STUDIA MATHEMATICA, T. XVII. (1958)

On some classes of linear spaces

by

S. MAZUR (Warszawa) and W. ORLICZ (Poznań)

Since the fundamental papers of F. Riesz ([5], [6]) the spaces L^a and l^a are reckoned among the classical examples of linear normed spaces in the functional analysis. The space l^a, where $a>0$, is the space of sequences $\{t_n\}$ such that the series $\sum_{\nu}^{\infty}|t_\nu|^a$ converges; L^a, where $a>0$, denotes the space of measurable functions in (a,b) for which the integral

$$\int_a^b |x(t)|^a dt$$

is finite. In the period between the wars and after World War II there appeared several papers dealing with generalizations of the spaces l^a and L^a. The idea of these generalizations is based upon the following. Let N be a non-negative function defined for all real values. One considers the class X^N of sequences for which the series

$$\varrho(x)=\sum_{\nu=1}^{\infty}N(t_\nu), \qquad x=\{t_n\},$$

converges. It may be proved under very general and natural supplementary conditions about the function N that X^N are Banach spaces. In the case $N(u)=|u|^a$, where $a\geqslant 1$, these conditions are satisfied, and l^a form a particular case of the spaces X^N. An analogous situation holds for the spaces of measurable functions in (a,b). Denoting by X^N the class of measurable functions in (a,b) for which the integral

$$\varrho(x)=\int_a^b N(x(t))\,dt$$

exists and is finite, we may prove under certain hypotheses on N that X^N is a Banach space. The spaces L^a with $a\geqslant 1$ are a particular case of the spaces X^N corresponding to $N(u)=|u|^a$.

In this paper we are concerned with the examination of the spaces X^N of sequences and those of functions from a very general standpoint. Under very slim hypotheses about the function N we deal with the following problems: a) in which case are the spaces X^N linear? b) which are the necessary and sufficient conditions for the function N in order to make it possible in the linear space X^N to define an F- or B-norm such that the relation $\|x_n\| \to 0$ as $n \to \infty$ be equivalent to $\varrho(x_n) \to 0$ as $n \to \infty$?

The main results of the paper are contained in theorems 3, 6, from which it follows that the well known sufficient conditions on N asserting that X^N is a B-space are in some sense necessary.

The paper consists of two paragraphs. In the first we deal with the spaces of sequences for which the obtained results have a more complete character than for the spaces of functions considered in the second paragraph.

Throughout this paper $N, M, \ldots$ denote non-negative functions defined for all real values; in § 1, $x, y, \ldots$ denote sequences $\{t_n\}, \{s_n\}, \ldots$ with real terms, in § 2, $x(t), y(t), \ldots$ stand for real measurable functions in (0,1). The spaces of sequences X^N and of measurable functions are always understood to be linear under the usual definitions of addition and multiplication by scalars. Equality of the measurable functions x and y means that $x(t) = y(t)$ for almost every t.

1.1. Given a function N, we shall write in this section

$$\varrho_N(x) = \sum_{\nu=1}^{\infty} N(t_\nu).$$

We shall also write $\varrho(x)$ instead of $\varrho_N(x)$ if the omission of the subscript will not cause misunderstanding about the involved function N. By X^N we shall denote the set of all sequences for which $\varrho_N(x) < \infty$. We shall frequently suppose that N satisfies the following condition:

(*) $\quad N(t_n) \to 0$ *as* $n \to \infty$ *if and only if* $t_n \to 0$ *as* $n \to \infty$.

The condition (*) implies in particular $N(0) = 0$ and $N(t) \neq 0$ for $t \neq 0$.

The functions N and M will be called *equivalent*, in symbols $N \sim M$, if the following property is satisfied:

There are constants $A > 0, B > 0, \varepsilon > 0$ *such that* $N(t) \leqslant BM(t)$ *for* $M(t) \leqslant \varepsilon$, $M(t) \leqslant AN(t)$ *for* $N(t) \leqslant \varepsilon$.

