $\langle a_1, b_1\rangle$ if $X \leqslant a_1 < b_1 < \infty$,*

4) *$\lim\limits_{x\to\infty} s(x) = s$.*

Then there exists a function $\bar{y}(x)$ satisfying for $x \geqslant a+1$ the difference equation

$$\Delta y(x-1) + \frac{\Delta g(x-1)}{g(x-1)} [y(x) - s(x)] = 0 \tag{12}$$

and tending to the limit s as $x \to \infty$. If $g(x)$ satisfies 1a) and $y(x)$ is a solution of (12), bounded in the interval $\langle x_0, x_0+1\rangle$ for some, sufficiently large $x_0 \geqslant a$ (more precisely, in the interval $\langle x_0, x_0+1\rangle$, in which $g(x)$ and $\bar{y}(x)$ are bounded), then $y(x)$ tends to s. If $g(x)$ satisfies 1b) and the functions $g(x)y(x)$ and $g(x)\bar{y}(x)$ are bounded in $\langle x_0, x_0+1\rangle$ for some $x_0 \geqslant a$, then

$$y(x) = \bar{y}(x) + O\big(1/g(x)\big) \quad \text{as} \quad x \to \infty.$$

Proof. If $y(x)$ is a solution of (12) and $y(x) = h(x)/g(x)$, $x \geqslant a$, then the function $h(x)$ satisfies the difference equation

$$\Delta h(x-1) = s(x) \Delta g(x-1).$$

If $g(x)$ satisfies the conditions 1a) then

$$h(x) = c(x) + \sum_{\nu=1}^{[x]} s(x-\nu+1) \Delta g(x-\nu),$$

and the general solution of (12) is

$$y(x) = \frac{c(x)}{g(x)} + \frac{1}{g(x)} \sum_{\nu=1}^{[x]} s(x-\nu+1) \Delta g(x-\nu).$$

If $g(x)$ satisfies 1b) then the series $\sum\limits_{\nu=0}^{\infty} \Delta g(x+\nu)$ and $\sum\limits_{\nu=0}^{\infty} s(x+\nu+1) \times \Delta g(x+\nu)$ are convergent for $x \geqslant a$ and

$$h(x) = -\sum_{\nu=0}^{\infty} \Delta h(x+\nu) = -\sum_{\nu=0}^{\infty} s(x+\nu+1) \Delta g(x+\nu),$$

$$y(x) = \frac{c_1(x)}{g(x)} - \frac{1}{g(x)} \sum_{\nu=0}^{\infty} s(x+\nu+1) \Delta g(x+\nu).$$

Here $c(x) = h(x-[x])$ and $c_1(x)$ are any periodic functions with a period $\omega = 1$; it is easy to state that in the case 1b) $h(x) \to 0$ as $x \to \infty$. In both cases we obtain the expressions for $\bar{y}(x)$ writing $c(x) = c_1(x) = 0$ and applying theorem B.

For the proof of the second part of the lemma we show, using 3), that $c(x)$ and $c_1(x)$ are bounded in the considered cases.

3.1.2. *Let $g(x) = \dfrac{\Gamma(x+z+1)}{\Gamma(x+1)}$, where $z = a+bi$, $x > 0$, $x+z \neq -1, -2, \ldots$ If $a > 0$, then $|g(x)| \to \infty$ and $\dfrac{1}{|g(x)|} \sum\limits_{\nu=1}^{[x]} |\Delta g(x-\nu)| \to \dfrac{|z|}{a}$ as $x \to \infty$, if $a < 0$, then $g(x) \to 0$ and $\dfrac{1}{|g(x)|} \sum\limits_{\nu=0}^{\infty} |\Delta g(x+\nu)| \to \dfrac{|z|}{-a}$ as $x \to \infty$.*

Proof. We notice that $g(x)$ satisfies the difference equation $\Delta g(x-1) = \dfrac{z}{x} g(x-1)$. From Stirling formula we obtain $|g(x)| \sim x^a$ as $x \to \infty$, whence

$$\lim_{x\to\infty} |g(x)| = \begin{cases} \infty & \text{in the case of } a > 0, \\ 0 & \text{in the case of } a < 0. \end{cases}$$

