THEOREM 6. *Let S be a compact semigroup and T the set of those $a \in S$ which have in S roots of every degree $n = 1, 2, 3, \ldots$ Then T is a closed non-empty subset of S such that every $a \in T$ has roots of every degree in T.*

Proof. If k/l, we clearly have $T_k \supset T_l$. Next we prove $f_k(T_l) = T_l$. If $c \in T_l$, then $c^\tau \in T_l$ for every integer $\tau \geqslant 1$, especially $c^k \in T_l$, therefore $f_k(T_l) \subset T_l$. Suppose conversely that $c \in T_l$. Then according to Theorem 5 there is a $d \in T_l$ with $c = d^l = (d^{l/k})^k$. Since $d \in T_l$ implies $d^{l/k} \in T_l$, we have $c \in f_k(T_l)$, whence $T_l \subset f_k(T_l)$. This proves our assertion. Now the sequence $T_{1!} \supset T_{2!} \supset T_{3!} \supset \ldots$ fulfils the suppositions of Lemma 1. Hence $T = \bigcap_{n=1}^{\infty} T_{n!} \neq \emptyset$ and $f_k(T) = T$ for every $k > 0$. This proves Theorem 6.

If S is commutative, T is obviously a semigroup. Let us call a semigroup U *complete* if every element from U has in U roots of every degree $k > 0$. We then have:

THEOREM 7. *In a compact commutative semigroup S the set of elements having roots of every degree $k > 0$ forms a complete closed subsemigroup.*

This theorem is known for compact abelian groups (T is then a group) and need not hold for discrete groups.

We use this opportunity to prove a further theorem on (non-necessarily commutative) complete compact semigroups.

THEOREM 8. *Let S be a complete compact semigroup. Let e be an idempotent from S and $H(e)$ the maximal group belonging to e. Then $H(e)$ is a complete closed group.*

Proof. Let $a \in H(e)$. Then according to the supposition there is an $x = x(n) \in S$ with $x^n = a$ for every $n \geqslant 1$. Let $n \geqslant 1$ be arbitrary, but fixed. We know that the set $\overline{\{x, x^2, x^3, \ldots\}}$ contains a unique idempotent. Since $\overline{\{x, x^2, x^3, \ldots\}} \supset \overline{\{x^n, x^{2n}, x^{3n}, \ldots\}} = \overline{\{a, a^2, a^3, \ldots\}}$ and a belongs to e, this idempotent is e. Hence x belongs to e. Now every element $x \in S$ belonging to e satisfies $xe = ex \in G^{(x)} \subset H(e)$. Since $a = x^n$, we have $ea = ex^n$, whence $a = (ex)^n$, i. e. $a = y^n$ with $y \in H(e)$. We have proved that for every n there is a $y = y(n) \in H(e)$ such that $y^n = a$. This proves that $H(e)$ is a complete group. The fact that $H(e)$ is closed is well known.

REFERENCES

[1] A. D. Wallace, *Inverses in Euclidean mobs*, Mathematical Journal of Okayama University 3 (1953), p. 23-28.

[2] — *The structure of topological semigroups*, Bulletin of the American Mathematical Society 61 (1955), p. 95-112.

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ON FINITE COCYCLES AND THE SPHERE THEOREM

BY

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1. Introduction. Let M be a closed (i. e. compact, unbounded), connected 3-manifold. We describe a 2-sphere in M as *tame* if, and only if, it is the image of $S \times 1/2$ in a homeomorphism of $S \times I$ into M, where S is a 2-sphere. We describe M as *reducible* if, and only if, it contains a tame 2-sphere, S, which is essential in M (i. e. the identical map $S \to M$ is not homotopic to a constant). In this case M may be "reduced" by cutting through S and filling in the holes. If M is reducible, then (cf. § 6 below; also [3]) $\pi_1(M)$ is either cyclic infinite or a free product with two non-trivial factors. In fact we shall prove (cf. [3]):

THEOREM (1.1). *For M to be reducible it is necessary and sufficient that $\pi_1(M)$ be either cyclic infinite or a non-trivial free product.*

We emphasize the fact that M need not be orientable. Specker [9] has proved that $\pi_2(M)$ is a free Abelian group whose rank is $0, 1$ or ∞ according as $\pi_1(M)$ has less than 2, 2 or ∞ ends [2]. If $\pi_1(M)$ is cyclic infinite or a non-trivial free product, it has 2 or ∞ ends. Therefore if M is orientable (1.1) follows from the triangulation theorem [4] and the sphere theorem [6, 10].

