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Some remarks on the convergence of functionals on bases

by

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1. In this paper X denotes a Banach space, unless explicitly stated otherwise.

We shall say that a subset B of the set $A \subset X$ is a *linear rational basis* (briefly — LRB) of the set A if $x \in A$ implies

$$(*) \qquad x = \sum_{i=1}^{m} a_i x_i,$$

where $x_i \in B$, a_i are rational numbers and m is a positive integer depending on x.

If the representation $(*)$ is unique for each $x \in A$ then the set B is called a *rational Hamel basis* of the set A.

We shall say that the subset B^* of the set $A \subset X$ is a *convex rational basis* (briefly — CRB) of the set A if there exists a point $a \in B^*$ and a real number $M > 0$ such that $x \in A$ implies

$$(**) \qquad x = \sum_{i=1}^{m} \beta_i (x_i - a) + a,$$

where $x_i \in B^*$, $\beta_i \geqslant 0$ are rational numbers satisfying the condition $\beta_1 + \ldots + \beta_m \leqslant M$ with a positive integer m, depending on x.

We observe that every CRB of the set A is an LRB of this set.

We say that the functional ξ defined in a convex set $D \subset X$ is a *convex functional in D* if for any $x, y \in D$ we have the inequality

$$(\overset{**}{*}) \qquad \xi(\lambda x + \mu y) \leqslant \lambda \xi(x) + \mu \xi(y),$$

where $\lambda \geqslant 0$ and $\mu \geqslant 0$ are arbitrary rational numbers satisfying the condition $\lambda + \mu = 1$.

If the functional ξ is continuous in D the inequality $(\overset{**}{*})$ is satisfied for real λ, μ.

1.1. We denote by $\bar{K}(x_0, r)$ the closure of the sphere $K(x_0, r)$. Further we use the following terminology.

An arbitrary functional ξ is *uniformly bounded in a set A* if there exists a constant G such that $x \in A$ implies $|\xi(x)| < G$, where G does not depend on x.

ξ is *locally uniformly bounded in a region* D[1]) if there exists for every $x \in D$ a neighbourhood K such that ξ is uniformly bounded in K.

The sequence $\{\xi_n\}$ is *bounded at the point* x if there exists a constant $G(x)$ such that $|\xi_n(x)| < G(x)$ for $n = 1, 2, \ldots$

$\{\xi_n\}$ is *bounded in a set* A if it is bounded at each point of this set.

$\{\xi_n\}$ is *uniformly bounded in a set* A if there exists a constant G such that $x \in A$ implies $|\xi_n(x)| < G$ for $n = 1, 2, \ldots$

$\{\xi_n\}$ is *locally uniformly bounded in a region* D if there exists for every $x \in D$ a neighbourhood K such that $\{\xi_n\}$ is uniformly bounded in K.

We shall also use the notion of the boundedness above (below) of a functional ξ and a sequence $\{\xi_n\}$. The meaning of this terminology will be analogical to that given above.

1.2. *Let the functional ξ defined and convex in a convex region D be uniformly bounded above in a sphere $K(x_0, r) \subset D$ by a constant G. Then it is uniformly bounded below in this sphere by a constant $2\xi(x_0) - G$.*

Since $y \in K(x_0, r)$ implies $\xi(y) < G$, we have for $x \in K(x_0, r)$

$$\xi(x) \geqslant 2\xi(x_0) - \xi(2x_0 - x) > 2\xi(x_0) - G.$$

1.3. *Let the functional ξ defined and convex in a convex region D be uniformly bounded above in a sphere $K(x_0, r) \subset D$. Then it is continuous at the point x_0.*

Suppose $\xi(x) < G$ for $x \in K(x_0, r)$. Then one may easily prove that we have the inequalities

$$\frac{G - \xi(x_0)}{n} > \xi(x_0 + h) - \xi(x_0) \geqslant \xi(x_0) - \xi(x_0 - h) > \frac{\xi(x_0) - G}{n},$$

where n and $h \in X$ are such that $x_0 \pm nh \in K(x_0, r)$ (see [4], p. 92-93). Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|h\| < \delta$ implies $|\xi(x_0) - \xi(x_0 + h)| < \varepsilon$.

1.4. *If the functional ξ is continuous, convex and uniformly bounded above in $\bar{K}(x_0, r)$, then it satisfies for every $0 < \gamma < 1$ the Lipschitz condition in the sphere $K(x_0, \gamma r)$.*

Suppose $\|y_1 - x_0\| = r$ and $\|y_2 - x_0\| = \gamma r$. Our assumptions imply the inequalities $2\xi(x_0) - G < \xi(y_1) < G$ and $-G < -\xi(y_2) < G - 2\xi(x_0)$. Hence $2\big(\xi(x_0) - G\big) < \xi(y_1) - \xi(y_2) < 2\big(G - \xi(x_0)\big)$.

Since $r(1-\gamma) \leqslant \|y_1 - y_2\| \leqslant r(1+\gamma)$, we have

$$\frac{2\big(\xi(x_0) - G\big)}{r(1+\gamma)} < \frac{\xi(y_1) - \xi(y_2)}{\|y_1 - y_2\|} < \frac{2\big(G - \xi(x_0)\big)}{r(1-\gamma)}.$$

Let x_1, x_2 be arbitrary two points of the sphere $K(x_0, \gamma r)$; we consider four different points $y_i = x_1 + t_i(x_2 - x_1)$ such that $\|y_i - x_0\| = r$ for $i = 1, 4$, $\|y_i - x_0\| = \gamma r$ for $i = 2, 3$. The values of the parameters t_1, t_2, t_3 and t_4, corresponding to the points y_1, y_2, y_3 and y_4 respectively, satisfy the conditions $t_1 t_4 < 0$ and $t_2 t_3 < 0$. We take $t_1 < 0$ and $t_2 < 0$. Since $y_3, y_4 \notin K(x_0, \gamma r)$ and $t_3 > 0$, $t_4 > 0$, we have $t_3 > 1$, $t_4 > 1$. We observe that $t_1 < t_2 < 0 < 1 < t_3 < t_4$. Since the real function $\varphi(t) = \xi\big(x_1 + t(x_2 - x_1)\big)$ is convex in the closed interval $\langle t_1, t_4 \rangle$, we have the inequalities (compare [4], p. 93-94)

$$\frac{\varphi(t_2) - \varphi(t_1)}{t_2 - t_1} \leqslant \varphi(1) - \varphi(0) \leqslant \frac{\varphi(t_4) - \varphi(t_3)}{t_4 - t_3}.$$

Hence follow the inequalities

$$\frac{\xi(y_2) - \xi(y_1)}{\|y_2 - y_1\|} \leqslant \frac{\xi(x_2) - \xi(x_1)}{\|x_2 - x_1\|} \leqslant \frac{\xi(y_4) - \xi(y_3)}{\|y_4 - y_3\|}.$$

The above inequalities imply

$$|\xi(x_2) - \xi(x_1)| < \frac{2\big(G - \xi(x_0)\big)}{r(1-\gamma)} \|x_2 - x_1\|$$

for arbitrary $x_1, x_2 \in K(x_0, \gamma r)$.

