On a stochastic model of a cascade

by

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1. The homogeneous regular Markov process with a denumerable multitude of states is defined as a stochastic process $\langle \Omega, \mathfrak{B}_\Omega, Pr \rangle$ satisfying the following conditions:

$1°$ the space Ω of the process consists of integral-valued step functions ω (*i. e.*, functions having only a finite set of points of discontinuity in every finite interval) taking on non-negative values, defined for $t \geqslant 0$, continuous on the right and assuming the same value at $t = 0$: $\omega(0) = n_0$;

$2°$ $\mathfrak{B}_\Omega$ is the smallest σ-field spanned upon the sets of form

$$(1) \qquad A(t, n) = \{\omega : \omega(t) = n\};$$

$3°$ there exists a family of functions $P_n^k(t)$ ($n, k = 0, 1, \ldots; t \geqslant 0$) called the *transition probabilities from the state n to the state k in the time interval t* satisfying the conditions:

$$(2) \qquad \sum_{k=0}^{\infty} P_n^k(t) = 1, \qquad P_n^k(t) \geqslant 0,$$

$$(3) \qquad P_n^k(t_1 + t_2) = \sum_{s=0}^{\infty} P_s^k(t_1) P_n^s(t_2),$$

$$(4) \qquad \lim_{t \to 0+} P_n^k(t) = P_n^k(0) = \delta_n^k,$$

(5) if $t_1 < t_2 < \ldots < t_r$, then $Pr\left(\bigcap_{i=1}^{r} \{\omega : \omega(t_i) = n_i\}\right) = \prod_{i=1}^{r} P_{n_{i-1}}^{n_i}(t_i - t_{i-1})$,

where $t_0 = 0$ (see, for instance, [3], p. 236).

It follows from theorems of Doob and Kolmogorov ([2], p. 468, [7], p. 53) that for the transition probabilities in the homogeneous regular Markov processes there exist limits

$$(6) \qquad a_n^k = \lim_{t \to 0+} \frac{P_n^k(t)}{t} \quad \text{for} \quad k \neq n,$$

$$(7) \qquad a_n^n = \lim_{t \to 0+} \frac{1 - P_n^n(t)}{t},$$

called the *intensities of the process*. The intensities of the process satisfy the following conditions:

$$(8) \qquad a_n^k \geqslant 0 \qquad (k = 0, 1, \ldots),$$

$$(9) \qquad a_n^n = \sum_{k \neq n} a_n^k.$$

When the intensities of the process are known, the transition probabilities of the process may be determined from the Kolmogorov equations (compare [4], p. 64)

$$(10) \qquad \frac{d}{dt} P_n^k(t) = -a_k^k P_n^k(t) + \sum_{s \neq k} a_s^k P_n^s(t)$$

with the initial values

$$(11) \qquad P_n^k(0) = \delta_n^k.$$

Thus it follows from condition (5) that the intensities determine the probability Pr in the process satisfying conditions 1° and 2°. It follows that if the intensities of two homogeneous regular Markov processes are the same and the realizations of both the processes assume the same value at zero, then the processes are identical.

The function $P(A/B)$ $(A, B \, \epsilon \, \mathcal{B}_\Omega)$ is called the *conditional probability* if it satisfies the equation

$$(12) \qquad Pr(A \cap B) = P(A/B) Pr(B),$$

and if, for fixed B, it is a normed σ-measure.

2. Let us consider a physical cascade composed of particles of the same kind in a homogeneous medium (e. g., the cascade of nucleons in cosmic radiation) in which the decomposition of the particles is a random event. For physical reasons we assume that

(i) the future of a particle does not depend on its past and depends only on its actual state,

(ii) the destiny of a particle and its progeny does not depend on the future of the actually existing particles.

It is customary to treat mathematically a cascade arising from a single particle as a homogeneous regular Markov process with a denumerable multitude of states. In this process the value of the function ω for the argument t is equal to the number of the particles in the cascade at the moment t; hence for all the realizations we have

$$(13) \qquad \omega(0) = 1.$$

One assumes as an intuitive consequence of (i) and (ii) that the transition probabilities $P_n^k(t)$ satisfy the supplementary condition

$$(14) \qquad \begin{aligned} & P_0^k(t) = \delta_0^k \qquad (k = 0, 1, \ldots), \\ & P_n^k(t) = \sum \prod_{i=1}^n P_1^{k_i}(t) \qquad (k = 0, 1, \ldots; n = 1, 2, \ldots), \end{aligned}$$

the summation being extended over all the systems of non-negative integers $k_1 + k_2 + \ldots + k_n = k$ (see, for instance, [1], p. 7; [5], p. 409). In terms of intensities condition (14) becomes

$$(15) \qquad a_n^k = n a_1^{k-n+1} \qquad (n, k = 0, 1, \ldots).$$

Then the Kolmogorov equations (10) change into

$$(16) \qquad \frac{d}{dt} P_n^k(t) = -k a_1^1 P_n^k(t) + \sum_{s=1}^{k-1} s a_1^{k-s+1} P_n^s(t) + (k+1) a_1^0 P_n^{k+1}(t).$$

In this model of the cascade we cannot, of course, deal with individual particles and their future, whence conditions (i) and (ii) have no direct interpretation. Thus in the precise treatment of the cascade processes conditions (14) or (15) are considered as axioms.

Professor E. Marczewski has proposed a construction of a mathematical model of a cascade in which it would be possible to deal with the individual particles and their future, and which would be based upon the conditions (i) and (ii). We shall now be concerned with the investigation of such a model.

3. Let $\varXi$ be the set of all the systems

$$\xi = \langle x_1, x_2, \ldots, x_n \rangle \qquad (n = 1, 2, \ldots),$$

where $x_1 = 1$, $x_k = \langle i, j \rangle$ $(k = 2, 3, \ldots; i = 1, 2, \ldots, j; j = 2, 3, \ldots)$.

Let V be the space of all the functions v defined on $\varXi$, whose values are intervals of the form $[a, b)$ lying on the positive semi-axes (i. e., $0 \leqslant a \leqslant b \leqslant \infty$) and satisfying the conditions

(a) $v(\langle 1 \rangle) = [0, a) \qquad (0 < a \leqslant \infty);$

(b) if $v(\langle x_1, \ldots, x_n \rangle) = [a, b)$, then $v(\langle x_1, \ldots, x_n, x_{n+} \rangle) = [b, c);$

(c) if $v(\langle x_1, \ldots, x_n, \langle i, j \rangle \rangle) \neq 0$ [1]), then $v(\langle x_1, \ldots, x_n, \langle k, j \rangle \rangle) \neq 0$ for $k = 1, 2, \ldots, j;$

$v(\langle x_1, \ldots, x_n, \langle s, m \rangle \rangle) = 0$ for $s = 1, 2, \ldots, m; m \neq j;$

(d) if $v(\langle x_1, \ldots, x_n \rangle) = 0$, then $v(\langle x_1, \ldots, x_n, x_{n+1} \rangle) = 0;$

(e) if $v(\langle x_1, \ldots, x_n \rangle) \neq 0$ for $n = 1, 2, \ldots$, then

$$\bigcup_{n=1}^\infty v(\langle x_1, \ldots, x_n \rangle) = [0, \infty).$$

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[1]) 0 denotes here the empty interval $[a, a)$.

The intuitive sense of the elements ξ and the function v is the following one:

The element $\xi = \langle 1 \rangle$ represents the particle generating the cascade; the element $\xi = \langle 1, \langle i, j \rangle \rangle$ represents the i-th particle arising by the decomposition into j parts of the particle generating the cascade; generally, the element $\xi = \langle x_1, \ldots, x_n, \langle i, j \rangle \rangle$ represents the i-th particle generated by the decomposition of the particle represented by the element $\langle x_1, \ldots, x_n \rangle$ into j new particles; $v(\xi) = [a, b]$ where $a < b$ means that the particle represented by the system ξ arose at the moment a and finished its life, i. e., was decomposed by the medium, at the moment b; $v(\xi) = [a, a] = 0$ means that a particle represented by ξ does not exist in the cascade. Hence the function v describes the history of all the particles in a given cascade. Condition (a) states that the cascade arises from a single particle at the moment $t = 0$, (d) states that the non-existing particle cannot generate new particles, finally, (e) states that in every finite time interval only a finite number of decompositions can happen.

The following lemmas will show that condition (e) in the definition of the space V may be replaced by

(e') for every $T < \infty$ the set $\{\xi : v(\xi) \cap [0, T) \neq 0\}$ is finite.

LEMMA 1. *Condition* (e') *is satisfied for every* $v \, \epsilon \, V$.

Proof. Arguing indirectly, suppose that there is a $v_0 \, \epsilon \, V$ such that (e') is not satisfied. Then there exists a number T_0 such that the set

$$\mathit{\Xi}_0 = \{\xi : v_0(\xi) \cap [0, T_0) \neq 0\}$$

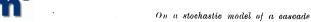is infinite. Conditions (b) and (d) lead to the following implication:

(17) if $\langle x_1, \ldots, x_n, x_{n+1} \rangle \, \epsilon \, \mathit{\Xi}_0$, then $\langle x_1, \ldots, x_n \rangle \, \epsilon \, \mathit{\Xi}_0$.

From (c) it follows that there is only a finite multitude of elements x_r appearing on the r-th place in the systems $\langle x_1, \ldots, x_{r-1}, x_r, x_{r+1}, \ldots, x_n \rangle$ of the set $\mathit{\Xi}_0$. The infinitude of the set $\mathit{\Xi}_0$ and condition (17) imply, for every n, the existence of a system $\langle x_1, \ldots, x_n \rangle$ belonging to $\mathit{\Xi}_0$. It follows that there exists a sequence $x_1^0, x_2^0, \ldots$ such that $\langle x_1^0, x_2^0, \ldots, x_n^0 \rangle \, \epsilon \, \mathit{\Xi}_0$ for every n ([8], p. 115). Let us write in accordance with condition (b)

(18) $v_0(\langle x_1^0, x_2^0, \ldots, x_n^0 \rangle) = \lceil a_n, a_{n+1} \rceil$ $(n = 1, 2, \ldots)$.

From the definition of the set $\mathit{\Xi}_0$ and of the sequence $x_1^0, x_2^0, \ldots$ we deduce the inequality $a_n < T_0$ $(n = 1, 2, \ldots)$, whence from (18) it follows that

$$\bigcup_{n=1}^{\infty} v_0(\langle x_1^0, \ldots, x_n^0 \rangle) \subset [0, T_0),$$

which, in virtue of the finiteness of T_0, contradicts condition (e). Thus the lemma is proved.

LEMMA 2. *Let the function v satisfy conditions* (a), (e') *and* (b); *then condition* (e) *is also satisfied.*

Proof. Let

(19) $v(\langle x_1, \ldots, x_n \rangle) \neq 0$ for $n = 1, 2, \ldots$

Conditions (a) and (b) imply

$$\bigcup_{n=1}^{\infty} v(\langle x_1, \ldots, x_n \rangle) = [0, T).$$

It follows by (19) that the set $\{\xi : v(\xi) \cap [0, T) \neq 0\}$ is infinite. Hence from (e') we deduce $T = \infty$, which proves the lemma.

4. Let $\mathcal{B}(Y, T)$ denote the smallest σ-field spanned upon the sets of the form

(20) $B(\xi, t) = \{v : v \, \epsilon \, V, \, t \, \epsilon \, v(\xi)\}$,

where $\xi \, \epsilon \, Y$ and $t \geqslant T$. The set $\{v : v \, \epsilon \, V, \, \Phi(v)\}$ will be denoted shortly by $\{\Phi(v)\}$; thus the set (20) will be denoted by $\{t \, \epsilon \, v(\xi)\}$.

We shall now define a family T_τ of transformations of sets belonging to the σ-field $\mathcal{B}(Y, T)$. For the sets of the form (20) we define

$$T_\tau \{t \, \epsilon \, v(\xi)\} = \{t - \tau \, \epsilon \, v(\xi)\}   \quad (\tau \leqslant T),$$

and then we extend this definition to all the sets of $\mathcal{B}(Y, T)$ by the formulae

$$T_\tau \bigcup_i A_i = \bigcup_i T_\tau A_i, \quad T_\tau A' = (T_\tau A)'.$$

By the definition of T_τ $(\tau \leqslant T)$ it follows for $A \, \epsilon \, \mathcal{B}(Y, T)$ that $T_\tau A \, \epsilon \, \mathcal{B}(Y, T - \tau)$.

Let $\xi = \langle x_1, \ldots, x_n \rangle$ and $\eta = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle$ $(m = 0, 1, \ldots)$; then we shall write $\eta \, \epsilon \, K(\xi)$ and $\eta - \xi = \langle 1, y_1, \ldots, y_m \rangle$ [2]).