Similarly to 2.1.2 we show that for fixed $x>0$ the sequence $b_n = |g(x+n)|$ is monotone for $a>0$, $n\geqslant 0$ and for $a<0$, $n>N$. Using theorem B for $|g(x)|$ instead of $g(x)$ we obtain in the case of $a>0$

$$\lim_{x\to\infty}\frac{\sum_{\nu=1}^{[x]}|\Delta g(x-\nu)|}{|g(x)|} = \lim_{x\to\infty}\frac{|\Delta g(x-1)|}{|g(x)|-|g(x-1)|} = \lim_{x\to\infty}\frac{|z|/x}{|1+z/x|-1} = \frac{|z|}{a}.$$

If $a<0$, then $|\Delta g(x+\nu)| = \left|\frac{z}{x+\nu}\, g(x+\nu)\right| \sim |z|(x+\nu)^{a-1}$, the series $\sum_{\nu=0}^{\infty}|\Delta g(x+\nu)|$ converges and $\lim_{x\to\infty}\sum_{\nu=0}^{\infty}|\Delta g(x+\nu)| = 0$; now we conclude the proof in the same way as in the case of $a>0$.

3.2. We assume that the function $f(x)$ is defined for $x\geqslant 0$ and is representable by the Newton-Gregory series for $x>x_0$, *i. e.*,

$$f(x) = \sum_{n=0}^{\infty}\binom{x}{n}\Delta^n f(0),$$

the series being for $x>x_0$ convergent in the ordinary sense; then for $x>x_0+1$ it is convergent absolutely and almost uniformly.

We define the *difference transform of Hausdorff type* of the function $f(x)$ by the formula

(13) $$t(x) = \sum_{n=0}^{\infty}\mu_n\binom{x}{n}\Delta^n f(0);$$

$\{\mu_n\}$ is a given sequence of complex numbers, such that the above series converges for $x>x_1$. We observe that $t(x)$ is defined for $x=0,1,2,\ldots$ also if $x_1>0$, since $\Delta^n t(0) = \mu_n\Delta^n f(0)$, $n=0,1,2,\ldots$ (see 2.2).

Hence, if $\mu_n\neq 0$, then the equality

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{\mu_n}\binom{x}{n}\Delta^n t(0) \quad \text{for} \quad x>x_0$$

immediately follows.

In symbolic form we write

$$\{t(x)\} = [\mu_n]\{f(x)\}, \qquad \{f(x)\} = [1/\mu_n]\{t(x)\}.$$

Suppose for a moment that $f(x)$ is defined for every real x. The Cesàro transform $c_k(x)$ of the function $f(x)$ defined by

(14) $$c_k(x) = \frac{\sum_{n=0}^{\infty}\binom{n+k-1}{k-1} f(x-n)}{\binom{x+k}{k}}$$

satisfies (13) with $\mu_n = 1\Big/\binom{n+k}{k}$ if the series in the numerator converges for $x\geqslant 0$ and if the function $c_k(x)$ is for $x>x_2$ representable by its Newton-Gregory series. Indeed

$$\Delta^k\left[\binom{x}{k}c_k(x-k)\right] = f(x)$$

and

$$\Delta^{n+k}\left[\binom{x}{k}c_k(x-k)\right] = \sum_{\nu=0}^{k}\binom{n+k}{\nu}\binom{x}{k-\nu}\Delta^{n+k-\nu}c_k(x-k+\nu) = \Delta^n f(x)$$

by means of the formula

$$\Delta^k f(x)g(x) = \sum_{\nu=0}^{k}\binom{k}{\nu}\Delta^\nu f(x)\,\Delta^{k-\nu}g(x+\nu).$$

For $x=0$ we obtain

$$\binom{n+k}{k}\Delta^n c_k(0) = \Delta^n f(0).$$

It may easily be shown by induction that the Hölder transform $h_k(x)$ defined as

(15) $$h_1(x) = c_1(x), \qquad h_k(x) = h_1[h_{k-1}(x)] \quad \text{for} \quad k=2,3,\ldots,$$

satisfies (13) with $\mu_n = 1/(n+1)^k$. We suppose here that the series in formulas defining $h_\nu(x)$ are convergent for every x if $\nu=1,2,\ldots,k-1$ and for $x>0$ if $\nu=k$ and that the functions $h_\nu(x)$ are for every x, respectively $x>x_2$, representable by its Newton-Gregory series.