2. *If the functional ξ defined and convex in a convex region $D \subset X$ is uniformly bounded above in a CRB of D then it satisfies the local Lipschitz condition and consequently is locally uniformly continuous in D.*

Let $r > 0$ be such that $K(a, r) \subset D$. Then there exists a number $\varrho > 0$ such that every $y \in K(a, \varrho)$ is of the form $y = s(x - a) + a$, where $x \in K(a, r)$ and $s = 1/M$. Moreover, the constant s may be supposed to be rational. Since $z \in B^*$ implies $\xi(z) < G$, we have

$$\xi(y) = \xi\big(s(x-a) + a\big) = \xi\Big(\sum_{i=1}^{m} s\beta_i(x_i - a) + a\Big)$$

$$\leqslant \sum_{i=1}^{m} s\beta_i \xi(x_i) + \Big(1 - \sum_{i=1}^{m} s\beta_i\Big)\xi(a) < G.$$

Hence it follows that $y \in K(a, \varrho)$ implies $\xi(y) < G$. Let x_0 belong to $D - K(a, \varrho)$. Then there exist numbers $\alpha > 0$, $\beta > 0$ and a point $b \in D$ such that $\alpha + \beta = 1$ and $x_0 = \alpha a + \beta b$. The point x_0 is an interior point of the cone

$$S = \{x : x = py + qb,\ y \in K(a, \varrho),\ p > 0,\ q > 0,\ p + q = 1\}.$$

In fact, there exists a $\delta > 0$ such that $\|x - x_0\| < \delta$ implies $\|y - a\| < \varrho$. Thus we have, for $x \in K(x_0, \delta)$, $\xi(x) < \max\big(G, \xi(b)\big)$. Hence 1.2, 1.3 and 1.4 imply 2.

[1]) *E. g.* an open set.

2.1. The assumption of the completeness of the space X is superfluous in 2.

2.2. Let η denote an additive functional defined in X as follows. If x belongs to a rational Hamel basis B of the sphere $K(0,1)$ then $\eta(x)$ equals an arbitrary negative number. For $x \in K(0,1)-B$ we put $\eta(x) = a_1\eta(x_1)+a_2\eta(x_2)+\ldots+a_m\eta(x_m)$, where a_i and $x_i \in B$ are given by (*). Since the functional η is additive, it is convex. Moreover, it is uniformly bounded above in an LRB of $K(0,1)$, but it is not continuous. It follows from 2 that the rational Hamel basis of the sphere $K(0,1)$ cannot be a CRB of the sphere $K(0,1)$.

3. *Let the functional ξ_n defined in a convex region D be continuous and convex in D for $n = 1, 2, \ldots$ Then if the sequence $\{\xi_n\}$ is bounded above in a CRB of D and*

(a) *if the sequence $\{\xi_n\}$ is bounded in a set dense in D then it is locally uniformly bounded in D, and ξ_n satisfies the local Lipschitz condition uniformly in n (consequently ξ_n are locally equicontinuous),*

(b) *if the sequence $\{\xi_n\}$ is convergent in a set dense in D then it is convergent in the whole region D to a convex and continuous functional ξ.*

Ad (a). Our assumption implies $\xi_n(x) < G$ for $n = 1, 2, \ldots, x \in B^*$. Retaining the notation from the section 2 we may conclude that for every $y \in K(a, \varrho)$ there exists an $x \in D$ such that $y = s(x-a)+a$. Hence

$$\begin{aligned}\xi_n(y) = \xi_n\big(s(x-a)+a\big) &= \xi_n\Big(\sum_{i=1}^m s\beta_i(x_i-a)+a\Big) \\ &\leqslant \sum_{i=1}^m s\beta_i\xi_n(x_i) + \Big(1-\sum_{i=1}^m s\beta_i\Big)\xi_n(a) \\ &< \sum_{i=1}^m s\beta_i G(x_i) + \Big(1-\sum_{i=1}^m s\beta_i\Big)G(a) \\ &\leqslant \max\big(G(x_1), \ldots, G(x_m), G(a)\big) = G(y).\end{aligned}$$

We consider the sphere $\bar{K}(a, \varrho^*) \subset K(a, \varrho)$. We write $F_r = \{x: \xi_n(x) \leqslant r,\ x \in K(a, \varrho^*),\ n = 1, 2, \ldots\}$ for $r = 1, 2, \ldots$ The sets F_r are closed and $\bar{K}(a, \varrho^*) = F_1 \subset F_2 \subset \ldots$ Thus there exists an r_0 such that $F_{r_0} \supset K(y_0, \varrho')$. Hence we have $\xi_n(x) < r_0+1 = G_1$ for $x \in K(y_0, \varrho')$ and $n = 1, 2, \ldots$ If $x_0 \in K(y_0, \varrho')$, then the sequence $\{\xi_n\}$ is uniformly bounded above in a neighbourhood of x_0 and the theorem is proved. Now let us suppose $x_0 \in D-K(y_0, \varrho')$. Let H denote a set dense in D such that $\{\xi_n\}$ is bounded in H. Then there exist a sphere $K(y_1, \varrho'') \subset K(y_0, \varrho')$, a point $z_0 \in H$ and numbers $\alpha > 0, \beta > 0$ such that $\alpha+\beta = 1$ and $x_0 = \alpha y_1+\beta z_0$.

Now we repeat the arguments of the last part of the proof in the section 2. We obtain $\xi_n(x) < \max\big(G_1, \xi_n(z_0)\big)$ for $n = 1, 2, \ldots$ and $x \in K(x_0, \delta)$. Since $\xi_n(z_0) < G(z_0)$, we have, for every n and $x \in K(x_0, \delta)$, $\xi_n(x) < \max\big(G_1, G(z_0)\big) = G(x_0)$.

Further let us observe that the sequence $\{\xi_n\}$ is locally uniformly bounded below in D. Indeed, there exists a point $z_1 \in H$ such that $\|z_1-x_0\| < \delta/4$ and the sequence $\{\xi_n\}$ is uniformly bounded below in the sphere $K(z_1, \delta/2)$. Since $x_0 \in K(z_1, \delta/2)$, there exists a sphere $\bar{K}(x_0, \delta_1) \subset K(z_1, \delta/2)$. Thus there exists for $x \in \bar{K}(x_0, \delta_1)$ a constant $G^*(x_0)$ such that $|\xi_n(x)| < G^*(x_0)$ for $n = 1, 2, \ldots$ From 1.4 it follows that

$$\frac{|\xi_n(x_1)-\xi_n(x_2)|}{\|x_1-x_2\|} < \frac{2\big(G^*(x_0)-\xi_n(x_0)\big)}{r(1-\gamma)}$$

for $n = 1, 2, \ldots$ and for arbitrary $x_1, x_2 \in K(x_0, \gamma\delta_1)$, where $0 < \gamma < 1$. Consequently,

$$|\xi_n(x_1)-\xi_n(x_2)| < \frac{4G^*(x_0)}{r(1-\gamma)}\|x_1-x_2\|$$

in the sphere $K(x_0, \gamma\delta_1)$.