$K(\xi)$ is a cascade arising from the particle ξ.

We have, in particular, $K(\langle 1 \rangle) = \mathit{\Xi}$ and $\eta - \langle 1 \rangle = \eta$. If $\eta \, \epsilon \, K(\xi)$, we put $L_\xi \{t \, \epsilon \, v(\eta)\} = \{t \, \epsilon \, v(\eta - \xi)\}$, and by the formulae

$$L_\xi \bigcup_i A_i = \bigcup_i L_\xi A_i, \quad L_\xi A' = (L_\xi A)'$$

we extend the definition of the transformation L_ξ to all the sets belonging to the σ-field $\mathcal{B}(K(\xi), T)$. Let us observe that $A \, \epsilon \, \mathcal{B}(K(\xi), T)$ implies $L_\xi A \, \epsilon \, \mathcal{B}(\mathit{\Xi}, t)$.

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[2]) We set $\langle x_1, \ldots, x_m, y_1, \ldots, y_m \rangle = \langle x_1, x_2, \ldots, x_n \rangle$ for $m = 0$.

The σ-field $\mathfrak{B}(\varXi, 0)$ will be denoted by $\mathfrak{B}_V$. Let P be a probability (= normed σ-measure) on the σ-field $\mathfrak{B}_V$, inducing the conditional probability (in the sense of section 1) and satisfying for $A \epsilon \mathfrak{B}(K(\xi), t)$ and $B \epsilon \mathfrak{B}(\varXi - K(\xi), 0)$ the following conditions:

(I) $\qquad\qquad P(A | t \epsilon v(\xi)) = P(L_\xi A | t \epsilon v(\langle 1 \rangle))$,

(II) $\qquad\qquad P(A | \{t \epsilon v(\xi)\} \cap B) = P(A | t \epsilon v(\xi))$,

(III) $\qquad\qquad P(A | t \epsilon v(\xi)) = P(T_\tau A | t - \tau \epsilon v(\xi))$,

where $t \geqslant \tau$.

The system $\langle V, \mathfrak{B}_V, P \rangle$ will be called a *cascade process*. Conditions (I) and (II) are transcriptions of conditions (i) and (ii), and condition (III) describes the homogeneity in time of the process.

5. Let $\chi(v(\xi), t)$ be the characteristic function (in the sense of the theory of sets) of the interval $v(\xi)$. Let us write for $v \epsilon V$ and $t \geqslant 0$

$$(21) \qquad \omega(v, t) = \sum_{\xi \epsilon \varXi} \chi(v(\xi), t).$$

Condition (e′) implies that $\omega(v, t)$ is a step-function of the variable t, taking on non-negative integer values. It is easily verified that the number of particles in the cascade described by the function v is equal at the instant t to $\omega(v, t)$.

Formula (21) defines a mapping of the space V onto the space of all the step-functions ω defined for $t \geqslant 0$, taking on non-negative integer values and equal to 1 at $t = 0$. Let us denote this mapping by $\varOmega$. Hence

$$\varOmega(v)(t) = \omega(v, t),$$

$$\varOmega^{-1}\{\omega : \omega(t) = n\} = \Big\{\sum_{\xi \epsilon \varXi} \chi(v(\xi), t) = n\Big\} = \bigcup_* \bigcap_{j=1}^n \{t \epsilon v(\xi_j)\} \cap \bigcap_* \{t \bar\epsilon v(\eta)\},$$

the union $\bigcup_*$ being extended to all non-ordered systems $\xi_1, \xi_2, \ldots, \xi_n$ ($\xi_i \epsilon \varXi$; $\xi_i \neq \xi_j$ for $i \neq j$) and the intersection $\bigcap_*$ being taken over all the elements $\eta \epsilon \varXi$ different from $\xi_1, \xi_2, \ldots, \xi_n$. We get

$$\varOmega^{-1}\{\omega : \omega(t) = n\} \epsilon \mathfrak{B}_V,$$

whence, denoting by $\mathfrak{B}_{\varOmega(V)}$ the smallest σ-field spanned upon the sets of the form (1), we see that $A \epsilon \mathfrak{B}_{\varOmega(V)}$ implies $\varOmega^{-1}A \epsilon \mathfrak{B}_V$. Thus the mapping $\varOmega$ defines in $\mathfrak{B}_{\varOmega(V)}$ a normed σ-measure which will be denoted by $\varOmega P$: $\varOmega P(A) = P(\varOmega^{-1}A)$ for $A \epsilon \mathfrak{B}_{\varOmega(V)}$. Thus the mapping transforms the cascade process $\langle V, \mathfrak{B}_V, P \rangle$ into the stochastic process $\langle \varOmega(V), \mathfrak{B}_{\varOmega(V)}, \varOmega P \rangle$.

THEOREM 1. *For every cascade process $\langle V, \mathfrak{B}_V, P \rangle$ the corresponding stochastic process $\langle \varOmega(V), \mathfrak{B}_{\varOmega(V)}, \varOmega P \rangle$ is a homogeneous regular Markov process with a denumerable multitude of states, satisfying conditions* (13) *and* (14).

THEOREM 2. *For every homogeneous regular Markov process with a denumerable multitude of states $\langle \varOmega, \mathfrak{B}_\varOmega, Pr \rangle$, satisfying conditions* (13) *and* (14) *there exists a cascade process $\langle V, \mathfrak{B}_V, P \rangle$ such that $\langle \varOmega(V), \mathfrak{B}_{\varOmega(V)}, \varOmega P \rangle$ is identical with $\langle \varOmega, \mathfrak{B}_\varOmega, Pr \rangle$.*

Theorems 1 and 2 join the models of stochastic cascades described in sections 2-4. The proofs of both theorems will be given in the sections to follow (they were announced without proof in the note [9]).

6. We need following lemmas.

LEMMA 3. *Let*

$$(22) \qquad K(\xi_i) \cap K(\xi_j) = 0 \qquad (i \neq j; i, j = 1, 2, \ldots, n);$$

then for

$$(23) \qquad A_i \epsilon \mathfrak{B}(K(\xi_i), t) \qquad (i = 1, 2, \ldots, n),$$

$$(24) \qquad B \epsilon \mathfrak{B}\Big(\varXi - \bigcup_{i=1}^n K(\xi_i), 0\Big)$$

we have

$$(25) \qquad P\Big(\bigcap_{i=1}^n A_i \cap \{t \epsilon v_*(\xi_i)\} \cap B\Big) = \Big[\prod_{i=1}^n P(A_i | t \epsilon v(\xi_i))\Big] \cdot P\Big(\bigcap_{i=1}^n \{t \epsilon v(\xi_i)\} \cap B\Big).$$

Proof. We shall prove this by induction with respect to n. For $n = 1$, formula (25) follows from condition (II). Suppose that (25) is valid for $n = k$, and let us write

$$B_0 = \bigcap_{i=1}^k A_i \cap \{t \epsilon v(\xi_i)\} \cap B.$$

Then we have

$$B_0 \epsilon \mathfrak{B}(\varXi - K(\xi_{k+1}), 0)$$

and

$$P\Big(\bigcap_{j=1}^{k+1} A_i \cap \{t \epsilon v(\xi_i)\} \cap B\Big) = P\big(A_{k+1} \cap \{t \epsilon v(\xi_{k+1})\} \cap B_0\big).$$

Hence by (II) we obtain

$$(26) \qquad P\Big(\bigcap_{i=1}^{k+1} A_i \cap \{t \epsilon v(\xi_i)\} \cap B\Big) = P\big(A_{k+1} | t \epsilon v(\xi_{k+1})\big) P\big(\{t \epsilon v(\xi_{k+1})\} \cap B_0\big).$$

Now, setting

$$(27) \qquad B_1 = \{t \epsilon v(\xi_{k+1})\} \cap B,$$

we get

$$P(\{t\,\epsilon\,v(\xi_{k+1})\}\cap B_0) = P\Big(\bigcap_{i=1}^{k} A_i \cap \{t\,\epsilon\,v(\xi_i)\}\cap B_1\Big),$$

and in virtue of (22), (23), and (24)

$$B_1 \epsilon \,\mathcal{B}\Big(\varXi - \bigcup_{i=1}^{k} K(\xi_i),\,0\Big).$$

By the hypothesis of induction we deduce hence

$$P(\{t\,\epsilon\,v(\xi_{k+1})\}\cap B_0) = \Big[\prod_{i=1}^{k} P\big(A_i\,|\,t\,\epsilon\,v(\xi_i)\big)\Big]\cdot P\Big(\bigcap_{i=1}^{k}\{t\,\epsilon\,v(\xi_i)\}\cap B_1\Big).$$

Putting this expression into formula (26) and taking into account formula (27) we obtain formula (25) for $n = k+1$, which proves the lemma.

LEMMA 4. *Let*

$$(28)\qquad K(\xi_i)\cap K(\xi_j) = 0 \qquad (i\neq j),$$

$$(29)\qquad t < T;$$

then for every k there exists a set

$$B \epsilon\,\mathcal{B}\Big(\varXi - \bigcup_{i=1}^{n} K(\xi_i),\,0\Big)$$

satisfying the equality

$$\bigcap_{i=1}^{n}\{T\,\epsilon\,v(\xi_i)\}\cap\{\omega(v,t) = k\} = \bigcap_{i=1}^{n}\{T\,\epsilon\,v(\xi_i)\}\cap B.$$

Proof. By hypothesis (29) and property (b) of the function v it follows that $T\,\epsilon\,v(\xi)$, $\eta\,\epsilon\,K(\xi)$, $\eta\neq\xi$ implies $t\,\epsilon\,v(\eta)$. This leads to

$$(30)\qquad \bigcap_{i=1}^{n}\{T\,\epsilon\,v(\xi_i)\}\cap\{\omega(v,t) = k\}$$

$$= \bigcup_{s=0}^{k}\bigcap_{i=1}^{n}\{T\,\epsilon\,v(\xi_i)\}\cap\Big\{\sum_{\eta\,\bar{\epsilon}\,\bigcup_{i=1}^{n} K(\xi_i)}\chi(v(\eta),t) = k-s\Big\}\cap\Big\{\sum_{i=1}^{n}\chi(v(\xi_i),t) = s\Big\}.$$

Let $\xi_i = \langle x_1^i,\ldots,x_{m_i}^i\rangle$; then by hypothesis (28)

$$(31)\qquad \langle x_1^i,\ldots,x_r^i\rangle\,\epsilon\,\varXi - \bigcup_{j=1}^{n} K(\xi_j) \qquad (r < m_i;\ i = 1,2,\ldots,n).$$

From inequality (29) and property (b) of the function v it follows that

$$(32)\qquad \{T\,\epsilon\,v(\xi_i)\}\cap\{t\,\epsilon\,v(\xi_i)\} = \{T\,\epsilon\,v(\xi_i)\}\cap\bigcap_{r=1}^{m_i-1}\{t\,\epsilon\,v(\langle x_1^i,\ldots,x_r^i\rangle)\}.$$

Setting

$$(33)\qquad E_i^0 = \bigcap_{r=1}^{m_i-1}\{t\,\epsilon\,v(\langle x_1^i,\ldots,x_r^i\rangle)\},\qquad E_i^1 = V - E_i^0,$$

we get, in virtue of formula (32),

$$(34)\qquad \Big\{\sum_{i=1}^{n}\chi(v(\xi_i),t) = s\Big\}\cap\bigcap_{i=1}^{n}\{T\,\epsilon\,v(\xi_i)\}$$

$$= \bigcap_{j=1}^{n}\{T\,\epsilon\,v(\xi_j)\}\cap\Big|\bigcup_{r_1+\ldots+r_n=k-s}\bigcap_{i=1}^{n} E_i^{r_i}\Big].$$

Formulae (31) and (33) lead directly to

$$(35)\qquad \bigcup_{r_1+\ldots+r_n=k-s}\bigcap_{i=1}^{n} E_i^{r_i}\,\epsilon\,\mathcal{B}\Big(\varXi - \bigcup_{i=1}^{n} K(\xi_i),\,0\Big).$$

Let us now write

$$B = \bigcup_{s=0}^{k}\Big\{\sum_{\eta\,\bar{\epsilon}\,\bigcup_{i=1}^{n} K(\xi_i)}\chi(v(\eta),t) = k-s\Big\}\cap\Big|\bigcup_{r_1+\ldots+r_n=k-s}\bigcap_{i=1}^{n} E_i^{r_i}\Big].$$

By formula (35) $B\,\epsilon\,\mathcal{B}\Big(\varXi - \bigcup_{i=1}^{n} K(\xi_i),\,0\Big).$

Equalities (30) and (34) imply that the set B satisfies our lemma.