In order to prove (1.1) we consider a certain Π-module $J(\Pi, G)$, which is associated with a given group Π and a given Abelian group G (see § 5 below). We write $J(\Pi, Z) = J(\Pi)$, where Z is the group of integers. According to Specker [9] there is an operator isomorphism $J(\pi_1(M)) \approx \pi_2(M)$. Assuming that Π is finitely presentable, we introduce a certain sub-set $\Sigma(\Pi, G) \subset J(\Pi, G)$. In general $\Sigma(\Pi, G)$ is not a sub-group of $J(\Pi, G)$ but it contains the element 0. If $\Pi = 1$ or $G = 0$, then $J(\Pi, G) = 0$. We write $\Sigma(\Pi, Z) = \Sigma(\Pi)$. We shall prove:

THEOREM (1.2). *In order that a finitely presentable group, Π, be either cyclic infinite or a non-trivial free product it is necessary and sufficient that $\Sigma(\Pi, G) \neq 0$ for a given $G \neq 0$.*

COROLLARY (1.3). *If $\Sigma(\Pi, G) \neq 0$, then $\Sigma(\Pi, G') \neq 0$, where G' is any non-zero Abelian group.*

THEOREM (1.4). *In order that a closed, connected 3-manifold M be reducible it is necessary and sufficient that $\Sigma(\pi_1(M)) \neq 0$.*

"Sufficiency" in (1.1) follows from "necessity" in (1.2) and "sufficiency" in (1.4). It will be seen that "necessity" is the easier half of (1.2). The slightly more difficult "sufficiency" in (1.2) seems to be of some algebraic interest. The group $J(\Pi, G)$ is defined in purely algebraic terms but $\Sigma(\Pi, G)$, though it depends only on Π and G, is defined geometrically.

2. The functors H_F^n. Let X be any topological space. If U is an open sub-set of X, then $H^n(U)$ will denote the relative cohomology group $H^n(X, X-U)$. Here the term "cohomology group" refers to some cohomology theory [1] which we need not specify. If $U \subset V \subset X$, where V is also open in X, then the inclusion map $i\colon U \subset V$, or alternatively $j\colon (X, X-V) \subset (X, X-U)$, induces an *injection* $i_* = j^*\colon H^n(U) \to H^n(V)$. We define

$$H_F^n(X) = \operatorname*{Lim}_{\to}\{H^n(U), i_*\} \tag{2.1}$$

for all open $U \subset X$ such that $\overline{U}$, the closure of U, is compact.

A map $f\colon X \to Y$, in a topological space Y, is called *proper* if, and only if, $f^{-1}B$ is compact for every compact $B \subset Y$. A *proper homotopy class* of maps $X \to Y$ is defined in terms of proper maps $X \times I \to Y$. Clearly H_F^n is a contravariant functor on the category of proper homotopy classes between topological spaces.

If X is a locally finite polyhedron then $H_F^n(X)$ may be identified with the cohomology group which is defined in terms of finite n-cochains, in a triangulation of X, with some coefficient group G (this follows from the excision theorem and the uniqueness theorem [1] for finite polyhedra). In this case we shall sometimes write $H_F^n(X) = H_F^n(X, G)$.

3. Π-simple elements. Let X be as before and let Π be a discrete group which operates effectively as a group of homeomorphisms $X \to X$. Then $H_F^n(X)$ is a right Π-module, with $a\tau = \tau^* a$ for any $a \in H_F^n(X)$, $\tau \in \Pi$. A sub-set $A \subset X$ will be called *Π-simple*, if, and only if, $A \cap \tau A = \emptyset$ for every $\tau \neq 1$ in Π. An element of $H_F^n(X)$ will be called *Π-simple* if, and only if, it has a representative in $H^n(U)$ for some open set $U \subset X$ whose closure is Π-simple and compact. We denote the set of Π-simple elements in $H_F^n(X)$ by $\Sigma^n(X)$, or by $\Sigma^n(X, G)$ if $H_F^n(X) = H_F^n(X, G)$ and we wish to indicate the dependence on G. We shall rely on the context to indicate the dependence of $\Sigma^n(X)$ on Π. Clearly $\tau\Sigma^n(X) = \Sigma^n(X)$, where $\tau\colon H_F^n(X) \to H_F^n(X)$ is a right operator in Π. In general $\Sigma^n(X)$ is not a sub-group of $H_F^n(X)$. However, the empty set is Π-simple and $\Sigma^n(X)$ contains the zero element in $H_F^n(X)$.

Let Π be without fixed points (i. e. $\tau x \neq x$ for every $x \in X$, $1 \neq \tau \in \Pi$). Let Π' operate similarly on a space X' and let $f\colon X \to X'$ be a proper map which is equivariant with respect to a monomorphism $\theta\colon \Pi \to \Pi'$ (i. e. $f \circ \tau = \theta\tau \circ f$). Clearly $\tau f^{-1}A' = f^{-1}(\theta\tau)A'$, where $A' \subset X'$. Since θ is a monomorphism it follows that, if A' is Π'-simple, then $f^{-1}A'$ is Π-simple. Hence we have:

THEOREM (3.1). *If $f\colon X \to X'$ is a proper map which is equivariant with respect to some monomorphism $\Pi \to \Pi'$, then $f^*\Sigma^n(X') \subset \Sigma^n(X)$.*

For example, let X be a closed sub-set of X' such that $\tau' X = X$ for every $\tau' \in \Pi'$ and let $\Pi = \Pi'$. Then (3.1) applies to the inclusion map $f\colon X \subset X'$.