Ad (b). The proof is immediately obtained from (a).

3.1. If we suppose X to be an arbitrary (F)-space, then lemmas 1.2 and 1.3 remain true. However, Theorem 2 must be formulated for spaces of type (F) in a less general form.

If the functional ξ, defined and convex in a convex region D, is uniformly bounded above in a CRB of D then it is locally uniformly continuous in D.

The method of the proof of the section 2 shows that ξ is locally uniformly bounded in D. Let x_0 denote an arbitrary point of D. Applying 1.3 (with the same notation) we see that for every point $x_1 \in K(x_0, r_1)$ (where $0 < r_1 < r$) the inequalities

$$\frac{G-\xi(x_1)}{n} > \xi(x_1+h)-\xi(x_1) \geqslant \xi(x_1)-\xi(x_1-h) > \frac{\xi(x_1)-G}{n}$$

hold for $h \in X$ such that $\|nh\| < r-r_1$. From 1.2 and the last inequalities follows the inequality

$$|\xi(x_1)-\xi(x_2)| < \frac{2\big(G-\xi(x_0)\big)}{n},$$

where $x_2 = x_1+h$. Thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x_1-x_2\| < \delta$ implies $|\xi(x_1)-\xi(x_2)| < \varepsilon$ for arbitrary $x_1, x_2 \in K(x_0, r_1)$.

3.2. The formulation of Theorem 3 for a space X of type (F) is similar to its formulation for a Banach space. In the case of (F)-spaces one must replace the local Lipschitz condition by the local equicontinuity in D of the functionals ξ_n, as follows from the method of the proof of Theorem 3 and from 1.2 and 1.3.

3.21. *Let ξ_n be a functional continuous and convex in D for $n = 1, 2, \ldots$ and let the sequence $\{\xi_n\}$ be convergent (bounded) in a set dense in D. Then the sequence $\{\xi_n\}$ is either convergent (locally uniformly bounded) in whole D or convergent (bounded above in) a set which is not CRB of D.*

3.22. If $D = X$ and if U_n is a linear operation from X to Y, X, Y being Banach spaces, then the functional $\xi_n(x) = \|U_n(x)\|$ is convex and continuous on X for $n = 1, 2, \ldots$ Then it is easily seen that if the sequence $\{\xi_n\}$ is bounded in an LRB of X then it is also bounded in X. Moreover, the functionals ξ_n are positive-homogeneous. Thus, since the sequence $\{\xi_n\}$ is locally uniformly bounded in a neighbourhood of zero (see section 3), it is also uniformly bounded in the unit sphere. Consequently, the well known Banach-Steinhaus theorem (see [3], p. 80) may be formulated more generally as follows:

A sequence of linear operations $\{U_n\}$ is either uniformly bounded in the unit sphere or bounded in a set which is not an LRB of X.

3.23. *Let $\xi = \xi(x_1, \ldots, x_k)$ denote a functional defined for $x_i \epsilon K(x_i^0, r_i) \subset X_i$, X_i being Banach spaces. If ξ is a continuous and convex functional of the variable x_i in $K(x_i^0, r_i)$ for $i = 1, 2, \ldots, k$, then it is a continuous functional of the point $(x_1, \ldots, x_k)$.*

Write $K = K(x_1^0, r_1) \times \ldots \times K(x_k^0, r_k)$. We choose for every point $(x_1^*, \ldots, x_k^*) \epsilon K$ an arbitrary sequence of points $(x_1^n, \ldots, x_k^n) \epsilon K$ such that $x_i^n \to x_i^*$ for $i = 1, 2, \ldots, k$. Put $\xi_n(x_1) = \xi(x_1, x_2^n)$. Then for every $x_1 \epsilon K(x_1^0, r_1)$, $\xi_n(x_1) \to \xi(x_1, x_2^*)$. According to section 3, ξ_n are equicontinuous at the point x_1^*. Hence $\xi(x_1^n, x_2^n) \to \xi(x_1^*, x_2^*)$. This implies, by an easy induction $\xi(x_1^n, \ldots, x_k^n) \to \xi(x_1^*, \ldots, x_k^*)$.

3.24. Let the functional ξ_n satisfy for $n = 1, 2, \ldots$ the same assumptions as the functional ξ in 3.23.

If $\xi_n(x_1, \ldots, x_k) \to \xi_0(x_1, \ldots, x_k)$ in K, then the ξ_n are equicontinuous at an arbitrary point $(x_1, \ldots, x_k) \epsilon K$.

Let $(x_1^*, \ldots, x_k^*)$ denote a fixed point belonging to K, and $(x_1^n, \ldots, x_k^n) \epsilon K$ for $n = 1, 2, \ldots$ an arbitrary point such that $x_i^n \to x_i^*$ for $i = 1, 2, \ldots, k$. We put $\eta_n(x_1, \ldots, x_i) = \xi_n(x_1, \ldots, x_i, x_{i+1}^*, \ldots, x_k^*)$. According to section 3, $\eta_n(x_1)$ are equicontinuous functionals of the variable x_1 at the point x_1^*. Thus $\eta_n(x_1^n) \to \eta_0(x_1^*) = \xi_0(x_1^*, \ldots, x_k^*)$. An easy induction shows that $\xi_n(x_1^n, \ldots, x_k^n) \to \xi_0(x_1^*, \ldots, x_k^*)$.

4. Now we shall give some remarks on subadditive functionals.

Given a positive integer k, we denote by S_k^* the class of functionals defined in X such that for arbitrary $x_i \epsilon X$ $(i = 1, \ldots, k+1)$ we have the inequality

$$(+) \qquad \xi(x_1 + \ldots + x_{k+1}) \leqslant \xi(x_1) + \ldots + \xi(x_{k+1}).$$

Let us put $S_k = S_k^* - \bigcup_{i=1}^{k-1} S_i^*$.

If $\xi \epsilon S_k$, we say that ξ is a *subadditive functional of order k*. Evidently, the classes S_k^* are non-empty. We prove that S_k are also non-empty.

4.1. *If η is an additive functional in X then $\xi(x) = |\sin(\eta(x) + \pi/k)|$ belongs to S_k.*

Indeed, since $|\sin t| \epsilon S_1$ we have

$$\xi(x_1 + \ldots + x_{k+1}) = |\sin(\eta(x_1) + \ldots + \eta(x_{k+1}) + \pi/k)|$$
$$\leqslant |\sin(\eta(x_1) + \pi/k)| + \ldots + |\sin(\eta(x_{k+1}) + \pi/k)| = \xi(x_1) + \ldots + \xi(x_{k+1}).$$

On the other hand, there exist $x_1, \ldots, x_i \epsilon X$ such that the inequality

$$\xi(x_1 + \ldots + x_i) \leqslant \xi(x_1) + \ldots + \xi(x_i) \quad \text{for} \quad i = 2, \ldots, k$$

is not satisfied. It suffices to put $\eta(x_j) = -\pi/k$ for $j = 1, 2, \ldots, i$.