LEMMA 5. *If $T > t > t_1 > \ldots > t_m$ then for every $k, n_1, \ldots, n_m$ and $n\geqslant 1$, $m\geqslant 0$ we have*

$$P\Big(\{\omega(v,T) = k\}\cap\{\omega(v,t) = n\}\cap\bigcap_{j=1}^{m}\{\omega(v,t_j) = n_j\}\Big)$$

$$= \Big[\sum_{k_1+\ldots+k_n=k}\prod_{i=1}^{n} P\big(\omega(v,T-t) = k_i\big)\Big]\cdot P\Big(\{\omega(v,t) = n\}\cap\bigcap_{j=1}^{m}\{\omega(v,t_j) = n_j\}\Big)$$

(we set $\bigcap_{j=1}^{m}\{\omega(v,t_j) = n_j\} = \Omega(V)$, as $m = 0$).

Proof. Using the definition of the function ω and properties (a) and (b) we may easily verify the following equality:

$$(36)\qquad \{\omega(v,T) = k\}\cap\{\omega(v,t) = n\}\cap\bigcap_{j=1}^{m}\{\omega(v,t_j) = n_j\}$$

$$= \bigcup_{*}\bigcup_{k_1+\ldots+k_n=k}\bigcap_{i=1}^{n}\Big\{\sum_{\eta\,\epsilon\,K(\xi_i)}\chi(v(\eta),T) = k_i\Big\}\cap\{T\,\epsilon\,v(\xi_i)\}\cap$$

$$\cap\bigcap_{\eta\,\bar{\epsilon}\,\bigcup_{i=1}^{n} K(\xi_i)}\{t\,\epsilon\,v(\eta)\}\cap\bigcap_{j=1}^{m}\{\omega(v,t_j) = n_j\},$$

the union $\bigcup_*$ being extended over all non-ordered systems $\xi_1, \xi_2, \ldots, \xi_n$ such that

$$(37) \qquad K(\xi_i) \cap K(\xi_j) = 0 \qquad \text{as} \qquad i \neq j.$$

By Lemma 4 there exist sets $B_j(\xi_1, \ldots, \xi_n)$ belonging to the σ-field $\mathcal{B}\big(\varXi - \bigcup_{i=1}^{n} K(\xi_i), 0\big)$ such that

$$(38) \qquad \bigcap_{i=1}^{n} \{t \, \varepsilon \, v(\xi_i)\} \cap \{\omega(v, t_j) = n_j\} = \bigcap_{i=1}^{n} \{t \, \varepsilon \, v(\xi_i)\} \cap B_j(\xi_1, \ldots, \xi_n).$$

Let us set

$$(39) \qquad B(\xi_1, \ldots, \xi_n) = \bigcap_{j=1}^{m} B_j(\xi_1, \ldots, \xi_n) \cap \bigcap_{\eta \, \bar{\varepsilon} \, \bigcup_{i=1}^{n} K(\xi_i)} \{t \, \bar{\varepsilon} \, v(\eta)\}.$$

We see from (36) and (38) that

$$(40) \qquad \bigcup_{*} \bigcap_{i=1}^{n} \{t \, \varepsilon \, v(\xi_i)\} \cap B(\xi_1, \ldots, \xi_n) = \{\omega(v, t) = n\} \cap \bigcap_{j=1}^{m} \{\omega(v, t_j) = n_j\}$$

and

$$(41) \qquad B(\xi_1, \ldots, \xi_n) \, \varepsilon \, \mathcal{B}\big(\varXi - \bigcup_{i=1}^{n} K(\xi_i), 0\big).$$

Setting

$$(42) \qquad A(\xi, s) = \Big\{ \sum_{\eta \, \varepsilon \, K(\xi)} \chi\big(v(\eta), T\big) = s \Big\},$$

and taking into account the fact that the summands on the right side of equality (36) are disjoint, we easily deduce from (36), (38), and (39) that

$$(43) \qquad P\big(\{\omega(v, T) = k\} \cap \{\omega(v, t) = n\} \cap \bigcap_{j=1}^{m} \{\omega(v, t_j) = n_j\}\big)$$

$$= \sum_{*} \sum_{k_1 + \ldots + k_n = k} P\big(\bigcap_{i=1}^{n} A(\xi_i, k_i) \cap \{t \, \varepsilon \, v(\xi_i)\} \cap B(\xi_1, \ldots, \xi_n)\big).$$

By definition (42) we get $A(\xi_i, k_i) \, \varepsilon \, \mathcal{B}\big(K(\xi_i), t\big)$. Hence formulae (37) and (41) and Lemma 3 lead to

$$(44) \qquad P\big(\bigcap_{i=1}^{n} A(\xi_i, k_i) \cap \{t \, \varepsilon \, v(\xi_i)\} \cap B(\xi_1, \ldots, \xi_n)\big)$$

$$= \Big[\prod_{i=1}^{n} P\big(A(\xi_i, k_i) \, | \, t \, \varepsilon \, v(\xi_i)\big)\Big] P\big(\bigcap_{i=1}^{n} \{t \, \varepsilon \, v(\xi_i)\} \cap B(\xi_1, \ldots, \xi_n)\big).$$

From conditions (I) and (III) (p. 242) we infer

$$(45) \qquad P\big(A(\xi_i, k_i) \, | \, t \, \varepsilon \, v(\xi_i)\big) = P\big(T_t L_{\xi_i} A(\xi_i, k_i) \, | \, 0 \, \varepsilon \, v(\langle 1 \rangle)\big).$$

By (42) and the definition of the transformations T_t and L_ξ we have

$$T_t L_{\xi_i} A(\xi_i, k_i) = \Big\{ \sum_{\eta \, \varepsilon \, \varXi} \chi\big(v(\eta), T - t\big) = k_i \Big\} = \{\omega(v, T - t) = k_i\}$$

and, by property (a) of the function v, $\{0 \, \varepsilon \, v(\langle 1 \rangle)\} = V$. Hence formula (45) may be written in the form

$$P\big(A(\xi_i, k_i) \, | \, t \, \varepsilon \, v(\xi_i)\big) = P\big(\omega(v, T - t) = k_i\big).$$

Thus from formulae (43) and (44) we get

$$P\big(\{\omega(v, T) = k\} \cap \{\omega(v, t) = n\} \cap \bigcap_{j=1}^{m} \{\omega(v, t_j) = n_j\}\big)$$

$$= \Big[\sum_{k_1 + \ldots + k_n = k} \prod_{i=1}^{n} P\big(\omega(v, T - t) = k_i\big)\Big] P\big(\bigcup_{*} \bigcap_{i=1}^{n} \{t \, \varepsilon \, v(\xi_i)\} \cap B(\xi_1, \ldots, \xi_n)\big);$$

this, together with equality (40), implies our lemma.

LEMMA 6. *If $T > t$, then for every $k, n_1, \ldots, n_m, t_1, \ldots, t_m$ and $m \geqslant 0$ we have*

$$P\big(\{\omega(v, T) = k\} \cap \{\omega(v, t) = 0\} \cap \bigcap_{j=1}^{m} \{\omega(v, t_j) = n_j\}\big)$$

$$= \delta_0^k P\big(\{\omega(v, t) = 0\} \cap \bigcap_{j=1}^{m} \{\omega(v, t_j) = n_j\}\big).$$

Proof. It follows from the definition of the function ω and from the properties (a), (b), and (d) of the function v for $T > t$ that

$$\{\omega(v, t) = 0\} \subset \{\omega(v, T) = 0\}.$$

Thus

$$\{\omega(v, T) = k\} \cap \{\omega(v, t) = 0\} = \begin{cases} \{\omega(v, t) = 0\} & \text{as} \quad k = 0, \\ 0 & \text{as} \quad k \neq 0, \end{cases}$$

whence the lemma follows directly.

LEMMA 7. *For every n we have*

$$\lim_{t \to T+} \{\omega(v, t) \geqslant n\} = \{\omega(v, T) \geqslant n\}.$$

Proof. Suppose that $v_0 \, \varepsilon \, \{\omega(v, T) \geqslant n\}$. By definition (21) of the function ω we see that there exists a system $\xi_1, \ldots, \xi_n$ $(\xi_i \, \varepsilon \, \varXi)$ such that

$$(46) \qquad \xi_i \neq \xi_j \qquad \text{if} \qquad i \neq j$$

and $T \, \varepsilon \, v_0(\xi_i)$ $(i = 1, 2, \ldots, n)$.

Let $v_0(\xi_i) = [a_i, b_i)$ $(i = 1, 2, \ldots, n)$. Then for every t satisfying the inequality

$$T < t < \min_{1 \leqslant i \leqslant n} b_i$$

we have the relation $t \epsilon v_0(\xi_i) \; (i = 1, 2, \ldots, n)$ which, in virtue of (46), leads to $\omega(v_0, t) \geqslant n$. Thus we have proved that

$$\{\omega(v, T) \geqslant n\} \subset \liminf_{t \to T+} \{\omega(v, t) \geqslant n\}.$$

Suppose now that

$$v_0 \epsilon \limsup_{t \to T+} \{\omega(v, t) \geqslant n\}.$$

Then there exists a sequence of numbers

$$(47) \qquad\qquad t_1 > t_2 > \ldots \to T$$

and a sequence of systems of elements of the set Ξ

$$\xi_{1,r}, \xi_{2,r}, \ldots, \xi_{n,r} \qquad (r = 1, 2, \ldots)$$

such that

$$(48) \qquad\qquad \xi_{i,r} \neq \xi_{j,r} \quad \text{for} \quad i \neq j, \; r = 1, 2, \ldots,$$

and

$$(49) \qquad\qquad t_r \epsilon v_0(\xi_{i,r}) \qquad (i = 1, 2, \ldots, n; r = 1, 2, \ldots).$$

By formulae (47) and (49)

$$v_0(\xi_{i,r}) \cap [0, t_1] \neq 0 \qquad (i = 1, 2, \ldots, n; r = 1, 2, \ldots),$$

whence from property (e′) it follows that among the systems $\xi_{1,r}, \xi_{2,r}, \ldots, \xi_{n,r}$ only a finite number is different. Thus there exists a subsequence $r_1, r_2, \ldots$ such that $\xi_{i,r} = \eta_i \; (i = 1, 2, \ldots, n; r = r_1, r_2, \ldots)$ where, because of (48),

$$(50) \qquad\qquad \eta_i \neq \eta_j \quad \text{as} \quad i \neq j.$$

Hence by (49) we infer that $t_r \epsilon v_0(\eta_i) \; (i = 1, 2, \ldots, n; r = r_1, r_2, \ldots)$, which, in view of (47), implies $T \epsilon v_0(\eta_i) \; (i = 1, 2, \ldots, n)$.

Thus by formula (50) we deduce that $\omega(v_0, T) \geqslant n$. Hence we have proved that

$$\limsup_{t \to T+} \{\omega(v, t) \geqslant n\} \subset \{\omega(v, T) \geqslant n\},$$
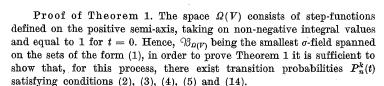
which, together with the inclusion proved before, leads to our lemma.

COROLLARY. *By lemma 7 we infer that*

$$\lim_{t \to 0+} \{\omega(v, t) = 1\} = \{\omega(v, 0) = 1\} = V,$$

which implies the equality

$$\lim_{t \to 0+} P(\omega(v, t) = k) = \delta_0^k.$$

Proof of Theorem 1. The space $\Omega(V)$ consists of step-functions defined on the positive semi-axis, taking on non-negative integral values and equal to 1 for $t = 0$. Hence, $\mathfrak{B}_{\Omega(V)}$ being the smallest σ-field spanned on the sets of the form (1), in order to prove Theorem 1 it is sufficient to show that, for this process, there exist transition probabilities $P_n^k(t)$ satisfying conditions (2), (3), (4), (5) and (14).

Let us write

$$(51) \qquad P_1^k(t) = \Omega P\{\omega : \omega(t) = k\} = P(\omega(v, t) = k) \qquad (k = 0, 1, \ldots),$$

$$(52) \quad P_n^k(t) = \sum_{k_1 + \ldots + k_n = k} \prod_{i=1}^{n} P_1^{k_i}(t), \; P_0^k(t) = \delta_0^k \quad (n = 1, 2, \ldots; k = 0, 1, \ldots).$$

Let $f_n(x, t)$ be the generating function of the sequence $P_n^0(t), P_n^1(t), \ldots$:

$$(53) \qquad\qquad f_n(x, t) = \sum_{k=0}^{\infty} P_n^k(t) x^k.$$

By definition (52)

$$(54) \qquad\qquad f_n(x, t) = [f_1(x, t)]^n,$$

whence

$$\sum_{k=0}^{\infty} P_n^k(t) = f_n(1, t) = [f_1(1, t)]^n = \left[\sum_{k=0}^{\infty} P(\omega(v, t) = k)\right]^n = 1,$$

which gives formula (2), the functions $P_n^k(t)$ being non-negative.

