Generalized stochastic processes

by

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The process describing the velocities of particles in the Brownian motion under the normal distribution of the trajectories of these particles does not enter into the classical theory of stochastic processes. Indeed, it is known that the trajectories of Brownian particles are non-differentiable with probability 1 (see, for instance, [1], p. 394-395). In the quantum field theory there are considered random events which cannot be described by means of usual functions and are characterized by distributions (generalized functions). I. M. Gelfand [4] and K. Ito [6], using the methods of functional analysis, have developed a theory of generalized stochastic processes (random functionals) which describe random events of this type. A still more general theory has been suggested by A. N. Kolmogorov (and not yet published). In the present paper we deal with such an extension (in some sense the narrowest one) of the concept of stochastic processes that every generalized stochastic process is differentiable. We apply here the method of representation of generalized processes by means of sequences of ordinary processes, which method is in fact implicitly involved in the works of physicists and is analogous to the method of J. Mikusiński of defining the distributions [9], [10]. By the use of this method we get elementary and almost trivial proofs of theorems on generalized stochastic processes, and such probabilistic concepts as correlation function are defined in the same way as in the ordinary stochastic processes. The present paper is devoted to three classes of generalized stochastic processes: processes with independent increments, processes with independent values, and stationary processes[1]). In another paper we intend to deal with the generalized Markov processes and with the concept of the generalized value of a generalized stochastic process at a fixed moment.

I.1. We shall first quote the simplest concepts of the classical theory of stochastic processes that will be needed in the sequel.

[1]) The principal results of this paper were presented to the IV-th Congress of Czechoslovakian Mathematicians at Prague in september 1955 and communicated without proof in the note [12].

Let Ω be the space of elementary events ω (the space of random parameter ω), and let P be the probability on a countably additive field $\mathfrak{B}$ of subsets of the space (the field of random events). In this paper the triple $\langle \Omega, \mathfrak{B}, P\rangle$ is arbitrary but fixed. Any measurable function of two arguments $f(\omega, t)$ with complex or real values $(-\infty < t < \infty, \omega \in \Omega)$ is called a *stochastic process*. We shall consider only stochastic processes whose almost all realizations are locally integrable functions, *i. e.*, for almost any $\omega_0 \in \Omega$ the function $f(\omega_0, t)$ is integrable with respect to t in any finite interval. Two processes are *identical* if they have the same realizations with the probability 1. A stochastic process is called *continuous* if almost all its realizations are continuous functions, and it is called *regular* if almost all its realizations are continuous on the right and have only discontinuities of the first kind. The class of continuous processes will be denoted by $\mathfrak{C}$, that of regular ones by $\mathfrak{R}$.

The sequence $\{f_n(\omega, t)\}$ of processes is called *convergent to the process* $f(\omega, t)$ *as* $n \to \infty$, in symbols $f_n(\omega, t) \rightrightarrows f(\omega, t)$, if for almost any $\omega_0 \in \Omega$ the sequence $\{f_n(\omega_0, t)\}$ of realizations converges almost uniformly to the realization $f(\omega_0, t)$ with respect to the variable t. It is known that if $f_n(\omega, t) \rightrightarrows f(\omega, t)$, then the distributions of the random variables $f_n(\omega, t_1)$, $f_n(\omega, t_2), \ldots, f_n(\omega, t_k)$ tend to the distribution of the random variables $f(\omega, t_1), f(\omega, t_2), \ldots, f(\omega, t_k)$. This fact will be used in the sequel.

The expected value of the stochastic process, *i. e.*, $\int_\Omega f(\omega, t)dP$ will be denoted by $\mathcal{E}f(\omega, t)$.

I.2. The sequence $\{f_n(\omega, t)\}$ $(f_n(\omega, t) \in \mathfrak{C})$ of stochastic processes is called *fundamental* if there exists a convergent sequence of processes $\{F_n(\omega, t)\}$ and a non-negative integer k such that $F_n^{(k)}(\omega, t) = f_n(\omega, t)$ $(n = 1, 2, \ldots)$. (By $G^{(k)}(\omega, t)$ we denote the k-th derivative of the function $G(\omega, t)$ with respect to t). It is easy to verify that every convergent sequence of processes is fundamental.

Two sequences of continuous processes $\{f_n(\omega, t)\}$, $\{g_n(\omega, t)\}$ are called *equivalent* if there exist sequences $\{F_n(\omega, t)\}$, $\{G_n(\omega, t)\}$ of processes convergent to the same limit and an integer k such that $F_n^{(k)}(\omega, t) = f_n(\omega, t)$, $G_n^{(k)}(\omega, t) = g_n(\omega, t)$ $(n = 1, 2, \ldots)$.

The relation of equivalence of fundamental sequences is reflexive, symmetrical, and transitive, whence it splits all the fundamental sequences into disjoint classes. These classes will be called *generalized stochastic processes* and will be denoted by capital Greek letters: $\Phi(\omega, t)$, $\Psi(\omega, t), \ldots$. If $\{f_n(\omega, t)\} \in \Phi(\omega, t)$, we shall write $\Phi(\omega, t) = [f_n(\omega, t)]$, and we shall say in this case that *the sequence* $\{f_n(\omega, t)\}$ *of continuous processes represents the generalized stochastic process* $\Phi(\omega, t)$.

I.3. Now we define the simplest operations on generalized stochastic processes.

Let $\Phi(\omega,t)$ and $\Psi(\omega,t)$ be two generalized stochastic processes. Suppose that $\Phi(\omega,t)=[f_n(\omega,t)]$, $\Psi(\omega,t)=[g_n(\omega,t)]$; then the sequence $\{f_n(\omega,t)+g_n(\omega,t)\}$ of continuous processes is fundamental. The *sum* of generalized stochastic processes $\Phi(\omega,t)$ and $\Psi(\omega,t)$ will be defined by the formula

$$\Phi(\omega,t)+\Psi(\omega,t)=[f_n(\omega,t)+g_n(\omega,t)].$$

It is easily seen that this sum does not depend upon the choice of the representations $\{f_n(\omega,t)\}$ and $\{g_n(\omega,t)\}$. Similarly the *conjugate* of the process is defined by the formula

$$\overline{[f_n(\omega,t)]}=[\overline{f_n(\omega,t)}],$$

and the *product by the complex scalar* λ is defined by

$$\lambda[f_n(\omega,t)]=[\lambda f_n(\omega,t)].$$

The product of two generalized stochastic processes is not defined in general. We shall restrict ourselves to the multiplications by certain particular processes. The continuous process $f(\omega,t)$ will be called a *multiplicator* if for any fundamental sequence $\{f_n(\omega,t)\}$ the sequence $\{f(\omega,t)f_n(\omega,t)\}$ is also fundamental.

It can be shown that every process whose almost all realizations are infinitely many times differentiable is a multiplicator.

If $f(\omega,t)$ is a multiplicator, we set $f(\omega,t)[f_n(\omega,t)]=[f(\omega,t)f_n(\omega,t)]$.

We shall show, for example, that the product defined here does not depend on the choice of the representation $\{f_n(\omega,t)\}$.

Let $[f_n(\omega,t)]=[g_n(\omega,t)]$; let us write $h_{2k}(\omega,t)=f_k(\omega,t)$, $h_{2k-1}(\omega,t)=g_k(\omega,t)$ $(k=1,2,\ldots)$. The sequence $\{h_n(\omega,t)\}$ is fundamental, whence by the definition of the multiplicator the sequence $\{f(\omega,t)h_n(\omega,t)\}$ is also fundamental. Therefore there exist a convergent sequence $\{H_n(\omega,t)\}$ of processes and a non-negative integer s such that

$$H_n^{(s)}(\omega,t)=f(\omega,t)h_n(\omega,t)\qquad(n=1,2,\ldots).$$

The sequences

$$F_n(\omega,t)=H_{2n}(\omega,t),\qquad G_n(\omega,t)=H_{2n-1}(\omega,t)\qquad(n=1,2,\ldots)$$

tend to the same limit and

$$F_n^{(s)}(\omega,t)=f(\omega,t)f_n(\omega,t),\qquad G_n^{(s)}(\omega,t)=f(\omega,t)g_n(\omega,t)\qquad(n=1,2,\ldots).$$

Thus the sequences $\{f(\omega,t)f_n(\omega,t)\}$ and $\{f(\omega,t)g_n(\omega,t)\}$ are equivalent, and we have proved that the product by a multiplicator does not depend upon the choice of the representant.

A translation of a generalized stochastic process $\Phi(\omega,t)$ by a real number h will be denoted by $\Phi(\omega,t+h)$ and is defined by the formula $\Phi(\omega,t+h)=[f_n(\omega,t+h)]$ if $\Phi(\omega,t)=[f_n(\omega,t)]$. It is easily seen that this element does not depend on the choice of the representant. It is also easily proved that the space of all generalized stochastic processes with addition and multiplication by complex scalars is a linear space.

A generalized stochastic process $\Phi(\omega,t)$ is called *real* if $\Phi(\omega,t)=\overline{\Phi(\omega,t)}$.

Every generalized real stochastic process has a representant composed of continuous real processes. Indeed, if $\Phi(\omega,t)=[f_n(\omega,t)]$, then $\overline{\Phi(\omega,t)}=[\overline{f_n(\omega,t)}]=\Phi(\omega,t)$ and $\Phi(\omega,t)=\left[\tfrac{1}{2}\left(f_n(\omega,t)+\overline{f_n(\omega,t)}\right)\right]$. The representation $\left[\tfrac{1}{2}\left(f_n(\omega,t)+\overline{f_n(\omega,t)}\right)\right]$ consists of real processes.

It is easily seen that every generalized stochastic process $\Phi(\omega,t)$ may be represented as $\Phi(\omega,t)=\Phi_1(\omega,t)+i\Phi_2(\omega,t)$ where the processes $\Phi_1(\omega,t)$ and $\Phi_2(\omega,t)$ are real. It is sufficient to write

$$\Phi_1(\omega,t)=\tfrac{1}{2}\left(\Phi(\omega,t)+\overline{\Phi(\omega,t)}\right),\qquad \Phi_2(\omega,t)=\tfrac{1}{2}i\left(\overline{\Phi(\omega,t)}-\Phi(\omega,t)\right).$$

I.4. We shall now define the differentiation of generalized stochastic processes. We first prove a lemma.

LEMMA 1. *Every generalized stochastic process has a representant composed of processes which are polynomials of t, i. e., are of the form $a_0(\omega)+a_1(\omega)t+\ldots+a_k(\omega)t^k$.*

Proof. Let

$$\Phi(\omega,t)=[f_n(\omega,t)]. \tag{1}$$

By the definition of the fundamental sequence there exist a convergent sequence of continuous processes $\{F_n(\omega,t)\}$ and a non-negative integer k such that

$$F_n^{(k)}(\omega,t)=f_n(\omega,t)\qquad(n=1,2,\ldots). \tag{2}$$

Let $F(\omega,t)$ be the limit of $\{F_n(\omega,t)\}$. Obviously $F(\omega,t)\in\mathfrak{C}$, whence for any fixed t_0 the function $F(\omega,t_0)$ is measurable with respect to ω. Let ω be fixed and let $B_{n,T}(\omega,t)$ be a polynomial which, in the interval $-T\leqslant t\leqslant T$, is equal to the n-th Bernstein polynomial of $F(\omega,t)$. In other words,

$$B_{n,T}(\omega,t)=\frac{1}{(2T)^n}\sum_{k=0}^{n}\binom{n}{k}F\left(\omega,T\left(\frac{2k}{n}-1\right)\right)(T+t)^k(T-t)^{n-k}.$$

The measurability of $F(\omega,t)$ for fixed t directly implies the measurability of $B_{n,T}(\omega,t)$ with respect to ω. Thus $B_{n,T}(\omega,t)\in\mathfrak{C}$.

Let us set $W_n(\omega,t) = B_{k_n,n}(\omega,t)$ $(n=1,2,\ldots)$ where k_n are chosen so that

(3) $$W_n(\omega,t) \rightrightarrows F(\omega,t).$$

Obviously, the process defined by the equality

(4) $$w_n(\omega,t) = W_n^{(k)}(\omega,t) \qquad (n=1,2,\ldots)$$

is a polynomial in t (of degree not exceeding $n-k$). By (3) and (4) it follows that the sequence $\{w_n(\omega,t)\}$ is fundamental, and by (2) and the definition of the process $F(\omega,t)$ the sequences $\{f_n(\omega,t)\}$ and $\{w_n(\omega,t)\}$ are equivalent. Hence by (1) we infer that $\Phi(\omega,t) = [w_n(\omega,t)]$. Thus the lemma is proved.

If $\{h_n(\omega,t)\}$ is a representation of a generalized stochastic process $\Phi(\omega,t)$ having almost all the realizations infinitely many times differentiable, then the sequence $\{h_n'(\omega,t)\}$ is fundamental. If $\{g_n(\omega,t)\}$ is another representation of $\Phi(\omega,t)$ also having almost all the realizations infinitely many times differentiable, then, as is easily seen, the sequences $\{h_n'(\omega,t)\}$ and $\{g_n'(\omega,t)\}$ are equivalent, whence the processes $[h_n'(\omega,t)]$ and $[g_n'(\omega,t)]$ are identical. They are called the *generalized derivative* of the generalized stochastic process $\Phi(\omega,t)$ and are denoted by $d\Phi(\omega,t)/dt$. From Lemma 1 it follows that every generalized process is differentiable.

We shall be now concerned with the question which ordinary stochastic processes may be considered as generalized stochastic processes. At first we identify every generalized stochastic process $\Phi(\omega,t)$ having the representation $f(\omega,t), f(\omega,t), \ldots$ with the continuous process $f(\omega,t)$. Thus the class $\mathfrak{C}$ is isomorphically embedded into the class of all generalized stochastic processes. This isomorphism retains the operations introduced in section I.3. If the process $F(\omega,t)$ has almost all the realizations locally integrable, then the process

$$F_0(\omega,t) = \int_0^t F(\omega,u)\,du$$

is continuous, *i. e.*, a generalized process. We make the generalized process $dF_0(\omega,t)/dt$ correspond to the process $F(\omega,t)$. This is a homomorphism in the class of all the processes with locally integrable realizations. Yet, in the class $\mathfrak{R}$ of regular processes it is an isomorphism retaining all the operations introduced so far. In particular the following implication holds: if $F(\omega,t)\in\mathfrak{C}$ and $F'(\omega,t)\in\mathfrak{R}$, then $dF(\omega,t)/dt = F'(\omega,t)$. Thus we see that the regular processes may be treated as generalized processes. In the sequel, speaking about ordinary stochastic processes we shall mean exclusively the regular processes.

From the definition of the derivative of the generalized stochastic process we immediately obtain the formulae:

$$\frac{d}{dt}\big(\Phi(\omega,t)+\Psi(\omega,t)\big) = \frac{d}{dt}\Phi(\omega,t)+\frac{d}{dt}\Psi(\omega,t),$$

$$\overline{\frac{d}{dt}\Phi(\omega,t)} = \frac{d}{dt}\overline{\Phi(\omega,t)},$$

$$\frac{d}{dt}\big(f(\omega,t)\Phi(\omega,t)\big) = f'(\omega,t)\Phi(\omega,t)+f(\omega,t)\frac{d}{dt}\Phi(\omega,t),$$

where $f'(\omega,t)$, and consequently $f(\omega,t)$ are multiplicators.

We shall also prove the following property of the generalized derivative:

If $d\Phi(\omega,t)/dt = 0$, then $\Phi(\omega,t)$ is a random variable independent of t, i. e., $\Phi(\omega,t) = a(\omega)$.

Proof. By Lemma 1 the process $\Phi(\omega,t)$ may be represented by the sequence $\{w_n(\omega,t)\}$ composed of polynomials of the variable t. If $d\Phi(\omega,t)/dt = 0$, then the sequences $w_1(\omega,t), w_2(\omega,t), \ldots$ and $0,0,\ldots$ are equivalent. Hence by the definition of the equivalence of sequences we deduce the existence of continuous processes $F_n(\omega,t)$, $G_n(\omega,t)$, $F(\omega,t)$ $(n=1,2,\ldots)$ and of an integer k such that

(5) $$F_n(\omega,t) \rightrightarrows F(\omega,t),$$

(6) $$G_n(\omega,t) \rightrightarrows F(\omega,t),$$

(7) $$F_n^{(k)}(\omega,t) = w_n'(\omega,t), \qquad G_n^{(k)}(\omega,t) = 0 \qquad (n=1,2,\ldots).$$

From the last formula it follows that almost all the realizations of the processes $G_n(\omega,t)$ $(n=1,2,\ldots)$ are polynomials of the variable t of degree less than k. Hence by (6) almost all the realizations of the process $F(\omega,t)$ are also polynomials of t of degree less than k. The sequence $\{w_n(\omega,t)\}$ being fundamental, there exist a continuous process $H(\omega,t)$ and a sequence $\{H_n(\omega,t)\}$ $(n=1,2,\ldots)$ of continuous processes satisfying, for a certain s, the conditions

(8) $$H_n^{(s)}(\omega,t) = w_n(\omega,t) \qquad (n=1,2,\ldots),$$

(9) $$H_n(\omega,t) \rightrightarrows H(\omega,t).$$

Without loss of generality we may suppose that $s=k-1$. Then the formulae (7) and (8) imply $H_n^{(k)}(\omega,t)-F_n^{(k)}(\omega,t) = 0$ $(n=1,2,\ldots)$. Hence almost all the realizations of the process $H_n(\omega,t)-F_n(\omega,t)$ $(n=1,2,\ldots)$ are polynomials of t of degree less than k. By (5) and (9) we see that almost all the realizations of the process $H(\omega, t)$ are polynomials of degree less than k. Thus

$$H^{(k-1)}(\omega, t) = a(\omega). \tag{10}$$

Formula (9) implies $H(\omega, t) = [H_n(\omega, t)]$; consequently, by (8), we have

$$\frac{d^{k-1}}{dt^{k-1}} H(\omega, t) = [H_n^{(k-1)}(\omega, t)] = [w_n(\omega, t)] = \Phi(\omega, t),$$

which gives the conclusion in virtue of (10).

I.5. Let $\Phi(\omega, t) = [f_n(\omega, t)]$; the element $\omega_0 \epsilon \Omega$ being fixed, we call the class of sequences $\Phi(\omega_0, t) = [f_n(\omega_0, t)]$ the *realization of the generalized stochastic process* $\Phi(\omega, t)$. By Mikusiński's [10] definition of distribution it follows directly that almost all the realizations of the generalized stochastic process are distributions. Moreover, it is easily seen that the realizations of the sum of the processes are sums of the realizations of the components, and the realizations of the derivative $d\Phi(\omega, t)/dt$ are distributional derivatives of the realizations of the process $\Phi(\omega, t)$ (Mikusiński's definition of distribution coincides with that of the generalized stochastic process in the case when the set Ω consists of one element. It is worth while to notice that the theorems on generalized processes in sections I.1.-I.6. are transferred, with a slight modification in the proofs, from the theorems on distributions (see [11])).

THEOREM 1. *Every generalized stochastic process is a generalized derivative of finite order of a continuous process.*

Proof. Let $\Phi(\omega, t)$ be an arbitrary generalized stochastic process and let

$$\Phi(\omega, t) = [f_n(\omega, t)], \tag{11}$$

where $\{f_n(\omega, t)\}$ is a representation composed of processes having almost all the realizations differentiable infinitely many times (*e. g.*, the representation of Lemma 1 composed of polynomials of the variable t). By the definition of the fundamental sequence there exist continuous processes $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ and a non-negative integer k such that

$$F_n(\omega, t) \rightrightarrows F(\omega, t) \tag{12}$$

and

$$F_n^{(k)}(\omega, t) = f_n(\omega, t) \qquad (n = 1, 2, \ldots). \tag{13}$$

By the last formula we infer that almost all the realizations of the process $F_n(\omega, t)$ $(n = 1, 2, \ldots)$ are differentiable infinitely many times, and from (12) it follows that $F(\omega, t) = [F_n(\omega, t)]$. Hence by the definition of the generalized derivative we get

$$\frac{d^k}{dt^k} F(\omega, t) = [F_n^{(k)}(\omega, t)].$$

In virtue of (11) and (13) it follows that

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t);$$

thus the theorem is proved.

I. 6. We now define the convergence in the space of the generalized stochastic processes. The sequence $\{\Phi_n(\omega, t)\}$ of generalized stochastic processes will be called *convergent for $n \to \infty$ to the generalized process* $\Phi(\omega, t)$, in symbols $\Phi_n(\omega, t) \to \Phi(\omega, t)$, if there exist continuous processes $F(\omega, t)$, $F_n(\omega, t)$ $(n = 1, 2, \ldots)$ and a non-negative integer k such that

$$F_n(\omega, t) \rightrightarrows F(\omega, t),$$

$$\frac{d^k}{dt^k} F_n(\omega, t) = \Phi_n(\omega, t) \qquad (n = 1, 2, \ldots),$$

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t).$$

It is easily seen that the limit of the sequence of generalized stochastic processes is uniquely determined. From the definition we directly infer that:

If $\Phi_n(\omega, t) \to \Phi(\omega, t)$, then

$$\frac{d}{dt}\Phi_n(\omega, t) \to \frac{d}{dt}\Phi(\omega, t) \quad \text{and} \quad \overline{\Phi_n(\omega, t)} \to \overline{\Phi(\omega, t)},$$

and, $f(\omega, t)$ being a multiplicator,

$$f(\omega, t)\Phi_n(\omega, t) \to f(\omega, t)\Phi(\omega, t).$$

If $\Phi_n(\omega, t) \to \Phi(\omega, t)$ and $\Psi_n(\omega, t) \to \Psi(\omega, t)$, then $\Phi_n(\omega, t) + \Psi_n(\omega, t) \to \Phi(\omega, t) + \Psi(\omega, t)$.