If η is a linear functional, then the functional ξ is continuous and uniformly bounded in X. On the other hand, if η is non-continuous everywhere in X (of Hamel type) then ξ is non-continuous and uniformly bounded in the whole space X.

4.2. Now we give a method of construction for some functionals of the class S_k.

We say that the set $\sum^{(k)} \subset X$ is a *semi-modulus of order k* if the following conditions are satisfied:

(1) $x_1, \ldots, x_{k+1} \epsilon \sum^{(k)}$ implies $(x_1 + \ldots + x_{k+1}) \epsilon \sum^{(k)}$,

(2) for every i $(i = 2, \ldots, k)$ there exist points $x_1, \ldots, x_i \epsilon \sum^{(k)}$ such that $(x_1 + \ldots + x_i) \notin \sum^{(k)}$.

It is easily seen that

$$(++) \qquad \xi(x) = \begin{cases} 1 & \text{for} \quad x \notin \sum^{(k)}, \\ 0 & \text{for} \quad x \epsilon \sum^{(k)} \end{cases}$$

implies $\xi \epsilon S_k$.

For a given $x_0 \epsilon X$, $x_0 \neq 0$, write

$$A^{(k)} = \{x : x = mx_0 + x_0/k,\ m = 0, \pm 1, \pm 2, \ldots\}$$

and choose $\sum^{(k)} = A^{(k)}$. Then the functional defined by (++) is of the class S_k. Moreover, the set of all points of discontinuity of ξ is exactly countable.

4.3. Functionals of the class S_k may also be constructed by applying infinite series (for the case $k=1$, see [5], p. 137).

We denote by $\sum_n^{(k)}$ for $n = 1, 2, \ldots$ a semi-modulus of order k such that for each positive integer N and $i = 2, \ldots, k$ there exist points

$$x_1(N, i), \ldots, x_i(N, i) \in \bigcap_{n=1}^{N} \sum\nolimits_n^{(k)}$$

such that

$$\big(x_1(N, i)+\ldots+x_i(N, i)\big) \notin \bigcup_{n=1}^{N} \sum\nolimits_n^{(k)}.$$

Let us replace in (++) $\sum^{(k)}$ by $\sum_n^{(k)}$ and ξ by ξ_n. Further, let the series $|a_1|+|a_2|+\ldots$ be convergent, where $|a_n| > 0$. Then the functional

$$\xi(x) = \sum_{n=1}^{\infty} |a_n|\, \xi_n(x)$$

is of the class S_k.

Evidently, $\xi \in S_k^*$. Now we prove that for $i = 2, \ldots, k$ there exist $x_1, \ldots, x_i$ such that the inequality

$$\xi(x_1+\ldots+x_i) \leqslant \xi(x_1)+\ldots+\xi(x_i)$$

is not satisfied. The inequality

$$\sum_{n=1}^{\infty} |a_n|\, \xi_n(x_1+\ldots+x_i) \leqslant \sum_{n=1}^{\infty} |a_n| \big(\xi_n(x_1)+\ldots+\xi_n(x_i)\big)$$

implies

$$\sum_{n=1}^{N} |a_n| + \sum_{n=N+1}^{\infty} |a_n|\, \xi_n\big(x_1(N, i)+\ldots+x_i(N, i)\big)$$
$$\leqslant \sum_{n=N+1}^{\infty} |a_n| \Big(\xi_n\big(x_1(N, i)\big)+\ldots+\xi_n\big(x_i(N, i)\big)\Big)$$

for $i = 2, \ldots, k$. Thus for sufficiently large N the last inequality is not satisfied for $i = 2, \ldots, k$. Hence $\xi \in S_k$.

4.31. *Let $\xi \in S_k^*$. Every functional of the form $\xi(x)+c\xi(x_0)$, where $c \geqslant [(k-1)/2-(1+(-1)^k)/4]$ and $\xi(x_0) = \xi(-x_0) \geqslant 0$ is of the class $S^*_{(3+(-1)^k)/2}$. Moreover, since $\xi(0) \geqslant 0$, the functional $\xi(x)+c\xi(0)$ is of the class S_1 for every $c \geqslant (k-1)/2$.*

These remarks immediately follow from (+).

4.32. *If $\xi \in S_k$ then $\xi \in S^*_{kn}$ for $n = 1, 2, \ldots$*

4.33. *Suppose that for $\xi \in S_k^*$ there exists a point $x_0 \in X$ such that ξ is continuous at x_0 and $\xi(x_0)+\xi(-x_0) = 0$. Then ξ is continuous in the whole space X.*

Since $\xi \in S_k$, we have

$$\xi(x+h) \leqslant \xi(x)+k\big(\xi(x_0+h/k)+\xi(-x_0)\big)$$

and

$$\xi(x) \leqslant \xi(x+h)+k\big(\xi(x_0-h/k)+\xi(-x_0)\big).$$

Hence

$$|\xi(x)-\xi(x+h)| \leqslant k\max\big(\xi(x_0+h/k)+\xi(-x_0),\ \xi(x_0-h/k)+\xi(-x_0)\big)$$

for an arbitrary point $x \in X$.

4.34. *If $\xi \in S_k$ and $\xi(x) = \xi(-x)$ for every $x \in X$, then $\xi \geqslant 0$* [2]).

It suffices to put in (+) $x_i = (-1)^i x$ for $i = 1, \ldots, k+1$.

4.4. *Let ξ_n be a functional continuous on X for $n = 1, 2, \ldots$ and let $\xi_n \in S_k^*$ for every n. Further, let the sequence $\{\xi_n\}$ be bounded above in a CRB of a sphere in X. Then there exists in X a sphere $K(x_0, r)$ such that the sequence $\{\xi_n\}$ is uniformly bounded above in $K(x_0, r)$.*

We denote by R the set of all points x such that the sequence $\{\xi_n\}$ is bounded above at x. Let us observe that there exist rational numbers $\beta_1 \geqslant 0, \ldots, \beta_m \geqslant 0$, satisfying the inequality $\beta_1+\ldots+\beta_m \leqslant 1$ and such that the set

$$E = \Big\{x\colon x = \sum_{i=1}^{m} \beta_i(x_i-a)+a,\ x_i \in B^*\Big\}$$

is of the second category in X (compare the proof in section 2). We choose such a positive integer p, which is a multiplicity of k, that the numbers $p\beta_i$ are non-negative integers. The set $E_p = \{y : y = px+a,\ x \in E\}$ is of the second category. Since $x \in B^*$ implies $\xi_n(x) < G(x)$ for $n = 1, 2, \ldots$, we have for $y \in E_p$ and $n = 1, 2, \ldots$

$$\xi_n(y) = \xi(px+a) = \xi_n\Big(\sum_{i=1}^{m} p\beta_i x_i + p\Big(1-\sum_{i=1}^{m}\beta_i\Big)a+a\Big)$$
$$\leqslant \sum_{i=1}^{m} p\beta_i \xi_n(x_i) + p\Big(1-\sum_{i=1}^{m}\beta_i\Big)\xi_n(a)+\xi_n(a)$$
$$< (1+p)\max\big(G(x_1), \ldots, G(x_m), G(a)\big) = G(y).$$

Thus $y \in R$. The set R is an F_σ and of the second category. Hence there exists a sphere $K(x_0, r) \subset R$ such that the sequence $\{\xi_n\}$ is uniformly bounded above in $K(x_0, r)$.