As another example, let Y, Y' be locally compact spaces which are locally and globally 0-connected and are locally 1-connected in the weak sense (i. e. each point in $Z = Y$ or Y' has a connected neighbourhood, V, such that the injection $\pi_1(V) \to \pi_1(Z)$ maps $\pi_1(V)$ to 1). Let X, X' be universal covers of Y, Y' and let $\Pi = \pi_1(Y)$, $\Pi' = \pi_1(Y')$, operating on X, X' in the usual way, with due regard to base-points. Let $g\colon Y \to Y'$ be a proper map which induces a monomorphism $\Pi \to \Pi'$ and let $f\colon X \to X'$ be the map determined by g. Then f satisfies the conditions of (3.1).

4. The case of a simplicial covering. Let X be a (connected) regular cover of a connected, locally finite, simplicial complex Y. Let $p\colon X \to Y$ be the projection and Π the group of covering transformations. That is to say, Π consists of all homeomorphisms $\tau\colon X \to X$ such that $p\tau = p$. The space X may be given the structure of a simplicial complex such that p, likewise each $\tau\colon X \to X$, is a simplicial map. When we refer to a sub-division, X', of X (e. g. the barycentric sub-division), we shall mean one with respect to which every $\tau \in \Pi$ is simplicial. Thus X' determines and is determined by a sub-division, Y', of Y such that p is simplicial with respect to X', Y'. By a polyhedral sub-set of X, or Y, we mean the sub-set covered by a sub-complex of some sub-division of X, or Y.

Let $p_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$, where $y_0 = px_0$, be the homomorphism induced by p. Then $p_*\pi_1(X)$ $(\pi_1(X) = \pi_1(X, x_0))$ is an invariant sub-group of $\pi_1(Y) = \pi_1(Y, y_0)$, because the covering $p\colon X \to Y$ is regular, and Π may be identified with $\pi_1(Y)/p_*\pi_1(X)$. We describe a map $g\colon S^1 \to Y$ as *inessential mod p* if, and only if, it can be lifted to a map $f\colon S^1 \to X$ such that $g = pf$. Let $a \in \pi_1(Y)$ be the element obtained by orientating S^1 and joining gs_0 to y_0 by a path in Y, where $s_0 \in S^1$. Then g is inessential mod p if, and only if, $a \in p_*\pi_1(X)$. We describe a sub-set $B \subset Y$ as *1-connected in Y, mod p*, if, and only if, every map $S^1 \to Y$, whose image is in B, is inessential mod p (in Y). If $\pi_1(X) = 1$ we may omit the qualification "mod p".

Let B be a locally 0-connected sub-set of Y which is 1-connected in Y, $\bmod p$. Then the inclusion map $i: B \subset Y$ can be lifted to a map $f: B \to X$ such that $pf = i$. Clearly fB is Π-simple.

Let A be a Π-simple subset of X. Then $p|A$ is a 1-1 map onto pA. If A is either compact or open in X, then $p|A$ is a homeomorphism. Therefore it has an inverse, $pA \to A$, and pA is obviously 1-connected in Y, $\bmod p$.

Let U be an open subset of X whose closure is compact and polyhedral. Then, by the excision theorem [1], $H^n(U)$ may be identified with $H^n(\overline{U}, \dot{U})$, where $\dot{U} = \overline{U} - U$. Every compact, Π-simple subset of X is contained in a compact, Π-simple, polyhedral subset. Therefore every element of $\Sigma^n(X)$ has a representative in $H^n(U)$ for some open $U \subset X$ whose (compact) closure is polyhedral and Π-simple.

Let $U \subset X$ be an open set such that $\overline{U}$ is polyhedral and compact. Let N be a closed simplicial neighbourhood of $\dot{U}$ in $\overline{U}$ such that $\dot{U}$ is a deformation retract of N. Let $U_0 = \overline{U} - N$. Then $\overline{U}_0 \subset U$ and $i_*: H^n(U_0) \approx H^n(U)$, where $i: U_0 \subset U$. Hence it follows that, if U is Π-simple, then every element of $H^n_F(X)$ with a representative in $H^n(U)$ is Π-simple.

Since $H^n_F(X)$ may be calculated in terms of finite cochains in X we have $i^*: H^n_F(X) \approx H^n_F(X^q)$ if $q > n$, where X^q denotes the q-section of X and $i: X^q \subset X$. I say that

$$i^*: \Sigma^1(X) = \Sigma^1(X^2). \tag{4.1}$$

Proof. It follows from (3.1) that $i^*\Sigma^1(X) \subset \Sigma^1(X^2)$. We have to prove that, if $b \in \Sigma^1(X^2)$, then $(i^*)^{-1} b \in \Sigma^1(X)$. Let $\beta \in H^1(U)$ be a representative of b, where U is a relatively open subset of X^2, whose (compact) closure is Π-simple and polyhedral. Let $V = pU \subset Y^2$ and let $p_0: (\overline{U}, \dot{U}) \to (\overline{V}, \dot{V})$ be the homeomorphism determined by p.