If $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ are continuous processes and $F_n(\omega, t) \rightrightarrows F(\omega, t)$, then $F_n(\omega, t) \to F(\omega, t)$.

If $\Phi(\omega, t) = [f_n(\omega, t)]$, then $f_n(\omega, t) \to \Phi(\omega, t)$.

Let $f_n(\omega, t)$ be continuous processes and let $f_n(\omega, t) \to \Phi(\omega, t)$; then $\Phi(\omega, t) = [f_n(\omega, t)]$.

Let $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ be regular processes. We shall prove the following implication:

For almost any $\omega_0 \in \Omega$ let the sequence $\{F_n(\omega_0, t)\}$ be convergent in the integral of the p-th power ($p \geqslant 1$, with respect to the variable t) in every finite interval to the function $F(\omega_0, t)$; then $F_n(\omega, t) \to F(\omega, t)$.

Proof. It is easy to verify that

$$\int_0^t F_n(\omega, u)\,du \rightrightarrows \int_0^t F(\omega, u)\,du,$$

whence, by the definition of convergence, $F_n(\omega, t) \to F(\omega, t)$, which concludes the proof.

We shall write, by definition,

$$\Phi(\omega, t) = \sum_{k=1}^{\infty} \Phi_k(\omega, t)$$

when

$$\sum_{k=1}^{n} \Phi_k(\omega, t) \to \Phi(\omega, t).$$

Then we have

$$\frac{d}{dt} \sum_{k=1}^{\infty} \Phi_k(\omega, t) = \sum_{k=1}^{\infty} \frac{d}{dt} \Phi_k(\omega, t). \tag{14}$$

Example. Let us consider the homogeneous Poisson process $F(\omega, t)$. It is well known that in this case

$$F(\omega, t) = \lim_{n\to\infty} \Big(\sum_{|k| \leqslant n} H\big(t - \nu_k(\omega)\big) + \mu_n(\omega) \Big), \tag{15}$$

where

$$H(t) = \begin{cases} 1 & \text{for} \quad t \geqslant 0, \\ 0 & \text{for} \quad t < 0, \end{cases}$$

and $\mu_n(\omega), \nu_k(\omega)$ are certain random variables. Since the derivative $dH(t)/dt$ is equal to the δ-function of Dirac, we infer by formulae (14) and (15) that

$$\frac{d}{dt} F(\omega, t) = \sum_k \delta\big(t - \nu_k(\omega)\big).$$

This formula may be generalized in a certain sense to every generalized stochastic process. Namely:

For every generalized process $\Phi(\omega, t)$ we have

$$\sum_{k=1}^{k_n} \lambda_{k,n}(\omega)\, \delta(t - \nu_{k,n}) \to \Phi(\omega, t),$$

the real numbers $\nu_{k,n}$ and the random variables $\lambda_{k,n}(\omega)$ being suitably chosen.

Proof. Suppose that $\Phi(\omega, t) = [f_n(\omega, t)]$; then there exist continuous processes $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ and a positive integer $k \geqslant 2$ such that

$$F_n^{(k)}(\omega, t) = f_n(\omega, t) \quad (n = 1, 2, \ldots), \tag{16}$$

$$F_n(\omega, t) \rightrightarrows F(\omega, t), \tag{17}$$

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t). \tag{18}$$

From (16) it follows that $F_n^{(k-1)}(\omega, t)$ $(n = 1, 2, \ldots)$ are continuous processes. Hence there exists a partition of the interval $-n \leqslant t \leqslant n$, $-n = \nu_{0,n} \leqslant \nu_{1,n} \leqslant \ldots \leqslant \nu_{k_n,n} = n$, such that

$$|F_n^{(k-1)}(\omega, t) - F_n^{(k-1)}(\omega, \nu_{s,n})| < a_n(\omega) \tag{19}$$

for $\nu_{s,n} \leqslant t \leqslant \nu_{s+1,n}$, $s = 0, 1, \ldots, k_n - 1$, where the sequence $\{a_n(\omega)\}$ converges almost everywhere to zero. Let us write

$$g_n(\omega, t) = \sum_{k=1}^{k_n} \lambda_{k,n}(\omega) H(t - \nu_{k,n}) + \lambda_n(\omega) \tag{20}$$

where

$$H(t) = \begin{cases} 1 & \text{for} \quad t \geqslant 0, \\ 0 & \text{for} \quad t < 0, \end{cases}$$

$$\lambda_{s,n}(\omega) = F_n^{(k-1)}(\omega, \nu_{s,n}) - F_n^{(k-1)}(\omega, \nu_{s-1,n}) \quad (s = 1, 2, \ldots, k_n;\ n = 1, 2, \ldots),$$

$$\lambda_n(\omega) = F_n^{(k-1)}(\omega, \nu_{0,n}) \quad (n = 1, 2, \ldots).$$

By (19) it follows that

$$|F_n^{(k-1)}(\omega, t) - g_n(\omega, t)| < a_n(\omega) \quad \text{for} \quad -n \leqslant t \leqslant n, \quad n = 1, 2, \ldots \tag{21}$$

Let us set

$$G_n(\omega, t) = F_n(\omega, t) - \frac{1}{(k-2)!} \int_0^t (t-u)^{k-2} \big(F_n^{(k-1)}(\omega, u) - g_n(\omega, u)\big)\,du; \tag{22}$$

then, for $|t| < A$ and $n \geqslant A$, it follows from (21) that

$$|G_n(\omega, t) - F_n(\omega, t)| < \frac{A^{k-1}}{(k-1)!}\, a_n(\omega),$$

whence by the definition of the sequence $\{a_n(\omega)\}$ we infer that $G_n(\omega,t)-F_n(\omega,t)\rightrightarrows 0$. Thus, in virtue of formula (17), $G_n(\omega,t)\rightrightarrows F(\omega,t)$, and this implies

$$\frac{d^k}{dt^k}G_n(\omega,t)\to\frac{d^k}{dt^k}F(\omega,t).$$

Taking into account formulae (18), (20), and (22) we see that

$$\sum_{k=1}^{k_n}\lambda_{k,n}(\omega)\,\delta(t-\nu_{k,n})\to\Phi(\omega,t),$$

which concludes the proof.

The *limit* of a family of generalized stochastic processes depending on a continuous parameter is defined as follows: $\Phi_h(\omega,t)\to\Phi(\omega,t)$ as $h\to h_0$ if, for every sequence $h_n\to h_0$, $\Phi_{h_n}(\omega,t)\to\Phi(\omega,t)$.

Let $\Phi(\omega,t)$ be an arbitrary generalized stochastic process. By Theorem 1 there exist a continuous process $F(\omega,t)$ and a non-negative integer k such that

$$\Phi(\omega,t)=\frac{d^k}{dt^k}F(\omega,t).$$

Let us set

$$G(\omega,t)=\int_0^t F(\omega,u)\,du;$$

then we have

$$\Phi(\omega,t)=\frac{d^{k+1}}{dt^{k+1}}G(\omega,t). \tag{23}$$

Almost all the realizations of the process $G(\omega,t)$ have continuous derivatives, whence as $h\to 0$

$$\frac{1}{h}\big(G(\omega,t+h)-G(\omega,t)\big)\rightrightarrows G'(\omega,t).$$

Differentiating k times we deduce hence, in virtue of (23),

$$\frac{1}{h}\big(\Phi(\omega,t+h)-\Phi(\omega,t)\big)\to\frac{d}{dt}\Phi(\omega,t)\quad\text{as}\quad h\to 0. \tag{24}$$

Writing

$$\Delta_h^{(k)}\Phi(\omega,t)=\sum_{s=0}^{k}(-1)^s\binom{k}{s}\Phi\big(\omega,t+(k-s)h\big),$$

we can prove analogously

$$\frac{1}{h^k}\Delta_h^{(k)}\Phi(\omega,t)\to\frac{d^k}{dt^k}\Phi(\omega,t)\quad\text{as}\quad h\to 0.$$

As a corollary we obtain, for arbitrary continuous processes $F(\omega,t)$, the equality

$$\frac{d^k}{dt^k}F(\omega,t)=[n^k\Delta_{1/n}^{(k)}F(\omega,t)].$$

We shall now prove an implication which will be needed in the sequel.

If $\Phi(\omega,t)=[f_n(\omega,t)]$, then for a certain sequence $h_n\to 0$ we have

$$\frac{d}{dt}\Phi(\omega,t)=\left[\frac{1}{h_n}\big(f_n(\omega,t+h_n)-f_n(\omega,t)\big)\right].$$

Proof. From the definition of a fundamental sequence follows the existence of continuous processes $F(\omega,t), F_n(\omega,t)$ $(n=1,2,\ldots)$ such that for a non-negative integer k we have

$$F_n^{(k)}(\omega,t)=f_n(\omega,t)\quad(n=1,2,\ldots), \tag{25}$$

$$\frac{d^k}{dt^k}F(\omega,t)=\Phi(\omega,t) \tag{26}$$

and

$$F_n(\omega,t)\rightrightarrows F(\omega,t). \tag{27}$$

As $h\to 0$ we have

$$\frac{1}{h}\left(\int_0^{t+h}F_n(\omega,u)\,du-\int_0^t F_n(\omega,u)\,du\right)\rightrightarrows F_n(\omega,t),$$

whence by (27) there exists a sequence $h_n\to 0$ such that

$$\frac{1}{h_n}\left(\int_0^{t+h_n}F_n(\omega,u)\,du-\int_0^t F_n(\omega,u)\,du\right)\rightrightarrows F(\omega,t).$$

Differentiating this formula $(k+1)$ times we obtain in view of (25) and (26)

$$\frac{1}{h_n}\big(f_n(\omega,t+h_n)-f_n(\omega,t)\big)\to\frac{d}{dt}\Phi(\omega,t),$$

whence

$$\frac{d}{dt}\Phi(\omega,t)=\left[\frac{1}{h_n}\big(f_n(\omega,t+h_n)-f_n(\omega,t)\big)\right].$$

We may similarly prove the more general implication:

If $\Phi(\omega, t) = [f_n(\omega, t)]$, *then for a certain sequence* $h_n \to 0$ *we have*

$$\frac{d^k}{dt^k}\Phi(\omega, t) = \left[\frac{1}{h_n^k}\Delta_{h_n}^{(k)} f_n(\omega, t)\right].$$

I.7. We shall now give the definition of the expected value for generalized stochastic processes. Let $\Phi(\omega, t)$ be a generalized stochastic process and let us assume for a non-negative integer k

$$\Phi(\omega, t) = \frac{d^k}{dt^k} F(\omega, t),$$

where $F(\omega, t) \in \mathfrak{C}$ and the function $\mathcal{E}|F(\omega, t)|$ is locally integrable. Then the *expected value* $E\Phi(\omega, t)$ *of the generalized process* $\Phi(\omega, t)$ is defined by the formula

$$E\Phi(\omega, t) = \frac{d^k}{dt^k}\mathcal{E}F(\omega, t) \tag{28}$$

The expected value thus defined does not depend on the choice of the integer k and the process $F(\omega, t)$. Indeed, suppose that there are non-negative integers $k_2 \geqslant k_1$ and continuous processes $F_1(\omega, t)$, $F_2(\omega, t)$ such that

$$\frac{d^{k_1}}{dt^{k_1}} F_1(\omega, t) = \Phi(\omega, t), \tag{29}$$

$$\frac{d^{k_2}}{dt^{k_2}} F_2(\omega, t) = \Phi(\omega, t), \tag{30}$$

and that for every $a < b$

$$\int_a^b \mathcal{E}|F_1(\omega, t)|\,dt < \infty, \qquad \int_a^b \mathcal{E}|F_2(\omega, t)|\,dt < \infty. \tag{31}$$

From formulae (29) and (30) it follows that

$$F_2(\omega, t) = \begin{cases} \dfrac{1}{(k_2 - k_2 - 1)!}\displaystyle\int_0^t (t-u)^{k_2-k_1-1} F_1(\omega, u)\,du + W(\omega, t) & \text{as} \quad k_2 > k_1, \\ F_1(\omega, t) + W(\omega, t) & \text{as} \quad k_2 = k_1, \end{cases} \tag{32}$$

where $W(\omega, t)$ is a process whose almost all realizations are polynomials of degree less than k_2. By (31) we may interchange the integration with respect to t and ω, whence we deduce from (32)

$$\mathcal{E}F_2(\omega, t) = \begin{cases} \dfrac{1}{(k_2 - k_1 - 1)!}\displaystyle\int_0^t (t-u)^{k_2-k_1-1}\mathcal{E}F_1(\omega, u)\,du + v(t) & \text{as} \quad k_2 > k_1, \\ \mathcal{E}F_1(\omega, t) + v(t) & \text{as} \quad k_2 = k_1, \end{cases}$$

where $v(t)$ is a polynomial of degree less than k_2. Differentiating this formula k_2 times with respect to t we get

$$\frac{d^{k_1}}{dt^{k_1}}\mathcal{E}F_1(\omega, t) = \frac{d^{k_2}}{dt^{k_2}}\mathcal{E}F_2(\omega, t).$$

Thus we have shown that the expected value does not depend on the choice of k and the process $F(\omega, t)$.

The expected value of a generalized stochastic process is, in general, a distribution.

Examples. (a) Let

$$\Phi(\omega, t) = \sum_{j=1}^{n} \delta\big(t - \nu_j(\omega)\big),$$

where $\nu_j(\omega)$ are random variables with densities $g_j(x)$ $(j = 1, 2, \ldots, n)$. Let us set

$$F(\omega, t) = \sum_{j=1}^{n} \max\big(0, t - \nu_j(\omega)\big) + \sum_{j=1}^{n} \min\big(0, \nu_j(\omega)\big).$$

The process $F(\omega, t)$ is continuous and the following equality holds:

$$\Phi(\omega, t) = \frac{d^2}{dt^2} F(\omega, t).$$

It is easily verified that $|F(\omega, t)| \leqslant n|t|$, which implies the local integrability of the expected value $\mathcal{E}|F(\omega, t)|$. It follows that the generalized process $\Phi(\omega, t)$ has an expected value, defined by the formula

$$E\Phi(\omega, t) = \frac{d^2}{dt^2}\mathcal{E}F(\omega, t).$$

Simple computations give

$$\mathcal{E}F(\omega, t) = \sum_{j=1}^{n}\int_0^t \int_{-\infty}^{y} g_j(x)\,dx\,dy,$$

whence

$$E\Phi(\omega, t) = \sum_{j=1}^{n} g_j(t).$$

(b) Let $\Psi(\omega, t)$ be the derivative of the homogeneous normal process $G(\omega, t)$ with the distribution function

$$P\big(G(\omega, t) < x\big) = \frac{1}{\sigma\sqrt{2\pi|t|}}\int_{-\infty}^{x} \exp\left(-\frac{(u - \lambda t)^2}{2\sigma^2|t|}\right) du,$$

i. e., with the expected value

$$\mathcal{E}G(\omega, t) = \lambda t. \tag{33}$$

The inequality

$$\mathcal{E}|G(\omega, t)| = \frac{\sqrt{2}}{\sigma\sqrt{\pi|t|}}\int_0^\infty u\exp\left(-\frac{(u-\lambda t)^2}{2\sigma^2|t|}\right)du \leqslant 2|t| + \sigma\sqrt{\frac{2|t|}{\pi}}$$

implies the local integrability of the function $\mathcal{E}|G(\omega, t)|$. Now $G(\omega, t)\in\mathfrak{C}$, which implies that the process $\Psi(\omega, t)$ has the expected value and that

$$E\Psi(\omega, t) = \frac{d}{dt}\mathcal{E}G(\omega, t),$$

i. e., in virtue of (33), $E\Psi(\omega, t) = \lambda$.

(c) We shall now give an example of a continuous process whose expected value is the δ-function of Dirac.

Let $\nu(\omega)$ be a random variable with continuous and positive density $g(x)$. Then the stochastic process defined by the formula

$$H(\omega, t) = \frac{\cos\nu(\omega)t}{2\pi g(\nu(\omega))}$$

is continuous. Let

$$F(\omega, t) = \frac{1-\cos\nu(\omega)t}{2\pi(\nu(\omega))^2 g(\nu(\omega))}.$$

The process $F(\omega, t)$ is continuous, takes on non-negative values and $F^{(2)}(\omega, t) = H(\omega, t)$. The equality

$$\mathcal{E}F(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1-\cos tx}{x^2}dx = \frac{1}{2}|t|$$

implies the local integrability of $\mathcal{E}|F(\omega, t)|$. Hence the process $H(\omega, t)$ has the expected value and

$$EH(\omega, t) = \frac{d^2}{dt^2}\mathcal{E}F(\omega, t) = \frac{1}{2}\frac{d^2}{dt^2}|t| = \delta(t).$$

From the definition we directly deduce the following properties of the expected value:

Let $E\Phi(\omega, t)$ exist; then there exists $E\frac{d}{dt}\Phi(\omega, t)$ and we have

$$E\frac{d}{dt}\Phi(\omega, t) = \frac{d}{dt}E\Phi(\omega, t).$$

If $E\Phi(\omega, t)$ and $E\Psi(\omega, t)$ exist, then there exists $E(\Phi(\omega, t)+\Psi(\omega, t))$ and we have

$$E(\Phi(\omega, t)+\Psi(\omega, t)) = E\Phi(\omega, t)+E\Psi(\omega, t).$$

Let $\Phi(\omega, t)$ have the expected value. Then for every positive integer s there exists a process $\Psi(\omega, t)$ having the expected value and satisfying the equality

$$\frac{d^s}{dt^s}\Psi(\omega, t) = \Phi(\omega, t).$$

(In the proof we must assume that

$$\Psi(\omega, t) = \begin{cases} \dfrac{1}{(s-k-1)!}\displaystyle\int_0^t (t-u)^{s-k-1}F(\omega, u)\,du & \text{as} \quad s > k, \\ F^{(k-s)}(\omega, t) & \text{as} \quad s \leqslant k, \end{cases}$$

where the process $F(\omega, t)$ and the integer k appear in the definition (28).)

If $E\Phi(\omega, t)$ exists and if the function $f(t)$ is differentiable infinitely many times, then $Ef(t)\Phi(\omega, t)$ exists and

$$Ef(t)\Phi(\omega, t) = f(t)E\Phi(\omega, t). \tag{34}$$

We shall prove, for example, the last implication. Suppose that

$$\Phi(\omega, t) = \frac{d^k}{dt^k}F(\omega, t), \quad F(\omega, t)\in\mathfrak{C},$$

and that the expected value $\mathcal{E}|F(\omega, t)|$ is locally integrable. Let us write

$$\begin{aligned} G(\omega, t) \\ = f(t)F(\omega, t) + \sum_{s=1}^{k}(-1)^s\binom{k}{s}\frac{1}{(s-1)!}\int_0^t (t-u)^{s-1}f^{(s)}(u)F(\omega, u)\,du. \end{aligned} \tag{35}$$

It is easily seen that $G(\omega, t)\in\mathfrak{C}$ and

$$\frac{d^k}{dt^k}G(\omega, t) = f(t)\Phi(\omega, t). \tag{36}$$

From (35) we deduce for $|t| < A$

$$\mathcal{E}|G(\omega, t)| \leqslant a\left(\mathcal{E}|F(\omega, t)| + \sum_{s=1}^{k}\binom{k}{s}\frac{A^{s-1}}{(s-1)!}\int_{-A}^{A}\mathcal{E}|F(\omega, u)|\,du\right),$$

where

$$a = \max_{\substack{0\leqslant s\leqslant k \\ -A\leqslant t\leqslant A}} |f^{(s)}(t)|.$$

This inequality implies the local integrability of the function $\mathcal{E}|G(\omega, t)|$. Hence from formula (36) it follows that the process $f(t)\Phi(\omega, t)$ has an expected value and

$$Ef(t)\Phi(\omega, t) = \frac{d^k}{dt^k}\mathcal{E}G(\omega, t).$$

Using formula (35) we deduce (34).

The following theorem gives the interrelations between the expected values E and $\mathcal{E}$:

THEOREM 2. *Let $f(\omega, t)\epsilon\mathfrak{R}$, then $E|f(\omega, t)|$ exists if and only if the function $\mathcal{E}|f(\omega, t)|$ is locally integrable. In this case $Ef(\omega, t) = \mathcal{E}f(\omega, t)$.*

Proof. Necessity. Suppose that $E|f(\omega, t)|$ exists. Let $F(\omega, t)$ be a continuous process such that the function $\mathcal{E}|F(\omega, t)|$ is locally integrable and that, for a certain $k \geqslant 1$, we have

$$(37)\qquad |f(\omega, t)| = \frac{d^k}{dt^k}F(\omega, t), \qquad E|f(\omega, t)| = \frac{d^k}{dt^k}\mathcal{E}F(\omega, t).$$

From (37) it follows that the difference

$$F(\omega, t) - \int_0^t\int_0^{x_k}\ldots\int_0^{x_2}|f(\omega, x_1)|\,dx_1dx_2\ldots dx_k$$

is a polynomial of the variable t of degree less than k. It follows that for every h the stochastic process

$$\Delta_h^{(k)}F(\omega, t) = \int_t^{t+h}\int_{x_k}^{x_k+h}\ldots\int_{x_2}^{x_2+h}|f(\omega, x_1)|\,dx_1dx_2\ldots dx_k$$

has a locally integrable expected value. The function $|f(\omega, t)|$ being non-negative, we may interchange the order of integration with respect to t and ω:

$$\mathcal{E}\Delta_h^{(k)}F(\omega, t) = \int_t^{t+h}\int_{x_k}^{x_k+h}\ldots\int_{x_2}^{x_2+h}\mathcal{E}|f(\omega, x_1)|\,dx_1dx_2\ldots dx_k.$$

We deduce therefrom that the integral

$$\int_{x_2}^{x_2+h}\mathcal{E}|f(\omega, x_1)|\,dx_1$$

is finite for almost every x_2 in the interval $t_0 \leqslant x_2 \leqslant t_0 + h$ (see, for instance, [5], § 36).