[2]) The inequality $\xi \geqslant \eta$ for two functionals ξ, η means $\xi(x) \geqslant \eta(x)$ for arbitrary $x \in X$.

5. *Let ξ_n be a functional continuous on X for $n = 1, 2, \ldots$, and let $\xi_n \epsilon S_k^*$ for every n. Further, let the sequence $\{\xi_n\}$ be bounded above in a set dense in X and in a CRB of a sphere in X. Then the sequence $\{\xi_n\}$ is uniformly bounded above in every bounded set.*

According to 4.4 there exists a sphere $K(x_0, r)$ such that $\xi_n(x) < G$ for $n = 1, 2, \ldots, x \epsilon K(x_0, r)$. We choose a point $x_1 \epsilon K(x_0, r)$ such that there exists a constant $G(-x_1)$ such that $\xi_n(-x_1) < G(-x_1)$ for $n = 1, 2, \ldots$ Further we choose a sphere $K(x_1, r_1) \subset K(x_0, r)$. If the set A is bounded, then the set $\{z: z = y - x_1, y \epsilon A\}$ is bounded too. We observe that there exists such a positive integer p, which is a multiplicity of k, that there exists for every $y \epsilon A$ a point $x \epsilon K(x_1, r_1)$ such that $y = p(x - x_1) + x_2$. Thus we have for $y \epsilon A$

$$\xi_n(y) = \xi_n\big(p(x - x_1) + x_1\big) \leqslant p\xi_n(x) + p\xi_n(-x_1) + \xi_n(x_1)$$
$$< (2p+1)\max\big(G, G(-x_1)\big).$$

5.1. From 4.4 we conclude the following:

Suppose the functional ξ_n to be continuous on X, $\xi_n \epsilon S_k^*$ and $\xi_n \leqslant \xi_{n+1}$ for $n = 1, 2, \ldots$ If the sequence $\{\xi_n\}$ is convergent in a CRB of a sphere in X, then there exists a sphere $K(x_0, r) \subset X$ such that $\{\xi_n\}$ is convergent in $K(x_0, r)$.

5.2. From 5 we conclude the following:

Let ξ_n be a functional continuous on X for $n = 1, 2, \ldots$, and let $\xi_n \epsilon S_k^*$, $\xi_n \leqslant \xi_{n+1}$ for every n. If the sequence $\{\xi_n\}$ is convergent in a set dense in X and in a CRB of a sphere in X, then it is convergent in the whole space X.

6. The method of proving 4.4 and 5 shows that the following theorem holds:

Suppose the functional ξ_n to be continuous on X, $\xi_n \epsilon S_k^*$ and $\xi_n(x) = \xi_n(-x)$, for $n = 1, 2, \ldots$, and $x \epsilon X$. Further let the sequence $\{\xi_n\}$ be bounded above in an LRB of a sphere in X. Then the sequence $\{\xi_n\}$ is uniformly bounded in every bounded set in X.

6.1. Theorem 6 implies the following:

Let ξ_n be a functional continuous on X for $n = 1, 2, \ldots$ and let $\xi_n \epsilon S_k^*$, $\xi_n \leqslant \xi_{n+1}$ and $\xi_n(x) = \xi_n(-x)$ for every n and $x \epsilon X$. Further suppose the sequence $\{\xi_n\}$ to be convergent in an LRB of a sphere in X. Then the sequence $\{\xi_n\}$ is convergent in the whole space X.

6.2. *Let us suppose the functional ξ_n to be continuous on X, $\xi_n \geqslant 0$ and $\xi_n \epsilon S_k^*$ for $n = 1, 2, \ldots$ Moreover, let $\xi_n(x) \to 0$ in a CRB of a sphere in X. Then $\xi_n(x) \to 0$ in a sphere in X.*

Denote by E the set of all points $x \epsilon X$ such that $\xi_n(x) \to 0$. Arguments analogical to that used in 4.4 show that the set E is of second category. Since E is $F_{\sigma\delta}$, 7.1 implies that the set $E' = \{x: x = (x_1+x_2)/2, x_1, x_2 \epsilon E\}$ contains a sphere. Hence the set $E'' = \{y: y = 2kx + a, x \epsilon E'\}$, where $a \epsilon B^*$, also contains a sphere. The inequality

$$\xi_n(y) \leqslant k\xi_n(x_1) + k\xi_n(x_2) + \xi_n(a)$$

implies $E'' \subset E$.

6.3. *Suppose the functional ξ_n to be continuous on X, $\xi_n \epsilon S_k^*$ and $\xi_n(x) = \xi_n(-x)$ for $n = 1, 2, \ldots, x \epsilon X$. Further $\xi_n(x) \to 0$ in an LRB of a sphere in X. Then $\xi_n(x) \to 0$ everywhere.*

Arguments analogical to that of 6.2 show that $\xi_n(x) \to 0$ in a sphere $K(x_0, r)$. Applying the condition $\xi_n(x) = \xi_n(-x)$ we use the arguments of 5 with $x_0 = x_1$.

6.31. We observe that the lemmas, theorems and corollaries 4.4, 5, 5.1, 5.2, 5.3, 6, 6.1, 6.2 and 6.3 remain true if we replace the assumption of the continuity of the functional ξ_n for $n = 1, 2, \ldots$, by the lower semi-continuity.

6.4. *Let the functional ξ_n be continuous on X for $n = 1, 2, \ldots$ and let $\xi_n \epsilon S_k^*$, $\xi_n(x) = \xi_n(-x)$ for every n and $x \epsilon X$. If $\xi_n(x) \to 0$ in the whole space X, then the ξ_n are equicontinuous at each point of the space X.*

First we prove the equicontinuity of ξ_n at zero. Choose an arbitrary $\varepsilon > 0$ and write $E_N = \{x: \xi_n(x) \leqslant \varepsilon/4k, n \geqslant N\}$. Since the set E_N is closed, there exists N_0 such that E_{N_0} contains a sphere $K(x_0, \delta_0)$. Let $h/k \epsilon K(0, \delta_0)$. Since $h = k(h/k + x_0) + k(-x_0) + 1\cdot 0$ and $n \geqslant N_1$ implies $\xi_n(0) < \varepsilon/4$, we obtain the inequalities

$$|\xi_n(0) - \xi_n(h)| \leqslant k\xi_n\left(\frac{h}{k} + x_0\right) + k\xi_n(x_0) + 2\xi_n(0) < \varepsilon$$

for $n \geqslant N = \max(N_0, N_1)$, $\|h/k\| < \delta_0$. Hence it follows that, for sufficiently small $\delta > 0$, $\|h\| < \delta$ implies $|\xi_n(0) - \xi_n(h)| < \varepsilon$ for $n = 1, 2, \ldots$ Applying the inequality

$$|\xi_n(x) - \xi_n(x+h)| \leqslant k\xi_n\left(\frac{h}{k}\right),$$

valid for arbitrary $x \epsilon X$, we obtain the equicontinuity of ξ_n at each point of the space X.