The transformation (13) is called *regular* if the hypothesis $\lim_{x\to\infty} f(x) = s$ implies $\lim_{x\to\infty} t(x) = s$. We observe that the abscissas of convergence x_0 and x_1 are not fixed and depend on the choice of the functions $f(x)$ and $t(x)$ resp. The essential assumption here is that $f(x)$ and $t(x)$ are defined for large x (if $x\neq 0,1,2,\ldots$).

3.2.1. If $\mu_n = \binom{n}{k}$ then $t(x) = \binom{x}{k}\Delta^k f(x-k)$ for $x>x_0+k$. We have namely

$$t(x) = \binom{x}{k}\sum_{n=0}^{\infty}\Delta^k\binom{x-k}{n}\Delta^n f(0) = \binom{x}{k}\Delta^k\left[\sum_{n=0}^{\infty}\binom{x-k}{n}\Delta^n f(0)\right] = \binom{x}{k}\Delta^k f(x-k),$$

since $\binom{n}{k}\binom{x}{n} = \binom{x}{k}\binom{x-k}{n-k} = \binom{x}{k}\Delta^k\binom{x-k}{n}$.

3.2.2. If $W(z)$ is a polynomial of degree k and $\mu_n = W(n)$ then for $x>x_0+k$ is $t(x) = L[f(x)]$, where $L(y) = \sum_{\nu=0}^{k}\lambda_\nu\binom{x}{\nu}\Delta^\nu y(x-\nu)$ and $\lambda_\nu = \Delta^\nu W(0)$.

For the proof we use 3.2.1 or the identity of Guderman

$$(16)\qquad \sum_{n=0}^{N}\varphi(n)\binom{x}{n}\Delta^n f^*(0)=\sum_{n=0}^{N}\Delta^n\varphi(0)\binom{x}{n}\Delta^n f^*(x-n)$$

(Ostrowski [7], p. 309), with $f^*(x)=\sum_{n=0}^{N}\binom{x}{n}\Delta^n f(0)$, supposing $N\to\infty$.

3.2.3. If $f(0)=1$, $f(x)=0$ for $x>0$, $r\neq 0,1,2,\ldots$ and $\mu_n=1/(n-r)$; then $\Delta^n f(0)=(-1)^n$ and from the well known formulas for the sum of the hypergeometric series $F(\alpha,\beta,\gamma,z)$ with $z=1$ it follows for $x>0$ that

$$t(x)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{n-r}\binom{x}{n}=\frac{\Gamma(x+1)\Gamma(-r)}{\Gamma(x+1-r)}\sim\Gamma(-r)x^r\quad\text{as}\quad x\to\infty.$$

3.2.4. *Let p be a positive integer. If* $\operatorname{re}a>0$ *then the transformation* (13) *with* $\mu_n=1/(an+1)^p$ *is regular.*

We may write the relation (13) in the form of a system of equations

$$[an+1]\{y_\nu(x)\}=\{y_{\nu+1}(x)\}\quad\text{for}\quad\nu=1,2,\ldots,p\quad\text{and}\quad x>x_0,$$

where $y_1(x)=t(x)$, $y_{p+1}(x)=f(x)$, which by 3.2.1 is equivalent to the system of difference equations

$$ax\Delta y_\nu(x-1)+y_\nu(x)=y_{\nu+1}(x)\quad\text{for}\quad x>x_0+p.$$

We conclude the proof as in 2.2.4 using 3.1.1 for $a=\max(0,x_0+p)$, $g(x)=\dfrac{\Gamma(x+1/a+1)}{\Gamma(x+1)}$ and 3.1.2. We observe that for $x>x_0+p+1$ every function $y_\nu(x)$ is continuous.

THEOREM 1B. *Let $W(z)$ and $W_1(z)$ be polynomials defined in theorem* 1A. *The transformation* (13) *with* $\mu_n=W_1(n)/W(n)$ *is regular if and only if* $W(z)\in\mathfrak{R}$.