The quantities t_0 and h being arbitrary, the expected value $\mathcal{E}|f(\omega, t)|$ is locally integrable.

Let us set

$$g(\omega, t) = \int_0^t f(\omega, u)\,du;$$

then $g(\omega, t)\epsilon\mathfrak{C}$, the expected value $\mathcal{E}|g(\omega, t)|$ is locally integrable, and $\mathcal{E}f(\omega, t) = d\mathcal{E}g(\omega, t)/dt$. We also have $Ef(\omega, t) = d\mathcal{E}g(\omega, t)/dt$, which implies $Ef(\omega, t) = \mathcal{E}f(\omega, t)$.

Sufficiency. Suppose that the function $\mathcal{E}|f(\omega, t)|$ is locally integrable. Put

$$h(\omega, t) = \int_0^t|f(\omega, u)|\,du.$$

The process $h(\omega, t)$ is continuous and for every $a < b$ we have

$$\int_a^b\mathcal{E}|h(\omega, t)|\,dt \leqslant \int_a^b\int_{-|t|}^{|t|}\mathcal{E}|f(\omega, u)|\,du\,dt < \infty.$$

Since

$$|f(\omega, t)| = \frac{d}{dt}h(\omega, t),$$

the expected value $E|f(\omega, t)|$ exists. Thus the theorem is proved.

II.1. In this section we shall concern ourselves with the independence of stochastic processes. The generalized stochastic processes $\Phi_1(\omega, t)$, $\Phi_2(\omega, t), \ldots, \Phi_r(\omega, t)$ are said to be *independent* if there exist representations $\{f_{s,n}(\omega, t)\}\,\epsilon\,\Phi_s(\omega, t)$ $(s = 1, 2, \ldots, r)$ such that for arbitrary $n_1, n_2, \ldots, n_r$ the processes

$$(38)\qquad f_{1,n_1}(\omega, t), f_{2,n_2}(\omega, t), \ldots, f_{r,n_r}(\omega, t)$$

are independent.

THEOREM 3. *Let the generalized processes $\Phi_1(\omega, t), \Phi_2(\omega, t), \ldots, \Phi_r(\omega, t)$ be independent; then for every system $m_1, m_2, \ldots, m_r$ of non-negative integers the processes*

$$\frac{d^{m_1}}{dt^{m_1}}\Phi_1(\omega, t), \frac{d^{m_2}}{dt^{m_2}}\Phi_2(\omega, t), \ldots, \frac{d^{m_r}}{dt^{m_r}}\Phi_r(\omega, t)$$

are also independent.

Proof. Let us consider representations $\{f_{s,n}(\omega, t)\}\,\epsilon\,\Phi_s(\omega, t)$ $(s = 1, 2, \ldots)$ such that for every $n_1, n_2, \ldots, n_r$ the processes (38) are independent. Then for every $n_1, n_2, \ldots, n_r$ and $h_1, h_2, \ldots, h_r$ the processes

$$\frac{1}{h_1^{m_1}}\Delta_{h_1}^{(m_1)}f_{1,n_1}(\omega, t), \ldots, \frac{1}{h^{m_r}}\Delta_{h_r}^{(m_r)}f_{r,n_r}(\omega, t)$$

are independent. Choosing the sequences $\{h_{s,n}\}$ $(s = 1, 2, \ldots, r)$ in an appropriate manner (see section I.6) we get

$$\frac{d^{m_s}}{dt^{m_s}}\Phi_s(\omega, t) = \left[\frac{1}{h_{s,n}^{m_s}}\Delta_{h_{s,n}}^{(m_s)} f_{s,n}(\omega, t)\right] \quad (s = 1, 2, \ldots, r).$$

This concludes the proof.

LEMMA 2. *Let the generalized processes $\Phi_1(\omega, t), \Phi_2(\omega, t), \ldots, \Phi_r(\omega, t)$ be independent. Then there exist continuous processes $F_1(\omega, t), F_2(\omega, t), \ldots, F_r(\omega, t)$ and a non-negative integer k such that*

$$\frac{d^k}{dt^k}F_s(\omega, t) = \Phi_s(\omega, t) \quad (s = 1, 2, \ldots, r)$$

and such that the processes

$$\Delta_h^{(k)}F_1(\omega, t), \Delta_h^{(k)}F_2(\omega, t), \ldots, \Delta_h^{(k)}F_r(\omega, t)$$

are independent for every h.

Proof. Let us consider the representations $\{f_{s,n}(\omega, t)\} \epsilon \Phi_s(\omega, t)$ $(s = 1, 2, \ldots, r)$ such that for every $n_1, n_2, \ldots, n_r$ the processes (38) are independent. From the definition of fundamental sequences follows the existence of continuous processes $F_s(\omega, t), F_{s,n}(\omega, t)$ $(s = 1, 2, \ldots, r;$ $n = 1, 2, \ldots)$ and of a non-negative integer k such that $F_{s,n}^{(k)}(\omega, t) = f_{s,n}(\omega, t)$ $(s = 1, 2, \ldots, r; n = 1, 2, \ldots)$ and

$$F_{s,n}(\omega, t) \rightrightarrows F_s(\omega, t) \quad (s = 1, 2, \ldots, r). \tag{39}$$

In particular we have

$$\frac{d^k}{dt^k}F_s(\omega, t) = \Phi_s(\omega, t) \quad (s = 1, 2, \ldots, r).$$

We shall show that the processes $F_s(\omega, t)$ satisfy our lemma. The independence of the processes (38) implies the independence of the processes

$$\int_0^t (t-u)^{k-1} f_{1,n_1}(\omega, u)\,du, \quad \ldots, \quad \int_0^t (t-u)^{k-1} f_{r,n_r}(\omega, u)\,du.$$

Hence the equality

$$\Delta_h^{(k)}F_{s,n}(\omega, t) = \Delta_h^{(k)}\frac{1}{(k-1)!}\int_0^t (t-u)^{k-1} f_{s,n}(\omega, u)\,du$$

gives the independence of the processes $\Delta_h^{(k)}F_{1,n}(\omega, t), \ldots, \Delta_h^{(k)}F_{r,n}(\omega, t)$ $(n = 1, 2, \ldots)$ for every h. Thus in virtue of (39) the processes $\Delta_h^{(k)}F_1(\omega, t), \ldots, \Delta_h^{(k)}F_r(\omega, t)$ are independent, which concludes the proof.

LEMMA 3. *Let the k-th differences of the continuous processes $F_1(\omega, t), F_2(\omega, t), \ldots, F_r(\omega, t)$,*

$$\Delta_h^{(k)}F_1(\omega, t), \quad \Delta_h^{(k)}F_2(\omega, t), \quad \ldots, \quad \Delta_h^{(k)}F_r(\omega, t)$$

be independent for every h. Then the processes

$$\frac{d^k}{dt^k}F_1(\omega, t), \quad \frac{d^k}{dt^k}F_2(\omega, t), \quad \ldots, \quad \frac{d^k}{dt^k}F_r(\omega, t)$$

are independent.

Proof. This follows directly from the possibility of the representation (see section I.6)

$$\frac{d^k}{dt^k}F_s(\omega, t) = [n^k\Delta_{1/n}^{(k)}F_s(\omega, t)] \quad (s = 1, 2, \ldots, r).$$

Lemmas 2 and 3 give

THEOREM 4. *The generalized stochastic processes $\Phi_1(\omega, t), \Phi_2(\omega, t), \ldots, \Phi_r(\omega, t)$ are independent if and only if there is a k such that these processes are generalized derivatives of the k-th order of certain continuous processes $F_1(\omega, t), F_2(\omega, t), \ldots, F_r(\omega, t)$, for which the k-th increments*

$$\Delta_h^{(k)}F_1(\omega, t), \quad \Delta_h^{(k)}F_2(\omega, t), \quad \ldots, \quad \Delta_h^{(k)}F_r(\omega, t)$$

form an independent system for every h.

We shall also need the following strengthening of Lemma 2:

LEMMA 4. *Let the generalized stochastic processes $\Phi_1(\omega, t), \Phi_2(\omega, t), \ldots, \Phi_r(\omega, t)$ be independent, and let the continuous processes $F_1(\omega, t), F_2(\omega, t), \ldots, F_r(\omega, t)$ satisfy, for a certain integer m,*

$$\frac{d^m}{dt^m}F_s(\omega, t) = \Phi_s(\omega, t) \quad (s = 1, 2, \ldots, r). \tag{40}$$

Then for every h the system

$$\Delta_h^{(m)}F_1(\omega, t), \quad \Delta_h^{(m)}F_2(\omega, t), \quad \ldots, \quad \Delta_h^{(m)}F_r(\omega, t)$$

consists of independent processes.

Proof. Suppose that the generalized processes $\Phi_1(\omega, t), \Phi_2(\omega, t), \ldots, \Phi_r(\omega, t)$ are independent. Hence by Lemma 2 there exist continuous processes $G_1(\omega, t), G_2(\omega, t), \ldots, G_r(\omega, t)$ and a non-negative integer k such that

$$\frac{d^k}{dt^k}G_s(\omega, t) = \Phi_s(\omega, t) \quad (s = 1, 2, \ldots, r), \tag{41}$$

and for every h the processes $\Delta_h^{(k)}G_1(\omega, t), \Delta_h^{(k)}G_2(\omega, t), \ldots, \Delta_h^{(k)}G_r(\omega, t)$ are independent.

Let us set for $s = 1, 2, \ldots, r$

$$H_s(\omega, t) = \begin{cases} \dfrac{1}{(m-k-1)!}\displaystyle\int_0^t (t-u)^{m-k-1} G_s(\omega, u)\,du & \text{as} \quad m > k, \\ G_s^{(k-m)}(\omega, t) & \text{as} \quad m \leqslant k. \end{cases} \tag{42}$$

Then we get

$$\Delta_h^{(m)} H_s(\omega, t) = \begin{cases} \Delta_h^{(m-k)} \dfrac{1}{(m-k-1)!}\displaystyle\int_0^t (t-u)^{m-k-1} \Delta_h^{(k)} G_s(\omega, u)\,du & \text{as } m > k, \\ \lim\limits_{n\to\infty}\left(\dfrac{n}{h}\right)^{k-m} \displaystyle\sum_{j_1,\ldots,j_m=0}^{n-1} \Delta_{h/n}^{(k)} G_s\left(\omega, t+\frac{h}{n}\sum_{i=1}^{m} j_i\right) & \text{as } m \leqslant k. \end{cases}$$

Hence for every h the processes $\Delta_h^{(m)} H_1(\omega, t)$, $\Delta_h^{(m)} H_2(\omega, t), \ldots, \Delta_h^{(m)} H_r(\omega, t)$ are independent. The formulae (40), (41), and (42) imply

$$\frac{d^m}{dt^m} H_s(\omega, t) = \frac{d^m}{dt^m} F_s(\omega, t) \qquad (s = 1, 2, \ldots, r),$$

which, in turn, leads to

$$\Delta_h^{(m)} H_s(\omega, t) = \Delta_h^{(m)} F_s(\omega, t) \qquad (s = 1, 2, \ldots, r),$$

which proves the lemma.

COROLLARY. *It follows from Lemma 4 that the independence of generalized processes is invariant under the passage to the limit.*

Indeed, let $\Phi_{1,n}(\omega, t), \Phi_{2,n}(\omega, t), \ldots, \Phi_{r,n}(\omega, t)$ be a sequence of systems composed of independent processes and let $\Phi_{s,n}(\omega, t) \to \Phi_s(\omega, t)$ $(s = 1, 2, \ldots, r)$. By the definition of convergence there exist continuous processes $F_s(\omega, t), F_{s,n}(\omega, t)$ $(s = 1, 2, \ldots, r;\ n = 1, 2, \ldots)$ and a non-negative integer k such that

$$\begin{gathered} \frac{d^k}{dt^k} F_s(\omega, t) = \Phi_s(\omega, t) \qquad (s = 1, 2, \ldots, r), \\ \frac{d^k}{dt^k} F_{s,n}(\omega, t) = \Phi_{s,n}(\omega, t) \qquad (s = 1, 2, \ldots, r;\ n = 1, 2, \ldots), \end{gathered} \tag{43}$$

$$F_{s,n}(\omega, t) \rightrightarrows F_s(\omega, t) \qquad (s = 1, 2, \ldots, r). \tag{44}$$

By Lemma 4 the processes $\Delta_h^{(k)} F_{1,n}(\omega, t), \Delta_h^{(k)} F_{2,n}(\omega, t), \ldots, \Delta_h^{(k)} F_{r,n}(\omega, t)$ are independent for every h, which, in virtue of (44), implies the independence (for every h) of the processes $\Delta_h^{(k)} F_1(\omega, t), \Delta_h^{(k)} F_2(\omega, t), \ldots, \Delta_h^{(k)} F_r(\omega, t)$. From this fact and from (43) we deduce in virtue of Lemma 3 that the processes $\Phi_1(\omega, t), \Phi_2(\omega, t), \ldots, \Phi_r(\omega, t)$ are independent.

II. 2. Our next object is a generalization of the concept of the independence of increments of generalized stochastic processes which for ordinary processes would coincide with the usual concept of the independence of increments. At first sight it seems that the generalized stochastic process with independent increments should be required to have a representation composed of processes (continuous of course) with independent increments. It turns out, however, that this postulate implies the normality of the elements of the representation (see, for instance, [1], p. 420). Hence among regular processes only the normal one has this property (see Theorem 7). Before introducing the definition of the independence of increments of a generalized stochastic process we shall deal with an auxiliary concept — the ε-independence of increments of ordinary processes.

The stochastic process $f(\omega, t)$ $(f(\omega, t) \in \mathfrak{R})$ is said *to have ε-independent increments* $(\varepsilon > 0)$ if for every system $(u_{j,s}, t_{j,s})$ $(j = 1, 2, \ldots, j_s;\ s = 1, 2, \ldots, r)$ of intervals distant one from another by more than ε the (vector valued) random variables

$$\begin{gathered} \langle f(\omega, t_{1,1}) - f(\omega, u_{1,1}), f(\omega, t_{2,1}) - f(\omega, u_{2,1}), \ldots, f(\omega, t_{j_1,1}) - f(\omega, u_{j_1,1})\rangle, \\ \langle f(\omega, t_{1,2}) - f(\omega, u_{1,2}), f(\omega, t_{2,2}) - f(\omega, u_{2,2}), \ldots, f(\omega, t_{j_2,2}) - f(\omega, u_{j_2,2})\rangle, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \langle f(\omega, t_{1,r}) - f(\omega, u_{1,r}), f(\omega, t_{2,r}) - f(\omega, u_{2,r}), \ldots, f(\omega, t_{j_r,r}) - f(\omega, u_{j_r,r})\rangle \end{gathered}$$

are independent.

Further on we shall need the following lemma:

LEMMA 5. *Let $f(\omega, t) \in \mathfrak{R}$; then the process $\Delta_h^{(k)} F(\omega, t)$ has, for every $h > 0$, kh-independent increments if and only if the process*

$$\Delta_h^{(k+1)} \int_0^t F(\omega, u)\,du$$

has $(k+1)h$-independent increments for every h.

Proof. Necessity. Suppose that the process $\Delta_h^{(k)} f(\omega, t)$ has kh-independent increments. Let us consider the system of intervals $(x_{j,s}, y_{j,s})$ $(j = 1, 2, \ldots, j_s;\ s = 1, 2, \ldots, r)$, which are distant from one another by more than $(k+1)h$. Let $y_{j,s} \leqslant u_{i,j,s} \leqslant y_{j,s} + h$ $(i = 1, 2, \ldots, i_{j,s};\ j = 1, 2, \ldots, j_s;\ s = 1, 2, \ldots, r)$. Then the intervals $(u_{i,j,s} + x_{j,s} - y_{j,s}, u_{i,j,s})$ for different s are distant by more than kh. Hence the random variables

$$\begin{gathered} \langle \Delta_h^{(k)} f(\omega, u_{1,1,1}) - \Delta_h^{(k)} f(\omega, u_{1,1,1} + x_{1,1} - y_{1,1}), \ldots\rangle, \\ \langle \Delta_h^{(k)} f(\omega, u_{1,1,2}) - \Delta_h^{(k)} f(\omega, u_{1,1,2} + x_{1,2} - y_{1,2}), \ldots\rangle, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \langle \Delta_h^{(k)} f(\omega, u_{1,1,r}) - \Delta_h^{(k)} f(\omega, u_{1,1,r} + x_{1,r} - y_{1,r}), \ldots\rangle \end{gathered}$$

are independent. Thus, in virtue of the equality

$$\Delta_h^{(k+1)} = \int_0^{y_{j,s}} f(\omega, u)\,du - \Delta_h^{(k+1)} \int_0^{x_{j,s}} f(\omega, u)\,du$$

$$= \int_{y_{j,s}}^{y_{j,s}+h} \left(\Delta_h^{(k)} f(\omega, u) - \Delta_h^{(k)} f(\omega, u+x_{j,s}-y_{j,s})\right) du$$

we deduce the independence of the random variables

$$\left\langle \Delta_h^{(k+1)} \int_0^{y_{1,1}} f(\omega, u)\,du - \Delta_h^{(k+1)} \int_0^{x_{1,1}} f(\omega, u)\,du, \ldots \right\rangle,$$

$$\left\langle \Delta_h^{(k+1)} \int_0^{y_{1,2}} f(\omega, u)\,du - \Delta_h^{(k+1)} \int_0^{x_{1,2}} f(\omega, u)\,du, \ldots \right\rangle,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left\langle \Delta_h^{(k+1)} \int_0^{y_{1,r}} f(\omega, u)\,du - \Delta_h^{(k+1)} \int_0^{x_{1,r}} f(\omega, u)\,du, \ldots \right\rangle.$$

Thus we have proved that the process

$$\Delta_h^{(k+1)} \int_0^t f(\omega, u)\,du$$

has $(k+1)h$-independent increments.

Sufficiency. Suppose now that for every $h > 0$ the process

$$\Delta_h^{(k+1)} \int_0^t f(\omega, u)\,du$$

has $(k+1)h$-independent increments, and let $(x_{j,s}, y_{j,s})$ $(j = 1, 2, \ldots, j_s;$ $s = 1, 2, \ldots, r)$ be a system of intervals distant from one another by more than kh. Let the smallest of these distances be equal to $kh+\varepsilon$ where $\varepsilon > 0$. If $n > h/\varepsilon$, then for different s the distances of the intervals

$$\left(x_{j,s} + \frac{h}{n} \sum_{i=1}^{k} n_i,\ y_{j,s} + \frac{h}{n} \sum_{i=1}^{k} n_i\right) \quad (n_i = 0, 1, \ldots, n-1)$$

are greater than $(k+1)hn^{-1}$. This implies the independence, for $n > h/\varepsilon$, of the random variables

$$\left\langle \Delta_{h/n}^{(k+1)} \int_0^{y_{1,1}+\frac{h}{n}\sum_{i=1}^{k} n_i} f(\omega, u)\,du - \Delta_{h/n}^{(k+1)} \int_0^{x_{1,1}+\frac{h}{n}\sum_{i=1}^{k} n_i} f(\omega, u)\,du, \ldots \right\rangle,$$

$$\left\langle \Delta_{h/n}^{(k+1)} \int_0^{y_{1,2}+\frac{h}{n}\sum_{i=1}^{k} n_i} f(\omega, u)\,du - \Delta_{h/n}^{(k+1)} \int_0^{x_{1,2}+\frac{h}{n}\sum_{i=1}^{k} n_i} f(\omega, u)\,du, \ldots \right\rangle,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left\langle \Delta_{h/n}^{(k+1)} \int_0^{y_{1,r}+\frac{h}{n}\sum_{i=1}^{k} n_i} f(\omega, u)\,du - \Delta_{h/n}^{(k+1)} \int_0^{x_{1,r}+\frac{h}{n}\sum_{i=1}^{k} n_i} f(\omega, u)\,du, \ldots \right\rangle.$$

Arguing as in the first part of the proof and taking into account the equality

$$\Delta_h^{(k)} f(\omega, t) = \lim_{n\to\infty} \sum_{n_1,\ldots,n_k=0}^{n-1} \Delta_{h/n}^{(k+1)} \int_0^{t+\frac{h}{n}\sum_{i=1}^{k} n_i} f(\omega, u)\,du$$

we infer that the random variables

$$\langle \Delta_h^{(k)} f(\omega, y_{1,1}) - \Delta_h^{(k)} f(\omega, x_{1,1}), \ldots \rangle,$$

$$\langle \Delta_h^{(k)} f(\omega, y_{1,2}) - \Delta_h^{(k)} f(\omega, x_{1,2}), \ldots \rangle,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\langle \Delta_h^{(k)} f(\omega, y_{1,r}) - \Delta_h^{(k)} f(\omega, x_{1,r}), \ldots \rangle$$

are independent. Thus the process $\Delta_h^{(k)} f(\omega, t)$ has kh-independent increments.

II.3. It is easily seen that a regular process has independent increments (in the ordinary sense) if and only if, for every $\varepsilon > 0$, it has ε-independent increments. Observing this fact we adopt the following definition:

The generalized stochastic process $\Phi(\omega, t)$ is said *to have independent increments* if there exists a representation $\{f_n(\omega, t)\} \in \Phi(\omega, t)$ such that for every $\varepsilon > 0$ the processes $f_n(\omega, t)$ have ε-independent increments for sufficiently large n.

We shall prove later that the independence of increments defined in this way coincides with the usual one in the class of regular processes.

From the definition of the independence of the increments we directly obtain the following implication:

If $\Phi(\omega, t)$ has independent increments, then $d\Phi(\omega, t)/dt$ has independent increments too.