6.41. Our proofs of the lemmas and theorems on subadditive functionals of order k may be applied to (F)-spaces without any change.

6.42. Let U_n denote linear operations from X to Y for $n = 1, 2, \ldots$, X, Y being (F)-spaces. We shall prove a known theorem (see [7]):

If the sequence $\{U_n\}$ is bounded at each point of X then U_n are equicontinuous at each point of X.

Since $t_n \to 0$ implies $U_n(t_n x) \to 0$ for arbitrary $x \epsilon X$, we have $\xi_n(x) = \|U_n(x)\| \to 0$ for every $x \epsilon X$. It follows from 6.4 that ξ_n are equicontinuous at every point $x \epsilon X$. Hence $x_n \to 0$ implies $U_n(t_n x_n) \to 0$, *e. g.*, for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that the inequalities $|t| \leqslant \delta$ and $\|x\| < \delta$ imply $\|U_n(tx)\| < \varepsilon$ for $n = 1, 2, \ldots$ Since, for sufficiently small $\eta > 0$, $\|x\| < \eta$ implies $\|x/\delta\| < \delta$, $\|x\| < \eta$ implies $\|U_n(x)\| < \varepsilon$ for $n = 1, 2, \ldots$

6.43. Let U_n be a linear operation from X to Y for $n = 1, 2, \ldots$, X, Y being (F)-spaces. Applying 6.3 we may formulate a well known theorem (see [7]) in the following generalized form:

The set of the points of boundedness of the sequence $\{U_n\}$ is either identical with the whole space or it is not an LRB of any sphere.

6.44. *There exists a sequence $\{\xi_n\}$ of functionals continuous on X and of the class S_k, such that $\{\xi_n\}$ is convergent to a non-continuous functional $\xi_0 \epsilon S_k$. (Consequently, there exists a sequence $\{\xi_n\}$ of continuous functionals of the class S_k convergent in the whole of space and such that the functionals ξ_n are not equicontinuous for a point $x \epsilon X$).*

We define the functionals ξ_n as follows:

$$\xi_n(x) = \sqrt[n]{\sin(\eta(x) + \pi/k)} \quad \text{for} \quad n = 1, 2, \ldots, \; x \epsilon X,$$

where η is a linear functional in X which vanishes only at zero. It is easily seen that $\xi_n \epsilon S_k$ for every n (compare 4.1) and that the set of all zeros of the functional ξ_n is identical with a semi-modulus of order k; it may be denoted by $\Sigma^{(k)}$. This set does not depend on n. Functionals ξ_n tend to the functional ξ_0 given by $(++)$ and $\sum^{(k)}$ in 4.2.

We shall return to the investigations of the general properties of convex functionals and subadditive functionals of order k in another paper.

7. Now we give some examples related to the previous theorems.

7.1. *If the set $E \subset X$ is of the second category and satisfies the Baire condition, then the set $\{x: x = (x_1 + x_2/2, x_1, x_2 \epsilon E\}$ contains a sphere.*

Since the set satisfies the Baire condition, there exist a point $a \epsilon E$ and a sphere $K(a, \varrho)$ such that the set $E \frown K(a, \varrho)$ is residual in $K(a, \varrho)$ (see [2]). Let us consider an arbitrary point $x \epsilon K(a, \varrho)$ and write $E(x) = \{y: 2x - y \epsilon E\}$. Notice that $E \frown E(x) \neq 0$. Thus $x \epsilon K(a, \varrho)$ implies $x = (x_1 + x_2)/2$ with $x_1, x_2 \epsilon E$.

7.2. *If $E \subset A \subset X$, A being a bounded set, and if E is of second category and satisfies the Baire condition, then E is a CRB of A.*

According to 7.1. there exists a sphere $K(a, \varrho)$ such that $a \epsilon E$ and $x \epsilon K(a, \varrho)$ implies $x = (x_1 + x_2)/2$, where $x_1, x_2 \epsilon E$. Since the set A is bounded, there exists a rational number $\beta \geqslant 0$ such that $y \epsilon A$ implies $y = \beta(x - a) + a$, where $x \epsilon K(a, \varrho)$, *e. g.*, $y \epsilon A$ implies $y = \beta(x_1 - a)/2 + \beta(x_2 - a)/2 + a$, where $x_1, x_2 \epsilon E$.

The method of our proof shows that 7.1 and 7.2 remain true for (F)-spaces.

7.21. From 2 and 7.2 we conclude the following:

Let the functional ξ defined and convex in a convex region D be uniformly bounded above in a subset of the second category satisfying the Baire condition. Then ξ is continuous in D.

7.22. From 7.21 we conclude the following:

If the functional ξ defined and convex in a convex region D satisfies the Baire condition, it is continuous in D.

7.23. Theorem 7.1 and the arguments of 2, 3.1, 1.3 show that 7.21 and 7.22 remain true if the space X is of type (F).

We say that the set $E^* \subset X$ is *homothetic* with the set E if $E^* = \{y: y = \tau x + b, x \epsilon E\}$, where $b \epsilon X$ and τ is a real number. Denote by C the Cantor ternary set. Put $\Gamma = \{x: \|x\| = t, x \epsilon X, t \epsilon C\}$. The set Γ is non-dense in X.

7.3. *If the set $\Gamma^* \subset K(x_0, r)$ is homothetic with the set Γ then Γ^* is a CRB of the sphere $K(x_0, r)$.*

We conclude from the triadic expansion of numbers of the closed interval $\langle 0, 1\rangle$ and from the definition of the set C that $t \epsilon \langle 0, 1\rangle$ implies $t = (t_1 + t_2)/2$, where $t_1, t_2 \epsilon C$. Hence it immediately follows that Γ^* is a CRB of $K(x_0, r)$.

7.31. We conclude from 7.3 that in the formulation of 7.21 the set of the second category satisfying the Baire condition can be replaced by Γ^*.

7.4. Further we denote by E_n the n-dimensional real Euclidean space. We define in E_n the norm $\|x\| = \max(|x_1|, \ldots, |x_n|)$, where $x = (x_1, \ldots, x_n)$. In the following theorems the notion of measurability will be understood in the sense of Lebesgue. We obtain the following result, analogical to 7.1:

If the measurable set $E \subset E_n$ is of finite positive measure then the set $\{x: x = (x_1 + x_2)/2, x_1, x_2 \epsilon E\}$ contains a sphere.