Let $\gamma = (p_0^*)^{-1}\beta \in H^1(V)$, where $p_0^*: H^1(V) \approx H^1(U)$ is the isomorphism induced by p_0. Then (4.1) will obviously follow when we have proved:

LEMMA (4.2). *There is an open set $V' \subset Y$, such that $\overline{V}'$ is a compact polyhedron and*

(a) $V' \cap Y^2 \subset V$,

(b) $\gamma \in j^* H^1(V')$, *where* $j: (Y^2, Y^2 - V) \subset (Y, Y - V')$,

(c) V' *is* 1-*connected in* Y, $\bmod p$.

Proof of (4.2). Let $N(V) \subset Y$ be the open set consisting of all the open simplexes of Y whose closures meet V. Since Y is locally finite and $\overline{V}$ is compact there are but a finite number of open simplexes in $N(V)$. Assume that, for some $q \geqslant 2$, there is a relatively open set, $V_q \subset Y^q \cap N(V)$, which satisfies the conditions imposed on V' in (4.2) when Y is replaced by Y^q. If $q = 2$ these conditions are satisfied by $V_q = V$ and if $q = \dim N(V)$, then they are satisfied for Y by $V' = V_q$. Let Y_q be a sub-division of Y such that $\overline{V}_q$ is a sub-complex of Y_q and let $\gamma = j^*\gamma_q$, where $\gamma_q \in H^1(V_q)$. Then $H^1(V_q)$ may be calculated combinatorially and γ_q is represented by a cocycle,

$$z_q \in Z^1(V_q, G) \subset Z^1(Y_q, G).$$

Our aim is to construct a V_{q+1}, s tisfying the analogous conditions, such that z_q can be extended to a cocycle $z_{q+1} \in Z^1(V_{q+1}, G)$. Some care is needed to ensure that V_{q+1} is 1-connected in Y, $\bmod p$.

Let σ be a (closed) $(q+1)$-simplex in Y and let S be the subcomplex of Y_q which covers $\dot{\sigma}$. Let $y \in Z^1(S, G)$ be the restriction of z_q to S. If $y \neq 0$ let S^* be the cell-complex, composed of blocks of simplexes of the barycentric subdivision of S, which is dual to S. Let $y^* \in Z_{q-1}(S^*, G)$ be the cycle dual to y and let P^* be the support of y^* (i. e. the union of the closed $(q-1)$-cells of S^* in which y^* has non-zero coefficients). Let $P_1^*, \ldots, P_r^*$ be the components of P^*. Then $y^* = y_1^* + \ldots + y_r^*$, where y_i^* is a $(q-1)$-cycle in P_i^*. Also $P^* \subset V_q$.

Let G^* be the character group of G and let T be any compact polyhedron in σ. Then $H_{q-1}(\sigma - T, G)$ and $H_1(T, T \cap \dot{\sigma}; G^*)$ are orthogonally paired, by linking coefficients, to the real numbers, mod 1. Hence it follows by induction on r that there are polyhedra $Q_r^*, \ldots, Q_r^* \subset \sigma$ such that

(a) $Q_i^* \cap \dot{\sigma} = P_i^*$ *and* $Q_i^* \cap Q_j^* = \emptyset$ *if* $i \neq j$,

(b) Q_i^* *carries a q-G-chain* z_i^* *such that* $\partial z_i^* = y_i^*$.

Moreover we may assume that $z_i^* \in C_q(E_\sigma^*, G)$, where E_σ^* is the cell-complex dual (1) to a (simplicial) sub-division, E_σ, of σ, which coincides with S in $\dot{\sigma}$, and that Q_i^* is the support of z_i^*.

Let W_i be the union of the open simplexes of E_σ which meet Q_i^*, and let $W_\sigma = W_1 \cup \ldots \cup W_r$. Let

$$z_i \in C^1(W_i, G) \subset C^1(E_\sigma, G)$$

be the cochain dual to z_i^* and let $z_\sigma = z_1 + \ldots + z_r$. Since $\partial z_i^* = y_i^*$ and $Q_i^* \cap S^* = P_i^*$ it follows that z_i is a cocycle which coincides with y_i in S. Therefore z_σ is an extension of y to a cocycle in W_σ. Since $\dot{\sigma}$ meets V_q, because $y \neq 0$, and since $V_q \subset N(V)$ it follows that $\sigma - \dot{\sigma} \subset N(V)$, whence $W_\sigma \subset N(V)$. Since $P^* \subset V_q$ it follows that $W_\sigma \cap \dot{\sigma} \subset V_q$.

(1) Let E'_σ, S' denote the barycentric sub-divisions of E_σ, S and let τ be a k-simplex of S. Then E_σ^* contains the $(q-k+1)$-cell, τ_1^*, which is the dual of τ in E'_σ and the $(q-k)$-cell, $\tau_2^* \subset \tau_1^*$, which is dual to τ in S'.