It is easily seen from this definition that

1.12. *For* N *and* M *satisfying the condition* (*), $N \sim M$ *if and only if simultaneously* $N(t) \leqslant BM(t)$ *and* $M(t) \leqslant AN(t)$ *in some neighbourhood of* 0.

In the sequel we shall need the following lemmata:

1.13. *Let* $M(t, s)$ *and* $N(t, s)$ *be non-negative functions defined for all real values of* t *and* s. *A necessary and sufficient condition that the convergence of the series* $\sum_{\nu=1}^{\infty} M(t_\nu, s_\nu)$ *imply the convergence of the series* $\sum_{\nu=1}^{\infty} N(t_\nu, s_\nu)$ *is the existence of two positive constants* C *and* ε *such that*

(+) $\quad N(t, s) \leqslant CM(t, s)$ *when* $M(t, s) < \varepsilon$.

Sufficiency being obvious, we prove only the necessity. First it is easily seen that $M(t, s) = 0$ implies $N(t, s) = 0$. Suppose (+) is not satisfied, then there must exist t_n, s_n such that $N(t_n, s_n) \geqslant nM(t_n, s_n)$ and $M(t_n, s_n) \leqslant 1/n^2$ for $n = 1, 2, \ldots$ We may suppose that $M(t_n, s_n) \neq 0$. Let us choose positive integers p_n such that $1/n^2 \leqslant p_n M(t_n, s_n) \leqslant 2/n^2$ for $n = 1, 2, \ldots$ Let $t_n' = t_r, s_n' = s_r$ for $p_0 + \ldots + p_{r-1} < n \leqslant p_0 + \ldots + p_r$ as $r = 1, 2, \ldots$ (we set $p_0 = 0$). Since

$$\sum_{\nu=1}^{\infty} M(t_\nu', s_\nu') < \infty, \qquad \sum_{\nu=1}^{\infty} N(t_\nu', s_\nu') = \infty$$

we are led to a contradiction.

Setting $M(t, s) = M(t), N(t, s) = N(t)$ for arbitrary t, s we find from 1.13 that

1.14. *A necessary and sufficient condition that* $\varrho_M(x) < \infty$ *imply* $\varrho_N(x) < \infty$ *is the existence of two constants* $B > 0, \varepsilon > 0$ *such that* $N(t) \leqslant BM(t)$ *for* $M(t) < \varepsilon$.

1.2. *Let* $\varphi_n(\omega)$ *denote for* $n = 1, 2, \ldots$ *non-negative measurable functions defined in* $(-\delta, \delta)$, *satisfying for a certain constant* $K > 0$ *the inequality*

(+) $\quad \varphi_n(\omega_1 + \omega_2) \leqslant K[\varphi_n(\omega_1) + \varphi_n(\omega_2)]$ *for* $\omega_1, \omega_2 \in E$,

E *being a measurable set in* $(-\delta, \delta)$ *such that* $|E| > \frac{7}{4}\delta$.

Then

A. *If* $\varphi_n(\omega) \to 0$ *as* $n \to \infty$ *for* $|\omega| < \delta$, *then for each* $\varepsilon > 0$ *the following inequality is satisfied for almost all* n*'s*:

$$\varphi_n(\omega) \leqslant \varepsilon \quad \textit{for} \quad |\omega| < \delta/4.$$

B. *If the sequence* $\varphi_n(\omega)$ *is bounded for each* $|\omega| < \delta$, *then there is a constant* $L > 0$ *such that*

$$\varphi_n(\omega) \leqslant L \quad \textit{for} \quad |\omega| < \delta/4 \textit{ and } n = 1, 2, \ldots$$

Ad A. Given $\varepsilon > 0$ let us set for $k = 1, 2, \ldots$

$$T_k^+ = \{\omega : \varphi_n(\omega) \leqslant \varepsilon \text{ for } n \geqslant k,\ \omega \in E \frown (0,\delta)\},$$
$$T_k^* = \{\omega : \varphi_n(\omega) \leqslant \varepsilon \text{ for } n \geqslant k,\ \omega \in E \frown (-\delta, 0)\}.$$