To prove that it is sufficient to observe that if $\{f_n(\omega, t)\}$ is the representation of the process occuring in the definition of the independence of the increments, then for a certain sequence h_n

$$\left\{\frac{1}{h_n}(f_n(\omega, t+h_n)-f_n(\omega, t))\right\}$$

is a representation of the process $d\Phi(\omega, t)/dt$ (see section I.6.) composed of processes having, for every $\varepsilon > 0$, ε-independent increments for sufficiently large n.

We shall consider first the relations between the generalized processes with independent increments and the ordinary ones with ε-independent increments.

LEMMA 6. *For every generalized process $\Phi(\omega, t)$ with independent increments there exists a continuous process $F(\omega, t)$ and a non-negative integer k such that*

$$\frac{d^k}{dt^k}F(\omega, t) = \Phi(\omega, t),$$

and such that for every $h > 0$ the process $\Delta_h^{(k)}F(\omega, t)$ has kh-independent increments.

Proof. Let $\{f_n(\omega, t)\}$ be a representation of the process $\Phi(\omega, t)$ such that for every $\varepsilon > 0$ the processes $f_n(\omega, t)$ have ε-independent increments for almost all n's. By the definition of the fundamental sequence there exist continuous processes $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ and a non-negative integer k such that

$$\begin{aligned} &\frac{d^k}{dt^k}F_n(\omega, t) = f_n(\omega, t) \quad (n = 1, 2, \ldots), \\ (45) \qquad &\frac{d^k}{dt^k}F(\omega, t) = \Phi(\omega, t), \end{aligned}$$

$$(46) \qquad F_n(\omega, t) \rightrightarrows F(\omega, t).$$

We shall prove that the process $F(\omega, t)$ satisfies the conditions of our lemma. For this sake it is sufficient, in virtue of formula (46), to show that for every $h > 0$ the processes $\Delta_h^{(k)}F_n(\omega, t)$ have kh-independent increments for sufficiently large n. Let $(x_{j,s}, y_{j,s})$ $(j = 1, 2, \ldots, j_s;$ $s = 1, 2, \ldots, r)$ be a system of intervals which are distant from one another by more than kh; let the smallest of these distances be equal to $kh+\varepsilon$ where $\varepsilon > 0$. From the equality

$$(47) \quad \Delta_h^{(k)}F_n(\omega, y_{j,s}) - \Delta_h^{(k)}F_n(\omega, x_{j,s}) = \int_{x_{j,s}}^{x_{j,s}+h}\int_{t_k}^{t_k+h}\cdots\int_{t_2}^{t_2+h}(f_n(\omega, t_1)-f_n(\omega, t_1+x_{j,s}-y_{j,s}))\,dt_1\,dt_2\ldots dt_k$$

resulting from (45) we see that the increments (47) are, for different s, integrals of the increments of the process $f_n(\omega, t)$ on the intervals whose mutual distances are at least ε. Arguing as in the proof of Lemma 5 we infer that for sufficiently large n the increments of the process $\Delta_h^{(k)}F_n(\omega, t)$ are kh-independent, which proves the Lemma 6.

LEMMA 7. *Let $F(\omega, t)\in\mathfrak{R}$. If, for every $h > 0$, the process $\Delta_h^{(k)}F(\omega, t)$ has kh-independent increments, then the generalized process $d^kF(\omega, t)/dt^k$ has independent increments.*

Proof. Let us write

$$G(\omega, t) = \int_0^t F(\omega, u)\,du.$$

The process $G(\omega, t)$ is continuous and

$$\frac{d^k}{dt^k}F(\omega, t) = [n^{k+1}\Delta_{1/n}^{(k+1)}G(\omega, t)]$$

(see section I.6.). By Lemma 5 the process $n^{k+1}\Delta_{1/n}^{(k+1)}G(\omega, t)$ has $(k+1)/n$-independent increments, which completes the proof.

The Lemmas 6 and 7 enable us to state the following necessary and sufficient condition for the independence of increments of generalized stochastic processes:

THEOREM 5. *A generalized stochastic process has independent increments if and only if, for a certain k, it is the generalized derivative of the k-th order of a continuous process $F(\omega, t)$ such that for every $h > 0$ the process $\Delta_h^{(k)}F(\omega, t)$ has kh-independent increments.*

From Lemma 5 follows the following strengthening of Lemma 6, which seems useful in applications:

LEMMA 8. *Let the generalized process $\Phi(\omega, t)$ have independent increments and let the continuous process $F(\omega, t)$ satisfy, for a certain k, the equality*

$$\frac{d^k}{dt^k}F(\omega, t) = \Phi(\omega, t).$$

Then, for every $h > 0$, the process $\Delta_h^{(k)}F(\omega, t)$ has kh-independent increments.

We shall now prove that *in the class of regular processes the generalized independence of increments coincides with the independence of increments in the ordinary sense.*

Proof. Let the regular process $F(\omega, t)$ have independent increments in the ordinary sense; then by Lemma 5 (with $k=0$) for every $h>0$ the process

$$\Delta_h^{(1)} \int_0^t F(\omega, u)\,du$$

has h-independent increments. Taking into account the continuity of the process

$$\int_0^t F(\omega, u)\,du$$

we deduce from Lemma 7 that the process

$$F(\omega, t) = \frac{d}{dt}\int_0^t F(\omega, u)\,du$$

has independent increments (in the generalized sense).

On the other hand, if the process $F(\omega, t)$ has the increments independent in the generalized sense, then by Lemma 8 for every $h>0$ the process

$$\Delta_h^{(1)} \int_0^t F(\omega, u)\,du$$

has h-independent increments. Hence by lemma 5 (with $k=0$) the process $F(\omega, t)$ has 0-independent increments, *i. e.*, has the increments independent in the ordinary sense.

The following theorem shows that the independence of increments of a generalized stochastic process is invariant under the passage to the limit $\to$:

THEOREM 6. *Let $\{\Phi_n(\omega, t)\}$ be a sequence of generalized processes with independent increments and let $\Phi_n(\omega, t) \to \Phi(\omega, t)$ as $n\to\infty$; then the process $\Phi(\omega, t)$ also has independent increments.*

Proof. By the definition of convergence there exist continuous processes $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ and a non-negative integer k such that

(48) $$\frac{d^k}{dt^k} F_n(\omega, t) = \Phi_n(\omega, t) \qquad (n = 1, 2, \ldots),$$

(49) $$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t),$$

(50) $$F_n(\omega, t) \rightrightarrows F(\omega, t).$$

Because of the independence of increments of the processes $\Phi_n(\omega, t)$ we deduce from (48) by Lemma 8 that for every $h>0$ the processes $\Delta_h^{(k)} F_n(\omega, t)$ $(n = 1, 2, \ldots)$ have kh-independent increments. Hence by formula (50) we obtain for every $h>0$ the kh-independence of the increments of the process $\Delta_k^{(h)} F(\omega, t)$. Thus in virtue of formula (49) and Lemma 7 we see that the increments of the process $\Phi(\omega, t)$ are independent, which completes the proof of our theorem.

THEOREM 7. *Let $\{f_n(\omega, t)\}$ be a sequence of normal processes and let $f_n(\omega, t) \to f(\omega, t)$ as $n\to\infty$. If the process $f(\omega, t)$ is regular, then it is normal (whence continuous).*

Proof. Using the fact that the generalized independence of increments is equivalent to independence in the ordinary sense in the class of regular processes, we infer by Theorem 6 that the process $f(\omega, t)$ has independent increments in the ordinary sense. Therefore, in order to prove our theorem it is sufficient to show that the increments of the process $f(\omega, t)$ have a normal distribution. The proof of this fact will be based on the following elementary properties of normal processes:

(a) the limit in probability of a sequence of normal random variables is a normal random variable;

(b) if $g(\omega, t)$ is a normal process, then for arbitrary a, b, and h the random variable

$$\int_a^{a+h} \cdot \int_{x_k}^{x_k+h} \cdots \int_{x_2}^{x_2+h} \big(g(\omega, x_1) - g(\omega, x_1+b)\big)\,dx_1\,dx_2 \ldots dx_k$$

is normal.

From the definition of the convergence of generalized stochastic processes follows the existence of continuous processes $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ and of a non-negative integer k such that

(51) $$F_n^{(k)}(\omega, t) = f_n(\omega, t) \qquad (n = 1, 2, \ldots),$$

(52) $$F^{(k)}(\omega, t) = f(\omega, t),$$

(53) $$F_n(\omega, t) \rightrightarrows F(\omega, t).$$

(In the formulae (51) and (52) the generalized derivative d^k/dt^k is replaced by the ordinary one, for the processes $f(\omega, t), f_n(\omega, t)$, $n = 1, 2, \ldots$, are regular.)

From (51) it follows that

(54) $$\Delta_h^{(k)} F_n(\omega, t_2) - \Delta_h^{(k)} F_n(\omega, t) = \int_{t_2}^{t_2+h} \int_{x_k}^{x_k+h} \cdots \int_{x_2}^{x_2+h} \big(F_n(\omega, x_1) - F_n(\omega, x_1+t_1-t_2)\big)\,dx_1\,dx_2 \ldots dx_k.$$

Hence, because of the property (b), the random variable (54) has a normal distribution. Thus by (53) and the property (a) the random variable $\Delta_h^{(k)}F(\omega,t_2)-\Delta_h^{(k)}F(\omega,t_1)$ also has a normal distribution. The process $f(\omega,t)$ being regular with independent increments, it has at most a denumerable multitude of points of discontinuity, *i. e.*, points t_0 such that

$$\lim_{t\to t_0-0} f(\omega,t)\neq f(\omega,t_0)$$

holds with a positive probability (see [1], p. 407-408). Suppose that t_1 and t_2 are points of continuity of the process $f(\omega,t)$. Then it follows from (52) that the random variables (which, as we have already shown, are normal)

$$\frac{1}{h^k}\left(\Delta_h^{(k)}F(\omega,t_2)-\Delta_h^{(k)}F(\omega,t_1)\right)$$

tend in probability to the increment $f(\omega,t_2)-f(\omega,t_1)$ as $h\to 0$. By property (a) the increment $f(\omega,t_2)-f(\omega,t_1)$ has a normal distribution if t_1 and t_2 are points of continuity of the process $f(\omega,t)$. The set of points of continuity and of the continuity on the right of almost all the realizations of the process $f(\omega,t)$ being dense, it follows that the increments of the process $f(\omega,t)$ over any arbitrary interval have a normal distribution. This completes the proof.

II.4. Among ordinary regular processes only the determined processes $f(\omega,t)=f(t)$ have independent values at different points; these are uninteresting from the point of view of the theory of probability. This is not so in the class of generalized stochastic processes. Generalized stochastic processes with independent values are defined similarly to those with independent increments. Viz., we first define ordinary processes with ε-independent values:

The process $f(\omega,t)$ belonging to the class $\mathfrak{K}$ is said *to have ε-independent values* if, for every system $t_{j,s}$ $(j=1,2,\ldots,j_s;\ s=1,2,\ldots,r)$ of points distant by more than ε from one another, the random variables

$$\langle f(\omega,t_{1,1}),f(\omega,t_{2,1}),\ldots,f(\omega,t_{j_1,1})\rangle,$$
$$\langle f(\omega,t_{1,2}),f(\omega,t_{2,2}),\ldots,f(\omega,t_{j_2,2})\rangle,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\langle f(\omega,t_{1,r}),f(\omega,t_{2,r}),\ldots,f(\omega,t_{j_r,r})\rangle$$

are independent.

Example. If the process $F(\omega,t)$ is regular and has independent increments, then the process $f(\omega,t)=F(\omega,t+\varepsilon)-F(\omega,t)$ $(\varepsilon>0)$ has ε-independent values.

A generalized stochastic process $\Phi(\omega,t)$ is said *to have independent values* if there exists a representation $\{f_n(\omega,t)\}\,\epsilon\,\Phi(\omega,t)$ such that for every $\varepsilon>0$ the processes $f_n(\omega,t)$ have ε-independent values, the integer n being sufficiently large.

Examples. (a) The derivative of a normal process $F(\omega,t)$ has independent values. As a representation appearing in the definition of the independence of values we may choose

$$\left\{n\left(F\left(\omega,t+\frac{1}{n}\right)-F(\omega,t)\right)\right\}.$$

(b) Let $\Phi(\omega,t)$ be a generalized process with independent values and let $\{f_n(\omega,t)\}$ be its representation composed of processes with values ε-independent for sufficiently large n's. Then, for every multiplicator $f(t)$, the process $f(t)\Phi(\omega,t)$ has independent values, and $\{f(t)f_n(\omega,t)\}$ is its representation composed of processes having ε-independent values for n sufficiently large.

(c) Let $f_1(t),f_2(t),\ldots,f_n(t)$ be multiplicators and let the process $F(\omega,t)$ be normal; then the process

$$\sum_{s=1}^{k} f_s(t)\frac{d^s}{dt^s}F(\omega,t)$$

has independent values. This follows directly from the representation

$$\sum_{s=1}^{k} f_s(t)\frac{d_s}{dt_s}F(\omega,t)=\left[\sum_{s=1}^{k} f_s(t)n^s\Delta_{1/n}^{(s)}F(\omega,t)\right].$$

The proofs of the following properties of generalized processes with independent values are the same as the proofs of analogous properties of processes with independent increments:

1. *Let the generalized process $\Phi(\omega,t)$ have independent values and let $F(\omega,t)$ be a continuous process satisfying, for a certain integer k, the equality*

$$\frac{d^k}{dt^k}F(\omega,t)=\Phi(\omega,t);$$

then, for every $h>0$, the process $\Delta_h^{(k)}F(\omega,t)$ has kh-independent values.

2. *Let $F(\omega,t)\,\epsilon\,\mathfrak{K}$. If for every $h>0$ the process $\Delta_h^{(k)}F(\omega,t)$ has kh-independent values, the generalized stochastic process*

$$\frac{d^k}{dt^k}F(\omega,t)$$

has independent values.

3. *The generalized and the ordinary independence of values are equivalent in the class of regular processes.*

4. *The independence of values of generalized stochastic processes is invariant under the passage to the limit* $\rightarrow$.

We shall now prove a theorem connecting the processes with independent values with those with independent increments.

THEOREM 8. *A generalized stochastic process has independent values if and only if it is the generalized derivative of the first order of a generalized process with independent increments.*

Proof. Necessity. Let $\{f_n(\omega, t)\}$ be a representation of the generalized stochastic process $\Phi(\omega, t)$ with independent values, such that for every $\varepsilon > 0$ the processes $f_n(\omega, t)$ have ε-independent values, n being sufficiently large. By the definition of fundamental sequences there exist continuous processes $F(\omega, t), F_n(\omega, t)$ $(n = 1, 2, \ldots)$ and a positive integer k such that

$$F_n^{(k)}(\omega, t) = f_n(\omega, t) \quad (n = 1, 2, \ldots), \tag{55}$$

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t), \tag{56}$$

$$F_n(\omega, t) \rightrightarrows F(\omega, t). \tag{57}$$

Let us set

$$\Psi(\omega, t) = [F_n^{(k-1)}(\omega, t)]. \tag{58}$$

From formulae (55)-(57) it follows directly that

$$\frac{d}{dt} \Psi(\omega, t) = \Phi(\omega, t).$$

We shall show that the process $\Psi(\omega, t)$ has independent increments. Let $(u_{j,s}, t_{j,s})$ $(j = 1, 2, \ldots, j_s;\ s = 1, 2, \ldots, r)$ be a system of intervals distant one from another by more than ε $(\varepsilon > 0)$. From the equality

$$F_n^{(k-1)}(\omega, t_{j,s}) - F_n^{(k-1)}(\omega, u_{j,s}) = \int_{u_{j,s}}^{t_{j,s}} f_n(\omega, u)\, du \tag{59}$$

resulting from (55) we see that the increments are integrals of the values of the process at the points distant from one another by more than ε. Arguing as in the proof of Lemma 5 we deduce thereof the ε-independence of the increments of the process $F_n^{(k-1)}(\omega, t)$ for n sufficiently large. Hence in view of (58) the process $\Psi(\omega, t)$ has independent increments.

Sufficiency. Let $\Psi(\omega, t)$ be a generalized process with independent increments and let $\{f_n(\omega, t)\}$ be its representation such that for every $\varepsilon > 0$ the processes $f_n(\omega, t)$ have ε-independent increments for sufficiently large n's. It is easily verified that if $h_n \to 0$, the processes

$$\frac{1}{h_n}\big(f_n(\omega, t+h_n) - f_n(\omega, t)\big)$$

have, for every $\varepsilon > 0$, ε-independent values for sufficiently large n's. Choosing the sequence $\{h_n\}$ in an appropriate manner we have (see section I.6.)

$$\frac{d}{dt}\Psi(\omega, t) = \left[\frac{1}{h_n}\big(f_n(\omega, t+h_n) - f_n(\omega, t)\big)\right],$$

which proves the independence of values of the process $d\Psi(\omega, t)/dt$. Thus the theorem is proved.

III.1. As in the case of one variable t, we define generalized stochastic processes depending on many variables $t_1, t_2, \ldots, t_r$. It will be manifest later that in the study of generalized stochastic processes dependent on one variable the use of stochastic processes dependent on more than one variable is inevitable (see section III). We shall now supply the definition of generalized stochastic processes dependent of many variables and present the simplest theorems dealing with these processes. The proofs of these theorems will be omitted, for they are identical with the proofs of the corresponding theorems dealing with processes dependent on one variable.

We shall deal first with ordinary stochastic processes dependent on r variables $t_1, t_2, \ldots, t_r$. The process $f(\omega, t_1, t_2, \ldots, t_r)$ is called *continuous* if almost all its realizations $f(\omega_0, t_1, t_2, \ldots, t_r)$ $(\omega_0 \in \Omega)$ are continuous functions of the variables $t_1, t_2, \ldots, t_r$. The class of continuous processes will be denoted by $\mathfrak{C}_r$. Two processes are *identical* if they have the same realization with the probability 1. The sequence of processes $\{f_n(\omega, t_1, t_2, \ldots, t_r)\}$ is said *to be convergent to the process* $f(\omega, t_1, t_2, \ldots, t_r)$, in symbols

$$f_n(\omega, t_1, t_2, \ldots, t_r) \rightrightarrows f(\omega, t_1, t_2, \ldots, t_r),$$

if, for almost every $\omega_0 \in \Omega$, the sequence of the realizations $\{f_n(\omega_0, t_1, \ldots, t_r)\}$ converges almost uniformly with respect to the variables $t_1, t_2, \ldots, t_r$ to the realization $f_n(\omega, t_1, t_2, \ldots, t_r)$. The sequence $\{f_n(\omega, t_1, t_2, \ldots, t_r)\}$ of continuous processes is called *fundamental* if there exists a convergent sequence of continuous processes $\{F_n(\omega, t_1, t_2, \ldots, t_r)\}$ and a system of non-negative integers $k_1, k_2, \ldots, k_r$ such that

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} F_n(\omega, t_1, t_2, \ldots, t_r) = f_n(\omega, t_1, t_2, \ldots, t_r) \quad (n = 1, 2, \ldots).$$

Two sequences of continuous processes $\{f_n(\omega, t_1, \ldots, t_r)\}$ and $\{g_n(\omega, t_1, \ldots, t_r)\}$ are said to be *equivalent* if there exist two sequences of continuous processes $\{F_n(\omega, t_1, \ldots, t_r)\}$ and $\{G_n(\omega, t_1, \ldots, t_r)\}$ convergent to the same limit, and a system of non-negative integers $k_1, k_2, \ldots, k_r$ such that

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} F_n(\omega, t_1, \ldots, t_r) = f_n(\omega, t_1, \ldots, t_r) \qquad (n = 1, 2, \ldots),$$

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} G_n(\omega, t_1, \ldots, t_r) = g_n(\omega, t_1, \ldots, t_r) \qquad (n = 1, 2, \ldots).$$

The relation of equivalence of fundamental sequences is reflexive, symmetric, and transitive, whence it splits the class of all fundamental sequences into disjoint sets; these sets are called, as in the case of one variable, *generalized stochastic processes depending on r variables* $t_1, t_2, \ldots, t_r$ and will be denoted by the symbols $\Phi(\omega, t_1, \ldots, t_r), \Psi(\omega, t_1, \ldots, t_r), \ldots$ If $\{f_n(\omega, t_1, \ldots, t_r)\} \in \Phi(\omega, t_1, \ldots, t_r)$, we write $\Phi(\omega, t_1, \ldots, t_r) = [f_n(\omega, t_1, \ldots, t_r)]$ and we shall say that *the sequence* $\{f_n(\omega, t_1, \ldots, t_r)\}$ *of continuous processes represents the generalized stochastic process* $\Phi(\omega, t_1, \ldots, t_r)$.