We have assumed that $y \neq 0$. If $y = 0$ let $W_\sigma = \emptyset$, $z_\sigma = 0$. Thus W_σ, z_σ are defined for every $(q+1)$-simplex, σ, of Y. Let $V_{q+1} = \bigcup_\sigma W_\sigma$. Since there are only a finite of open simplexes of L in $N(V)$ and since $W_\sigma = \emptyset$ if $\sigma - \dot{\sigma} \notin N(V)$ it follows that $\overline{V}_{q+1}$ is compact (and polyhedral). If σ, τ are different $(q+1)$-simplexes of Y then z_σ, z_τ both coincide with z_q in $\sigma \cap \tau$. Therefore a cocycle $z_{q+1} \in Z^1(V_{q+1}, G)$ is defined by the condition $z_{q+1}|Y_q = z_q$, $z_{q+1}|\sigma = z_\sigma$, for every $\sigma = \sigma^{q+1}$ in Y. Since $W_\sigma \cap \dot{\sigma} \subset V_q$ it follows that $V_{q+1} \cap Y^2 \subset V_q \cap Y^2 \subset V$, in accordance with (4.2a). Also z_{q+1} extends z_q and it follows that (4.2b) is satisfied.

Clearly $W_i \cap \dot{\sigma}$ is an open simplicial neighbourhood of P_i^* in $\dot{\sigma}$ and is therefore 0-connected. Since $Q_j^* \cap Q_i^* = \emptyset$ if $i \neq j$ it follows that $W_i \cap W_j = \emptyset$ if $i \neq j$. Therefore, if points $a, b \in W_\sigma \cap \dot{\sigma}$ are joined by a path in W_σ, then a, b are in the same set $W_i \cap \dot{\sigma}$. Therefore the path is homotopic, with its end points held fixed, to a path in $W_\sigma \cap \dot{\sigma} \subset V_q$. Hence it follows that a given map $S^1 \to V_{q+1}$, which we may assume to be piecewise linear, is homotopic in Y^{q+1} to a map in V_q. Since V_q is 1-connected in Y, mod p, so is V_{q+1}. Therefore (4.2c) is satisfied and (4.2) follows by induction on q. This completes the proof of (4.1).

THEOREM (4.3). *If* $\dim X < 2n$, *then* $\Sigma^n(X) = H_F^n(X)$.

The proof is similar to that of (4.1) but much simpler since it only involves placing the dual chains in general position. The details are left to the reader.

In general, if $b \in \Sigma^n(X)$ then the cup-product $b \smile b\tau$, defined in terms of any pairing $G \times G \to G'$, is obviously zero, provided $\tau \neq 1$. On considering a double covering of a Klein bottle by a torus, with $n = 1$, one sees that this is not always the case.

Let $H_F^n(X) = H_F^n(X, G)$ and let $\gamma: G \to G'$ be a homomorphism in an Abelina group G'. Then, obviously,

$$\gamma_* \Sigma^n(X, G) \subset \Sigma^n(X, G'), \tag{4.4}$$

where $\gamma_*: H_F^n(X, G) \to H_F^n(X, G')$ is the homomorphism induced by γ.

5. The set $\Sigma(\Pi, G)$. Let Π is a given group and G a given (additive) Abelian group. Then G^Π is a left Π-module, in which addition is defined by addition of values and $\tau: G^\Pi \to G^\Pi$ is defined by $(\tau f)(\xi) = f(\xi\tau)$ for all $f: \Pi \to G$, $\tau, \xi \in \Pi$. Let $G(\Pi) \subset G^\Pi$ be the sub-module consisting of all finitely non-zero functions. That is to say, $f \in G(\Pi)$ if, and only if, $f(\xi) = 0$ for all but a finite number of elements $\xi \in \Pi$ (thus $G(\Pi)$ is the group-ring over G if G is a ring). Let $F(\Pi, G)$ be the sum (direct if Π is an infinite group) of $G(\Pi)$ and the sub-module of constant functions. Thus $f \in F(\Pi, G)$ if, and only if, $f(\xi) = g_f$ for all but a finite number of elements ξ and some $g_f \in G$. Let

$$A(\Pi, G) = \{f \in G^\Pi | \tau f - f \in G(\Pi) \quad \text{for each} \quad \tau \in \Pi\}. \tag{5.1}$$

Clearly $F(\Pi, G)$, $A(\Pi, G)$ are sub-modules of G^Π and $F(\Pi, G) \subset A(\Pi, G)$. We define (cf. [9])

$$J(\Pi, G) = A(\Pi, G)/F(\Pi, G). \tag{5.2}$$

The groups G^Π, $F(\Pi, G)$, $A(\Pi, G)$ are also right Π-modules, with $f\tau: \Pi \to G$ defined by $(f\tau)(\xi) = f(\tau\xi)$. Therefore $J(\Pi, G)$ is a right Π-module with $x\tau$ defined in the obvious way for all $x \in J(\Pi, G)$, $\tau \in \Pi$.

Let Π' be any group and $h: \Pi \to \Pi'$ a homomorphism whose kernel is finite. Then $h^{-1}\Lambda'$ is a finite set if Λ' is a finite sub-set of Π'. Hence it follows that h induces, by composition, a homomorphism

$$J(h): J(\Pi', G) \to J(\Pi, G). \tag{5.3}$$

Similarly a homomorphism $G \to G'$ induces a homomorphism $J(\Pi, G) \to J(\Pi, G')$. Let $\mathfrak{F}$ be a category of groups and homomorphisms with finite kernels and let $\mathfrak{G}$ be the category of all homomorphisms between Abelian groups. Then J is clearly a functor $J: \mathfrak{F} \times \mathfrak{G} \to \mathfrak{G}$, which is contravariant in $\mathfrak{F}$, covariant in $\mathfrak{G}$.