Let us denote by T_k^- the set symmetrical to T_k^+ with respect to the point 0. Since $\lim_k T_k^+ = (0,\delta)\frown E$, $\lim_k T_k^* = (-\delta, 0)\frown E$ we infer that $\lim_k |T_k^*| = |(0,\delta)\frown E|$, $\lim_k |T_k^-| = |(-\delta,0)\frown E|$, $\lim_k |T_k^*| = |(-\delta,0)\frown E|$. It follows that it is possible to choose k_0 so large that for the set $S^- = T_{k_0}^* \frown T_{k_0}^-$ and for the set S^+ symmetrical to it (with respect to 0) the inequalities $|S^-| > \frac{3}{4}\delta$, $|S^+| > \frac{3}{4}\delta$ respectively are satisfied. Every translation S_ω^+ of the set S^+ by the length ω such that $|\omega| \leqslant \delta/4$ has common points with S^+. It follows that

$$(++) \qquad \omega = \omega_1 - \omega_2 \quad \text{where} \quad \omega_1 \in S^+ \subset T_{k_0}^+,\ -\omega_2 \in S^- \subset T_{k_0}^* \text{ if } |\omega| < \delta/4.$$

By (+) we have $\varphi_n(\omega) \leqslant K[\varphi_n(\omega_1) + \varphi_n(-\omega_2)] \leqslant K\cdot 2\varepsilon$ for $|\omega| < \delta/4$ and $n \geqslant k_0$.

Ad B. Let us set for $k = 1, 2, \ldots$

$$T_k^+ = \{\omega : \varphi_n(\omega) \leqslant k \text{ for } n \geqslant 1,\ \omega \in E \frown (0,\delta)\},$$
$$T_k^* = \{\omega : \varphi_n(\omega) \leqslant k \text{ for } n \geqslant 1,\ \omega \in E \frown (-\delta, 0)\}.$$

Starting with these sets let us define the sets S^+ and S^- as above. Then (++) remains true, whence by (+) it follows

$$(\substack{+\,+\\+}) \quad \varphi_n(\omega) \leqslant K[\varphi_n(\omega_1) + \varphi_n(-\omega_2)] \leqslant K\cdot 2k_0 \quad \text{for} \quad |\omega| \leqslant \delta/4,\ n \geqslant 1.$$

1.3. (a) *We have $X^N = X^M$ if and only if $N \sim M$;*

(b) *The space X^N is identical with the space of all sequences if and only if $N(t) = 0$ for all t.*

(c) *The space X^N is identical with the space of the sequences $\{t_n\}$ for which $t_n = 0$ for almost all n if and only if $N(0) = 0$ and there is $\delta > 0$ such that $N(t) > \delta$ for $t \neq 0$.*

Ad (a). It follows immediately by 1.14 and by the definition of equivalence.

Ad (b). Sufficiency being trivial, let us suppose that $N(t) \neq 0$; then for the sequence $x = (t, t, \ldots)$ we have $\varrho_N(x) = \infty$, whence X^N could not contain all the sequences.

Ad (c). The sufficiency is trivial. Now let X^N consist only of sequences whose almost all elements are equal to zero: then the sequence $(0, 0, \ldots)$ is in X^N, whence $N(0) = 0$. Let us now suppose that $N(t_n) \to 0$ where $t_n \neq 0$. Then for a subsequence of t_n we have $\varrho_N(x) < \infty$ and X^N contains a sequence with infinitely many terms different from 0.

1.4. *The following conditions are necessary and sufficient that the space X^N be linear:*

(a) *There are constants $C > 0$, $\varepsilon > 0$ such that*

$$N(t+s) \leqslant C[N(t) + N(s)] \quad \textit{for} \quad N(t) \leqslant \varepsilon,\ N(s) \leqslant \varepsilon;$$

(b) *For each ω there are constants $D_\omega > 0$, $\varepsilon_\omega > 0$ such that*

$$N(\omega t) \leqslant D_\omega N(t) \quad \textit{for} \quad N(t) < \varepsilon_\omega;$$

(c) $N(0) = 0$.