Proof. We put for $x>x_0$

$$t_0(x)=l_0f(x),\qquad \{t_\nu(x)\}=\left[\frac{1}{(1-n/r_\nu)^{p_\nu}}\right]\{f(x)\}\quad\text{for}\quad\nu=1,2,\ldots,k,$$

where r_ν, l_ν and p_ν are defined in the proof of theorem 1A. If $t(x)=\sum_{\nu=0}^{k}l_\nu t_\nu(x)$, then

$$\{t(x)\}=\left[l_0+\sum_{\nu=1}^{k}\frac{l_\nu}{(1-n/r_\nu)^{p_\nu}}\right]\{f(x)\}=\left[\frac{W_1(n)}{W(n)}\right]\{f(x)\}.$$

From the hypothesis $\operatorname{re}r_\nu<0$ for $\nu=1,2,\ldots,k$ and $\lim_{x\to\infty}f(x)=s$ it follows that $\lim_{x\to\infty}t_\nu(x)=\lim_{x\to\infty}t(x)=s$ by 3.2.4. This proves that in the considered case the transformation (13) with $\mu_n=W_1(n)/W(n)$ is regular.

Suppose now that $\operatorname{re}r_1\geqslant 0$. Assuming $\Delta^n f(0)=(-1)^nW(n)/(n-r_1)$ we obtain for $x>k-1$

$$f(x)=\sum_{n=0}^{\infty}\binom{x}{n}\Delta^n f(0)=0,$$

$$t(x)=\sum_{n=0}^{\infty}(-1)^n\frac{W_1(n)}{n-r_1}\binom{x}{n}=W_1(r_1)\sum_{n=0}^{\infty}(-1)^n\frac{1}{n-r_1}\binom{x}{n}.$$

We suppose that $f(x)$ is linear in the intervals $\langle\nu-1,\nu)$, $\nu=1,2,\ldots,k-1$. Since by 3.2.3 we have $t(x)\sim W_1(r_1)\Gamma(-r_1)x^{r_1}$ as $x\to\infty$, we see that in this case the transformation (13) is not regular.

We observe that the following variant of theorem 1B is also true:

Supposing that $W_1(n)\neq 0$ instead of $W(n)\neq 0$, $n=0,1,2,\ldots$, the hypothesis $\lim_{x\to\infty}t(x)=s$, where $t(x)$ is defined by (13) *with $\mu_n=W(n)/W_1(n)$, implies $\lim_{x\to\infty}f(x)=s$ if and only if $W(z)\in\mathfrak{R}$.*

After theorem 1B we have only to prove that if $W(m)=0$ for some positive integer m, then there exists a function $f(x)$ divergent and such that its transform $t(x)$ converges as $x\to\infty$. We take, for example, $f(x)=\binom{x}{m}$; hence $t(x)=0$ for every x. This completes the proof.

From this we obtain

THEOREM 2B. *Let $W(z)$ denote the polynomial defined in theorem* 2A, *furthermore suppose that $t(x)$ satisfies* (13) *with $\mu_n=1/W(n)$ and $p(x)=af(x)+(1-a)t(x)$ for $x>x_0$, $a\neq 0$. In order that the hypothesis $\lim_{x\to\infty}p(x)=s$ should imply $\lim_{x\to\infty}f(x)=s$ it is necessary and sufficient that $W(z)-1+1/a\in\mathfrak{R}$.*

In the case of transforms $c_k(x)$ and $h_k(x)$, which are representable by its series of Newton-Gregory, we obtain for positive integers k

THEOREM 3B. *Suppose that a is real, $p(x)=af(x)+(1-a)c_k(x)$, where $c_k(x)=\sum_{n=0}^{\infty}\dfrac{1}{\binom{n+k}{k}}\binom{x}{n}\Delta^n f(0)$ $(x>x_0)$. The hypothesis $\lim_{x\to\infty}p(x)=s$ implies $\lim_{k\to\infty}f(x)=s$ if and only if $a>a_k'$.*

Let $q(x) = af(x)+(1-a)h_k(x)$, *where* $h_k(x) = \sum_{n=0}^{\infty}\frac{1}{(n+1)^k}\binom{x}{n}\Delta^n f(0)$ $(x > x_0)$. *The hypothesis* $\lim_{x\to\infty} q(x) = s$ *implies* $\lim_{x\to\infty} f(x) = s$ *if and only if* $a > a_k''$.