Now let Π be finitely presentable and let Y be a finite, connected, 2-dimensional simplicial complex, with a base point $y_0 \in Y^0$, such that $\Pi \approx \pi_1(Y, y_0)$. Let X be the universal cover of Y, whose points are homotopy classes, rel. end points, of paths $(I, 0) \to (Y, y_0)$. Let the class of the constant path, $I \to y_0$, be taken as the base-point $x_0 \in X$. Then [9] an isomorphism $\Phi: \Pi \approx \pi_1(Y, y_0)$ determines an equivariant isomorphism

$$\Phi_*: J(\Pi, G) \approx H_F^1(X, G). \tag{5.4}$$

Let

$$\Sigma(\Pi, G) = \Phi_*^{-1}\Sigma^1(X, G). \tag{5.5}$$

Let (Y', y_0'), $\Phi': \Pi' \approx \pi_1(Y', y_0')$, (X', x_0') have similar meanings to (Y, y_0), Φ, (X, x_0). Then there is a map $g: (Y, y_0) \to (Y', y_0')$ which induces

$$\Phi'\Phi^{-1}: \pi_1(Y, y_0) \approx \pi_1(Y', y_0'),$$

and any two such maps are homotopic on Y^1. Let $f: (X, x_0) \to (X', x_0')$ be obtained by lifting g. Then we have

$$J(\Pi, G) \xrightarrow{\Phi'_*} H_F^1(X'; G) \xrightarrow{f_*} H_F^1(X; G)$$

and (5.4) is natural in the sense that $\Phi_* = f^* \Phi'_*$. If $f' : (X', x'_0) \to (X, x_0)$ is similarly defined in terms of $\Phi\Phi'^{-1}$, then $f'^* = (f^*)^{-1}$ and it follows from (3.1) that

$$f^* \Sigma^1(X', G) = \Sigma^1(X, G).$$

Hence it follows that $\Sigma(\Pi, G)$ defined by (5.5) does not depend on the choice of Y, Φ. Similarly the homomorphisms $J(\Pi', G) \to J(\Pi, G)$, $J(\Pi, G) \to J(\Pi, G')$ induced by a monomorphism $\Pi \to \Pi'$ and a homomorphism $G \to G'$ carry $\Sigma(\Pi', G)$ into $\Sigma(\Pi, G)$ and $\Sigma(\Pi, G)$ into $\Sigma(\Pi, G')$.

The group $J(\Pi, G)$ is isomorphic to the reduced Čech 0-G-cohomology group of the (compact) space of ends of Π. If $G = Z$, the group of integers, then $J(\Pi, G)$ is free Abelian [9]. We write $J(\Pi, Z) = J(\Pi)$, $\Sigma(\Pi, Z) = \Sigma(\Pi)$. In general $J(\Pi)$ is not a free Π-module.

6. Proof of (1.2). Let (X, x_0), (Y, y_0) be as in the paragraph containing (5.4). Let Π be either cyclic infinite or a non-trivial free product. In the first case we take Y to be a circle and in the second we take $Y = Y_1 \cup Y_2$, where $Y_1 \cap Y_2 = y_0$ and $\pi_1(Y_i) \neq 1$. In either case $X = X_1 \cup X_2$, where X_1, X_2 are infinite simplicial complexes and $X_1 \cap X_2 = x_0$. Let U'_1 be the (relatively) open simplicial neighbourhood of x_0 in X_1 and let $U_1 = U'_1 - x_0$. Then $\overline{U}_1$ is compact and Π-simple. Let $0 \neq g \epsilon G$ and let $c \epsilon C^0(X, G)$ be the infinite cochain defined by $c(v) = 0$ or g according as the vertex v is in $X - X_2 = X_1 - x_0$ or in X_2. Then δc is a (finite) cocycle in $Z^1(U_1, G)$. If $\delta c = \delta c'$, where c' is a finite 0-cochain, then $\delta(c - c') = 0$. Since X is connected this implies $c(v) = c'(v) + g_0$ for every vertex v of X and some $g_0 \epsilon G$. But this is absurd since it implies $g_0 = 0$ and $g_0 = g \neq 0$. Therefore the cohomology class of δc in $H^1_F(X, G)$ is a non-zero element in $\Sigma^1(X, G)$.

Conversely, let $0 \neq b \epsilon \Sigma^1(X, G)$ and let $\beta \epsilon H^1(U, G)$ be a representative of b, where $U \subset X$ is open and $\overline{U}$ is compact, polyhedral and Π-simple. Let $U_1, \ldots, U_n$ be the components of U. Then $\beta = \beta_1 + \ldots + \beta_n$, where $\beta_i \epsilon H^1(U_i, G)$, and $b = b_1 + \ldots + b_n$, where $b_i \epsilon H^1_F(X, G)$ is the element represented by β_i. Clearly $b_i \epsilon \Sigma^1(X, G)$ and $b_i \neq 0$ for at least one value of i. We may therefore assume to begin with that U, and hence $\dot{U}$ are connected.