Addition, multiplication by a complex scalar and, generally, multiplication by a multiplicator, the conjugate process, and translation are defined by the formulae:

$$[f_n(\omega, t_1, \ldots, t_r)] + [g_n(\omega, t_1, \ldots, t_r)] = [f_n(\omega, t_1, \ldots, t_r) + g_n(\omega, t_1, \ldots, t_r)],$$

$$f(\omega, t_1, \ldots, t_r)[f_n(\omega, t_1, \ldots, t_r)] = [f(\omega, t_1, \ldots, t_r) f_n(\omega, t_1, \ldots, t_r)],$$

$$\overline{[f_n(\omega, t_1, \ldots, t_r)]} = [\overline{f_n(\omega, t_1, \ldots, t_r)}],$$

and if $\Phi(\omega, t_1, \ldots, t_r) = [f_n(\omega, t_1, \ldots, t_r)]$, then, for every system $h_1, \ldots, h_r$ of real numbers,

$$\Phi(\omega, t_1+h_1, \ldots, t_r+h_r) = [f_n(\omega, t_1+h_1, \ldots, t_r+h_r)].$$

The proof of every generalized stochastic process $\Phi(\omega, t_1, \ldots, t_r)$ having a representation composed of processes which are polynomials of the variables $t_1, t_2, \ldots, t_r$ proceeds as for Lemma 1. This enables us to define the generalized derivative

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}}$$

of an arbitrary generalized process $\Phi(\omega, t_1, \ldots, t_r)$ by the formula

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} \Phi(\omega, t_1, \ldots, t_r) = \left[\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} h_n(\omega, t_1, \ldots, t_r)\right],$$

where $\{h_n(\omega, t_1, \ldots, t_r)\}$ is a representation of the process $\Phi(\omega, t_1, \ldots, t_r)$ having almost all the realizations differentiable infinitely many times with respect to the variables $t_1, t_2, \ldots, t_r$. One can prove that if $F(\omega, t_1, \ldots, t_r) \in \mathfrak{C}_r$ and

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} F(\omega, t_1, \ldots, t_r) = 0,$$

then

$$F(\omega, t_1, \ldots, t_r) = \sum_{j=1}^{r}\sum_{s=0}^{k_j-1} f_{s,j}(\omega, t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_r) t_j^s.$$

From the definition of generalized partial derivatives immediately follows the equality

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} \Phi(\omega, t_1, \ldots, t_r) = \frac{\partial^{k_1+\ldots+k_r}}{\partial t_{j_1}^{k_{j_1}}\ldots\partial t_{j_r}^{k_{j_r}}} \Phi(\omega, t_1, \ldots, t_r)$$

for every permutation $j_1, j_2, \ldots, j_r$ of the integers $1, 2, \ldots, r$.

The realizations of the generalized stochastic process $\Phi(\omega, t_1, \ldots, t_r) = [f_n(\omega, t_1, \ldots, t_r)]$, i. e., the classes of sequences $\Phi(\omega_0, t_1, \ldots, t_r) = [f_n(\omega_0, t_1, \ldots, t_r)]$ are distributions depending on r variables $t_1, t_2, \ldots, t_r$ (in Mikusiński's sense). The analogue of Theorem 1 is

THEOREM 9. *For every generalized process* $\Phi(\omega, t_1, \ldots, t_r)$ *there exist a continuous process* $F(\omega, t_1, \ldots, t_r)$ *and a system* $k_1, k_2, \ldots, k_r$ *of non-negative integers such that*

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} F(\omega, t_1, \ldots, t_r) = \Phi(\omega, t_1, \ldots, t_r).$$

The sequence $\{\Phi_n(\omega, t_1, \ldots, t_r)\}$ of generalized processes is said *to converge as* $n \to \infty$ *to the generalized process* $\Phi(\omega, t_1, \ldots, t_r)$, in symbols $\Phi_n(\omega, t_1, \ldots, t_r) \to \Phi(\omega, t_1, \ldots, t_r)$, if there exist continuous processes $F(\omega, t_1, \ldots, t_r), F_n(\omega, t_1, \ldots, t_r)$ $(n = 1, 2, \ldots)$ and a system of non-negative integers $k_1, k_2, \ldots, k_r$ such that

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} F_n(\omega, t_1, \ldots, t_r) = \Phi_n(\omega, t_1, \ldots, t_r) \qquad (n = 1, 2, \ldots),$$

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}} F(\omega, t_1, \ldots, t_r) = \Phi(\omega, t_1, \ldots, t_r),$$

and

$$F_n(\omega, t_1, \ldots, t_r) \rightrightarrows F(\omega, t_1, \ldots, t_r).$$

From the definition of convergence follow directly the properties:

If $\Phi_n(\omega, t_1, \ldots, t_r) \to \Phi(\omega, t_1, \ldots, t_r)$, then

$$\frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}}\Phi_n(\omega, t_1, \ldots, t_r) \to \frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}}\Phi(\omega, t_1, \ldots, t_r),$$

$$\overline{\Phi_n(\omega, t_1, \ldots, t_r)} \to \overline{\Phi(\omega, t_1, \ldots, t_r)},$$

and for every multiplicator $f(\omega, t_1, \ldots, t_r)$:

$$f(\omega, t_1, \ldots, t_r)\Phi_n(\omega, t_1, \ldots, t_r) \to f(\omega, t_1, \ldots, t_r)\Phi(\omega, t_1, \ldots, t_r).$$

The addition of generalized processes is continuous with respect to convergence.

We shall now define a product of generalized processes depending on different variables, called in the sequel the *direct product*. Let $\Phi(\omega, t_1, \ldots, t_s) = [f_n(\omega, t_1, \ldots, t_s)]$, $\Psi(\omega, t_{s+1}, \ldots, t_r) = [g_n(\omega, t_{s+1}, \ldots, t_r)]$. Then we set

$$\Phi(\omega, t_1, \ldots, t_s)\Psi(\omega, t_{s+1}, \ldots, t_r) = [f_n(\omega, t_1, \ldots, t_s)g_n(\omega, t_{s+1}, \ldots, t_r)].$$

It is easily seen that the product defined in this fashion does not depend on the choice of the representation. The direct product $\Phi(\omega, t_1, \ldots, t_s)\Psi(\omega, t_{s+1}, \ldots, t_r)$ is a generalized stochastic process depending on the variables $t_1, t_2, \ldots, t_r$. In particular, the direct product $\Phi(\omega, t_1)\overline{\Phi(\omega, t_2)}$, often appearing in the sequel, is a generalized process depending on the variables t_1 and t_2.

Let us suppose that for a certain system $k_1, k_2, \ldots, k_r$ we have

$$\Phi(\omega, t_1, \ldots, t_r) = \frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}}F(\omega, t_1, \ldots, t_r),$$

where $F(\omega, t_1, \ldots, t_r)\in\mathfrak{C}_r$ and the function $\mathcal{E}|F(\omega, t_1, \ldots, t_r)|$ is locally integrable with respect to the variables $t_1, t_2, \ldots, t_r$. In this case, as for the processes of one variable, the expected value of the generalized process is defined by the formula

$$E\Phi(\omega, t_1, \ldots, t_r) = \frac{\partial^{k_1+\ldots+k_r}}{\partial t_1^{k_1}\ldots\partial t_r^{k_r}}\mathcal{E}F(\omega, t_1, \ldots, t_r).$$

The expected value $E\Phi(\omega, t_1, \ldots, t_r)$ is, in general, a distribution depending on r variables $t_1, \ldots, t_r$.

III.2. We shall now be concerned with generalized stochastic processes $\Phi(\omega, t)$ for which the expected value $E\Phi(\omega, t_1)\overline{\Phi(\omega, t_2)}$ exists.

LEMMA 9. *If the expected value $E\Phi(\omega, t_1)\overline{\Phi(\omega, t_2)}$ exists, then there exists a continuous process $F(\omega, t)$ such that the function $\mathcal{E}|F(\omega, t)|^2$ is locally integrable and for a certain k we have*

$$\frac{d^k}{dt^k}F(\omega, t) = \Phi(\omega, t).$$

Proof. Suppose that the expected value $E\Phi(\omega, t_1)\overline{\Phi(\omega, t_2)}$ exists. Taking the indefinite integral of appropriate multiplicity with respect to t_1 and t_2 of the continuous process appearing in the definition of the expected value, we deduce the existence of a continuous process $H(\omega, t_1, t_2)$ and of a k such that the expected values $\mathcal{E}H(\omega, t_1, t_2)$ and $\mathcal{E}|H(\omega, t_1, t_2)|$ are continuous functions of t_1 and t_2, and

$$\frac{\partial^{2k}}{\partial t_1^k\partial t_2^k}H(\omega, t_1, t_2) = \Phi(\omega, t_1)\overline{\Phi(\omega, t_2)}. \tag{60}$$

Without loss of generality one may suppose that for the same integer k there exists a continuous process $G(\omega, t)$ satisfying the equality

$$\frac{d^k}{dt^k}G(\omega, t) = \Phi(\omega, t). \tag{61}$$

Let $x_1, x_2, \ldots, x_k$ ($x_i \neq x_j$ for $i \neq j$) be an arbitrary system of real numbers and set

$$F(\omega, t) = G(\omega, t) - \sum_{j=1}^{k} G(\omega, x_j)\frac{(t-x_1)\ldots(t-x_{j-1})(t-x_{j+1})\ldots(t-x_k)}{(x_j-x_1)\ldots(x_j-x_{j-1})(x_j-x_{j+1})\ldots(x_j-x_k)}.$$

It is easily seen that

$$\frac{d^k}{dt^k}F(\omega, t)\Phi(\omega, t), \qquad F(\omega, x_j) = 0 \qquad (j = 1, 2, \ldots, k). \tag{62}$$

From (62) we infer that

$$\frac{\partial^{2k}}{\partial t_1^k\partial t_2^k}F(\omega, t_1)\overline{F(\omega, t_2)} = \Phi(\omega, t_1)\overline{\Phi(\omega, t_2)},$$

Consequently, in view of (60),

$$F(\omega, t_1)F(\omega, t_2) = H(\omega, t_1, t_2) + \sum_{j=0}^{k-1}(A_j(\omega, t_1)t_2^j + B_j(\omega, t_2)t_1^j). \tag{63}$$

Substituting in this equality $t_1 = t$, $t_2 = x_s$ $(s = 1, 2, \ldots, k)$ and taking into account the equality (62) we see that the functions $A_j(\omega, t)$ satisfy the system of equations

$$\sum_{j=0}^{k-1} A_j(\omega, t) x_s^j = -H(\omega, t, x_s) - \sum_{j=0}^{k-1} B_j(\omega, x_s) t^j \qquad (s = 1, 2, \ldots, k).$$

Hence it follows that the functions $A_j(\omega, t)$ are of the form

$$A_j(\omega, t) = D_j(\omega, t) + \sum_{s=0}^{k-1} d_{s,j}(\omega) t^s \qquad (j = 0, 1, \ldots, k-1),$$

where $\mathcal{E}|D_j(\omega, t)|$ is continuous. In the same way we obtain

$$B_j(\omega, t) = R_j(\omega, t) + \sum_{s=0}^{k-1} r_{s,j}(\omega) t^s \qquad (j = 0, 1, \ldots, k-1),$$

where $\mathcal{E}|R_j(\omega, t)|$ is continuous. Substituting the expression obtained in formula (63) we get

$$(64) \qquad F(\omega, t_1)\overline{F(\omega, t_2)} = H^*(\omega, t_1, t_2) + \sum_{i,j=0}^{k-1} c_{i,j}(\omega) t_1^i t_2^j,$$

where $\mathcal{E}H^*(\omega, t_1, t_2)$ is locally bounded. Setting $t_1 = x_s$, $t_2 = x_r$ $(s, r = 1, 2, \ldots, k)$ and taking into account the equality (62) we infer that $\mathcal{E}|c_{ij}(\omega)| < \infty$ $(i, j = 0, 1, \ldots, k-1)$. Consequently, from (64) it follows that $\mathcal{E}|F(\omega, t)|^2$ is locally integrable, which proves the Lemma.

THEOREM 10. *The existence of the expected value $E\Phi(\omega, t_1)\overline{\Phi(\omega, t_2)}$ implies the existence of the expected value $E\Phi(\omega, t)$.*

Proof. For the proof, the process $\Phi(\omega, t)$ must be represented in the form

$$\Phi(\omega, t) = \frac{d^k}{dt^k} F(\omega, t),$$

where $F(\omega, t)$ is the process appearing in Lemma 9. Since the function $\mathcal{E}|F(\omega, t)|^2$ is locally integrable, the function $\mathcal{E}|F(\omega, t)|$ is locally integrable too. Thus the process $\Phi(\omega, t)$ has the expected value $d^k \mathcal{E}F(\omega, t)/dt^k$, which proves our theorem.

The class of all the generalized processes $\Phi(\omega, t)$ for which the expected value $E\Phi(\omega, t_1)\overline{\Phi(\omega, t_2)}$ exists will be denoted by $\mathfrak{K}$. The following implication is easily verified:

If $\Phi(\omega, t) \in \mathfrak{K}$, then we have $d^s\Phi(\omega, t)/dt^s \in \mathfrak{K}$ for every positive integer s and $f(t)\Phi(\omega, t) \in \mathfrak{K}$ for every multiplicator $f(t)$.

We shall see later that the class $\mathfrak{K}$ behaves in the same way as the class of ordinary processes with a finite expected value of the square.

THEOREM 11. *Suppose that $\Phi(\omega, t) \in \mathfrak{K}$ and $\Psi(\omega, t) \in \mathfrak{K}$; then there exists the expected value $E\Phi(\omega, t_1)\Psi(\omega, t_2)$.*

Proof. Let us represent the processes $\Phi(\omega, t)$ and $\Psi(\omega, t)$ in the form

$$\Phi(\omega, t) = \frac{d^{k_1}}{dt^{k_1}} F_1(\omega, t), \qquad \Psi(\omega, t) = \frac{d^{k_2}}{dt^{k_2}} F_2(\omega, t),$$

where the processes $F_1(\omega, t)$ and $F_2(\omega, t)$ appear in Lemma 9. Let us set $G(\omega, t_1, t_2) = F_1(\omega, t_1)F_2(\omega, t_2)$. Then we have the equality

$$\frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} G(\omega, t_1, t_2) = \Phi(\omega, t_1)\Psi(\omega, t_2).$$

From the inequality

$$\mathcal{E}|G(\omega, t_1, t_2)| \leqslant \mathcal{E}|F_1(\omega, t_1)|^2 + \mathcal{E}|F_2(\omega, t_2)|^2$$

and from the local integrability of the functions $\mathcal{E}|F_1(\omega, t)|^2$ and $\mathcal{E}|F_2(\omega, t)|^2$ follows the local integrability of the function $\mathcal{E}|G(\omega, t_1, t_2)|$ with respect to t_1 and t_2. Hence $\Phi(\omega, t_1)\Psi(\omega, t_2)$ has the expected value defined by the formula

$$E\Phi(\omega, t_1)\Psi(\omega, t_2) = \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} \mathcal{E}G(\omega, t_1, t_2).$$

COROLLARY. *From Theorem 11 it follows that $\Phi(\omega, t) \in \mathfrak{K}$ together with $\Psi(\omega, t) \in \mathfrak{K}$ imply $\Phi(\omega, t) + \Psi(\omega, t) \in \mathfrak{K}$.*

THEOREM 12. *Let the processes $\Phi(\omega, t)$ and $\Psi(\omega, t)$ belonging to the class $\mathfrak{K}$ be independent; then*

$$E\Phi(\omega, t_1)\Psi(\omega, t_2) = E\Phi(\omega, t_1)E\Phi(\omega, t_2).$$

Proof. Let us represent the processes $\Phi(\omega, t)$ and $\Psi(\omega, t)$ in the form

$$\Phi(\omega, t) = \frac{d^{k_1}}{dt^{k_1}} F_1(\omega, t), \qquad \Psi(\omega, t) = \frac{d^{k_2}}{dt^{k_2}} F_2(\omega, t),$$

where the processes $F_1(\omega, t)$ and $F_2(\omega, t)$ satisfy Lemma 9. An argument similar to that used in the proof of the foregoing theorem gives the equalities

$$E\Phi(\omega, t) = \frac{d^{k_1}}{dt^{k_1}} \mathcal{E}F_1(\omega, t), \qquad E\Psi(\omega, t) = \frac{d^{k_2}}{dt^{k_2}} \mathcal{E}F_2(\omega, t),$$

$$E\Phi(\omega, t_1)\Psi(\omega, t_2) = \frac{\partial^{k_1+k_2}}{\partial t_1^{k_1}\partial t_2^{k_2}} \mathcal{E}F_1(\omega, t_1)F_2(\omega, t_2),$$

whence, as $h_1 \to 0$ and $h_2 \to 0$, we have

$$\frac{1}{h_1^{k_1} h_2^{k_2}} \mathcal{E}\left(\Delta_{h_1}^{(k_1)} F_1(\omega, t_1)\right) \mathcal{E}\left(\Delta_{h_2}^{(k_2)} F_2(\omega, t_2)\right) \to E\Phi(\omega, t_1) E\Psi(\omega, t_2), \tag{65}$$

$$\frac{1}{h_1^{k_1} h_2^{k_2}} \mathcal{E}\left(\Delta_{h_1}^{(k_1)} F_1(\omega, t_1) \Delta_{h_2}^{(k_2)} F_2(\omega, t_2)\right) \to E\Phi(\omega, t_1) E\Psi(\omega, t_2). \tag{66}$$

The processes $\Phi(\omega, t)$ and $\Psi(\omega, t)$ being independent by hypothesis, we infer by Lemma 4 that the increments $\Delta_{h_1}^{(k_1)} F_1(\omega, t_1)$ and $\Delta_{h_2}^{(k_2)} F_2(\omega, t_2)$ are independent. The left-hand sides of formulae (65) and (66) are identical, whence the theorem follows.

III.3. In the sequel we shall need the following lemmas concerning processes of the class $\mathfrak{K}$:

LEMMA 10. *Let $\Phi(\omega, t) \in \mathfrak{K}$. If for a certain integer k there exists a continuous function $a(t_1, t_2)$ satisfying the equality*

$$\frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} a(t_1, t_2) = E\Phi(\omega, t_1) \overline{\Phi(\omega, t_2)},$$

then there exists a continuous process $F(\omega, t)$ with a continuous expected value $\mathcal{E}|F(\omega, t)|^2$ and such that

$$\frac{d^{k+1}}{dt^{k+1}} F(\omega, t) = \Phi(\omega, t).$$

Proof. From Lemma 9 follows the existence of a continuous process $G(\omega, t)$ with a locally integrable expected value $\mathcal{E}|G(\omega, t)|^2$ and satisfying, for a certain s, the equality

$$\frac{d^s}{dt^s} G(\omega, t) = \Phi(\omega, t). \tag{67}$$

In the case $s \leqslant k$ it is easily seen that the process

$$F(\omega, t) = \frac{1}{(k-s)!} \int_0^t (t-u)^{k-s} G(\omega, u)\, du$$

satisfies the requirememts of the lemma. Thus it is sufficient to consider only the case $s > k$. We may suppose that

$$G(\omega, x_j) = 0 \quad (j = 1, 2, \ldots, s), \tag{68}$$

where $x_1, x_2, \ldots, x_s$ ($x_i \neq x_j$ for $i \neq j$) be a system of real numbers.

From the hypothesis and formula (67) we have

$$\frac{\partial^{2s}}{\partial t_1^s \partial t_2^s} \mathcal{E}G(\omega, t_1)\overline{G(\omega, t_2)} = \frac{\partial^{2s}}{\partial t_1^s \partial t_2^s} \int_0^{t_1}\int_0^{t_2} \frac{[(t_1-u_1)(t_2-u_2)]^{s-k-1}}{((s-k-1)!)^2} a(u_1, u_2)\, du_1 du_2$$
$$= \frac{\partial^{2s}}{\partial t_1^s \partial t_2^s} b(t_1, t_2),$$

where

$$\frac{\partial^{2(s-k)}}{\partial t_1^{s-k} \partial t_2^{s-k}} b(t_1, t_2)$$

is continuous. Hence

$$\mathcal{E}G(\omega, t_1)\overline{G(\omega, t_2)} = b(t_1, t_2) + \sum_{j=0}^{s-1} \left(A_j(t_1) t_2^j + B_j(t_2) t_1^j\right).$$

Substituting in this equality $t_1 = t$, $t_2 = x_r$ ($r = 1, 2, \ldots, s$) and taking into account the equality (68) we see that the functions $A_j(t)$ satisfy the system of equations

$$\sum_{j=0}^{s-1} A_j(t) x_r^j = -b(t, x_r) - \sum_{j=0}^{s-1} B_j(x_r) t^j \quad (r = 1, 2, \ldots, s).$$

Hence it follows that $\frac{d^{s-k}}{dt^{s-k}} A_j(t)$ is continuous. In the same way we obtain that $\frac{d^{s-k}}{dt^{s-k}} B_j(t)$ is continuous. Consequently,

$$a_0(t_1, t_2) = \frac{\partial^{2(s-k)}}{\partial t_1^{s-k} \partial t_2^{s-k}} \mathcal{E}G(\omega, t_1)\overline{G(\omega, t_2)}$$

is continuous.