Let us observe that there exists for every $1 \geqslant a > \frac{1}{2}$ a point $a \epsilon E$ and a real number $\varrho > 0$ such that $\mu(E \frown K(a, \varrho)) = a\mu(E)$ where $\mu(E)$ denotes the measure of the set E. Put $\beta = 1 - \sqrt[n]{\frac{3}{2} - a}$. For each $x \epsilon K(a, \beta\varrho)$

we write $E^*(x) = \{y: 2x-y\epsilon E^*\}$, where $E^* = E\cap K(a,\varrho)$ and $K(x) = \{y: 2x-y\epsilon K(a,\varrho)\}$.

Since

$$\mu(E^*\cap K(x)) \geqslant \mu(E^*)-\mu(K(a,\varrho)-K(x))$$
$$= \mu(E^*)-\mu(K(a,\varrho))+\mu(K(a,\varrho)\cap K(x))$$
$$> \mu(E^*)-\mu(K(a,\varrho))+(2r(1-\beta))^n = (2r)^n/2$$

and similarly

$$\mu(E^*(x)\cap K(a,\varrho)) > (2r)^n/2;$$

we have

$$\mu(E^*\cap E^*(x)) \geqslant \mu(E^*\cap E^*(x))-\mu(K(a,\varrho)\cap K(x))+ +\mu\big((E^*\cup E^*(x))\cap(K(a,\varrho)\cap K(x))\big)$$
$$= \mu(E^*(x)\cap K(a,\varrho))+\mu(E^*\cap K(x))- -\mu(K(a,\varrho)\cap K(x)) > (2r)^n-(2r)^n = 0.$$

Since $E^*\cap E^*(x)\neq 0$, we have proved that $x\epsilon K(a,\varrho)$ implies $x = (x_1+x_2)/2$, where $x_1, x_2\epsilon E$.

7.5. If we apply 7.4, arguments similar to that of 7.2 show that:

An arbitrary measurable set $E\subset E_n$ of positive measure contained in a sphere $K(x_0,r)$ is a CRB of this sphere.

7.51. It follows from 2 and 7.5 that:

If the function $f(x_1,\dots,x_n)$ of n variables, defined and convex in a convex set $D\subset E_n$, is uniformly bounded above in a set $E\subset D$ of positive measure then $f(x_1,\dots,x_n)$ is continuous in D.

Especially, we obtain for $n=1$ the theorem given by A. Ostrowski (see [11]).

7.52. We conclude from 7.51 that:

If the function defined in 7.51 is measurable in D then it is continuous in D.

For $n=1$ we obtain the theorem of W. Sierpiński (see [12]).

7.53. It follows from 7.52 especially that:

If a function $f(x_1,\dots,x_n)$ defined and additive in E_n is measurable in E_n then it is linear [3]).

7.54. We conclude from 7.3 that:

There exist sets of measure zero contained in a sphere $K(x_0,r)$, which are CRB of this sphere.

7.6. We say that the set $E\subset X$ is *totally asymmetric* if there exists for every $x_0\epsilon E$ a sphere $K(x_0,r)$ such that $x\epsilon E\cap K(x_0,r)$ implies $2x_0-x\notin E$.

[3]) In the case $n=1$ Theorem 7.53 was first proved by M. Fréchet. This theorem has been the object of many proofs, see for example [1] and [13].

7.61. Since in the proof of 7.1 the point a is taken from the set E, we find that:

If the space X is an (F)-space then every set satisfying the Baire condition and contained in a totally asymmetric set is of the first category.

In the case $X = E_1$ we obtain the theorem given by S. Marcus (see [6]).

7.62. As in 7.61 we find from 7.4 that:

Every measurable set in E_n contained in a totally asymmetric set is of measure zero.

Especially we obtain for $n=1$ a theorem given by S. Marcus (see [6]).

7.7. *Let $\varphi(x)$ denote a function defined in E_1, periodic with the period l and integrable in $(0,l)$. If $|\lambda_n|\to\infty$ then for arbitrary $x_0\epsilon E_1$ and $h>0$ we have the relation*

$$\lim_{n\to\infty}\frac{1}{2h}\int_{x_0-h}^{x_0+h}|\varphi(\lambda_n t)|dt = \frac{1}{l}\int_0^l|\varphi(t)|dt.$$

7.8. Let $[x]$ denote the greatest integer $\leqslant x$. Let us observe that the function $\xi(x) = x+1/k-[x+1/k]$ is of the class S_k; it may be obtained by arguments similar to that of 4.1.

8. *If the series*

$$\sum_{n=1}^{\infty}|a_n|\left(\lambda_n x+\frac{1}{k}-\left[\lambda_n+\frac{1}{k}\right]\right),\qquad |\lambda_n|\to\infty,$$

is convergent in a CRB of an interval (α,β) then

$$\sum_{n=1}^{\infty}|a_n|<\infty.$$

Write

$$\xi_n(x) = \sum_{i=1}^{n}|a_i|\left(\lambda_i x+\frac{1}{k}-\left[\lambda_i x+\frac{1}{k}\right]\right).$$

Evidently $\xi_n\epsilon S_k^*$ and ξ_n is lower semi-continuous for $n=1,2,\dots$ Since the sequence $\{\xi_n\}$ is convergent in CRB of the interval (α,β), according to 4.4 and 6.31, it is uniformly bounded above in an interval (x_0-h, x_0+h). Applying 7.7 we obtain the convergence of the series

$$\sum_{n=1}^{\infty}|a_n|.$$

8.1. *If the series*

$$\sum_{n=1}^{\infty}(|a_n\cos\lambda_n x|+|b_n\sin\lambda_n x|), \qquad |\lambda_n|\to\infty,$$

is convergent in an LRB of an interval (α,β) *then*

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)<\infty.$$

Put

$$\xi_n(x)=\sum_{i=1}^{n}(|a_i\cos\lambda_i x|+|b_i\sin\lambda_i x|).$$

The functional ξ_n satisfies the assumptions of the section 6 for $n=1,2,\ldots$ Hence the sequence $\{\xi_n\}$ is uniformly bounded in an interval (x_0-h, x_0+h). If we apply 7.7 as above, we obtain the convergence of the series

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|).$$

8.11. A certain modification of the known method of proof yields, if we apply 8.1 and 7.7 (compare [14], p. 133), a more general result[4]):

If the series

$$\sum_{n=1}^{\infty}|a_n\cos\lambda_n x+b_n\sin\lambda_n x|, \qquad |\lambda_n|\to\infty,$$

is convergent in an LRB of an interval (α,β) *then*

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)<\infty.$$

It may be pointed out that the results 8.11 and 8.1 are obtained without the use of the Lebesgue integral.

8.2. *Let* η_n *be a linear functional defined in* X *for* $n=1,2,\ldots$ *and let* $\|\eta_n\|\to\infty$. *If*

$$\big(a_n\cos\eta_n(x)+b_n\sin\eta_n(x)\big)\to 0$$

in LRB of a sphere, then

$$(|a_n|+|b_n|)\to 0.$$

Write $\xi_n(x)=a_n\cos\eta_n(x)+b_n\sin\eta_n(x)$ and $\xi_n^*(x)=b_n\cos\eta_n(x)-a_n\sin\eta_n(x)$. If we put $x=z+y$, where $x,y,z\in X$, then we obtain

$$\xi_n(x)=\xi_n(z)\cos\eta_n(y)+\xi_n^*(z)\sin\eta_n(y).$$

4) Special classes LRB were used in considering absolute convergence of trigonometrical series by W. Niemytzki in [8].