Let $V = pU$, where $p : X \to Y$ is the projection. Let P be a component of $Y - V$ and suppose that $P \cup \overline{V}$ is 1-connected in Y. Let $V_1 = P \cup V$. Then V_1 is open and $\overline{V}_1 = P \cup \overline{V}$. Therefore the homeomorphism $\overline{V} \to \overline{U}$ inverse to $p|\overline{U}$ may be extended to a homeomorphism, f, of $\overline{V}_1$ into a compact, Π-simple subset of X. Moreover $fV_1 = U_1$, say, is open in X and $f\overline{V}_1 = \overline{U}_1$. The element $b \epsilon \Sigma^1(X, G)$ is equally well represented by $i_*\beta$, where $i_* : H^1(U) \to H^1(U_1)$ is the injection. Therefore U, V may be replaced by U_1, V_1. Since Y is a finite polyhedron and $\overline{V}$ is polyhedral the set $Y - V$ has but a finite number, k, of components. The set $Y - V_1$ has $k-1$ components. Therefore the theorem will follow by induction on k when we have proved it on the assumption that $P \cup \overline{V}$ is not 1-connected in Y for any component, P, of $Y - V$. So we assume this to be the case.

Suppose that $\dot{U}$ and hence $X - U = A$, say, are connected. Then $\overline{H}^0(A, G) = 0$, where $\overline{H}^0$ indicates a reduced cohomology group, defined in terms of infinite cochains. Also $H^1(X, G) = 0$ since X is 1-connected. Therefore it follows from the exactness of the cohomology sequence of (X, A) that $H^1(U, G) = 0$, contrary to hypothesis. Therefore $\dot{U}$, likewise $\dot{V}$, are not connected.

Assume that no two components of $\dot{V}$ are in the same component of $Y - V$. Since $\dot{V}$ is not connected it follows that $Y - V$ has at least two components. Let P_1 be one of them and P_2 the union of all the others. Let $Q_\lambda = P_\lambda \cup \overline{V}$ $(\lambda = 1, 2,)$. Then Q_λ is not 1-connected in Y. Let $\iota_\lambda : \pi_1(\overline{V}) \to \pi_1(Q_\lambda)$ be the injection. Then [7, § 52] there is a presentation for $\pi_1(Y)$ which consists of a presentation for $\pi_1(Q_1)$ together with a presentation for $\pi_1(Q_2)$ and the additional relators $(\iota_1 a)(\iota_2 a)^{-1}$, for every $a \epsilon \pi_1(\overline{V})$. Since $\overline{V}$ is 1-connected in Y these additional relators may be replaced by $\iota_1 a$, $\iota_2 a$. Therefore

$$\pi_1(Y) \approx \big(\pi_1(Q_1)/\Gamma_1\big) * \big(\pi_1(Q_2)/\Gamma_2\big),$$

where $\Pi * \Pi'$ indicates the free product of given groups Π, Π' and Γ_λ is the smallest invariant sub-group of $\pi_1(Q_\lambda)$ which contains $\iota_\lambda \pi_1(\overline{V})$. Moreover this isomorphism is such that the free factor $\pi_1(Q_\lambda)/\Gamma_\lambda$ corresponds to the image of $\pi_1(Q_\lambda)$ in the injection $\pi_1(Q_\lambda) \to \pi_1(Y)$. This image is not 1, since Q_λ is not 1-connected in Y. Therefore $\pi_1(Y)$ is a non-trivial free product.

Finally let two distinct components of $\dot{V}$ be in the same component, P, of $Y - V$. Let Q be the union of $\overline{V}$ and all the other components of $Y - V$. Since Y, $\overline{V}$ are connected, so is Q and $Y = P \cup Q$, $P \cap Q$ is not connected. Hence it follows from a variant of the theorem in [7], to which we have just referred, that $\pi_1(Y)$ is a free product of the form $\Gamma * Z$. Therefore it is either cyclic infinite or a non-trivial free product, according as $\Gamma = 1$, or $\Gamma \neq 1$. This completes the proof.

7. Proof of (1.4). We observed at the beginning of § 1 that, if M is reducible, then $\pi_1(M)$ is either cyclic infinite or a non-trivial free product. Therefore "necessity" in (1.4) follows from (1.2).

Conversely, let $\Sigma\big(\pi_1(M)\big) \neq 0$, let Y be a triangulation [4] of M and let X in § 4 be a universal cover of Y. We take X^2, Y^2 to be the

X, Y of § 5. Then $\Sigma^1(X^2) \neq 0$ and it follows from (4.1) that $\Sigma^1(X) \neq 0$. Let U and $\beta \epsilon H^1(U) = H^1(U, Z)$ be as in § 6, β being a representative of some $b \neq 0$ in $\Sigma^1(X)$. Let $z \epsilon Z^1(U) \subset Z^1(X)$ be a (finite) cocycle representing β, let X^* be the cell complex dual to X and let $z^* \epsilon Z_2(X^*)$ be the cycle dual to z. Then z^* does not bound a (finite) chain in $C_3(X^*)$ because $b \neq 0$. Moreover [5, p. 296] z^*, treated as a singular cycle, is homologous to a 2-cycle carried by a non-singular, compact, orientable, polyhedral 2-manifold, P, in U. Since z^* is non-bounding it follows that some component, A, of P is non-bounding in X. Then A is a compact connected, orientable, non-bounding, Π-simple 2-manifold in X.