Let us define the family of processes

$$H_h(\omega, t) = \frac{\Delta_h^{(s-k)} G(\omega, t)}{h^{s-k}}. \tag{69}$$

Since

$$\mathcal{E}H_{h_1}(\omega, t_1)\overline{H_{h_2}(\omega, t_2)} = \frac{\Delta_{h_1}^{(s-k)} \Delta_{h_2}^{(s-k)}}{h_1^{s-k} h_2^{s-k}} \mathcal{E}G(\omega, t_1)\overline{G(\omega, t_2)},$$

then, in virtue of the continuity of the function $a_0(t_1, t_2)$,

$$\mathcal{E}H_{h_1}(\omega, t_1)\overline{H_{h_2}(\omega, t_2)} \rightrightarrows a_0(t_1, t_2) \tag{70}$$

as $h_1, h_2 \to 0$. Thus from the equality

$$\mathcal{E}|H_{h_1}(\omega, t) - H_{h_2}(\omega, t)|^2 = \mathcal{E}|H_{h_1}(\omega, t)|^2 + \mathcal{E}|H_{h_2}(\omega, t)|^2 -$$
$$- \mathcal{E}H_{h_1}(\omega, t)\overline{H_{h_2}(\omega, t)} - \mathcal{E}\overline{H_{h_1}(\omega, t)} H_{h_2}(\omega, t)$$

we obtain, as $h_1, h_2 \to 0$, $\mathcal{E}|H_{h_1}(\omega, t)-H_{h_2}(\omega, t)|^2 \rightrightarrows 0$. This implies the existence of a process $H(\omega, t)$ such that

$$\mathcal{E}|H_h(\omega, t)-H(\omega, t)|^2 \rightrightarrows 0 \quad \text{as} \quad h \to 0. \tag{71}$$

Formula (70) leads to $\mathcal{E}|H(\omega, t)|^2 = a_0(t, t)$, whence the expected value $\mathcal{E}|H(\omega, t)|$ is a continuous function. From (71) it follows that for $m = 1, 2, \ldots$ the sequence

$$\Big\{\int_{-m}^{m} |H_h(\omega, t)-H(\omega, t)|^2 dt\Big\}$$

tends to zero as $h \to 0$. Thus there exists a sequence $h_1, h_2, \ldots \to 0$ such that for almost every $\omega \epsilon \Omega$ and for $m = 1, 2, \ldots$ the sequence

$$\Big\{\int_{-m}^{m} |H_{h_n}(\omega, t)-H(\omega, t)|^2 dt\Big\}$$

tends to zero as $n \to \infty$. It follows (see section I.6, p. 275) that

$$H_{h_n}(\omega, t) \to H(\omega, t). \tag{72}$$

By definition (69) we obtain for $h \to 0$

$$\frac{d^k}{dt^k} H_h(\omega, t) = \frac{1}{h^{s-k}} \Delta_h^{(s-k)} \frac{d^k}{dt^k} G(\omega, t) \to \frac{d^s}{dt^s} G(\omega, t),$$

which in virtue of (67) and (72) leads to

$$\frac{d^k}{dt^k} H(\omega, t) = \Phi(\omega, t).$$

The stochastic process $H(\omega, t)$ has a continuous expected value $\mathcal{E}|H(\omega, t)|^2$, whence almost all of its realizations are locally integrable. Setting

$$F(\omega, t) = \int_0^t H(\omega, u)\, du$$

we obtain a process satisfying our lemma.

LEMMA 11. *Let $\{a_n(t_1, t_2)\}$ be a sequence of continuous functions satisfying the conditions*

$$\frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} a_n(t_1, t_2) = \frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} \overline{a_n(t_2, t_1)} \quad (n = 1, 2, \ldots),$$

$$a_n(t_1, t_2) \rightrightarrows 0.$$

Then there exists a sequence of continuous functions $\{b_n(t_1, t_2)\}$ such that

$$b_n(t_1, t_2) = \overline{b_n(t_2, t_1)} \quad (n = 1, 2, \ldots),$$

$$\frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} b_n(t_1, t_2) = \frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} a_n(t_1, t_2) \quad (n = 1, 2, \ldots),$$

$$b_n(t_1, t_2) \rightrightarrows 0.$$

Proof. In the proof we must assume that

$$b_n(t_1, t_2) = \tfrac{1}{2}\big(a_n(t_2, t_1) + \overline{a_n(t_2, t_1)}\big).$$

LEMMA 12. *Let $\Phi_n(\omega, t) \epsilon \mathfrak{R}$ $(n = 1, 2, \ldots)$. If $E\Phi_n(\omega, t_1)\overline{\Phi_n(\omega, t_2)} \to 0$, then there exists a sequence $\{F_n(\omega, t)\}$ of continuous processes with continuous expected values $\mathcal{E}|F_n(\omega, t)|^2$ such that $\mathcal{E}|F_n(\omega, t)|^2 \rightrightarrows 0$, and such that for a certain k*

$$\frac{d^k}{dt^k} F_n(\omega, t) = \Phi_n(\omega, t) \quad (n = 1, 2, \ldots).$$

Proof. Suppose that $E\Phi_n(\omega, t_1)\overline{\Phi_n(\omega, t_2)} \to 0$.

From the definition of convergence $\to$ follows the existence of a sequence of continuous functions $\{a_n(t_1, t_2)\}$ such that

$$a_n(t_1, t_2) \rightrightarrows 0 \tag{76}$$

and such that, for a certain k,

$$\frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} a_n(t_1, t_2) = E\Phi_n(\omega, t_1)\overline{\Phi_n(\omega, t_2)} \quad (n = 1, 2, \ldots). \tag{77}$$

From this equality and from formula (76) it follows that the functions $a_n(t_1, t_2)$ $(n = 1, 2, \ldots)$ satisfy the hypotheses of Lemma 11. Hence without loss of generality we may suppose that

$$a_n(t_1, t_2) = \overline{a_n(t_2, t_1)} \quad (n = 1, 2, \ldots) \tag{78}$$

holds. We may also suppose without loss of generality that the functions $a_n(t_1, t_2)$ $(n = 1, 2, \ldots)$ have continuous partial derivatives with respect to both variables (in the contrary case we should consider the indefinite integrals of these functions). Applying Lemma 10 to the function

$$\frac{\partial^2}{\partial t_1 \partial t_2} a_n(t_1, t_2)$$

and the processes $\Phi_n(\omega, t)$ $(n = 1, 2, \ldots)$ we deduce the existence of a sequence of continuous processes $\{G_n(\omega, t)\}$ with continuous expected values $\mathcal{E}|G_n(\omega, t)|^2$ $(n = 1, 2, \ldots)$ and satisfying the equalities

$$\frac{d^k}{dt^k} G_n(\omega, t) = \Phi_n(\omega, t) \quad (n = 1, 2, \ldots). \tag{79}$$

Let $x_1, x_2, \ldots, x_k$ $(x_i \neq x_j$ for $i \neq j)$ be an arbitrary system of real numbers and set

$$F_n(\omega, t) = G_n(\omega, t) - \sum_{j=1}^{k} G_n(\omega, x_j) \frac{(t-x_1)\ldots(t-x_{j-1})(t-x_{j+1})\ldots(t-x_k)}{(x_j-x_1)\ldots(x_j-x_{j-1})(x_j-x_{j+1})\ldots(x_j-x_k)}.$$

It is easily seen that the processes $F_n(\omega, t)$ are continuous and have continuous expected values $\mathcal{E}|F_n(\omega, t)|^2$. We also have

$$F_n(\omega, x_j) = 0 \quad (j = 1, 2, \ldots, k;\ n = 1, 2, \ldots). \tag{80}$$

From (79) we infer that

$$\frac{d^k}{dt^k} F_n(\omega, t) = \Phi_n(\omega, t) \quad (n = 1, 2, \ldots). \tag{81}$$

To prove our lemma it is sufficient to show that $\mathcal{E}|F_n(\omega, t)|^2 \rightrightarrows 0$. From (77) and (81) using simple arguments we get

$$\mathcal{E} F_n(\omega, t_1)\overline{F_n(\omega, t_2)} = a_n(t_1, t_2) + \sum_{s=0}^{k-1} \left(U_{s,n}(t_1) t_2^s + V_{s,n}(t_2) t_1^s\right).$$

Interchanging in this equality t_1 and t_2 and passing to the conjugate value, we obtain from (78)

$$\mathcal{E} F_n(\omega, t_1)\overline{F_n(\omega, t_2)} = a_n(t_1, t_2) + \sum_{s=0}^{k-1} \left(\overline{V_{s,n}(t_1)} t_2^s + \overline{U_{s,n}(t_2)} t_1^s\right).$$

Hence

$$\mathcal{E} F_n(\omega, t_1)\overline{F_n(\omega, t_2)} = a_n(t_1, t_2) + \sum_{s=0}^{k-1} \left(W_{s,n}(t_1) t_2^s + W_{s,n}(t_2) t_1^s\right), \tag{82}$$

where

$$W_{s,n}(t) = \tfrac{1}{2}\left(U_{s,n}(t) + \overline{V_{s,n}(t)}\right) \quad (s = 0, 1, \ldots, k-1;\ n = 1, 2, \ldots).$$

Substituting in formula (82) $t_1 = t$, $t_2 = x_j$ $(j = 1, 2, \ldots, k)$ and taking into account equalities (80) we see that the functions $W_{s,n}(t)$ $(s = 0, 1, \ldots, k-1)$ satisfy the system of equations

$$\sum_{s=0}^{k-1} W_{s,n}(t) x_j^s = -a_n(t, x_j) - \sum_{s=0}^{k-1} \overline{W_{s,n}(x_j)} t^s \quad (j = 1, 2, \ldots, k).$$

Hence it follows from (76) that the functions $W_{s,n}(t)$ are of the form

$$W_{s,n}(t) = \sum_{j=0}^{k-1} \lambda_{j,s,n} t^j + D_{s,n}(t) \quad (s = 0, 1, \ldots, k-1;\ n = 1, 2, \ldots)$$

where $\lambda_{j,s,n}$ $(j, s = 0, 1, \ldots, k-1;\ n = 1, 2, \ldots)$ are constant coefficients and $D_{s,n}(t) \rightrightarrows 0$ $(s = 0, 1, \ldots, k-1)$ as $n \to \infty$. Substituting the expression obtained in formula (82) we get

$$\mathcal{E} F_n(\omega, t_1)\overline{F_n(\omega, t_2)} = \sum_{j=0}^{k-1} \sum_{s=0}^{k-1} (\lambda_{s,j,n} + \bar{\lambda}_{j,s,n}) t_1^s t_2^j + D_n(t_1, t_2) \tag{83}$$

where

$$D_n(t_1, t_2) \rightrightarrows 0. \tag{84}$$

Setting $t_1 = t$, $t_2 = x_r$ $(r = 1, 2, \ldots, k)$ in equality (83) and taking into account formulae (80) and (84) we infer that for $n \to \infty$

$$\sum_{s=0}^{k-1} \left(\sum_{j=0}^{k-1} (\lambda_{j,s,n} + \bar{\lambda}_{j,s,n}) x_r^j\right) t^s \rightrightarrows 0 \quad (r = 1, 2, \ldots, k),$$

whence

$$\lim_{n\to\infty} \sum_{j=0}^{k-1} (\lambda_{s,j,n} + \bar{\lambda}_{j,s,n}) x_r^j = 0 \quad (s = 0, 1, \ldots, k-1;\ r = 1, 2, \ldots).$$

It follows that

$$\lim_{n\to\infty} (\lambda_{s,j,n} + \bar{\lambda}_{j,s,n}) = 0 \quad (s, j = 0, 1, \ldots, k-1). \tag{85}$$

Formula (83) implies

$$\mathcal{E}|F_n(\omega, t)|^2 = \sum_{j=0}^{k-1} \sum_{s=0}^{k-1} (\lambda_{s,j,n} + \bar{\lambda}_{j,s,n}) t^{j+1} + D_n(t, t),$$

whence, by (84) and (85), $\mathcal{E}|F_n(\omega, t)|^2 \rightrightarrows 0$, which proves the Lemma.

LEMMA 13. *Let $E\Phi_n(\omega, t_1)\overline{\Phi_n(\omega, t_2)} \to 0$; then there exists a subsequence $k_1, k_2, \ldots$ such that $\Phi_{k_n}(\omega, t) \to 0$.*

Proof. If $E\Phi_n(\omega, t_1)\overline{\Phi_n(\omega, t_2)} \to 0$, then applying Lemma 12 we see that there exists a sequence of continuous processes $\{F_n(\omega, t)\}$ with continuous expected values $\mathcal{E}|F_n(\omega, t)|^2$ and a k such that

$$\mathcal{E}|F_n(\omega, t)|^2 \rightrightarrows 0, \tag{86}$$

$$\frac{d^k}{dt^k} F_n(\omega, t) = \Phi_n(\omega, t) \quad (n = 1, 2, \ldots). \tag{87}$$

It follows from (86) that for $m = 1, 2, \ldots$ the sequence

$$\Big\{\int_{-m}^{m} \mathcal{E}\,|F_n(\omega, t)|^2 dt\Big\}$$

converges to zero as $n \to \infty$. This implies the existence of a sequence of indices $k_1, k_2, \ldots$ such that for $m = 1, 2, \ldots$ and almost every $\omega \in \Omega$ the sequence

$$\Big\{\int_{-m}^{m} |F_{k_n}(\omega, t)|^2 dt\Big\}$$

converges to zero as $n \to \infty$. We know that this convergence implies $F_{k_n}(\omega, t) \to 0$ (see section I.6, p. 275). We obtain the lemma differentiating this formula k times and applying formula (87).

III.4. It is possible to define in the class $\mathfrak{R}$ a convergence which corresponds to the convergence in mean of ordinary stochastic processes.

The sequence $\{\Phi_n(\omega, t)\}$ of generalized stochastic processes is said *to converge in mean to the generalized stochastic process* $\Phi(\omega, t)$, in symbols $\Phi_n(\omega, t) \longrightarrow \Phi(\omega, t)$, if

$$E\big(\Phi_n(\omega, t_1) - \Phi(\omega, t_1)\big)\big(\overline{\Phi_n(\omega, t_2) - \Phi(\omega, t_2)}\big) \to 0.$$

It follows from Lemma 13 that the limit in mean $\Phi(\omega, t)$ is uniquely determined. Directly from the definition we obtain the following implication:

If $\Phi_n(\omega, t) \longrightarrow \Phi(\omega, t)$, then

$$\overline{\Phi_n(\omega, t)} \longrightarrow \overline{\Phi(\omega, t)}, \qquad \frac{d}{dt}\Phi_n(\omega, t) \longrightarrow \frac{d}{dt}\Phi(\omega, t),$$

and, $f(t)$ being a multiplicator,

$$f(t)\Phi_n(\omega, t) \longrightarrow f(t)\Phi(\omega, t).$$

One may also prove that $\Phi_n(\omega, t) \longrightarrow \Phi(\omega, t)$ with $\Psi_n(\omega, t) \longrightarrow \Psi(\omega, t)$ imply

$$\big(\Phi_n(\omega, t) + \Psi_n(\omega, t)\big) \longrightarrow \big(\Phi(\omega, t) + \Psi(\omega, t)\big).$$

From lemma 13 follows directly

THEOREM 13. *Let* $\Phi_n(\omega, t) \longrightarrow \Phi(\omega, t)$, *then there exists a subsequence* $k_1, k_2, \ldots$ *such that* $\Phi_{k_n}(\omega, t) \to \Phi(\omega, t)$.

There exist, however, sequences of generalized processes convergent in mean but not in the sense of the convergence $\to$. As an example we may

take the sequence $\Phi_n(\omega, t) = a_n(\omega)$ $(n = 1, 2, \ldots)$ where $a_n(\omega)$ are random variables satisfying the condition

$$\lim_{n\to\infty} \mathcal{E}\,|a_n(\omega)|^2 = 0$$

and such that the sequence $\{a_n(\omega)\}$ does not converge almost everywhere. Since $E\,\Phi_n(\omega, t_1)\overline{\Phi_n(\omega, t_2)} = \mathcal{E}\,|a_n(\omega)|^2$, we have $\Phi_n(\omega, t) \longrightarrow 0$.

Suppose, on the contrary, that $\Phi_n(\omega, t) \to 0$. Then there exists a sequence of continuous processes $F_n(\omega, t)$ almost uniformly convergent for almost every ω and satisfying for a certain k the equality:

$$\frac{d^k}{dt^k} F_n(\omega, t) = a_n(\omega) \qquad (n = 1, 2, \ldots).$$

It follows that

$$F_n(\omega, t) = \frac{a_n(\omega)}{k!} + \sum_{j=0}^{k-1} \beta_{j,n}(\omega) t^j.$$

Hence, the sequence $\{F_n(\omega, t)\}$ converging almost uniformly for almost any ω, the sequence of random variables $\{a_n(\omega)\}$ converges almost everywhere. This leads to a contradiction, which proves that the sequences $\{\Phi_n(\omega, t)\}$ does not converge in the sense $\to$.

THEOREM 14. *The relation* $\Phi_n(\omega, t) \longrightarrow \Phi(\omega, t)$ *is equivalent to the existence of a sequence of continuous processes* $F_n(\omega, t)$ $(n = 1, 2, \ldots)$ *such that the functions* $\mathcal{E}\,|F_n(\omega, t) - F(\omega, t)|^2$ $(n = 1, 2, \ldots)$ *are continuous,* $\mathcal{E}\,|F_n(\omega, t) - F(\omega, t)|^2 \rightrightarrows 0$, *and for a certain* k *satisfy the equalities*

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t), \qquad \frac{d^k}{dt^k} F_n(\omega, t) = \Phi_n(\omega, t) \qquad (n = 1, 2, \ldots).$$

Proof. The necessity of the condition follows directly from Lemma 12. We now prove the sufficiency. Suppose that, for a sequence of continuous processes $F_n(\omega, t)$ $(n = 1, 2, \ldots)$, the expected values $\mathcal{E}\,|F_n(\omega, t) - F(\omega, t)|^2$ are continuous functions satisfying $\mathcal{E}\,|F_n(\omega, t) - F(\omega, t)|^2 \rightrightarrows 0$.

It follows that

$$\mathcal{E}\big(F_n(\omega, t_1) - F(\omega, t_1)\big)\big(\overline{F_n(\omega, t_2) - F(\omega, t_2)}\big) \rightrightarrows 0.$$

Differentiating k times with respect to t_1 and t_2 we get

$$E\Big(\frac{d^k}{dt_1^k} F_n(\omega, t_1) - \frac{d^k}{dt_1^k} F(\omega, t_1)\Big)\Big(\overline{\frac{d^k}{dt_2^k} F_n(\omega, t_2) - \frac{d^k}{dt_2^k} F(\omega, t_2)}\Big) \to 0,$$

which proves our Theorem.

THEOREM 15. *Let $\Phi_n(\omega, t) \to 0$; then $E\Phi_n(\omega, t) \to 0$.*

Proof. Suppose that $\Phi_n(\omega, t) \to 0$. By the foregoing theorem there exists a sequence of continuous processes $\{F_n(\omega, t)\}$ with a continuous expected values $\mathcal{E}|F_n(\omega, t)|^2$ and such that

$$\mathcal{E}|F_n(\omega, t)|^2 \rightrightarrows 0, \tag{88}$$

and for a certain k

$$\frac{d^k}{dt^k} F_n(\omega, t) = \Phi_n(\omega, t) \quad (n = 1, 2, \ldots). \tag{89}$$

Thus we see that the expected values $\mathcal{E}|F_n(\omega, t)|$ $(n = 1, 2, \ldots)$ are locally integrable functions and, in virtue of (88), $\mathcal{E}F_n(\omega, t) \rightrightarrows 0$. Hence by (89)

$$E\Phi_n(\omega, t) = \frac{d^k}{dt^k} \mathcal{E}F_n(\omega, t) \to 0,$$

which proves the theorem.

Let $f(\omega, t)$ be an ordinary process, and let $\{\Phi_\lambda(\omega, t)\}$ be a family of generalized stochastic processes. Suppose that for every sequence $a = \lambda_{0,n} < \lambda_{1,n} < \ldots < \lambda_{k_n,n} = b$ of partitions of the interval (a, b) satisfying the condition

$$\lim_{n\to\infty} \max_{0 \leqslant j \leqslant k_n - 1} (\lambda_{j+1,n} - \lambda_{j,n}) = 0$$

the sums

$$\sum_{j=0}^{k_n-1} \Phi_{\lambda_{j,n}}(\omega, t)\big(f(\omega, \lambda_{j+1,n}) - f(\omega, \lambda_{j,n})\big)$$

converge in mean. If this limit does not depend upon the choice of the partitions, it will be denoted by

$$\int_a^b \Phi_\lambda(\omega, t)\, df(\omega, \lambda).$$

The integral

$$\int_{-\infty}^{\infty} \Phi_\lambda(\omega, t)\, df(\omega, \lambda)$$

will be defined as the limit in mean as $n \to \infty$ of the integrals

$$\int_{-n}^{n} \Phi_\lambda(\omega, t)\, df(\omega, \lambda).$$

By the definition of the integral and from the properties of the convergence in mean of generalized stochastic processes we directly obtain the following implications:

If the integral

$$\int_a^b \Phi_\lambda(\omega, t)\, df(\omega, \lambda)$$

exists, then the integral

$$\int_a^b \frac{d}{dt} \Phi_\lambda(\omega, t)\, df(\omega, \lambda)$$

exists and

$$\int_a^b \frac{d}{dt} \Phi_\lambda(\omega, t)\, df(\omega, \lambda) = \frac{d}{dt} \int_a^b \Phi_\lambda(\omega, t)\, df(\omega, \lambda).$$

If the integrals

$$\int_a^b \Phi_\lambda(\omega, t)\, df(\omega, \lambda), \qquad \int_a^b \Psi_\lambda(\omega, t)\, df(\omega, \lambda)$$

exist, then for every couple c_1, c_2 of complex numbers the integral

$$\int_a^b \big(c_1 \Phi_\lambda(\omega, t) + c_2 \Psi_\lambda(\omega, t)\big)\, df(\omega, \lambda)$$

exists and

$$\int_a^b \big(c_1 \Phi_\lambda(\omega, t) + c_2 \Psi_\lambda(\omega, t)\big)\, df(\omega, \lambda) = c_1 \int_a^b \Phi_\lambda(\omega, t)\, df(\omega, \lambda) + c_2 \int_a^b \Psi_\lambda(\omega, t)\, df(\omega, \lambda).$$

If the integrals

$$\int_a^b \Phi_\lambda(\omega, t)\, df_1(\omega, \lambda) \quad \text{and} \quad \int_a^b \Phi_\lambda(\omega, t)\, df_2(\omega, \lambda)$$

exist, then for every couple c_1, c_2 of complex numbers the integral

$$\int_a^b \Phi_\lambda(\omega, t)\, d\big(c_1 f_1(\omega, \lambda) + c_2 f_2(\omega, \lambda)\big)$$

exists and

$$\int_a^b \Phi_\lambda(\omega, t)\, d\big(c_1 f_1(\omega, \lambda) + c_2 f_2(\omega, \lambda)\big) = c_1 \int_a^b \Phi_\lambda(\omega, t)\, df_1(\omega, \lambda) + c_2 \int_a^b \Phi_\lambda(\omega, t)\, df_2(\omega, \lambda).$$

Example. Let $f(\omega,t)$ be a continuous process belonging to the class $\mathfrak{K}$. Let us write

$$H(t)=\begin{cases}1 & \text{for} \quad t\geqslant 0,\\ 0 & \text{for} \quad t<0.\end{cases}$$

Then for every n we have

$$\int_{-n}^{n} H(t-\lambda)\,df(\omega,\lambda)=\begin{cases}0 & \text{for} \quad t<-n,\\ f(\omega,t)-f(\omega,-n) & \text{for} \quad -n\leqslant t\leqslant n,\\ f(\omega,n)-f(\omega,-n) & \text{for} \quad t>n.\end{cases}$$

Thus we see that

$$\int_{-n}^{n} H(t-\lambda)\,df(\omega,\lambda)+f(\omega,-n)\to f(\omega,t) \quad \text{as} \quad n\to\infty,$$

whence differentiating with respect to t we obtain

$$\int_{-n}^{n}\delta(t-\lambda)\,df(\omega,\lambda)\to\frac{d}{dt}f(\omega,t).$$

Hence

$$\int_{-\infty}^{\infty}\delta(t-\lambda)\,df(\omega,\lambda)=\frac{d}{dt}f(\omega,t).$$

III.5. Let $\Phi(\omega,t)\in\mathfrak{K}$. Then by Theorem 10 the expected value $E\Phi(\omega,t)$ exists; let us denote it by $m_\Phi(t)$. As for ordinary processes, the distribution

$$E\big(\Phi(\omega,t_1)-m_\Phi(t_1)\big)\big(\overline{\Phi(\omega,t_2)}-\overline{m_\Phi(t_2)}\big)$$

will be called the *correlation distribution of the generalized process* $\Phi(\omega,t)$ and will be denoted by $B_\Phi(t_1,t_2)$.