Using definitions (51) and (52) we may write the conclusion of lemmas 5 and 6 in the following form:

$$(55) \qquad P\left(\{\omega(v, T) = k\} \cap \{\omega(v, t) = n\} \cap \bigcap_{j=1}^{m} \{\omega(v, t_j) = n_j\}\right)$$

$$= P_n^k(T-t) P\left(\{\omega(v, t) = n\} \cap \bigcap_{j=1}^{m} \{\omega(v, t_j) = n_j\}\right)$$

where $T > t > t_1 > \ldots > t_m$. Applying repeatedly lemmas 5 and 6 to the second factor on the right side of formula (55), we obtain (5).

By (51) and (55) we get

$$P_1^k(T+t) = \sum_{s=0}^{\infty} P\left(\{\omega(v, T+t) = k\} \cap \{\omega(v, t) = s\}\right) = \sum_{s=0}^{\infty} P_s^k(T) P_1^s(t).$$

Multiplying this equality by x^k and summing up from $k = 0$ to ∞ we get in virtue of (53)

$$f_1(x, T+t) = \sum_{s=0}^{\infty} f_s(x, T) P_1^s(t),$$

whence by (54) it follows that

$$f_1(x, t+T) = \sum_{s=0}^{\infty} [f_1(x, T)]^s P_1^s(t) = f_1(f_1(x, T), t).$$

Thus we get

$$f_n(x, t+T) = [f_1(x, t+T)]^n = [f_1(f_1(x, T), t)]^n = f_n(f_1(x, T), t),$$

i. e.,

$$\sum_{k=0}^{\infty} P_n^k(t+T)x^k = \sum_{s=0}^{\infty} [f_1(x, T)]^s P_n^s(t) = \sum_{s=0}^{\infty} f_s(x, T) P_n^s(t)$$

$$= \sum_{k=0}^{\infty} \Big[\sum_{s=0}^{\infty} P_s^k(T) P_n^s(t) \Big] x^k,$$

whence formula (3) follows.

The corollary of Lemma 7, and definitions (51) and (52) imply

$$\lim_{t\to 0+} P_n^k(t) = \sum_{k_1+\ldots+k_n=k} \prod_{i=1}^{n} \delta_1^{k_i} = \delta_n^k = P_n^k(0);$$

condition (4) is thus fulfilled. Finally, definition (52) directly implies condition (14), which concludes the proof.

7. Suppose we are given a homogeneous regular Markov process $\langle \Omega, \mathcal{B}_\Omega, Pr \rangle$ with a denumerable multitude of states, satisfying conditions (13) and (14). Let the numbers

$$(56) \qquad\qquad \pi_1^0, \pi_1^1, \pi_1^2, \ldots$$

be the intensities of this process (the remaining ones are already defined by formula (15)). In order to prove Theorem 2 it is sufficient to show that there exists a process $\langle V, \mathcal{B}_V, P \rangle$ inducing the process $\langle \Omega(V), \mathcal{B}_{\Omega(V)}, \Omega P \rangle$ with the intensities (56) (compare section 1).

Suppose that $\pi_1^1 = 0$; then for almost every $\omega \epsilon \Omega$ we have $\omega(t) \equiv 1$, for $\omega(0) = 1$ (compare [2], p. 457). Defining the measure of the process $\langle V, \mathcal{B}_V, P \rangle$ by the formula

$$P(A) = \begin{cases} 1 & \text{as} \quad v_0 \epsilon A, \\ 0 & \text{as} \quad v_0 \notin A, \end{cases}$$

where $A \epsilon \mathcal{B}_V$ and $v_0(\langle 1 \rangle) = [0, \infty)$, $v_0(\xi) = 0$ for $\xi \neq \langle 1 \rangle$, we infer that for almost any $v \epsilon V$ the equality $\omega(v, t) \equiv 1$ is satisfied, whence the processes $\langle \Omega(V), \mathcal{B}_{\Omega(V)}, \Omega P \rangle$ and $\langle \Omega, \mathcal{B}_\Omega, Pr \rangle$ are identical, which proves Theorem 2 in the case in question. Thus it is sufficient to deal in the sequel with the case

$$(57) \qquad\qquad \pi_1^1 > 0.$$

Let W be the space of all pairs $w = \langle w_1, w_2 \rangle$ of functions defined on Ξ, taking on non-negative real values including ∞. Every element

$v \epsilon V$ may be considered as belonging to W; to do this we define the pair $\langle w_1, w_2 \rangle$ as

$$\left. \begin{aligned} w_1(\xi) &= a_\xi \\ w_-(\xi) &= b_\xi \end{aligned} \right\} \quad \text{when} \quad v(\xi) = [a_\xi, b_\xi).$$

Thus we have

$$(58) \qquad\qquad V \subset W.$$

Let $\mathcal{B}_W$ be the smallest σ-field spanned upon the sets of form

$$(59) \qquad C(\xi, t, T) = \{w : w \epsilon W, w_1(\xi) < t, w_-(\xi) < T\}.$$

Hence by the definition of the σ-field $\mathcal{B}_V$ and inclusion (58) we obtain

$$(60) \qquad\qquad \mathcal{B}_V = \mathcal{B}_W \cap V$$

(compare [6], p. 25).

In the case (57) the proof consists of several parts. First, by aid of intensities (56) we define a normed σ-measure P_0 on the σ-field $\mathcal{B}_W$; then we prove that the set V is measurable and $P_0(V) = 1$. Hence making use of the well-known relativization of the σ-measure P_0 to the set V we obtain, in view of (60), a σ-measure P defined on the σ-field $\mathcal{B}_V$. We prove that P satisfies conditions (I), (II), and (III), whence we obtain a cascade process $\langle V, \mathcal{B}_V, P \rangle$. Finally, we prove that the process $\langle \Omega(V), \mathcal{B}_{\Omega(V)}, \Omega P \rangle$ generated by this process has intensities (56), and this concludes the proof of Theorem 2.

8. The space W may be considered as the product

$$(61) \qquad\qquad W = \underset{\xi \epsilon \Xi}{\mathcal{P}} (\mathcal{X}_\xi \times \mathcal{Y}_\xi),$$

where the factors $\mathcal{X}_\xi$ and $\mathcal{Y}_\xi$ ($\xi \epsilon \Xi$) are sets composed of non-negative real numbers including ∞.

Let $\mathcal{B}(Y)$ ($Y \subset \Xi$) denote the smallest σ-field spanned on the sets of form (59) where $\xi \epsilon Y$. Thus $\mathcal{B}(Y)$ is the σ-field of cylinders of the product (61), whose bases are Borel sets contained in the product

$$(62) \qquad\qquad \underset{\xi \epsilon Y}{\mathcal{P}} (\mathcal{X}_\xi \times \mathcal{Y}_\xi).$$

Let the set Y consist of the elements $\xi_1, \xi_2, \ldots, \xi_n$. The function $F(t_{\xi_1}, t_{\xi_2}, \ldots, t_{\xi_n}; T_{\xi_1}, T_{\xi_2}, \ldots, T_{\xi_n})$ of $2n$ real variables will be denoted simply as $F_Y(t_\xi, T_\xi)$.

We now define the operators $\Delta_{\xi_i}^{(1)}(h)$ and $\Delta_{\xi_i}^{(2)}(h)$ by the following formulae:

$$\Delta_{\xi_i}^{(1)}(h)F_Y(t_\xi, T_\xi) = F(t_{\xi_1}, \ldots, t_{\xi_{i-1}}, t_{\xi_i}+h, t_{\xi_{i+1}}, \ldots, t_{\xi_n}; T_{\xi_1}, \ldots, T_{\xi_n}) -$$
$$- F(t_{\xi_1}, \ldots, t_{\xi_n}; T_{\xi_1}, \ldots, T_{\xi_n}),$$

$$\Delta_{\xi_i}^{(2)}(h)F_Y(t_\xi, T_\xi) = F(t_{\xi_1}, \ldots, t_{\xi_n}; T_{\xi_1}, \ldots, T_{\xi_{i-1}}, T_{\xi_i}+h, T_{\xi_{i+1}}, \ldots, T_{\xi_n}) -$$
$$- F(t_{\xi_1}, \ldots, t_{\xi_n}; T_{\xi_1}, \ldots, T_{\xi_n}).$$

Let us write

$$\Delta_Y(h_\xi, k_\xi) = \prod_{\xi \in Y} \Delta_\xi^{(1)}(h_\xi) \Delta_\xi^{(2)}(k_\xi).$$

If the set $Y \subset \varXi$ is finite, then the *distribution function* in the product (62) (which is a closed Euclidean space with the number of dimensions equal to the doubled multitude of the set Y) is the term given to every function $F_Y(t_\xi, T_\xi)$ satisfying the following conditions:

1. the function $F_Y(t_\xi, T_\xi)$ is continuous on the left with respect to every argument t_ξ, T_ξ;

2. for every argument t_ξ, T_ξ and non-negative h_ξ, k_ξ the inequality $\Delta_Y(h_\xi, k_\xi)F_Y(t_\xi, T_\xi) \geqslant 0$ [3] holds.

3. $F_Y(t_\xi, T_\xi) = 1$ as $t_\xi = T_\xi = \infty$ for $\xi \in Y$, $F_Y(t_\xi, T_\xi) = 0$ as $t_\xi \leqslant 0$ or $T_\xi \leqslant 0$ for a certain $\xi \in Y$.

The class of all distribution functions in the product (62) will be denoted by $\mathcal{D}(Y)$. We quote here the well-known properties of the distribution functions:

(d$_1$) If $Y_1 \cap Y_2 = 0$, $F_1 \in \mathcal{D}(Y_1)$ and $F_2 \in \mathcal{D}(Y_2)$, then $F_1 \cdot F_2 \in \mathcal{D}(Y_1 \cup Y_2)$.

(d$_2$) If $F_Y(t_\xi, T_\xi) \in \mathcal{D}(Y)$, then $F_Y(t_\xi-t, T_\xi-t) \in \mathcal{D}(Y)$ for every $t \geqslant 0$.

(d$_3$) If $F_t \in \mathcal{D}(Y)$ for $t \geqslant 0$ and if $f(t)$ is a distribution function on the semi-axis $t \geqslant 0$, then

$$\int_0^\infty F_t \, df(t) \in \mathcal{D}(Y).$$

This implies in particular that

(d$_4$) Every convex combination of distribution functions is a distribution function: if $F_1, \ldots, F_n \in \mathcal{D}(Y)$, $\lambda_1 \geqslant 0, \ldots, \lambda_n \geqslant 0$ and $\lambda_1 + \ldots + \lambda_n = 1$, then $\lambda_1 F_1 + \ldots + \lambda_n F_n \in \mathcal{D}(Y)$.

If $\xi = \langle x_1, \ldots, x_n \rangle$, then the integer n is called the *length of the element* ξ. The *length of a finite set* is defined as the maximal length of its elements.

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[3] $\Delta_Y(h_\xi, k_\xi)F_Y(t_\xi, T_\xi)$ denotes the volume of the cube $\mathcal{P}_{\xi \in Y}([t_\xi, t_\xi+h_\xi] \times [T_\xi, T_\xi+k_\xi])$.

9. For every finite set $Y \subset \varXi$ we shall define, by aid of intensities (56), a distribution function $\nu_Y(t_\xi, T_\xi) \in \mathcal{D}(Y)$. This will be done by induction with respect to the length of the set Y.

(α) Let the length of the set Y be equal to 1, i. e., let $Y = (\langle 1 \rangle)$. Then we set

$$(63) \qquad \nu_Y(t_\xi, T_\xi) = \begin{cases} 1 - \exp(-\pi_1^1 T_{\langle 1 \rangle}) & \text{as} \quad t_{\langle 1 \rangle} > 0 \text{ and } T_{\langle 1 \rangle} > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

It is easily verified that conditions 1, 2, and 3 (p. 252) are verified, whence $\nu_Y(t_\xi, T_\xi) \in \mathcal{D}((\langle 1 \rangle))$.

(β) Suppose now that the length of the set Y is greater than 1 and that the distribution function $\nu_Y(t_\xi, T_\xi)$ is already defined for all sets with smaller lengths.

Let us consider first the case

$$(64) \qquad \langle 1 \rangle \in Y.$$

Then the set Y may be represented as the union of disjoint summands

$$Y = \bigcup_j \bigcup_i Y_{i,j}$$

where

$$(65) \qquad Y_{i,j} = Y \cap K(\langle 1, \langle i, j \rangle \rangle).$$

It follows from (65) that $\xi \in Y_{i,j}$ implies $\xi - \langle 1, \langle i, j \rangle \rangle \in \varXi$. For non-void $Y_{i,j}$ let us put

$$(66) \qquad Y_{i,j}^* = \{\xi - \langle 1, \langle i, j \rangle \rangle : \xi \in Y_{i,j}\}.$$

We see from this formula that the length of the set $Y_{i,j}^*$ is less than that of Y.