$\{a_k'\}$ *and* $\{a_k''\}$ *are the sequences of theorem* 3A.

3.3. The theorem of l'Hospital, as an analogue of theorems A and B, will be used in the following formulation:

THEOREM C. *Let I denote a neighbourhood of the point x_0 and let $s(x), f(x), g(x)$ be given functions. The hypothesis* $\lim_{x\to x_0} s(x) = s$ *implies* $\lim_{x\to x_0}\frac{f(x)}{g(x)} = s$ *if*

1a) $|g(x)|\to\infty$ *as* $x\to x_0$ *and* $\left|\int_y^x |g'(t)|\,dt\right| \leqslant K|g(x)|$ *for* $x, y\in I$ *or*

1b) $f(x)\to 0$, $g(x)\to 0$ *as* $x\to x_0$ *and* $g(x)\neq 0$, $\left|\int_x^{x_0} |g'(t)|\,dt\right| \leqslant K|g(x)|$ *for* $x\in I$,

2) $f'(x) = s(x)g'(x)$ *in* I,

3) $f'(x)$ *is integrable in* I.

In this theorem we may suppose that $f'(x)$ and $g'(x)$ exist in I except a finite number of points.

3.3.1. *Suppose that*

1) $g(x)$ *satisfies the conditions* 1a) *or* 1b) *of theorem* C *with some* $x_0\in(a, b)$ *and some* $I\subset(a, b)$,

2) $g(x)\neq 0$ *for* $x\in(a, b)$,

3) $g'(x)$ *exists in* (a, b),

4) *the function* $g'(x)s(x)$ *is continuous in* (a, b)

5) $\lim_{x\to x_0} s(x) = s$.

Then there exists a function $\bar{y}(x)$ satisfying in (a, b) the differential equation

$$y'(x)+\frac{g'(x)}{g(x)}[y(x)-s(x)] = 0 \tag{17}$$

and tending to the limit s as $x\to x_0$. In the case 1a) *every solution $y(x)$ of* (17) *tends to s, in the case* 1b) $y(x) = \bar{y}(x)+c/g(x)$.

Since the general integral of the differential equation (17) is of the form

$$y(x) = \frac{1}{g(x)}\int_a^x s(t)g'(t)\,dt, \quad \text{where} \quad a, x\in(a, b),$$

it suffices to apply theorem C. We remark that in the case 1b) the integral $\int_a^{x_0} s(t)g'(t)\,dt$ converges and $\bar{y}(x) = \frac{1}{g(x)}\int_{x_0}^{x} s(t)g'(t)\,dt$.

3.4. We suppose in this section that for every real x

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}.$$

We define the *differential Hausdorff transform* $t(x)$ of the function $f(x)$ by the formula

$$t(x) = \sum_{n=0}^{\infty}\mu_n f^{(n)}(0)\frac{x^n}{n!}; \tag{18}$$

$\{\mu_n\}$ is a given sequence of complex numbers, such that the above series converges for every x.(1) In symbolic form we write

$$[t(x)] = [\mu_n][f(x)].$$

If $\mu_n\neq 0$, from (18) follows immediately

$$[f(x)] = [1/\mu_n][t(x)];$$

furthermore

$$\lambda_1[\mu_n^{(1)}][f(x)]+\lambda_2[\mu_n^{(2)}][f(x)] = [\lambda_1\mu_n^{(1)}+\lambda_2\mu_n^{(2)}][f(x)],$$

$$[\mu_n^{(2)}][[\mu_n^{(1)}][f(x)]] = [\mu_n^{(1)}\mu_n^{(2)}][f(x)].$$

We notice that the Cesàro mean $C_k(x)$ defined as

$$C_k(x) = \frac{k}{x^k}\int_0^x (x-t)^{k-1}f(t)\,dt \quad \text{for} \quad x\neq 0, \quad k = 1, 2, \ldots$$

satisfies (18) with $\mu_n = 1\Big/\binom{n+k}{k}$. Namely

$$C_k(x) = \frac{k}{x^k}\int_0^x (x-t)^{k-1}\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}\,dt$$

$$= \frac{k}{x^k}\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\int_0^x (x-t)^{k-1}t^n\,dt = \sum_{n=0}^{\infty}\frac{1}{\binom{n+k}{n}}f^{(n)}(0)\frac{x^n}{n!}.$$