As a consequence of Lemma 13 we obtain

THEOREM 16. *Let* $B_\Phi(t_1,t_2)=0$; *then* $\Phi(\omega,t)=m_\Phi(t)$.

On the other hand, Theorem 15 implies

THEOREM 17. *Let* $\Phi_n(\omega,t)\to\Phi(\omega,t)$; *then* $B_{\Phi_n}(t_1,t_2)\to B_\Phi(t_1,t_2)$.

From the definition of the correlation distribution it follows directly that

$$B_{\frac{d}{dt}\Phi}(t_1,t_2)=\frac{\partial^2}{\partial t_1\partial t_2}B_\Phi(t_1,t_2),$$

$$B_\Phi(t_1,t_2)=\overline{B_\Phi(t_2,t_1)},\qquad B_{\overline{\Phi}}(t_1,t_2)=\overline{B_\Phi(t_1,t_2)}.$$

Examples. (a) Let $F(\omega,t)$ be an arbitrary stochastic process with independent increments. Suppose that the variance $\sigma^2(t)$ of the increment $F_0(\omega,t)=F(\omega,t)-F(\omega,0)$ is a locally integrable function. Then $F_0(\omega,t)\in\mathfrak{K}$, whence

$$\Phi(\omega,t)=\frac{d}{dt}F(\omega,t)=\frac{d}{dt}F_0(\omega,t)\in\mathfrak{K}.$$

Moreover we have

$$B_\Phi(t_1,t_2)=\frac{\partial^2}{\partial t_1\partial t_2}B_{F_0}(t_1,t_2).$$

Since

$$B_{F_0}(t_1,t_2)=\begin{cases}\sigma^2\big(\min(|t_1|,|t_2|)\big) & \text{as} \quad t_1t_2\geqslant 0,\\ 0 & \text{as} \quad t_1t_2<0,\end{cases}$$

we have

$$B_\Phi(t_1,t_2)=\delta(t_1-t_2)\frac{d}{dt_1}\sigma^2(t_1). \tag{90}$$

(b) Let us consider the Brownian motion of a particle in a fluid. Suppose that the surrounding medium acts on the particle in the following way: 1° the particle meets with friction proportional to the velocity of the particle; 2° in consequence of random collisions of the molecules of the fluid with the Brownian particle, there arises a supplementary random force which is supposed to have independent values at different moments, its impulse being supposed to have a normal distribution in every finite time interval. Thus the velocity $v(\omega,t)$ of the Brownian particle is a stochastic process determined by the equation of motion (called the *equation of Langevin*)

$$\frac{d}{dt}v(\omega,t)=-av(\omega,t)+\Phi(\omega,t),$$

where a is a constant and $\Phi(\omega,t)$ is the generalized derivative of a normal process, *i. e.*, a generalized stochastic process with independent values (concerning the theory of Langevin's equation, see [3]). We find that in this theory the velocity $v(\omega,t)$ is a continuous process (in the theory of Einstein-Smoluchowski, assuming that the trajectories of a Brownian motion form a normal process, velocity is not even an ordinary process). We shall determine the distribution of this velocity using the methods of generalized stochastic processes.

Suppose that $F(\omega,t)$ is a normal process with the mean $\mu(t)$ and the variance $\sigma^2(t)$ and that

$$\Phi(\omega,t)=\frac{d}{dt}F(\omega,t). \tag{91}$$

Let us also suppose that the functions $\mu(t)$ and $\sigma^2(t)$ are locally integrable. To simplify the computations, we may suppose without loss of generality that $F(\omega, 0) = 0$. From the foregoing example (formula (90)) we obtain

$$B_\Phi(t_1, t_2) = \delta(t_1-t_2)\frac{d}{dt}\sigma^2(t_1). \tag{92}$$

Let us write

$$V(\omega, t) = e^{at}v(\omega, t), \tag{93}$$

$$\Psi(\omega, t) = e^{at}\Phi(\omega, t); \tag{94}$$

then Langevin's equation assumes the form

$$\frac{d}{dt}V(\omega, t) = \Psi(\omega, t). \tag{95}$$

The process $\Psi(\omega, t)$ also has independent values (see section II.4, example (b)), whence by Theorem 8 it is the derivative of a generalized process $G(\omega, t)$ with independent increments:

$$\frac{d}{dt}G(\omega, t) = \Psi(\omega, t). \tag{96}$$

From (91) and (94) it follows that

$$\frac{d}{dt}\left(e^{at}F(\omega, t)-a\int_0^t e^{au}F(\omega, u)\,du\right) = \Psi(\omega, t),$$

which, together with (96), implies

$$G(\omega, t) = e^{at}F(\omega, t)-a\int_0^t e^{au}F(\omega, u)\,du+a(\omega).$$

Hence follows the continuity of the process $G(\omega, t)$, which, in virtue of the independence of the increments of this process, implies that this process is normal (see, for instance, [1], p. 420). By (95) and (96) we infer that $V(\omega, t) = G(\omega, t)+b(\omega)$. Hence the process $V(\omega, t)$ is also normal. Hence, in virtue of formula (93), follows the continuity of the process $v(\omega, t)$. From (92) and (94) it follows that

$$B_\Psi(t_1, t_2) = e^{a(t_1+t_2)}\delta(t_1-t_2)\frac{d}{dt_1}\sigma^2(t_1) = e^{2at_1}\delta(t_1-t_2)\frac{d}{dt_1}\sigma^2(t_1). \tag{97}$$

Denoting the variance of the process $V(\omega, t)-V(\omega, 0)$ by $\sigma_0^2(t)$, we obtain from formula (90)

$$B_\Psi(t_1, t_2) = \delta(t_1-t_2)\frac{d}{dt_1}\sigma_0^2(t_1).$$

From this formula and from (97) we obtain after simple computations

$$\sigma_0^2(t) = e^{2at}\sigma^2(t)-2a\int_0^t e^{2au}\sigma^2(u)\,du.$$

Taking into account formulae (91), (94) and (96), we see that the mean value of the increment $V(\omega, t)-V(\omega, 0)$ is equal to

$$e^{at}\mu(t)-a\int_0^t e^{au}\mu(u)\,du.$$

Finally, using f rmula (93), we get

$$P\big(v(\omega, t)-v(\omega, 0)\,e^{-at}< x\big) =$$

$$\frac{1}{\sqrt{2\pi\left(\sigma^2(t)-2a\int_0^t e^{2a(u-t)}\sigma^2(u)\,du\right)}}\int_{-\infty}^{x}\exp\left\{-\frac{\left(y-\mu(t)+a\int_0^t e^{a(u-t)}\mu(u)\,du\right)^2}{2\left(\sigma^2(t)-2a\int_0^t e^{2a(u-t)}\sigma^2(u)\,du\right)}\right\}dy.$$

IV.1. Stationary generalized stochastic processes may be defined in the same way as stationary ordinary ones.

A generalized stochastic process $\Phi(\omega, t)$ belonging to the class $\mathfrak{K}$ is said to be *stationary* if

1° the expected value $m_\Phi(t)$ is constant,

2° the correlation distribution $B_\Phi(t_1, t_2)$ depends only on the difference t_1-t_2, *i. e.*, $B_\Phi(t_1, t_2) = [b_n(t_1-t_2)]$.

It is easily verified that if $B_\Phi(t_1, t_2) = \{b_n(t_1-t_2)\}$, then the sequence $\{b_n(t)\}$ is fundamental (considered as a sequence of functions of one variable t), whence it represents a distribution depending on one variable; this distribution will be written as $B_\Phi(t)$. We obviously have $B_\Phi(t_1, t_2) = B_\Phi(t_1-t_2)$.

$B_\Phi(t)$ will also be called the *correlation distribution of the stationary process* $\Phi(\omega, t)$.

Examples. (a) In order to show that the ordinary stationary process $F(\omega, t)$ is also stationary in the generalized sense it is sufficient to show that

$$m_F(t) = \mathcal{E}F(\omega, t),$$

$$B_F(t_1, t_2) = \mathcal{E}\big(F(\omega, t_1)-\mathcal{E}F(\omega, t_1)\big)\big(\overline{F(\omega, t_2)}-\overline{\mathcal{E}F(\omega, t_2)}\big).$$

To prove these equalities it is sufficient to establish the local integrability of the functions on the right side (for the expected value $\mathcal{E}$ is then equal to the expected value E). The process $F(\omega, t)$ being stationary (in the ordinary sense), $\mathcal{E}|F(\omega,t)|^2 = \mathcal{E}|F(\omega,0)|^2$. Thus we see that every ordinary process stationary in the ordinary sense is stationary in the generalized sense.

(b) Now we shall supply an example of a continuous process which is stationary in the general but not in the ordinary sense. For this purpose we shall set forth a new hypothesis about the space of elementary events: we suppose that there exists a partition of Ω into measurable disjoint sets:

$$\Omega = \bigcup_{n=1}^{\infty} \Omega_n$$

with $P(\Omega_n) > 0$ $(n = 1, 2, \ldots)$. We define a sequence of random variables:

$$a_n(\omega) = \begin{cases} \sqrt{\dfrac{P(\Omega_{2n-1})}{P(\Omega_{2n-1})P(\Omega_{2n}) + P(\Omega_{2n})^2}} & \text{if} \quad \omega \in \Omega_{2n}, \\ \sqrt{\dfrac{P(\Omega_{2n})}{P(\Omega_{2n-1})P(\Omega_{2n}) + P(\Omega_{2n-1})^2}} & \text{if} \quad \omega \in \Omega_{2n-1}, \\ 0 & \text{elsewhere.} \end{cases} \tag{98}$$

Let us write

$$f(\omega,t) = \sum_{n=1}^{\infty} a_n(\omega) e^{int}.$$

By formula (98) we infer that $f(\omega,t) = a_k(\omega)e^{ikt}$, for $\omega \in \Omega_{2k} \cup \Omega_{2k-1}$, which implies the continuity of the process $f(\omega,t)$. We shall show that this process is stationary in the generalized sense.

Let

$$F(\omega,t) = \sum_{n=1}^{\infty} \frac{a_n(\omega)}{in} e^{int}. \tag{99}$$

The process $F(\omega,t)$ is also continuous and we have

$$\frac{d}{dt}F(\omega,t) = f(\omega,t). \tag{100}$$

Formulae (98) and (99) lead to

$$\mathcal{E}F(\omega,t_1)\overline{F(\omega,t_2)} = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{in(t_1-t_2)}, \tag{101}$$

$$\mathcal{E}F(\omega,t) = 0. \tag{102}$$

By (101) we see that the expected value $\mathcal{E}F(\omega,t_1)\overline{F(\omega,t_2)}$ is continuous. From formula (100) we deduce the existence of $m_f(t)$ and $B_f(t_1,t_2)$. Taking into account (101) and (102) we have

$$m_f(t) = 0, \qquad B_f(t_1,t_2) = \sum_{n=1}^{\infty} e^{in(t_1-t_2)}, \tag{103}$$

consequently the process $f(\omega,t)$ is stationary in the generalized sense. If the process $f(\omega,t)$ were stationary in the ordinary sense, arguing as in the foregoing example we should prove that the right side in formula (103) is an ordinary function, which is not true.

(c) Let $F(\omega,t)$ be a homogeneous normal process. Then the mean value of the increment $F(\omega,t)-F(\omega,0)$ is equal to μt and the variance to $\sigma^2 t$. Thus the example (a) of section III.5 implies, for the process

$$\Phi(\omega,t) = \frac{d}{dt}F(\omega,t)$$

the equalities

$$m_\Phi(t) = \mu, \qquad B_\Phi(t_1,t_2) = \sigma^2\delta(t_1-t_2).$$

Consequently, the generalized process $\Phi(\omega,t)$ is stationary.

(d) Let the process $\Phi(\omega,t)$ be stationary; then the process $d\Phi(\omega,t)/dt$ is also stationary. This follows from the equalities

$$m_{\frac{d}{dt}\Phi}(t) = \frac{d}{dt}m_\Phi(t), \qquad B_{\frac{d}{dt}\Phi}(t_1,t_2) = \frac{\partial^2}{\partial t_1 \partial t_2}B_\Phi(t_1,t_2).$$

IV.2. We shall recall some results concerning ordinary stochastic processes with stationary k-th increments. These results will lead directly to the theorems on generalized stationary processes.

An ordinary stochastic process $F(\omega,t)$ has stationary k-th increments if

1° for every h the expected value $\mathcal{E}\Delta_h^{(k)}F(\omega,t)$ does not depend on t,

2° the expected value $\mathcal{E}\Delta_{h_1}^{(k)}F(\omega,t_1)\overline{\Delta_{h_2}^{(k)}F(\omega,t_2)}$ is a continuous function depending only on h_1, h_2, and t_1-t_2 (In the particular case of $k=0$ we obtain ordinary stationary processes.)

A. M. Yaglom ([7], p. 152) has given the following spectral representation of processes with stationary k-th increments:

(*) *For every process $F(\omega,t)$ with stationary k-th increments there exists a process $z(\omega,t)$ with non-correlated increments (i. e., satisfying, for* *all* $t_1 < t_2 < t_3 < t_4$, $\mathcal{E}\big(z(\omega, t_1) - z(\omega, t_2)\big)\big(\overline{z(\omega, t_3)} - \overline{z(\omega, t_4)}\big) = 0$ *and a system* $a_0(\omega), a_1(\omega), \ldots, a_{k-1}(\omega)$ *of random variables such that*

$$(104)\quad F(\omega, t) = \int_{-\infty}^{\infty} \left(e^{i\lambda t} - \frac{1}{1+|\lambda|^k} \sum_{j=0}^{k-1} \frac{(i\lambda t)^j}{j!}\right) \frac{(1+i\lambda)^k}{(i\lambda)^k} dz(\omega, \lambda) + \sum_{j=0}^{k-1} a_j(\omega) t^j$$

(this integral is understood in the sense of almost uniform mean convergence with respect to t, whence also in the sense of the convergence $\rightarrow$).

Every bounded non-decreasing function $H_F(t)$, continuous on the right and satisfying, at the points of continuity $t_1 > t_2$, the relation

$$(105)\quad \mathcal{E}|z(\omega, t_1) - z(\omega, t_2)|^2 = H_F(t_1) - H_F(t_2),$$

and such that $H_F(0) = 0$ is called the *spectral function of the process* $F(\omega, t)$; this function is uniquely determined. A. M. Yaglom has shown that every bounded, non-decreasing function, continuous at the right, and vanishing at zero, is a spectral function of a process with stationary k-th differences (see [7], p. 154).

We shall now prove some lemmas connecting generalized stationary processes with continuous processes having stationary k-th increments.

LEMMA 14. *For every generalized stationary proces $\Phi(\omega, t)$ there exist a non-negative integer k and a continuous process $F(\omega, t)$ with stationary k-th increments such that*

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t).$$

Proof. The process $\Phi(\omega, t)$ is in the clas $\mathfrak{K}$, for it is stationary. Making use of Lemma 9 we deduce the existence of a continuous process $F(\omega, t)$ with a continuous expected value $\mathcal{E}|F(\omega, t)|^2$ and satisfying, for a certain k, the equality

$$\frac{d^k}{dt^k} F(\omega, t) = \Phi(\omega, t),$$

whence also the equality

$$(106)\quad \frac{\partial^{2k}}{\partial t_1^k \partial t_2^k} \mathcal{E}F(\omega, t_1)\overline{F(\omega, t_2)} = B_\Phi(t_1, t_2) + |m_\Phi|^2.$$

We can suppose without loss of generality that for the same k there exists a continuous function $b(t)$ such that

$$(107)\quad \frac{d^{2k}}{dt^{2k}} b(t) = B_\Phi(t).$$

We shall show that the process $F(\omega, t)$ has stationary k-th increments. From the equality

$$\frac{d^k}{dt^k} \mathcal{E}F(\omega, t) = m_\Phi,$$

valid in this case for every h, it follows that $\mathcal{E}\Delta_h^{(k)} F(\omega, t) = m_\Phi h^k$. The continuity of the function $\mathcal{E}|F(\omega, t)|^2$ implies the continuity of the function $\mathcal{E}\Delta_{h_1}^{(k)} F(\omega, t_1) \overline{\Delta_{h_2}^{(k)} F(\omega, t_2)}$ with respect to h_1, h_2, t_1 and t_2. From formulae (106) and (107) we obtain

$$\mathcal{E}\Delta_{h_1}^{(k)} F(\omega, t_1) \Delta_{h_2}^{(k)} F(\omega, t_2) = (-1)^k \Delta_{h_1}^{(k)} \Delta_{h_2}^{(k)} b(t_1 - t_2) + m_\Phi^2 h_1^k h_2^k,$$

from which it follows that the process $F(\omega, t)$ has stationary k-th increments.

LEMMA 15. *Let $F(\omega, t)$ be a continuous process having stationary k-th increments; then $d^k F(\omega, t)/dt^k$ is a stationary process.*

Proof. This immediately follows from the representation

$$m_{\frac{d^k}{dt^k}F}(t) = [n^k \mathcal{E}\Delta_{1/n}^{(k)} F(\omega, t)],$$

$$B_{\frac{d^k}{dt^k}F}(t_1, t_2) = [n^{2k} \mathcal{E}\Delta_{1/n}^{(k)} F(\omega, t_1) \overline{\Delta_{1/n}^{(k)} F(\omega, t_2)} - |m_{d^kF/dt^k}|^2].$$

Joining Lemmas 14 and 15 we get

THEOREM 18. *A generalized stochastic process is stationary if and only if it is the generalized derivative of the k-th order of a continuous process with stationary k-th increments, k being a non-negative integer.*

COROLLARY. *Using theorem* 18 *and the theorem* (*) *of Yaglom we immediately obtain the spectral representation of the stationary generalized processes given by K. Ito* [6].

Indeed, let the stationary process $\Phi(\omega, t)$ be the derivative of the k-th order of the continuous process $F(\omega, t)$ with stationary k-th increments. From formula (104), differentiating k times, we obtain

$$(108)\quad \Phi(\omega, t) = \int_{-\infty}^{\infty} e^{i\lambda t} dZ(\omega, \lambda),$$

where the process

$$Z(\omega, t) = \int_0^t (1+iu)^k dz(\omega, u)$$

has non-correlated increments. The function

$$H_\Phi(t) = \int_0^t (1+u^2)^k dH_F(u)$$

is called the *spectral function of the generalized process* $\Phi(\omega, t)$. It is easily seen that the function $H_\Phi(t)$ is non-decreasing, continuous on the right, vanishing at zero, and satisfies, at the points of continuity $t_1 > t_2$,

$$\mathcal{E}|Z(\omega, t_1)-Z(\omega, t_2)|^2 = H_\Phi(t_1)-H_\Phi(t_2). \tag{109}$$

The spectral function of a stationary generalized process is unbounded in general. There exists however a number m such that

$$\lim_{|t|\to\infty} \frac{H_\Phi(t)}{t^m} = 0. \tag{110}$$

Every non-decreasing function $H(t)$ continuous on the right, vanishing at zero and satisfying relation (110) is, for a certain m, the spectral function of a stationary generalized process. This follows from the fact that, in this case, the function

$$F(t) = \int_0^t \frac{dH(u)}{(1+u^2)^m}$$

is a spectral function of an ordinary process with stationary m-th increments (compare the theorem (*)).

IV.3. Now we shall investigate the correlation distributions of generalized stationary processes. For ordinary stationary processes the correlation function is the Fourier-Stieltjes transform of the spectral function of the process. An analogous theorem holds for generalized processes (compare [6], [4]). We adopt the following definition: If the sequence of functions

$$\Big\{ \int_{-n}^{n} g(t, u)\,df(u)\Big\}$$

is fundamental, then the distribution

$$\Big[\int_{-n}^{n} g(t, u)\,df(u)\Big]$$

will be denoted by

$$\int_{-\infty}^{\infty} g(t, u)\,df(u)$$

and will be called the *generalized Stieltjes integral.*

THEOREM 19. *The correlation distribution of a stationary generalized process is the generalized Fourier transform of the spectral function of this process, i. e.,*

$$B_\Phi(t) = \int_{-\infty}^{\infty} e^{iut}\,dH_\Phi(u)$$

holds.