Let $\varrho_Z(t_\xi, T_\xi)$ be the unit distribution, i. e., let

$$\varrho_Z(t_\xi, T_\xi) = \begin{cases} 1 & \text{as} \quad t_\xi > 0 \text{ and } T_\xi > 0 \text{ for } \xi \in Z, \\ 0 & \text{elsewhere.} \end{cases}$$

In the case (64) the distribution function $\nu_Y(t_\xi, T_\xi)$ will be defined by the formula

$$(67) \qquad \nu_Y(t_\xi, T_\xi)$$
$$= \sum_{j \in M} \pi_1^j \int_0^\infty \exp(-\pi_1^1 t) \varrho_{Y - \bigcup_i Y_{i,j}}(t_\xi-t, T_\xi-t) \prod_i \nu_{Y_{i,j}^*}(t_\xi-t, T_\xi-t) \, dt +$$
$$+ \left(\pi_1^1 - \sum_{j \in M} \pi_1^j\right) \int_0^\infty \exp(-\pi_1^1 t) \varrho_Y(t_\xi-t, T_\xi-t) \, dt,$$

where $M = \{j: \bigcup_i Y_{i,j} \neq 0\}$. In this formula we consider only non-void summands $Y_{i,j}$, whence in view of (66) only non-void sets $Y_{i,j}^*$. From the hypothesis of induction it follows that the distribution functions $\nu_{Y_{i,j}^*}(t_\xi, T_\xi)$ are already defined for $Y_{i,j}^* \neq 0$, since the length of $Y_{i,j}^*$ is less that that of Y. It follows from formula (66) that changing suitably the numbering of the variables we may consider $\nu_{Y_{i,j}^*}(t_\xi, T_\xi)$ as an element of $\mathcal{D}(Y_{i,j})$. The sets $Y_{i,j}, \ldots, Y_{j,j}, Y - \bigcup_i Y_{i,j}$ being disjoint, we infer by (d$_1$) and (d$_2$) that for $t \geq 0$

$$(68) \qquad F_j(t) = \varrho_{Y - \bigcup_i Y_{i,j}}(t_\xi - t, T_\xi - t) \prod_i \nu_{Y_{i,j}^*}(t_\xi - t, T_\xi - t) \epsilon \mathcal{D}(Y), \quad j \epsilon M.$$

From inequality (57) we deduce that the function $f(t) = 1 - \exp(-\pi_1^1 t)$ is a distribution function on the semi-axis $t \geq 0$, whence by (d$_3$) and (68) we get

$$(69) \qquad G_Y^j(t_\xi, T_\xi) = \pi_1^1 \int_0^\infty \exp(-\pi_1^1 t) F_j(t) \, dt = \int_0^\infty F_j(t) \, df(t) \epsilon \mathcal{D}(Y) \quad (j \epsilon M),$$

$$(70) \qquad G_Y^0(t_\xi, T_\xi) = \pi_1^1 \int_0^\infty \exp(-\pi_1^1 t) \varrho_Y(t_\xi - t, T_\xi - t) \, dt \, \epsilon \, \mathcal{D}(Y).$$

Setting $\lambda_j = \pi_1^j / \pi_1^1$ for $j \epsilon M$, and

$$\lambda_0 = 1 - \sum_{j \epsilon M} \pi_1^j / \pi_1^1$$

we have $\lambda_j \geq 0$ $(j \epsilon M)$, $\lambda_0 \geq 0$, and

$$\lambda_0 + \sum_{j \epsilon M} \lambda_j = 1$$

(we take into account the inequality $j \geq 2$ valid for $j \epsilon M$ and formulae (8) and (9)). By aid of (d$_4$) and formulae (67), (68), (69), and (70) we get

$$\nu_Y(t_\xi, T_\xi) = \lambda_0 G_Y^0 + \sum_{j \epsilon M} \lambda_j G_Y^j \epsilon \mathcal{D}(Y),$$

whence formula (67) defines a distribution function in the case (64).

From (67) we deduce

$$\Delta_Y(h_\xi, k_\xi) \nu_Y(t_\xi - T, T_\xi - T)$$
$$= \sum_j \pi_1^j \int_0^\infty \exp(-\pi_1^1 t) \Delta_{Y - \bigcup_i Y_{i,j}}(h_\xi, k_\xi) \varrho_{Y - \bigcup_i Y_{i,j}}\big(t_\xi - (T+t), T_\xi - (T+t)\big) \times$$
$$\times \prod_i \Delta_{Y_{i,j}^*}(h_\xi, k_\xi) \nu_{Y_{i,j}^*}\big(t_\xi - (T+t), T_\xi - (T+t)\big) \, dt +$$
$$+ \Big(\pi_1^1 - \sum_j \pi_1^j\Big) \int_0^\infty \exp(-\pi_1^1 t) \Delta_Y(h_\xi, k_\xi) \varrho_Y\big(t_\xi - (T+t), T_\xi - (T+t)\big) \, dt.$$

Replacing in the integrals on the right side of the last formula $T + t$ by a new variable we get, after simple computations,

$$(71) \qquad \Delta_Y(h_\xi, k_\xi) \nu_Y(t_\xi - T, T_\xi - T) \leq \exp(\pi_1^1 T) \Delta_Y(h_\xi, k_\xi) \nu_Y(t_\xi, T_\xi)$$

for the sets satisfying condition (64).

Suppose now that

$$(72) \qquad Y = Z \cup \langle 1 \rangle \qquad \text{where} \qquad \langle 1 \rangle \bar\epsilon Z.$$

In this case we set

$$(73) \qquad \nu_Y(t_\xi, T_\xi) = \nu_Z(t_\xi, T_\xi) - \exp(-\pi_1^1 T_{\langle 1 \rangle}) \nu_Z(t_\xi - T_{\langle 1 \rangle}, T_\xi - T_{\langle 1 \rangle})$$

if $t_{\langle 1 \rangle} > 0$ and $T_{\langle 1 \rangle} > 0$; in the remaining cases we set $\nu_Y(t_\xi, T_\xi) = 0$. The distribution function $\nu_Z(t_\xi, t_\xi)$ is already defined, for the set Z satisfies condition (64). Hence follows also the continuity on the left in each variable of the function defined by (73). From (73) for $t_{\langle 1 \rangle} > 0$ and $T_{\langle 1 \rangle} > 0$ we deduce the equality

$$\Delta_Y(h_\xi, k_\xi) \nu_Y(t_\xi, T_\xi) = \exp(-\pi_1^1 T_{\langle 1 \rangle}) \Delta_Z(h_\xi, k_\xi) \nu_Z(t_\xi - T_{\langle 1 \rangle}, T_\xi - T_{\langle 1 \rangle}) -$$
$$- \exp\big(-\pi_1^1 (T_{\langle 1 \rangle} + k_{\langle 1 \rangle})\big) \Delta_Z(h_\xi, k_\xi) \nu_Z\big(t_\xi - (T_{\langle 1 \rangle} + k_{\langle 1 \rangle}), T_\xi - (T_{\langle 1 \rangle} - k_{\langle 1 \rangle})\big),$$

for $t_{\langle 1 \rangle} > 0$, $T_{\langle 1 \rangle} + k > 0$ and $T_{\langle 1 \rangle} \leq 0$ the equality

$$\Delta_Y(h_\xi, k_\xi) \nu_Y(t_\xi, T_\xi) = \Delta_Z(h_\xi, k_\xi) \nu_Z(t_\xi, T_\xi) -$$
$$- \exp\big(-\pi_1^1 (T_{\langle 1 \rangle} + k_{\langle 1 \rangle})\big) \Delta_Z(h_\xi, k_\xi) \nu_Z\big(t_\xi - (T_{\langle 1 \rangle} + t_{\langle 1 \rangle}), T_\xi - (T_{\langle 1 \rangle} + t_{\langle 1 \rangle})\big),$$

and in the remaining cases $\Delta_Y(h_\xi, k_\xi) \nu_Y(t_\xi, T_\xi) = 0$.

Inequality (71) being true for the set Z, we obtain for arbitrary t_ξ, T_ξ and non-negative h_ξ, k_ξ

$$\Delta_Y(h_\xi, k_\xi) \nu_Y(t_\xi, T_\xi) \geq 0.$$

We see from (73) that $\nu_Y(t_\xi, T_\xi) = \nu_Z(t_\xi, T_\xi)$ for $t_\xi = T_\xi = \infty$, $\xi \epsilon Y$. Hence $\nu_Y(t_\xi, T_\xi) = 1$ if $t_\xi = T_\xi = \infty$ for $\xi \epsilon Y$. If $t_\xi \leq 0$ or $T_\xi \leq 0$ for a certain $\xi \epsilon Z$, then $\nu_Z(t_\xi, T_\xi) = 0$ and $\nu_Z(t_\xi - T, T_\xi - T) = 0$ for $T > 0$, which together with (73) leads to $\nu_Y(t_\xi, T_\xi) = 0$ if $t_\xi \leq 0$ or $T_\xi \leq 0$ for a certain $\xi \epsilon Y$. Thus we have shown that formula (73) defines a distribution function, which completes the inductive definition of $\nu_Y(t_\xi, T_\xi)$ for all Y. By definitions (63), (67), and (73) it is easily seen that for $Y \cap Z = 0$ the following compatibility condition is satisfied:

$$(74) \qquad \nu_{Y \cup Z}(t_\xi, T_\xi) = \nu_Y(t_\xi, T_\xi) \qquad \text{if} \qquad t_\xi = T_\xi = \infty \text{ for } \xi \epsilon Z.$$

The distribution function $\nu_Y(t_\xi, T_\xi)$ determines in the product (62) a normed σ-measure μ_Y defined on all Borel sets, i. e., on the sets of the σ-field $\mathfrak{B}(Y)$. By the compatibility condition (74), $A \epsilon \mathfrak{B}(Y) \cap \mathfrak{B}(Z)$ implies

$\mu_Y(A) = \mu_Z(A)$. Setting $\mu(A) = \mu_Y(A)$ when $A \epsilon \mathfrak{B}(Y)$, we define a function μ on the field of cylinders whose bases are finite dimensional Borel sets, i. e., on

$$\bigcup_Y \mathfrak{B}(Y),$$

where Y are finite subsets of $\mathcal{Z}$. The function μ is, by definition, a σ-measure on every $\mathfrak{B}(Y)$ (Y is a finite set), and $\mu(W) = 1$.

By a theorem of Kolmogorov (see, for instance, [6], p. 212) it follows that there exists a unique extension of the function μ to a normed σ-measure, which will be denoted by P_0; this measure is defined on the smallest σ-field containing the field

$$\bigcup_Y \mathfrak{B}(Y)$$

of the finite dimensional cylinders, whence on the σ-field $\mathfrak{B}_W$ (compare section 7).

By the definition of P_0 and by formulae (63), (67), and (73) we get

(75)
$$P_0\big(w_1(\langle 1 \rangle) = a, w_2(\langle 1 \rangle) < T\big) = \begin{cases} 1 - \exp(-\pi_1^1 T) & \text{if } T > 0 \text{ and } a = 0, \\ 0 & \text{elsewhere,} \end{cases}$$

(76)
$$P_0\big(\bigcap_{\xi \epsilon Y} \{w_1(\xi) < t_\xi, w_2(\xi) < T_\xi\} \cap \{w_2(\langle 1 \rangle) < T\}\big)$$
$$= P_0\big(\bigcap_{\xi \epsilon Y} \{w_1(\xi) < t_\xi, w_2(\xi) < T_\xi\}\big) -$$
$$- \exp(-\pi_1^1 T) P_0\big(\bigcap_{\xi \epsilon Y} \{w_1(\xi) < t_\xi - T, w_2(\xi) < T_\xi - T\}\big)$$

for $T > 0$ and the sets Y satisfying condition (64),

(77)
$$P_0\big(\bigcap_{\xi \epsilon Y} \{w_1(\xi) < t_\xi, w_2(\xi) < T_\xi\}\big)$$
$$= \sum_j \pi_1^j \int_0^\infty \exp(-\pi_1^1 t)\, \delta\big(\bigcap_{\xi \epsilon Y - \bigcup_i Y_{i,j}} \{w_1(\xi) < t_\xi - t, w_2(\xi) < T_\xi - t\}\big) \times$$
$$\times \prod_i P_0\big(\bigcap_{\xi \epsilon Y_{i,j}} \{w_1(\xi - \langle 1, \langle i, j \rangle \rangle) < t_\xi - t, w_2(\xi - \langle 1, \langle i, j \rangle \rangle) < T_\xi - t\}\big) dt +$$
$$+ \big(\pi_1^1 - \sum_j \pi_1^j\big) \int_0^\infty \exp(-\pi_1^1 t)\, \delta\big(\bigcap_{\xi \epsilon Y} \{w_1(\xi) < t_\xi - t, w_2(\xi) < T_\xi - t\}\big) dt$$

for the sets Y satisfying (64). The sets $Y_{i,j}$ are defined by formula (65) and the σ-measure δ by

$$\delta(A) = \begin{cases} 1 & \text{as} \quad w^0 \epsilon A, \\ 0 & \text{as} \quad w^0 \notin A, \end{cases}$$

where $w^0 = \langle w_1^0, w_2^0 \rangle$ is a pair of two functions identically equal to 0.