(1) We observe that this definition is not less general than the definition of the integral Hausdorff transformation (Rogosinski [8] II), since, for example, in the case $\mu_n = 1/(n-r)$, $r\neq r > 0$ the integral $\int_0^1 f(xt)t^{-r-1}\,dt$ is divergent if $|f(x)| > 0$ for $0\leqslant x\leqslant\delta$ and the integral Hausdorff transform of $f(x)$ cannot be defined.

In the case of Hölder mean $H_k(x)$ defined as

$$H_1(x) = \frac{1}{x}\int_0^x f(t)\,dt, \quad H_k(x) = H_1\{H_{k-1}(x)\} \quad \text{for} \quad k \geqslant 2, \quad x \neq 0$$

we may prove by induction that $\mu_n = 1/(n+1)^k$.

The transformation (18) is called *regular* if the hypothesis $\lim\limits_{x\to\infty} f(x) = s$ implies $\lim\limits_{x\to\infty} t(x) = s$.

3.4.1. It is easy to state that if $\mu_n = \binom{n}{k}$ then $t(x) = \frac{x}{k!} f^{(k)}(x)$.

3.4.2. If $W(z)$ is a polynomial of degree k and $\mu_n = W(n)$ then $t(x) = L[f(x)]$, where $L(y) = \sum\limits_{\nu=0}^{k} \lambda_\nu \frac{x^\nu}{\nu!} y^{(\nu)}$, $\lambda_\nu = \Delta^\nu W(0)$.

The proof follows from 3.4.1, since $W(n) = \sum\limits_{\nu=0}^{n} \binom{n}{\nu} \Delta^\nu W(0)$. We may also use the formula of Guderman (Ostrowski [7], p. 309):

$$\sum_{n=0}^{\infty} \varphi(n) f^{(n)}(0)\frac{x^n}{n!} = \sum_{n=0}^{\infty} \Delta^n\varphi(0)\frac{x^n}{n!} f^{(n)}(x).$$

3.4.3. If $f(x) = e^{-x}$, $r \neq 0, 1, 2, \ldots$ and $\mu_n = 1/(n-r)$, then for $x > 0$

$$t(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n-r}\cdot\frac{x^n}{n!} = \Gamma(-r, x)x^r \sim \Gamma(-r)x^r \quad \text{as} \quad x \to \infty,$$

where $\Gamma(-r, x) = \Gamma(-r) - \int_x^\infty e^{-t}t^{-r-1}\,dt$, cf. for example F. Lösch and F. Schoblik [3], p. 108.

3.4.4. *Let p be a positive integer. If* $\mathrm{re}\, a > 0$ *then the transformation* (18) *with* $\mu_n = 1/(an+1)^p$ *is regular.*

Under our hypothesis we may write the relation (18) in the form of a system of equations

$$[an+1][y_\nu] = [y_{\nu+1}], \quad \nu = 1, 2, \ldots, p,$$

where $y_1 = t(x)$, $y_{p+1} = f(x)$, which by 3.4.1 is equivalent to the system of differential equations

$$axy'_\nu + y_\nu = y_{\nu+1}.$$

We then prove in the same way as 2.2.4 using 3.3.1 for $a = 0$ and $x_0 = \infty$; we observe that the function $g(x) = x^{1/a}$ satisfies for large x the conditions 1a) of theorem C.

THEOREM 1C. *Let $W(z)$ and $W_1(z)$ be polynomials defined in theorem* 1A. *The transformation* (18) *with* $\mu_n = W_1(n)/W(n)$ *is regular if and only if* $W(z) \in \mathfrak{R}$.

If r_ν are the roots of the equation $W(z) = 0$ and $\mathrm{re}\, r_\nu < 0$ for $\nu = 1, 2, \ldots, k$, then in virtue of 3.4.4 the proof of regularity of the transformation (18) is like that of theorem 1B.