Proof. Let us represent the generalized stationary process $\Phi(\omega, t)$ in the form (108). We define a sequence of ordinary processes

$$F_n(\omega, t) = \int_{-n}^{n} e^{i\lambda t}\,dZ(\omega, \lambda) \quad (n = 1, 2, \ldots) \tag{111}$$

for which the relation $F_n(\omega, t) \twoheadrightarrow \Phi(\omega, t)$ is valid. Hence by theorem 17

$$B_{F_n}(t_1, t_2) \to B_\Phi(t_1, t_2). \tag{112}$$

The formulae (109) and (111) imply

$$B_{F_n}(t_1, t_2) = \int_{-n}^{n} e^{iu(t_1-t_2)}\,dH_\Phi(u), \quad n = 1, 2, \ldots,$$

from which, taking into account the continuity with respect to t_1 and t_2 of the functions on the right side of this formula and formula (112), we obtain

$$B_\Phi(t_1, t_2) = \Big[\int_{-n}^{n} e^{iu(t_1-t_2)}\,dH_\Phi(u)\Big].$$

Hence

$$B_\Phi(t) = \Big[\int_{-n}^{n} e^{iut}\,dH_\Phi(u)\Big] = \int_{-\infty}^{\infty} e^{iut}\,dH_\Phi(u),$$

which proves our theorem.

This theorem together with the hitherto quoted properties of the spectral function implies the following necessary and sufficient condition:

The distribution $B(t)$ is a correlation distribution of a stationary generalized process if and only if it is the Fourier-Stieltjes transform of a non-decreasing function $H(t)$ satisfying for a certain m the condition

$$\lim_{|t|\to\infty} \frac{H(t)}{t^m} = 0.$$

Example. The correlation distribution $B_\Phi(t)$ of the derivative of a homogeneous normal process is equal to $\sigma^2\delta(t)$ (compare the example (c), section IV.1). Since

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iut}\,du,$$

the spectral function of the process $\Phi(\omega, t)$ is equal to $\sigma^2 t/2\pi$. The derivative of the spectral function, *i. e.*, the spectral density is therefore equal to $\sigma^2/2\pi$. Hence the process $\Phi(\omega, t)$ is the so called "white noice", *i. e.*, a stationary process with a constant spectral density. Similarly we obtain

$$B_{\frac{d^k}{dt^k}\Phi}(t) = (-1)^k\sigma^2\delta^{(2k)}(t) = \frac{\sigma^2}{2\pi}\int_{-\infty}^{\infty} e^{iut}u^{2k}\,du. \tag{113}$$

A generalization of formula (113) to arbitrary generalized processes with independent values has been given by I. M. Gelfand [4]. The following theorem gives an analogue of formula (113) for generalized stationary processes with independent values:

THEOREM 20. *The distribution $B(t)$ is the correlation distribution of a stationary generalized process with independent values if and only if the equality*

$$B(t) = \sum_{s=0}^{m}(-i)^s c_s\delta^{(s)}(t)$$

holds, where $c_0, c_1, \ldots, c_m$ are reals such that the polynomial

$$\sum_{s=0}^{m} c_s x^s$$

takes on non-negative values for real x.

Proof. Necessity. Without loss of generality we may suppose that the generalized stationary process $\Phi(\omega, t)$ with independent values satisfies the condition $m_\Phi = 0$. From the property 1 of section II.4 and from Lemma 14 follows the existence of a continuous process $F(\omega, t)$ having, for a certain k, stationary k-th increments and satisfying the equalities

$$\frac{d^k}{dt^k}F(\omega, t) = \Phi(\omega, t),$$

$$\frac{\partial^{2k}}{\partial t_1^k\partial t_2^k}\mathcal{E}F(\omega, t_1)\overline{F(\omega, t_2)} = B_\Phi(t_1 - t_2), \tag{114}$$

$$\frac{d^k}{dt^k}\mathcal{E}F(\omega, t) = m_\Phi = 0, \tag{115}$$

and such that for every $h > 0$ the processes $\Delta_h^{(k)}F(\omega, t)$ have kh-independent values. It follows from formula (114) as $h \to 0$ that

$$\frac{1}{h^{2k}}\mathcal{E}\Delta_h^{(k)}F(\omega, t_1)\overline{\Delta_h^{(k)}F(\omega, t_2)} \to B_\Phi(t_1 - t_2). \tag{116}$$

The k-th increments of the process $F(\omega, t)$ being stationary, we can set

$$b_n(t_1 - t_2) = n^{2k}\mathcal{E}\Delta_{1/n}^{(k)}F(\omega, t_1)\overline{\Delta_{1/n}^{(k)}F(\omega, t_2)}. \tag{117}$$

The functions $b_n(t)$ $(n = 1, 2, \ldots)$ are continuous, whence formula (116) implies

$$B_\Phi(t) = [b_n(t)]. \tag{118}$$

We deduce from the kh-independence of the process $\Delta_h^{(k)}F(\omega, t)$ and from formulae (115) and (117) that

$$b_n(t) = 0 \quad \text{for} \quad |t| > k/n \quad (n = 1, 2, \ldots). \tag{119}$$

From the definition of fundamental sequences it follows that there exist continuous functions $g(t), g_n(t)$ $(n = 1, 2, \ldots)$ such that

$$g_n(t) \rightrightarrows g(t) \tag{120}$$

and such that for a certain $r > 1$

$$g_n^{(r)}(t) = b_n(t) \quad (n = 1, 2, \ldots). \tag{121}$$

Hence from (118) it follows that

$$\frac{d^r}{dt^r}g(t) = B_\Phi(t). \tag{122}$$

We infer from formulae (120) and (121) that $\Delta_h^{(r)}g_n(t) \rightrightarrows \Delta_h^{(r)}g(t)$ for every h, whence, in virtue of (119), $\Delta_h^{(r)}g(t) = 0$ for $t, h > 0$ and for $t, h < 0$. Thus there exist two polynomials $V_1(t)$ and $V_2(t)$ of degree less than r such that

$$g(t) = V_1(t)H(t) + V_2(t)H(-t) + g(0) \tag{123}$$

where $H(t)$ is a function equal to 1 for $t \geqslant 0$ and equal to 0 for $t < 0$. From the equality

$$\frac{d}{dt}\big(t^sH(t)\big) = st^{s-1}H(t) \quad (s = 1, 2, \ldots)$$

and formulae (122) and (123) we obtain directly

$$B_\Phi(t) = \sum_{s=0}^{m} a_s\delta^{(s)}(t), \tag{124}$$

where $a_0, a_1, \ldots, a_m$ are certain complex numbers. From

$$B_\Phi(t) = \overline{B_\Phi(-t)},$$

$$\delta^{(s)}(t) = (-1)^s\delta^{(s)}(-t), \qquad \delta^{(s)}(t) = \overline{\delta^{(s)}(t)} \quad (s = 0, 1, \ldots)$$

and from (124) it follows that

$$B_\Phi(t) = \sum_{s=0}^{m}(-1)^s\bar{a}_s\delta^{(s)}(t),$$

whence

$$B_\Phi(t) = \sum_{s=0}^{m} (-i)^s c_s \delta^{(s)}(t), \tag{125}$$

where the reals $c_0, c_1, \ldots, c_m$ are defined by the formula

$$c_s = \frac{i^s}{2}(a_s + (-1)^s \bar{a}_s) \qquad (s = 0, 1, \ldots).$$

From formula (125), using the equality

$$\delta^{(s)}(t) = \frac{i^s}{2\pi} \int_{-\infty}^{\infty} e^{iut} d\left(\frac{u^{s+1}}{s+1}\right) \qquad (s = 0, 1, \ldots),$$

we obtain

$$B_\Phi(t) = \int_{-\infty}^{\infty} e^{iut} d\left(\sum_{s=0}^{m} \frac{c_s u^{s+1}}{2\pi(s+1)}\right).$$

From this equality and Theorem 19 we easily deduce that

$$\sum_{s=0}^{m} \frac{c_s x^{s+1}}{2\pi(s+1)}$$

is the spectral function of the process $\Phi(\omega, t)$, whence in particular it does not decrease. Differentiating with respect to the variable x, we obtain for all real x's

$$\sum_{s=0}^{m} c_s x^s \geqslant 0;$$

thus the necessity is proved.

Sufficiency. Let $c_0, c_1, \ldots, c_m$ be a system of reals such that for every real x

$$\sum_{s=0}^{m} c_s x^s \geqslant 0. \tag{126}$$

We shall show that the distribution

$$\sum_{s=0}^{m} (-i)^s c_s \delta^{(s)}(t) \tag{127}$$

is the correlation distribution of a generalized stationary process with independent values. We shall use the fact that there exists a homogeneous normal process (see [2]).

Let $\Phi(\omega, t)$ be the generalized derivative of this process. Taking into account example (c) of section II.4 we see that for every system of numbers $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$ the process

$$\Psi(\omega, t) = \sum_{j=0}^{n} \lambda_j \frac{d^j}{dt^j} \Phi(\omega, t) \tag{128}$$

has independent values. Let us suppose that the variance of the considered normal process is equal to 1, *i. e.*, that $B_\Phi(t) = \delta(t)$. From (128) we obtain $m_\Psi(t) = \lambda_0 m_\Phi$.

The following equality is easily verified:

$$B_\Psi(t) = \sum_{s=0}^{2n} \Bigg(\sum_{\substack{k+j=s \\ 0 \leqslant k, j \leqslant n}} (-1)^j \lambda_k \bar{\lambda}_j \Bigg) \delta^{(s)}(t). \tag{129}$$

Hence the process $\Psi(\omega, t)$ is stationary. We shall show that it is possible to choose complex numbers $\lambda_0, \lambda_1, \ldots, \lambda_n$ in formula (128) so that $B_\Psi(t)$ be equal to distribution (127). Without loss of generality we may suppose that m is equal to the degree of the polynomial on the left side of inequality (126). In virtue of (126), the number m is even and $c_m \geqslant 0$. We define the number n in formula (128) as $n = \frac{1}{2}m$.

From (126) we also deduce that every real root of the polynomial

$$\sum_{s=0}^{m} c_s x^s$$

is of even multiplicity. The coefficients are real, whence if a non-real number is a root of this polynomial, its conjugate complex number is also a root. Thus there exist n complex numbers $z_1, \ldots, z_n$ such that

$$z_1, z_2, \ldots, z_n, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n \tag{130}$$

are all the roots of the polynomial

$$\sum_{s=0}^{m} c_s x^s$$

(each root counted with its multiplicity).

Let $\tau_k(y_1, y_2, \ldots, y_r)$ $(k \geqslant 0)$ be the k-th fundamental symmetric polynomial of variables $y_1, y_2, \ldots, y_r$. This means that $\tau_0(y_1, \ldots, y_r) = 1$, and for $k \geqslant 1$ the polynomial $\tau_k(y_1, y_2, \ldots, y_r)$ is the sum of all the products of the form $y_{p_1} y_{p_2} \ldots y_{p_k}$, where the indices $p_1, p_2, \ldots, p_n$ run through the values $1, 2, \ldots, r$ and no two of them are equal. It is known that the coefficients of the polynomial

$$\sum_{s=0}^{m} c_s x^s$$

may be expressed by its roots (130) as follows:

$$c_s = (-1)^{m-s} c_m \tau_{m-s}(z_1, z_2, \ldots, z_n, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n) \quad (s = 0, 1, \ldots, m). \tag{131}$$

The following relation is easily verified:

$$\sum_{\substack{k+j=s \\ 0 \leqslant k, j \leqslant n}} \tau_{n-k}(z_1, z_2, \ldots, z_n) \overline{\tau_{n-j}(z_1, z_2, \ldots, z_n)} = \tau_{m-s}(z_1, z_2, \ldots, z_n, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n) \tag{132}$$

where $n = m/2$. Let us set in formula (128)

$$\lambda_j = i^{n+j} \sqrt{c_m} \tau_{n-j}(z_1, z_2, \ldots, z_n) \quad (j = 0, 1, \ldots, n);$$

then we infer from formula (129) that

$$B_\Psi(t) = \sum_{s=0}^{2n} \Big(\sum_{\substack{k+j=s \\ 0 \leqslant k, j \leqslant n}} i^s c_m \tau_{n-k}(z_1, z_2, \ldots, z_n) \overline{\tau_{n-j}(z_1, z_2, \ldots, z_n)} \Big) \delta^{(s)}(t).$$

Making use of formula (132) and the equality $n = m/2$ we get

$$B_\Psi(t) = \sum_{s=0}^{m} (-i)^s (-1)^{m-s} c_m \tau_{m-s}(z_1, z_2, \ldots, z_n, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n) \delta^{(s)}(t),$$

which, in virtue of (131), gives

$$B_\Psi(t) = \sum_{s=0}^{m} (-i)^s c_s \delta^{(s)}(t),$$

which proves the theorem.

VI.4. Let $\Phi(\omega, t)$ be an arbitrary generalized stochastic process; then there exists a generalized stochastic process $\Psi(\omega, t)$ satisfying

$$\frac{d}{dt} \Psi(\omega, t) = \Phi(\omega, t).$$

Then, for every couple a, b of real numbers, we set

$$\int_{t+a}^{t+b} \Phi(\omega, u)\, du = \Psi(\omega, t+b) - \Psi(\omega, t+a) \tag{133}$$

(this expression does not depend on the choice of the process $\Psi(\omega, t)$). This definition is due to Z. Zieleźny (in connection with this see the paper [8]). From the definition we directly infer the following:

1. Let the expected value $E\Phi(\omega, t)$ exist; then the expected value

$$E \int_{t+a}^{t+b} \Phi(\omega, u)\, du$$

exists and we have

$$E \int_{t+a}^{t+b} \Phi(\omega, u)\, du = \int_{t+a}^{t+b} E\Phi(\omega, u)\, du.$$

2. Let $\Phi_n(\omega, t) \to \Phi(\omega, t)$; then

$$\int_{t+a}^{t+b} \Phi_n(\omega, u)\, du \to \int_{t+a}^{t+b} \Phi(\omega, u)\, du.$$

The same implication holds for the convergence $\twoheadrightarrow$.

LEMMA 16. *Let $\Phi(\omega, t)$ be a generalized stationary process and let*

$$\Psi_T(\omega, t) = \frac{1}{2T} \int_{t-T}^{t+T} \Phi(u, u)\, du;$$

then as $T \to \infty$ we have

$$\frac{1}{2T} \int_{t-T}^{t+T} B_\Phi(u)\, du \to H_\Phi(+0) - H_\Phi(-0),$$

$$B_{\Psi_T}(t_1, t_2) \to H_\Phi(+0) - H_\Phi(-0).$$

Proof. We infer from Theorem 19, as $n \to \infty$, that

$$\int_{-n}^{n} e^{iut} dH_\Phi(u) \to B_\Phi(t),$$

whence by property 2

$$\int_{-n}^{n} \frac{\sin uT}{uT} e^{iut} dH_\Phi(u) \to \frac{1}{2T} \int_{t-T}^{t+T} B_\Phi(u)\, du \quad \text{as} \quad n \to \infty.$$

The continuity with respect to t of the function on the left side of this formula gives

$$\frac{1}{2T} \int_{t-T}^{t+T} B_\Phi(u)\, du = \int_{-\infty}^{\infty} \frac{\sin uT}{uT} e^{iuT} dH_\Phi(u). \tag{134}$$

Let us represent the stationary process $\Phi(\omega, t)$ in the form (108) and let us define the sequence of ordinary processes

$$\Phi_n(\omega, t) = \int_{-n}^{n} e^{i\lambda t} dZ(\omega, \lambda) \quad (n = 1, 2, \ldots). \tag{135}$$

Then we have $\Phi_n(\omega,t)\to\Phi(\omega,t)$. Thus, writing

$$\Psi_{n,T}(\omega,t)=\frac{1}{2T}\int_{t-T}^{t+T}\Phi_n(\omega,u)\,du \qquad (n=1,2,\ldots), \tag{136}$$

we deduce from property 2 for $n\to\infty$ that $\Psi_{n,T}(\omega,t)\to\Psi_T(\omega,t)$.

By Theorem 17 we see that $B_{\Psi_{n,T}}(t_1,t_2)\to B_{\Psi_T}(t_1,t_2)$ as $n\to\infty$. Since in virtue of (109), (135), and (136)

$$B_{\Psi_{n,T}}(t_1,t_2)=\int_{-n}^{n}\frac{\sin^2 uT}{(uT)^2}e^{iu(t_1-t_2)}\,dH_\Phi(u),$$

we have

$$B_{\Psi_T}(t_1,t_2)=\int_{-\infty}^{\infty}\frac{\sin^2 uT}{(uT)^2}e^{iu(t_1-t_2)}\,dH_\Phi(u). \tag{137}$$

From the definition of the spectral function it follows that there is an m such that

$$\lim_{|t|\to\infty}\frac{H_\Phi(t)}{t^m}=0.$$

Let $k=2m$; then the function

$$F(t)=\int_0^t\frac{dH_\Phi(u)}{1+u^k}$$

is bounded, non-decreasing and satisfies the equality

$$F(+0)-F(-0)=H_\Phi(+0)-H_\Phi(-0). \tag{138}$$

Let us set

$$J_T(u,T)=(-i)^k\frac{\sin uT}{uT}\frac{(1+u^k)}{(1+u^{k+2})}\left(u^2e^{iut}+\frac{e^{iut}-\sum_{j=0}^{k-1}\frac{(iut)^j}{j!}}{u^k}\right),$$

$$f_T(t)=\int_{-\infty}^{\infty}J_T(u,t)\,dF(u),$$

$$g_T(t_1,t_2)=\int_{-\infty}^{\infty}J_T(u,t_1)\overline{J_T(u,t_2)}\frac{dF(u)}{1+u^k}.$$

Taking into account (138) we obtain as $T\to\infty$

$$f_T(t)\rightrightarrows\big(H_\Phi(+0)-H_\Phi(-0)\big)\frac{t^k}{k!}, \tag{139}$$

$$g_T(t_1,t_2)\rightrightarrows\big(H_\Phi(+0)-H_\Phi(-0)\big)\frac{t_1^k}{k!}\frac{t_2^k}{k!}, \tag{140}$$

and from (134) and (137) we infer that

$$\frac{d^k}{dt^k}f_T(t)=\frac{1}{2T}\int_{t-T}^{t+T}B_\Phi(u)\,du,\qquad \frac{\partial^{2k}}{\partial t_1^k\partial t_2^k}g_T(t_1,t_2)=B_{\Psi_T}(t_1,t_2).$$

The conclusion of our Lemma then follows by formulae (139) and (140).

Let $\Phi(\omega,t)$ be a stationary process; then by property 1 of the distribution integral the expected value of the process

$$\Psi_T(\omega,t)=\frac{1}{2T}\int_{t-T}^{t+T}\Phi(\omega,u)\,du$$

is equal to m_Φ. Hence the convergence of the correlation distributions $B_{\Psi_T}(t_1,t_2)\to 0$ as $T\to\infty$ is equivalent to the convergence $\Psi_T(\omega,t)\to m_\Phi$ as $T\to\infty$. Hence from Lemma 16 we deduce the following ergodic theorem for generalized stochastic processes:

THEOREM 21. *Let $\Phi(\omega,t)$ be a generalized stationary process. The relation*

$$\frac{1}{2T}\int_{t-T}^{t+T}\Phi(\omega,u)\,du\to m_\Phi \quad \textit{as} \quad T\to\infty$$

is equivalent to the relation

$$\frac{1}{2T}\int_{t-T}^{t+T}B_\Phi(u)\,du\to 0 \quad \textit{as} \quad T\to\infty.$$

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Some remarks on the convergence of functionals on bases

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1. In this paper X denotes a Banach space, unless explicitly stated otherwise.

We shall say that a subset B of the set $A \subset X$ is a *linear rational basis* (briefly — LRB) of the set A if $x \epsilon A$ implies

$$(*)\qquad x = \sum_{i=1}^{m} a_i x_i,$$

where $x_i \epsilon B$, a_i are rational numbers and m is a positive integer depending on x.

If the representation $(*)$ is unique for each $x \epsilon A$ then the set B is called a *rational Hamel basis* of the set A.

We shall say that the subset B^* of the set $A \subset X$ is a *convex rational basis* (briefly — CRB) of the set A if there exists a point $a \epsilon B^*$ and a real number $M > 0$ such that $x \epsilon A$ implies

$$(**)\qquad x = \sum_{i=1}^{m} \beta_i (x_i - a) + a,$$

where $x_i \epsilon B^*$, $\beta_i \geqslant 0$ are rational numbers satisfying the condition $\beta_1 + \ldots + \beta_m \leqslant M$ with a positive integer m, depending on x.

We observe that every CRB of the set A is an LRB of this set.

We say that the functional ξ defined in a convex set $D \subset X$ is a *convex functional in D* if for any $x, y \epsilon D$ we have the inequality

$$({}^{**}_{\;*})\qquad \xi(\lambda x + \mu y) \leqslant \lambda \xi(x) + \mu \xi(y),$$

where $\lambda \geqslant 0$ and $\mu \geqslant 0$ are arbitrary rational numbers satisfying the condition $\lambda + \mu = 1$.

If the functional ξ is continuous in D the inequality $({}^{**}_{\;*})$ is satisfied for real λ, μ.

1.1. We denote by $\overline{K}(x_0, r)$ the closure of the sphere $K(x_0, r)$. Further we use the following terminology.

An arbitrary functional ξ is *uniformly bounded in a set A* if there exists a constant G such that $x \epsilon A$ implies $|\xi(x)| < G$, where G does not depend on x.