The following formulae define the transformations $\mathcal{T}_t$ and $\mathcal{L}_\xi$ of the space W:

(78) $\mathcal{T}_t w = \langle u_1, u_2 \rangle$ where $u_1(\xi) = w_1(\xi) - t, u_2(\xi) = w_2(\xi) - t;$

(79) $\mathcal{L}_\xi w = \langle u_1, u_2 \rangle$ where $u_1(\eta - \xi) = w_1(\eta), u_2(\eta - \xi) = w_2(\eta)$

for $\eta \epsilon K(\xi)$.

LEMMA 8. *If* $A \epsilon \mathfrak{B}\big(\mathcal{Z} - (\langle 1 \rangle)\big)$, *then*

(80) $P_0\big(A \cap \{w_2(\langle 1 \rangle) < T\}\big) = P_0(A) - \exp(-\pi_1^1 T) P_0(\mathcal{T}_T A)$

for $T > 0$.

Proof. Let $\mathfrak{K}$ be the class of all the sets A belonging to the σ-field $\mathfrak{B}\big(\mathcal{Z} - (\langle 1 \rangle)\big)$ and satisfying our Lemma. By formula (76) we see that $Y \subset \mathcal{Z} - (\langle 1 \rangle)$ implies

(81) $\bigcap_{\xi \epsilon Y} \{t_\xi^{(1)} \leqslant w_1(\xi) < t_\xi^{(2)}, T_\xi^{(1)} \leqslant w_2(\xi) < T_\xi^{(2)}\} \epsilon \mathfrak{K}.$

It follows that the finite disjoint unions of the sets of form (81) belong to the class $\mathfrak{K}$, whence $\mathfrak{K}$ contains the smallest field spanned upon the sets (59) for $\xi \neq \langle 1 \rangle$. From (80) we directly deduce that the limit of a monotone sequence of sets of the class $\mathfrak{K}$ also belongs to the class $\mathfrak{K}$. Thus the class $\mathfrak{K}$ is monotone (compare, for instance, [6], p. 26). Hence it follows by the definition of the σ-field $\mathfrak{B}(\mathcal{Z} - (\langle 1 \rangle))$ (compare [6], p. 27), which implies that $\mathfrak{K} = \mathfrak{B}(\mathcal{Z} - (\langle 1 \rangle))$, which implies the lemma.

LEMMA 9. *If* $A_{ij} \epsilon \mathfrak{B}\big(K(\langle 1, \langle i, j \rangle \rangle)\big)$, *then*

(82) $P_0\big(\bigcap_{j \epsilon M} \bigcap_i A_{ij}\big) = \sum_{j \epsilon M} \pi_1^j \int_0^\infty \exp(-\pi_1^1 t)\, \delta\big(\bigcap_{s \neq j} \bigcap_i \mathcal{T}_t A_{is}\big) \times$
$$\times \prod_i P_0(\mathcal{T}_t \mathcal{L}_{\langle 1, \langle i,j \rangle \rangle} A_{ij})\, dt + \big(\pi_1^1 - \sum_{j \epsilon M} \pi_1^j\big) \int_0^\infty \exp(-\pi_1^1 t)\, \delta\big(\mathcal{T}_t \bigcap_{i,j} A_{ij}\big)\, dt.$$

The proof of this lemma is similar to the proof of the foregoing one. Formula (77) implies equality (82) for the sets A_{ij} of form (81). Thus (82) is valid for finite disjoint unions of the sets of form (81). Then we verify that the class of the sets A_{ij} (depending, of course, on i and j) satisfying the lemma is monotone, whence it is equal to $\mathfrak{B}\big(K(\langle 1, \langle i, j \rangle \rangle)\big)$, which concludes the proof.

10. We shall now prove some lemmas which imply the measurability of the set V together with the equality $P_0(V) = 1$.

It follows directly from (75) that the set

(83) $V_1 = \{w_1(\langle 1 \rangle) = 0, w_2(\langle 1 \rangle) > 0\}$

is of full measure:

(84) $$P_0(V_1) = 1.$$

For $\xi \epsilon \Xi$, $n, m = 2, 3, \ldots, r$, $s = 1, 2, \ldots, r \neq s$, $n \neq m$, $k = 1, 2, \ldots, n$, let us introduce the following notation:

(85) $$A_{m,r,s,n,k}(\xi)$$

$$= \{w_2(\xi) < w_1(\xi)\} \cup \{w_1(\xi) = w_2(\xi) < w_2(\langle \xi, r, m \rangle \cup \{w_2(\xi) \neq w_1(\langle \xi, r, m \rangle)\})\} \cup$$

$$\cup \{w_2(\langle \xi, r, m \rangle) > w_1(\langle \xi, r, m \rangle), w_2(\langle \xi, s, m \rangle) = w_1(\langle \xi, s, m \rangle)\} \cup$$

$$\cup \{w_2(\langle \xi, r, m \rangle) > w_1(\langle \xi, r, m \rangle), w_2(\langle \xi, k, n \rangle) > w_1(\langle \xi, k, n \rangle)\},$$

where $\langle \xi, r, m \rangle$ denotes the element $\langle x_1, \ldots, x_n, \langle r, m \rangle \rangle$ for $\xi = \langle x_1, \ldots, x_n \rangle$. The definitions of the transformations $\mathcal{T}_t$ and $\mathcal{L}_\eta$ imply

(86) $$\mathcal{T}_t A_{m,r,s,n,k} = A_{m,r,s,n,k}(\xi),$$

(87) $$\mathcal{L}_\eta A_{m,r,s,n,k} = A_{m,r,s,n,k}(\xi - \eta) \quad \text{for} \quad \xi \epsilon K(\eta).$$

LEMMA 10. *For every $\xi \epsilon \Xi$ we have $P_0(A_{m,r,s,n,k}(\xi)) = 0$.*

Proof. We shall prove this by induction with respect to the length of the element ξ.

(α) Let the length of the element ξ be equal to 1, *i. e.*, $\xi = \langle 1 \rangle$. From (83) and (84) we infer that

(88) $$P_0(w_2(\langle 1 \rangle) < w_1(\langle 1 \rangle)) = 0,$$

(89) $$P_0(w_1(\langle 1 \rangle) = w_2(\langle 1 \rangle) < w_2(\langle 1, \langle r, m \rangle \rangle)) = 0.$$

Inserting $A = \{w_1(\langle 1, \langle r, m \rangle \rangle) > t\}$ in formula (80) we obtain

$$P_0(w_2(\langle 1 \rangle) < t < w_1(\langle 1, \langle r, r \rangle \rangle))$$

$$= P_0(w_1(\langle 1, \langle r, m \rangle \rangle) > t) - \exp(-\pi_1^1 t) P_0(w_1(\langle 1, \langle r, m \rangle \rangle) > 0).$$

It follows from formula (82) with $A_{rm} = \{w_1(\langle 1, \langle r, m \rangle \rangle) > T\}$ $(T \geqslant 0)$, $M = \{m\}$, $i = r$, and from (75) that

$$P_0(w_1(\langle 1, \langle r, m \rangle \rangle) > T) = \pi_1^m \int_0^\infty \exp(-\pi_1^1 t) P_0(w_1(\langle 1 \rangle) > T - t) dt$$

$$= \pi_1^m \int_0^\infty \exp(-\pi_1^1 t) dt = \frac{\pi_1^m}{\pi_1^1} \exp(-\pi_1^1 T),$$

whence

$$P_0(w_2(\langle 1 \rangle) < t < w_1(\langle 1, \langle r, m \rangle \rangle)) = \frac{\pi_1^m}{\pi_1^1} \exp(-\pi_1^1 t) - \frac{\pi_1^m}{\pi_1^1} \exp(-\pi_1^1 t) = 0.$$

This immediately implies

(90) $$P_0(w_2(\langle 1 \rangle) \neq w_1(\langle 1, \langle r, m \rangle \rangle)) = 0.$$

We obtain from (77) by aid of formulae (86) and (87)

$$P_0(w_2(\langle 1 \rangle) > w_1(\langle 1, \langle r, m \rangle \rangle), w_2(\langle 1, \langle s, m \rangle \rangle) = w_1(\langle 1, \langle s, m \rangle \rangle))$$

$$= \pi_1^m \int_0^\infty P_0(w_2(\langle 1 \rangle) > w_2(\langle 1 \rangle)) P_0(w_2(\langle 1 \rangle) = w_1(\langle 1 \rangle)) \exp(-\pi_1^1 t) dt = 0,$$

$$P_0(w_2(\langle 1, \langle r, m \rangle \rangle) > w_1(\langle 1, \langle r, m \rangle \rangle), w_2(\langle 1, \langle k, n \rangle \rangle) > w_1(\langle 1, \langle k, n \rangle \rangle))$$

$$= \pi_1^m \int_0^\infty P_0(w_2(\langle 1 \rangle) > w_1(\langle 1 \rangle)) \delta(w_2(\langle 1, \langle k, n \rangle \rangle))$$

$$> w_1(\langle 1, \langle k, n \rangle \rangle)) \exp(-\pi_1^1 t) dt +$$

$$+ \pi_1^m \int_0^\infty P_0(w_2(\langle 1 \rangle) > w_1(\langle 1 \rangle)) \delta(w_2(\langle 1, \langle r, m \rangle \rangle))$$

$$> w_1(\langle 1, \langle r, m \rangle \rangle)) \exp(-\pi_1^1 t) dt = 0,$$

whence in view of formulae (88), (89), and (90) we obtain our Lemma.

(β) Suppose now that

(91) $$\xi = \langle 1, \langle i, j \rangle, \ldots \rangle \neq \langle 1 \rangle$$

and that our Lemma is valid for elements of length less than that of ξ. It follows from (91) that

$$A_{m,r,s,n,k}(\xi) \epsilon \mathfrak{B}(K(\langle 1, \langle i, j \rangle \rangle)),$$

whence in virtue of formulae (82), (86), and (87)

(92) $$P_0(A_{m,r,s,n,k}(\xi)) = \pi_1^j \int_0^\infty P_0(A_{m,r,s,n,k}(\xi - \langle 1, \langle i, j \rangle \rangle)) \exp(-\pi_1^1 t) dt +$$

$$+ (\pi_1^1 - \pi_1^j) \int_0^\infty \delta(A_{m,r,s,n,k}(\xi)) \exp(-\pi_1^1 t) dt.$$

By the hypothesis of induction

(93) $$P_0(A_{m,r,s,n,k}(\xi - \langle 1, \langle i, j \rangle \rangle)) = 0,$$

since the length of element $\xi - \langle 1, \langle i, j \rangle \rangle$ is less than the length of ξ. By the definition of the set $A_{m,r,s,n,k}(\xi)$ and the measure δ we also have $\delta(A_{m,r,s,n,k}(\xi)) = 0$, which, in view of (92) and (93), implies the validity of the lemma for the element ξ. This concludes the proof.

COROLLARY. *Let us write*

$$A = \bigcup_{\xi \epsilon \Xi} \bigcup_{m=2}^{\infty} \bigcup_{\substack{n=2 \\ n \neq m}}^{\infty} \bigcup_{k=1}^{n} \bigcup_{r=1}^{m} \bigcup_{\substack{s=1 \\ s \neq r}}^{m} A_{m,r,s,n,k}(\xi).$$

We may easily verify from (85) that the set A consists of functions w which do not satisfy conditions (b), (c), *or* (d) *(compare section 3). It follows by lemma 10 that*

$$(95) \qquad P_0(A) = 0.$$

Let us now recollect the well-known properties of homogeneous regular Markov processes with a denumerable multitude of states, satisfying conditions (13) and (14), i. e., of the processes describing a stochastic cascade. Suppose that the intensities are determined by formula (56).

Let $Q_k(n, T)$ be the probability that the function $\omega(t)$ (the realization of the process) has n jumps in an interval of length T, under the hypothesis that at the origin of the interval the function $\omega(t)$ is equal to k. Then we have

$$Q_k(0, T) = \exp(-\pi_k^k T),$$

$$(96) \qquad Q_k(1, T) = \sum_{j \neq k} \pi_k^j \int_0^T \exp(-\pi_k^k t) \exp\big(-\pi_k^j(T-t)\big) dt,$$

$$Q_k(n+1, T) = \sum_{j \neq k} \pi_k^j \int_0^T \exp(-\pi_k^k t) Q_j(n, T-t) dt$$

(compare [2], p. 458; for more general processes compare [3], p. 268).