Let $B = pA$, where $p: X \to M$ is the projection. Then B is 1-connected in M and is therefore 2-sided. If B is a 2-sphere it is tame, being polyhedral, and essential in M since A is non-bounding in X. Therefore M is reducible in this case. So we assume that B is not a 2-sphere. Let $\dot{I}^2 \to B$ be an essential, piecewise linear map. Since B is 1-connected in M this can be extended to a piecewise linear map $f: I^2 \to M$. Since B is 2-sided we may assume that there is a neighbourhood, $N \subset I^2$, of $\dot{I}^2$ such that $f(N - \dot{I}^2)$ lies entirely on one side of B. We may further assume that $f^{-1}B$ is a set of non-singular closed curves in I^2. Let C be one which contains no other in its interior and let $D \subset I^2$ be the disc bounded by C. If the circuit $f|C: C \to B$ is essential in B we replace f by $f|D$. Otherwise $f|C$ can be extended to a map $g: D \to B$ and we replace f by $f_1: I^2 \to M$, where $f_1 s = fs$ or gs according as $s \epsilon I^2 - D$ or $s \epsilon D$. We then deform f_1 slightly so as to free $f_1 D$ from B and thus reduce number of components of $f^{-1}B$. It follows from induction on the latter that there is a map $k: I^2 \to M$ such that $k|\dot{I}^2$ is an essential circuit in B and $k^{-1}B = \dot{I}^2$ (cf. [3]).

Let us cut M along B, thus converting it into a bounded manifold M_1. Then $\dot{M}_1 = B_1 \cup B_2$, where B_1, B_2 are two copies of B, and k determines a map $k_1: I^2 \to M_1$, such that $k_1|I$ is an essential circuit in one of B_1, B_2, say in B_1. There is, therefore [5], a non-singular (polyhedral) circuit $C_1 \subset B_1$, which is essential in B_1 and bounds a singular 2-cell $E_1 \subset M_1$. We may obviously assume that $E_1 - C_1 \subset M_1 - B_1$ and it follows from Dehn's lemma, proved recently by C. D. Papakyriakopoulos [6] (see also [8]), that there is a non-singular 2-cell which satisfies these conditons. Therefore B contains an essential, non-singular circuit C which bounds a non-singular 2-cell $E \subset M$ such that $E \cap B = C$. The homeomorphism $B \to A$, inverse to $p|A$, can obviously be extended to a homeomorphism $g: B \cup E \to X$ such that $pgy = y$ for every $y \epsilon B \cup E$. Clearly $A \cup gE$ is Π-simple. We now cut through gE and separate the two sides of the cut. We thus transform A into a connected, Π-simple manifold A', if $\dot{E}$ is non-bounding on B, or into two disjoint, Π-simple manifolds A_1, A_2, if $\dot{E}$ bounds on B. Let h be the genus of A. Then A' is of genus $h-1$ in the first case. In the second case, if h_λ is the genus of A_λ, then $h_\lambda > 0$, since $\dot{E}$ is essential in B, and $h_1 + h_2 = h$. In the first case A, and in the second case one, at least, of A_1, A_2 is non-bounding in X. Therefore the theorem follows by induction on h.

REFERENCES

[1] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton 1952.

[2] H. Hopf, *Enden offener Räume und unendliche diskontinuierliche Gruppen*, Commentarii Mathematici Helvetici 16 (1943), p. 81-100.

[3] H. Kneser, *Geschlossene Flächen in dreidimensionalen Mannigfaltigkeiten*, Sitzungsberichte der Deutschen Mathematiker-Vereinigung 38 (1929), p. 248-260.

[4] E. E. Moise, *The triangulation theorem and Hauptvermutung*, Annals of Mathematics 56 (1952), p. 96-114.

[5] C. D. Papakyriakopoulos, *On solid tori*, Proceedings of the London Mathematical Society (3), 7 (1957), p. 281-299.

[6] — *On Dehn's lemma and the asphericity of knots*, Annals of Mathematics 66 (1957), p. 1-26.

[7] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig 1934.

[8] A. Shapiro and J. H. C. Whitehead, *A proof and extension of Dehn's lemma*, Bulletin of the American Mathematical Society 64 (1958), p. 174-178.

[9] E. Specker, *Die erste Cohomologiegruppe von Überlagerungen und Homotopieeigenschaften dreidimensionaler Mannigfaltigkeiten*, Commentarii Mathematici Helvetici 22 (1949), p. 303-333.

[10] J. H. C. Whitehead, *On 2-spheres in 3-manifolds*, Bulletin of the American Mathematical Society 64 (1958), p. 161-166.

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