Let z be a fixed point of B. Our assumptions imply $\xi_n(z)\to 0$. Since $\xi_n(x)\to 0$ for $x\in B$, we have $\xi_n^*(z)\sin\eta_n(y)\to 0$ for $y\in B(z)$, where $B(z)=\{y: y=x-z, x\in B\}$. The set $B(z)$ is an LRB of a sphere. Since the sequence $\{|\xi_n^{**}|\}$, where $\xi_n^{**}(y)=\xi_n^*(z)\sin\eta_n(y)$, satisfies the assumption of 6.3, we obtain $\xi_n^{**}(y)\to 0$ for every $y\in X$. Hence it follows that $\xi_n(x)\to 0$ for every $x\in X$. If we put $x=0$ we obtain $a_n\to 0$. Write $\xi_n'(x)=|b_n\sin\eta_n(x)|$. Then the sequence $\{\xi_n'\}$ satisfies the assumptions of 6.3. Hence $\xi_n'(x)\to 0$ for every $x\in X$. According to 6.4 the functionals ξ_n' are equicontinuous at the point $x=0$. If we put $x_n=y_n/\|\eta_n\|$, where $|\eta_n(y_n)|\geqslant\|\eta_n\|/2$ and $\|y_n\|=1$, then we obtain $1\geqslant|\eta_n(x_n)|\geqslant 1/2$. Since $\xi_n'(x_n)=|b_n\sin\eta_n(x_n)|\to 0$, we have $b_n\to 0$.

8.3. Denote by $\{\vartheta_n\}$ an arbitrary sequence of real numbers. Retaining the notation of 8.2 we obtain the following theorem:

The set

$$E(\vartheta)=\{x: (\eta_n(x)+\vartheta_n-[\eta_n(x)+\vartheta_n])\to\vartheta, \|\eta_n\|\to\infty\}, \quad \textit{where} \quad 0\leqslant\vartheta\leqslant 1,$$

is not an LRB of any sphere.

Suppose that $E(\vartheta)$ is an LRB of a sphere. Then $\sin\pi(\eta_n(x)+\vartheta_n-\vartheta)\to 0$ in this LRB, which is contrary to 8.2.

8.31. From 8.3 we immediately conclude the following:

Let the function $\varphi(x)$ *periodic with period* 1 *be continuous and let* $\varphi(x)$ *vanish only at* ϑ *in the interval* $(0,1)$, *where* $0\leqslant\vartheta\leqslant 1$. *If the sequence* $\{a_n\varphi(\eta_n(x)+\vartheta_n)\}$ *is convergent to zero in an LRB of a sphere then* $a_n\to 0$ (compare [9]).

9. Finally we give an application to the theory of (F)-spaces. In connexion with other applications of subadditive functionals concerning the theory of (F)-spaces we refer the reader to the paper [10].

Let X *be the* (F)*-space under the norm* $\|\ \|$ *satisfying for a number* $\alpha>0$ *the condition*

$$\sup_{|t|>0}\frac{\|tx\|}{|t|^\alpha}<\infty \quad \textit{for} \quad x\in X.$$

Then the norm $\|\ \|^*$ *where*

$$\|x\|^*=\sup_{|t|>0}\frac{\|tx\|}{|t|^\alpha}$$

is equivalent to the norm $\|\ \|$ *and satisfies the condition* $\|tx\|^*=|t|^\alpha\|x\|^*$.

It follows from the inequality $\|x\|^*\geqslant\|x\|$ that convergence with respect to the norm $\|\ \|^*$, implies convergence with respect to the norm $\|\ \|$. Let $\{t_n\}$ denote the sequence of all rational numbers different from zero. Put $\xi_n(x) = \|t_n x\|/|t_n|^a$ for $n = 1, 2, \ldots$ The functionals ξ_n satisfy the assumptions of 6. Thus there exists a constant G such that $\xi_n(x) < G$ for $x \epsilon K(0,1)$. The continuity of the norm $\|\ \|$ implies $\|tx\|/|t|^a < G$ for real t and for $x \epsilon K(0,1)$. Let ε be an arbitrary positive number. We choose a number $\tau > 0$ such that $G < \tau^a \varepsilon$. There exists a $\delta > 0$ such that $\|x\| < \delta$ implies $\|\tau x\| < 1$. Consequently, the inequality $\|t\tau x\|/|t|^a < G$ is satisfied. Hence $\|x\| < \delta$ implies $\|x\|^* < \varepsilon$.

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Problème de l'analyticité par rapport à un opérateur linéaire*

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1. Définitions et premier axiome. Cas intermédiaires. Ce qui suit constitue un schéma général abstrait de plusieurs résultats obtenus par l'auteur dans la théorie des fonctions polyharmoniques, ou dans la théorie de l'opérateur hyperbolique itéré ou encore dans la théorie des fonctions polycaloriques.

Le problème de l'analyticité a été posé d'une manière concrète dans une communication récente au III^e Congrès Unional des Mathématiciens Sovietiques [6].

Le cadre le plus convenable, au schéma proposé me paraît être une algèbre normée, par exemple une algèbre de Banach, possédant un élément unitaire. D'ailleurs, dans les deux premiers §, les résultats sont valables pour une algèbre quelconque.

Soit, donc, $\mathcal{B}$ une algèbre commutative à élément unitaire e. Nous utiliserons au cours de ce travail des *opérateurs linéaires* $A, B, \ldots, L, \ldots$, c'est-à-dire additifs et homogènes sur le corps K des nombres complexes.

Pour un opérateur linéaire quelconque A, nous poserons $A^0 = e$, $A^1 = A$, $A^n = A(A^{n-1})$, $n = 1, 2, 3, \ldots$.

Avec cela, la signification d'une opération telle que $A^m B^n \ldots L^p$ est claire.

À tout opérateur linéaire A nous attacherons l'opérateur bilinéaire B, défini par l'égalité

$$B(x, y) = A(xy) - xAy - yAx, \tag{1}$$

où $x \epsilon \mathcal{B}$, $y \epsilon \mathcal{B}$. Manifestement, $B(x, y) = B(y, x)$.

Il peut arriver que l'on ait $B(x, y) = \emptyset$, quels que soient les éléments x, y. Alors A s'appelle un *opérateur de dérivation*, ou plus simplement une *dérivée algébrique* de l'élément auquel il s'applique [1]. Nous laisserons de côté ce cas, qui a été amplement étudié[1]) récemment.

Dans la suite nous considérerons un opérateur A linéaire, pour lequel sont vérifiées certaines conditions.

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* Les résultats de ce Mémoire ont été présentés au Congrès des Mathématiciens Autrichiens, Vienne, 17-22 septembre 1956.

[1]) On consultera, à ce propos, avec profit le Mémoire [1] de M. J. Mikusiński.