For the processes satisfying condition (14) we have

$$(97) \qquad Q_j(n, T) = \sum_{n_1 + \ldots + n_j = n} \prod_{i=1}^{j} Q_1(n_i, t)$$

(this equality follows for $n = 0$ from (15); the general proof proceeds by induction). Inserting $k = 1$ in formulae (96) and taking into account that $Q_0(n, t) = 0$ for $n \neq 0$ we obtain in view of (97)

$$(98) \qquad Q_1(0, T) = \exp(-\pi_1^1 T),$$

$$(99) \qquad Q_1(1, T) = \sum_{j \neq 1} \pi_1^j \int_0^T \exp(-\pi_1^1 t) \exp\big(-j\pi_1^1(T-t)\big) dt,$$

$$(100) \quad Q_1(n+1, T)$$
$$= \sum_{j=2}^{\infty} \pi_1^j \int_0^T \exp(-\pi_1^1 t) \Big[\sum_{n_1 + \ldots + n_j = n} \prod_{i=1}^{j} Q_1(n_i, T-t) \Big] dt \quad \text{for} \quad n \geq 1.$$

The regular processes satisfy the equality

$$(101) \qquad \sum_{n=0}^{\infty} Q_1(n, T) = 1 \qquad (T \geq 0),$$

for the space of the process consists of step functions.

For every positive integer n and $T \geq 0$ write

$$(102) \quad B_n(T) = \bigcup_{\xi_1, \ldots, \xi_n \in \varXi} \bigcap_{r=1}^{n} \{w_1(\xi_r) < w_2(\xi_r) < T\} \cap$$
$$\cap \bigcup_{\eta \neq \xi_1, \ldots, \xi_n} \big[\{w_1(\eta) = w_2(\eta)\} \cup \{w_2(\eta) \geq T\}\big],$$

$$(103) \qquad V_2 = \bigcap_{k=1}^{\infty} \bigcup_{n=0}^{\infty} B_n(k).$$

We see that, if the function w satisfies conditions (a), (b), (c), and (d) (p. 239) and belongs to the set V_2, then it also satisfies condition (e).

LEMMA 11. *We have $P_0(V_2) = 1$.*

Proof. In order to prove this it suffices to show, in view of (103), that for every $T \geq 0$

$$P_0\big(\bigcup_{n=0}^{\infty} B_n(T)\big) = 1.$$

i. e., the sets $B_n(T)$ being disjoint for different indices n, that

$$(104) \qquad \sum_{n=0}^{\infty} P_0\big(B_n(T)\big) = 1.$$

The function w satisfying conditions (a), (b), (c), and (d), i. e., belonging to the set $V_1 \cap [W-A]$, is in the set $B_r(T)$ if and only if $w_2(\langle 1 \rangle) \geq T$. Thus we deduce from formulae (83) and (95) that the sets $B_0(T)$ and $\{w_2(\langle 1 \rangle) \geq T\}$ are almost equal, whence in virtue of (75)

$$(105) \qquad P_0\big(B_0(T)\big) = \exp(-\pi_1^1 T).$$

Similarly, we may verify that the sets $B_1(T)$ and

$$(106) \quad \bigcup_{j=2}^{\infty} \bigcap_{i=1}^{j} \{w_2(\langle 1, \langle i, j \rangle \rangle) \geq T, w_2(\langle 1 \rangle) < T\} \cup$$
$$\cup \bigcap_{j=2}^{\infty} \bigcap_{i=1}^{j} \{w_1(\langle 1, \langle i, j \rangle \rangle) = w_2(\langle 1, \langle i, j \rangle \rangle), w_2(\langle 1 \rangle) < T\}$$

are almost equal. Taking into account, in view of condition (b), the fact that the summands in (106) are almost disjoint we obtain according to formula (77)

$$(107) \quad P_0\big(B_1(T)\big) = \sum_{j=2}^{\infty} P_0\big(\bigcap_{i=1}^{j} \{w_2(\langle 1, \langle i, j \rangle \rangle) \geq T, w_2(\langle 1 \rangle) < T\}\big) +$$
$$+ P_0\big(\bigcap_{j=2}^{\infty} \bigcap_{i=1}^{j} \{w_1(\langle 1, \langle i, j \rangle \rangle) =$$
$$= w_2(\langle 1, \langle i, j \rangle \rangle), w_2(\langle 1 \rangle) < T\}\big)$$
$$= \sum_{j \neq 1} \pi_1^j \int_0^T \exp(-\pi_1^1 t) \exp\big(-j\pi_1^1(T-t)\big) dt.$$

Suppose now that $n \geqslant 1$. If $w \, \epsilon \, V_1 \cap [W-A]$ and $w \, \epsilon \, B_{n+1}(T)$, then $w_2(\langle 1 \rangle) < T$, whence the sets $B_{n+1}(T)$ and

$$(108) \qquad \bigcup_{\xi_1, \ldots, \xi_n \neq \langle 1 \rangle} \bigcap_{r=1}^{n} \{w_1(\xi_r) < w_2(\xi_r) < T\} \cup$$

$$\cup \bigcap_{\eta \neq \xi_1, \ldots, \xi_n, \langle 1 \rangle} [\{w_1(\eta) \neq w_2(\eta)\} \cup \{w_2(\eta) \geqslant T\}]$$

are almost identical. Setting

$$(109) \qquad B_k^{(i,j)}(T) = \bigcup_{\xi_1, \ldots, \xi_k \, \epsilon \, K(\langle 1, \langle i, j \rangle \rangle)} \bigcap_{r=1}^{k} \{w_1(\xi_r) < w_2(\xi_r) < T\} \cap$$

$$\cap \bigcap_{\substack{\eta \, \epsilon \, K(\langle 1, \langle i, j \rangle \rangle) \\ \eta \neq \xi_1, \ldots, \xi_k}} [\{w_1(\eta) = w_2(\eta)\} \cup \{w_2(\eta) \geqslant T\}]$$

we may represent the set (108) as the union

$$\bigcup_{j=2}^{\infty} \bigcup_{n_1 + \ldots + n_j = n} \bigcap_{i=1}^{j} B_{n_i}^{(i,j)}(T),$$

in which the summands are almost disjoint, conditions (a), (b), (c), and (d) being satisfied almost everywhere. Thus we have

$$(110) \qquad P_0\big(B_{n+1}(T)\big) = \sum_{j=2}^{\infty} \sum_{n_1 + \ldots + n_j = n} P_0\big(\bigcap_{i=1}^{j} B_{n_i}^{(i,j)}(T)\big).$$

By definition (109)

$$B_k^{(i,j)}(T) \, \epsilon \, \mathfrak{B}\big(K(\langle 1, \langle i, j \rangle \rangle)\big),$$

$$\mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} B_k^{(i,j)}(T) = B_k(T), \qquad \delta\big(\bigcap_{i=1}^{j} B_{n_i}^{(i,j)}(t)\big) = 0.$$

On the other hand, it follows from (102) that $\mathcal{T}_t B_k(T) = B_k(T-t)$, $B_k(t) = 0$ for $t < 0$. Taking into account these formulae we obtain from (82) the probability

$$P_0\big(\bigcap_{i=1}^{j} B_{n_i}^{(i,j)}(T)\big) = \pi_1^j \int_0^T \exp(-\pi_1^1 t) \prod_{i=1}^{j} P_0\big(B_{n_i}(T-t)\big) dt.$$

Inserting the expression obtained into formula (110) we get for $n \geqslant 1$

$$(111) \qquad P_0\big(B_{n+1}(T)\big) = \sum_{j=2}^{\infty} \pi_1^j \int_0^T \exp(-\pi_1^1 t) \Big[\sum_{n_1 + \ldots + n_j = n} \prod_{i=1}^{j} P_0\big(B_{n_i}(T-t)\big)\Big] dt.$$

We see from (98), (99), (105), and (107) that $P_0\big(B_0(T)\big) = Q_1(0, T)$, $P_0\big(B_1(T)\big) = Q_1(1, T)$, whence in view of the recurrence formulae (100) and (111) $P_0 B_n(T) = Q_1(n, T)$ $(n = 0, 1, \ldots)$. Hence by (101) we obtain (104), which **proves** the lemma.

We see from (83), (94), and (103) that the function w belongs to the set $V_1 \cap V_2 \cap [W-A]$ if and only if it satisfies conditions (a), (b), (c), (d), and (e) (section 3) and $w_1(\xi) \leqslant w_2(\xi)$ for all $\xi \, \epsilon \, \Xi$; thus the pair $\langle w_1(\xi), w_2(\xi) \rangle$ may be considered as the interval $[w_1(\xi), w_2(\xi))$. Hence we have $V = V_1 \cap V_2 \cap [W-A]$. By formulae (83), (94), and (103) we deduce hence the measurability of the set V: $V \, \epsilon \, \mathfrak{B}_W$.

It follows from (84), Lemma 11, and the corollary to Lemma 10 that $P_0(V) = 1$. The relativization of the σ-measure P_0 to the set V gives a normed σ-measure P defined on the σ-field $\mathfrak{B}_V$ (compare section 7).

11. We shall now prove that the σ-measure P generates a conditional probability (in the sense of section 1) satisfying conditions (I), (II), and (III) (compare section 4).

Let us set for $A \, \epsilon \, \mathfrak{B}\big(k(\xi), t\big)$ and $B \, \epsilon \, \mathfrak{B}\big(\Xi - K(\xi), 0\big)$

$$(112) \qquad P\big(A | \{t \, \epsilon \, v(\xi)\} \cap B\big) = P(\mathcal{T}_t \mathcal{L}_\xi A).$$

We observe that, t, ξ, B being fixed, the function $P\big(A | \{t \, \epsilon \, v(\xi)\} \cap B\big)$ is a normed σ-measure on $\mathfrak{B}\big(K(\xi), t\big)$. The transformations $\mathcal{T}_t$ and $\mathcal{L}_\xi$ coincide on the σ-field $\mathfrak{B}\big(K(\xi), t\big)$ with the transformations T_t and L_ξ respectively. Hence it follows directly from definition (112) that the function $P\big(A | \{t \, \epsilon \, v(\xi)\} \cap B\big)$ satisfies conditions (I), (II), and (III). In order to prove that it is equal to the conditional probability induced by the probability P, it is sufficient to prove that for $A \, \epsilon \, \mathfrak{B}\big(K(\xi), t\big)$ and $B \, \epsilon \, \mathfrak{B}\big(\Xi - K(\xi), 0\big)$ the equality

$$(113) \qquad P\big(A \cap \{t \, \epsilon \, v(\xi)\} \cap B\big) = P(\mathcal{T}_t \mathcal{L}_\xi A) P\big(\{t \, \epsilon \, v(\xi)\} \cap B\big)$$

is valid.

Suppose that equality (113) is proved for the sets $A \, \epsilon \, \mathfrak{B}\big(K(\xi), t\big)$ of the form

$$(114) \qquad \bigcap_{i=1}^{m} \{v_1(\xi_i) < a_i, v_2(\xi_i) < b_i\},$$

where $v_1(\eta)$ and $v_2(\eta)$ denote the left and the right bound of the interval $v(\eta)$ respectively. Then equality (113) is valid for the sets A of the form

$$(115) \qquad \bigcap_{i=1}^{m} \{a_i^{(1)} \leqslant v(\xi_i) < a_i^{(2)}, b_i^{(1)} \leqslant v(\xi_i) < b_i^{(2)}\}$$

and for finite disjoint unions of sets of this form. It is easily seen that equality (113) is valid for the limit of the monotone sequence $A_1, A_, \ldots$ of sets if it is valid for each of the sets A_i. The finite disjoint unions of the sets (115) give the smallest field containing the sets of form (20). Thus we see that the class of the sets $A \, \epsilon \, \mathfrak{B}\big(K(\xi), t\big)$ satisfying (113) is monotone and contains the smallest field spanned upon the sets of

form (20) contained in $\mathscr{B}(K(\xi), t)$, whence it is identical with the σ-field $\mathscr{B}(K(\xi), t)$ (compare the proof of Lemma 8). It follows that it is sufficient to prove that equality (113) is valid for the sets A of form (114). Analogously, we may state that it is sufficient to prove equality (113) for the sets B of form (114). The proof for these sets will be carried out by induction with respect to the length of the element ξ.