Suppose now that $\mathrm{re}\, r_1 \geqslant 0$. Assuming $f^{(n)}(0) = (-1)^n W(n)/(n-r_1)$ we obtain $f(x) = W_2(x)e^{-x}$, where $W_2(x)$ is a polynomial of degree $k-1$ and by 3.4.3

$$t(x) = \sum_{n=0}^{\infty} (-1)^n \frac{W_1(n)}{n-r_1}\cdot\frac{x^n}{n!} = W_3(x)e^{-x} + W_1(r_1)\Gamma(-r_1, x)$$
$$\sim W_1(r_1)\Gamma(-r_1)x^{r_1} \quad \text{as} \quad x \to \infty,$$

where $W_3(x)$ is a polynomial of degree $l-1$ if $l > 0$ and $W_3(x) = 0$ if $l = 0$.

We see that $f(x) \to 0$, whereas $t(x)$ does not converge as $x \to \infty$. This proves that the transformation is not regular.

Analogously to the case of theorem 1B, we may state the following variant of theorem 1C:

Supposing that $W_1(n) \neq 0$ *instead of* $W(n) \neq 0$, $n = 0, 1, 2, \ldots$, *the hypothesis* $\lim\limits_{x\to\infty} t(x) = s$, *where $t(x)$ is defined by* (18) *with* $\mu_n = W(n)/W_1(n)$, *implies* $\lim\limits_{x\to\infty} f(x) = s$ *if and only if* $W(z) \in \mathfrak{R}$.

The proof is similar to that applied in the case of difference transforms: we take $f(x) = x^m$ instead of $\binom{x}{m}$.

From this we obtain

THEOREM 2C. *Let $W(z)$ denote the polynomial defined in theorem* 2A; *furthermore suppose that $t(x)$ satisfies* (18) *with* $\mu_n = 1/W(n)$ *and that* $p(x) = af(x) + (1-a)t(x)$, $a \neq 0$. *In order that the hypothesis* $\lim\limits_{x\to\infty} p(x) = s$ *should imply* $\lim\limits_{x\to\infty} f(x) = s$ *it is necessary and sufficient that* $W(z) - 1 + 1/a \in \mathfrak{R}$.

In the case of transforms $C_k(x)$ and $H_k(x)$ of the function $f(x)$ representable by its Maclaurin expansion we obtain for positive integers k

THEOREM 3C. *Suppose that a is real. The hypothesis* $\lim\limits_{x\to\infty} p(x) = s$, *where* $p(x) = af(x) + (1-a)C_k(x)$ *implies* $\lim\limits_{x\to\infty} f(x) = s$ *if and only if* $a > a'_k$; *the hypothesis* $\lim\limits_{x\to\infty} q(x) = s$, *where* $q(x) = af(x) + (1-a)H_k(x)$, *implies*

$\lim\limits_{x\to\infty} f(x) = s$ *if and only if* $a > a_k''$. $\{a_k'\}$ *and* $\{a_k''\}$ *are the sequences of theorem* 3A.

We also easily obtain

THEOREM 4C. *Let* $W(z)$ *and* $W_1(z)$ *be the polynomials defined in theorem* 1A *and let* $p(x) = \sum_{\nu=0}^{l} \lambda_\nu \frac{x^\nu}{\nu!} f^{(\nu)}(x) + t(x)$, *where* $t(x)$ *satisfies* (18) *with* $\mu_n = W_1(n)/W(n)$. *The hypothesis* $\lim\limits_{x\to\infty} p(x) = s$ *implies* $\lim\limits_{x\to\infty} f(x) = s$ *if and only if*

$$W(z) \sum_{\nu=0}^{l} \lambda_\nu \binom{z}{\nu} + W_1(z) \in \mathfrak{R}.$$

From this the following Tauberian theorem results: *if* $\lim\limits_{x\to\infty} C_k(x) = s$ *and* $\lim\limits_{x\to\infty} xf'(x) = 0$ *then* $\lim\limits_{x\to\infty} f(x) = s$.

References

[1] G. H. Hardy, *Divergent series*, Oxford 1949.

[2] F. Hausdorff, *Summationsmethoden und Momentfolgen I, II*, Math. Zeitschr. 9 (1921), p. 74-109 and 280-299.