To prove (a) let us set $M(t,s) = N(t) + N(s)$, $N(t,s) = N(t+s)$. Since $\varrho_N(x) < \infty$, $\varrho_N(y) < \infty$ implies $\varrho_N(x+y) < \infty$, it is sufficient to apply 1.13 to $M(t,s)$ and $N(t,s)$. To prove (b) we apply 1.14 to the functions $N(\omega t)$ and $N(t)$ considering that from $\varrho_N(x) < \infty$ follows $\varrho_N(\omega x) < \infty$. (c) follows from the trivial remark that X^N contains the sequence $(0, 0, \ldots)$ if and only if $N(0) = 0$.

1.5. *Let the function N be measurable and let the space X^N be linear. Then one of the following three cases holds:*

(a) $\quad N(t) = 0$ *for every* t;

(b) $\quad N(0) = 0$, *there is a* $\delta > 0$ *such that* $N(t) \geqslant \delta$ *as* $t \neq 0$;

(c) $\quad N$ *satisfies the condition* (*).

Let us suppose that the cases (a) and (b) are not satisfied. Suppose that for t_n we have $|t_n| \geqslant \varrho > 0$ and $N(t_n) \to 0$. The space X^N being linear, we have by 1.4 (b) $N(\omega t_n) \to 0$. for $|\omega| < 1$. In $(-1, 1)$ there is a set E of measure greater than $7/4$ such that $N(\omega t_n) < \varepsilon$ for almost all n's for $\omega \in E$. Applying 1.4 (a) and 1.2,A to $\varphi_n(\omega) = N(\omega t_n)$ with n sufficiently large, we get for m sufficiently large

$$N(\omega t_m) < \varepsilon \quad \text{for} \quad |\omega| < 1/4,$$

and since $|t_m| \geqslant \varrho$, we infer that $N(t) < \varepsilon$ for $|t| < \varrho/4$, consequently $N(t) = 0$ for $|t| < \varrho/4$. This implies together with 1.4 (b) that $N(t) = 0$ for arbitrary t, which is contrary to the hypothesis. Thus $N(t_n) \to 0$ as $n \to \infty$ implies $t_n \to 0$. To prove that, conversely, $t_n \to 0$ as $n \to \infty$ implies $N(t_n) \to 0$ let us notice that if (b) is not satisfied, then for certain $\bar{t}_n \neq 0$ we have $N(\bar{t}_n) \to 0$ as $n \to \infty$. By the preceding $\bar{t}_n \to 0$ as $n \to \infty$. Applying to $N(\omega \bar{t}_n)$ the same argument as formerly to $N(\omega t_n)$ we can prove $N(\omega \bar{t}_m) < \varepsilon$ for $|\omega| < 1/4$ and m sufficiently large. Let $t_n \to 0$ as $n \to \infty$. Then since for almost all n we have $t_n = \omega_n \bar{t}_m$ where $|\omega_n| < 1/4$, we see that $N(t_n) < \varepsilon$.

1.51. The following example shows that without the hypothesis of measurability of N Theorem 1.5 is no longer true. Let $f(z)$ be a complex function of the complex variable discontinuous and satisfying the equations $f(z_1+z_2) = f(z_1) + f(z_2)$, $f(z_1 z_2) = f(z_1) f(z_2)$ for arbitrary z_1, z_2. Let us set $N(t) = |f(t)|$; for real t. The space X^N is linear, for $N(t+s) \leqslant N(t) + N(s)$, $N(st) = N(s)N(t)$. For the function N none of the conditions (a), (b), (c) of 1.5 is satisfied, whence we can easily show by aid of 1.3 (a) that there exists no measurable function M such that $X^N = X^M$. The condition 1.5 (a) does not hold for N for, if it were so, the function $f(z)$ would vanish along the real axis, whence also for every z, which is evidently impossible. The condition 1.5 (b) is not satisfied for $f(t) = at+bti$ for rational t where a and b are real constants. The condition 1.5 (c) is not satisfied. Indeed, if N satisfies the condition (*), the function $f(t)$ is continuous at 0 and since $f(t_1+t_2) = f(t_1)+f(t_2)$, we infer that $f(t) = at+bti$ for all real t. Hence $f(z) = f(x)\pm f(y)i$ when $z = x+iy$; therefore f would be continuous.