(α) Suppose first that the length of the element ξ is equal to 1, $i.\ e.$, that $\xi = \langle 1 \rangle$. Then the σ-field $\mathscr{B}(\varXi - K(\xi), 0) = \mathscr{B}(\varXi - \varXi, 0)$ consists of the empty set and the entire set V. If B is equal to the empty set, then (113) is trivially satisfied, for both sides of the formula are equal to zero. Suppose now that $B = V$; then the equality to be proved may be written as

$$(116) \qquad P\big(A \frown \{t\,\epsilon\,v(\langle 1 \rangle)\}\big) = P(\mathcal{C}_t A) P\big(t\,\epsilon\,v(\langle 1 \rangle)\big).$$

Every set $A\,\epsilon\,\mathscr{B}_r$ of form (114) may be represented as the intersection $A = A_0 \frown \{v_2(\langle 1 \rangle) < T\}$ where $A_0\,\epsilon\,\mathscr{B}(\varXi - (\langle 1 \rangle), 0)$.

If $T \leqslant t$, then $\{v_2(\langle 1 \rangle) < T\} \frown \{t\,\epsilon\,v(\langle 1 \rangle)\} = 0$ and $\mathcal{C}_t\{v_2(\langle 1 \rangle) < T\}$ $\subset \{v_2(\langle 1 \rangle) \leqslant 0\} = 0$, whence (116) is satisfied, for both sides of the equality are equal to zero. Suppose now that $T > t$. Then

$$(117) \qquad \begin{aligned} A \frown \{t\,\epsilon\,v(\langle 1 \rangle)\} &= A_0 \frown \{v_2(\langle 1 \rangle) < T\} - A_0 \frown \{v_2(\langle 1 \rangle) \leqslant t\}, \\ \mathcal{C}_t A &= \mathcal{C}_t A_0 \frown \{v_2(\langle 1 \rangle) < T-t\}. \end{aligned}$$

By formula (80) we get hence

$$\begin{aligned} P\big(A \frown \{t\,\epsilon\,v(\langle 1 \rangle)\}\big) &= P\big(A_0 \frown \{v_2(\langle 1 \rangle) < T\}\big) - P\big(A_0 \frown \{v_2(\langle 1 \rangle) \leqslant t\}\big) \\ &= P(\mathcal{C}_t A_0)\exp(-\pi_1^1 t) - P(\mathcal{C}_T A_0)\exp(-\pi_1^1 T), \\ P(\mathcal{C}_t A) &= P\big(\mathcal{C}_t A_0 \frown \{v_2(\langle 1 \rangle) < T-t\}\big) \\ &= P(\mathcal{C}_t A_0) - P(\mathcal{C}_T A)\exp\big(-\pi_1^1(T-t)\big). \end{aligned}$$

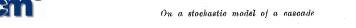It follows that $P\big(A \frown \{t\,\epsilon\,v(\langle 1 \rangle)\}\big) = P(\mathcal{C}_t A)\exp(-\pi_1^1 t)$, which gives (116) in virtue of (75). Thus equality (113) is proved in the case $\xi = \langle 1 \rangle$.

(β) Suppose now that

$$(118) \qquad \xi = \langle 1, \langle i, j \rangle, \dots \rangle \neq \langle 1 \rangle$$

and that (113) is valid for elements of a length less than that of ξ. It follows from notation (118) that $A\,\epsilon\,\mathscr{B}(K(\xi), t)$ implies

$$(119) \qquad \begin{aligned} &A\,\epsilon\,\mathscr{B}\big(K(\langle 1, \langle i, j \rangle \rangle), 0\big), \\ &\mathcal{C}_t \mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} A\,\epsilon\,\mathscr{B}\big(K(\xi - \langle 1, \langle i, j \rangle \rangle), t-\tau\big). \end{aligned}$$

Every set $B\,\epsilon\,\mathscr{B}(\varXi - K(\xi), 0)$ of form (114) may be represented as the intersection

$$B = \bigcap_m \bigcap_{r=1}^m B_{r,m} \frown \{v_2(\langle 1 \rangle) < T\} \quad \text{where} \quad B_{r,m}\,\epsilon\,\mathscr{B}\big(K(\langle 1, \langle r, m \rangle \rangle), 0\big).$$

If $T \geqslant t$, then $\{v_2(\langle 1 \rangle) < T\} \frown \{t\,\epsilon\,v(\xi)\} = \{t\,\epsilon\,v(\xi)\} \frown \{v_2(\langle 1 \rangle) < t\}$. Thus we may suppose in the proof that

$$(120) \qquad T < t.$$

We get by (80) and (82)

$$(121) \qquad P\big(A \frown \{t\,\epsilon\,v(\xi)\} \frown B\big) = P\big(A \frown \{t\,\epsilon\,v(\xi)\}\big) \frown \bigcap_m \bigcap_{r=1}^m B_{r,m} -$$
$$- P\big(\mathcal{C}_T A \frown \{t - T\,\epsilon\,v(\xi)\} \frown \bigcap_m \bigcap_{r=1}^m \mathcal{C}_T B_{r,m}\big)\exp(-\pi_1^1 T)$$
$$= \pi_1^j \int_0^t \exp(-\pi_1^1 \tau) P\big(\mathcal{C}_t \mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} A \frown \{t - \tau\,\epsilon\,v(\xi - \langle 1, \langle i, j \rangle \rangle)\} \frown$$
$$\frown \mathcal{C}_\tau \mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} B_{i,j}\big) \prod_{s \neq i} P(\mathcal{C}_\tau \mathcal{L}_{\langle 1, \langle s, j \rangle \rangle} B_{s,j}) d\tau +$$
$$+ \sum_{m \neq j} \pi_1^m \int_0^t \exp(-\pi_1^1 \tau) \prod_{r=1}^m P(\mathcal{C}_t \mathcal{L}_{\langle 1, \langle r, m \rangle \rangle} B_{r,m}) d\tau -$$
$$- \exp(-\pi_1^1 T)\pi_1^j \int_0^{t-T} \exp(-\pi_1^1 \tau) P(\mathcal{C}_{T+\tau} \mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} A) \frown$$
$$\frown \{t - T - \tau\,\epsilon\,v(\xi - \langle 1, \langle i, j \rangle \rangle)\} \frown \mathcal{C}_{T+\tau} \mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} B_{i,j}) d\tau -$$
$$- \exp(-\pi_1^1 T) \sum_{m \neq j} \pi_1^m \int_0^{t-T} \exp(-\pi_1^1 \tau) \prod_{r=1}^m P(\mathcal{C}_{T+\tau} \mathcal{L}_{\langle 1, \langle r, m \rangle \rangle} B_{r,m}) d\tau,$$

$$(122) \qquad P\big(\{t\,\epsilon\,v(\xi)\} \frown B\big) = P\big(\{t\,\epsilon\,v(\xi)\} \frown \bigcap_m \bigcap_{r=1}^m B_{r,m}\big) -$$
$$- \exp(-\pi_1^1 T) P\big(\{t - T\,\epsilon\,v(\xi)\} \frown \bigcap_m \bigcap_{r=1}^m \mathcal{C}_T B_{r,m}\big)$$
$$= \pi_1^j \int_0^t \exp(-\pi_1^1 \tau) P\big(\{t - \tau\,\epsilon\,v(\xi - \langle 1, \langle i, j \rangle \rangle)\} \frown \mathcal{C}_\tau \mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} B_{i,j}\big) \times$$
$$\times \prod_{s \neq i} P(\mathcal{C}_\tau \mathcal{L}_{\langle 1, \langle s, j \rangle \rangle} B_{s,j}) d\tau +$$
$$+ \sum_{m \neq j} \pi_1^m \int_0^t \exp(-\pi_1^1 \tau) \prod_{r=1}^m P(\mathcal{C}_t \mathcal{L}_{\langle 1, \langle r, m \rangle \rangle} B_{r,m}) d\tau -$$
$$- \exp(-\pi_1^1 T)\pi_1^j \int_0^{t-T} \exp(-\pi_1^1 \tau) P\big(\{t - T - \tau\,\epsilon\,v(\xi - \langle 1, \langle i, j \rangle \rangle)\} \frown$$
$$\frown \mathcal{C}_{T+\tau} \mathcal{L}_{\langle 1, \langle i, j \rangle \rangle} B_{i,j}\big) \cdot \prod_{s \neq i} P(\mathcal{C}_{T+\tau} \mathcal{L}_{\langle 1, \langle s, j \rangle \rangle} B_{s,j}) d\tau -$$
$$- \exp(-\pi_1^1 T) \sum_{m \neq j} \pi_1^m \int_0^{t-T} \exp(-\pi_1^1 \tau) \prod_{r=1}^m P(\mathcal{C}_{T+\tau} \mathcal{L}_{\langle 1, \langle r, m \rangle \rangle} B_{r,m}) d\tau.$$

By the hypothesis of induction

$$(123) \quad P\big(\mathcal{T}_\tau \mathcal{L}_{\langle 1, \langle i, j \rangle\rangle} A \frown \{t - \tau \, \varepsilon \, v(\xi - \langle 1, \langle i, j \rangle\rangle)\} \frown \mathcal{T}_\tau \mathcal{L}_{\langle 1, \langle i, j \rangle\rangle} B_{i,j}\big)$$

$$= P(\mathcal{T}_{t-\tau} \mathcal{T}_\tau A) P\big(\{t - \tau \, \varepsilon \, v(\xi - \langle 1, \langle i, j \rangle\rangle)\} \frown \mathcal{T}_\tau \mathcal{L}_{\langle 1, \langle i, j \rangle\rangle} B_{i,j}\big),$$

for the length of the element $\xi - \langle 1, \langle i, j \rangle\rangle$ is less than the length of ξ and (119) is satisfied. For $\tau < t$ the equality $\mathcal{T}_{t-\tau} \mathcal{T}_\tau A = \mathcal{T}_t A$ is satisfied. Making use of this equality in formula (123), inserting the expression obtained into (121), and extracting the common factor $P(\mathcal{T}_t A)$, we obtain as the second factor the expression on the right side of formula (122). This proves equality (113), which concludes the proof of the Lemma.

It follows from (113) that the function defined by (112) is equal to the conditional probability induced by the probability P and satisfying conditions (I), (II), and (III). Thus the system $\langle V, \mathcal{B}_V, P \rangle$ is a cascade process.

12. We shall now prove that the intensities of the process $\langle \Omega(V), \mathcal{B}_{\Omega(V)}, \Omega P \rangle$ (which by Theorem 1 is a homogeneous regular Markov process with a denumerable multitude of states, satisfying conditions (13) and (14)) are equal to $\pi_1^0, \pi_1^1, \pi_1^2, \ldots$

By definition (21) of the function $\omega(v, t)$ it follows that

$$(124) \quad E_0 = \bigcap_{j=2}^{j} \bigcap_{i=1}^{i} \{v_1(\langle 1, \langle i, j \rangle\rangle) = v_2(\langle 1, \langle i, j \rangle\rangle), v_2(\langle 1 \rangle) \leqslant T\} \subset$$

$$\subset \{\omega(v, T) = 0\},$$

$$(125) \quad E_1 = \{v_2(\langle 1 \rangle) > T\} \subset \{\omega(v, T) = 1\},$$

$$(126) \quad E_k = \bigcap_{i=1}^{k} \{v_2(\langle 1, \langle i, k \rangle\rangle) > T, v_1(\langle 1, \langle i, k \rangle\rangle) \leqslant T\} \subset$$

$$\subset \{\omega(v, T) = k\} \quad \text{for} \quad k = 2, 3, \ldots$$

From formulae (75), (80), and (82) we get the probabilities

$$P(E_0) = \pi_1^0 / \pi_1^1 (1 - \exp(-\pi_1^1 T)), \quad P(E_1) = \exp(-\pi_1^1 T),$$

$$P(E_k) = \pi_1^k \int_0^T \exp(-\pi_1^1 t) \exp(-k \pi_1^1 (T - t)) \, dt \quad (k \geqslant 2),$$

whence, in view of inclusions (124), (125), and (126),

$$(127) \quad \begin{aligned} a_1^k &= \lim_{T \to 0+} \frac{P(\omega(v, T) = k)}{T} \geqslant \pi_1^k \quad \text{for} \quad k \neq 1, \\ a_1^1 &= \lim_{T \to 0+} \frac{1 - P(\omega(v, T) = 1)}{T} \leqslant \pi_1^1. \end{aligned}$$

Thus we have the inequality

$$a_1^1 = \sum_{k \neq 1} a_1^k \geqslant \sum_{k \neq 1} \pi_1^k = \pi_1^1,$$

which, together with (127), implies $a_1^1 = \pi_1^1$. It follows from this equality and the foregoing inequalities that $a_1^k = \pi_1^k \ (k = 0, 1, \ldots)$.

Thus we have proved that intensities of the process $\langle \Omega(V), \mathcal{B}_{\Omega(V)}, \Omega P \rangle$ are equal to $\pi_1^0, \pi_1^1, \pi_1^2, \ldots$ The remark in section 7 implies Theorem 2.

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