[3] J. Karamata, *Sur quelques inversions d'une proposition de Cauchy et leurs généralisations*, Tôhoku Math. J. 36 (1933), p. 22-28.

[4] F. Lösch und F. Schoblik, *Die Fakultät*, Leipzig 1951.

[5] O. Perron, *Über nichthomogene lineare Differentialgleichungen*, Math. Zeitschr. 6 (1920), p. 161-166.

[6] H. R. Pitt, *Mercerian theorems*, Proc. Cambridge Phil. Soc. 34 (1938), p. 510-520.

[7] A. Ostrowski, *Vorlesungen über Differential- und Integralrechnung*, Bd. II, Basel 1951.

[8] W. W. Rogosinski, *On Hausdorff's methods of summability I, II*, Proc. Cambridge Phil. Soc. 38 (1942), p. 166-192 and 344-363.

[9] H. Späth, *Asymptotisches Verhalten der Lösungen linearer Differentialgleichungen*, Math. Zeitschr. 30 (1929), p. 487-513.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 25. 4. 1956

ANNALES
POLONICI MATHEMATICI
V. (1958)

Estimation du domaine d'existence de l'intégrale d'un système en involution d'équations aux dérivées partielles du premier ordre dans le cas de variables complexes

par W. PAWELSKI (Gdańsk)

Comme le titre l'indique, la présente note a pour but de généraliser dans le cas de variables complexes le problème analogue déjà résolu dans le cas de variables réelles [1].

Les démonstrations des théorèmes seront indiquées à grands traits, vu leur analogie avec celles de mes travaux antérieurs [1] et [2].

Considérons le système d'équations

$$(1)\qquad \frac{\partial z}{\partial t_\alpha} + H_\alpha\left(t_\beta, x_i, \frac{\partial z}{\partial x_i}, z\right) = 0, \qquad \alpha, \beta = 1, 2, \ldots, m;\ i = 1, 2, \ldots, n$$

remplissant les conditions de compatibilité

$$(2)\qquad \frac{\partial H_\beta}{\partial t_\alpha} - H_\alpha \frac{\partial H_\beta}{\partial z} - \frac{\partial H_\alpha}{\partial t_\beta} + H_\beta \frac{\partial H_\alpha}{\partial z} + \sum_{i=1}^{\infty} \left\{ \frac{\partial H_\alpha}{\partial q_i}\left(\frac{\partial H_\beta}{\partial x_i} + q_i \frac{\partial H_\beta}{\partial z}\right) - \frac{\partial H_\beta}{\partial q_i}\left(\frac{\partial H_\alpha}{\partial x_i} + q_i \frac{\partial H_\alpha}{\partial z}\right)\right\} \equiv 0.$$

Admettons que les fonctions H_α des variables complexes t_β, x_i, q_i, z soient analytiques dans l'ensemble défini par les conditions

$$(3)\qquad |t_\alpha - t_\alpha^0| \leqslant c, \quad |x_i - x_i^0| \leqslant c, \quad |z - z_0| \leqslant c, \quad |q_i - q_i^0| \leqslant c$$

et que la fonction $\omega(x_1, \ldots, x_n)$ soit analytique dans l'ensemble

$$(4)\qquad |x_i - x_i^0| \leqslant c.$$

Soit M un nombre positif constant tel que

1° les valeurs absolues

$$(5)\qquad |H_\alpha|, \left|\frac{\partial H_\alpha}{\partial x_i}\right|, \left|\frac{\partial H_\alpha}{\partial z}\right|, \left|\frac{\partial H_\alpha}{\partial q_i}\right|, \left|\frac{\partial^2 H_\alpha}{\partial x_i \partial x_j}\right|, \left|\frac{\partial^2 H_\alpha}{\partial z^2}\right|, \left|\frac{\partial^2 H_\alpha}{\partial q_i \partial q_j}\right|, \left|\frac{\partial^2 H_\alpha}{\partial x_i \partial z}\right|, \left|\frac{\partial^2 H_\alpha}{\partial z \partial q_i}\right|, \left|\frac{\partial^2 H_\alpha}{\partial x_i \partial q_j}\right|$$