Theorems 1.3, 1.5 explain the importance of the condition (*): Under the hypothesis of measurability of N, except the two extreme trivial cases of linear sequence spaces listed in 1.3 (a), (b), X^N is a linear space only if the condition (*) is satisfied. If N satisfies the condition (*) (without being measurable), then the cases 1.5 (a), (b) evidently do not hold, whence the condition (*) excludes the above-mentioned extreme cases. The last remarks justify the need of supposing (*) in the sequel.

1.6. *Let N satisfy the condition (*); if*

(a) *there exist constants $C > 0$, $\varepsilon > 0$ such that $N(t+s) \leqslant C[N(t)+N(s)]$ for* $|t| < \varepsilon$, $|s| < \varepsilon$;

(b) *for every ω there are constants $D_\omega > 0$, $\varepsilon_\omega > 0$ such that $N(\omega t) \leqslant D_\omega N(t)$ for $|t| < \varepsilon_\omega$,*

then

(c) *for every $\varrho > 0$ there exists $D > 0$, $\delta > 0$ such that $N(\omega t) \leqslant DN(t)$ for $|t| < \delta$, $|\omega| < \varrho$.*

Let us define M as follows:

$$M(t) = \begin{cases} \overline{\lim\limits_{s\to t}} N(s) & \text{as} \quad |t| < \varepsilon, \\ 1 & \text{as} \quad |t| \geqslant \varepsilon. \end{cases}$$

Continuity at 0 of the function N implies, together with (a), $M(t) < \infty$ for $|t| < \varepsilon$. M is a measurable function equivalent to N. To prove that $M \sim N$ let us observe that $M(t) \geqslant N(t)$ in some neighbourhood of 0. The inequality $N(s) \leqslant C[N(t)+N(s-t)]$ satisfied for $|t| < \varepsilon$, $|s-t| < \varepsilon$ implies $M(t) = \overline{\lim\limits_{s\to t}} N(s) \leqslant C[N(t)+\overline{\lim\limits_{s\to t}} N(s-t)] = CN(t)$ for $|t| < \varepsilon$. It is sufficient to apply 1.12.

Let us suppose that our theorem does not hold. Then there exist $t_n \neq 0$, $t_n \to 0$ as $n \to \infty$ and ω_n such that $|\omega_n| < \varrho$, $N(\omega_n t_n)/N(t_n) \to \infty$ as $n \to \infty$. Let us observe that the condition (*) implies $N(t_n) \neq 0$, since $t_n \neq 0$. Let us define for $n = 1, 2, \ldots$ and $|\omega| < \varrho$ the functions

$$\varphi_n(\omega) = M(\omega t_n)/M(t_n).$$

We have $M(t_n) \neq 0$, for $M \sim N$. The definition of M implies that the conditions (a), (b) remain valid when N is replaced by M. It follows that for n sufficiently large the functions $\varphi_n(\omega)$ satisfy the inequality (+) of 1.2, with some constant K and $\delta = \varrho$, $E = (-\delta, \delta)$. Applying (b) to M we deduce that the sequence $\varphi_n(\omega)$ is bounded for $|\omega| < \varrho$; moreover the functions $\varphi_n(\omega)$ are measurable. By 1.2, B we have $\varphi_n(\omega) \leqslant L$ for $|\omega| < \varrho/4$ and sufficiently large n. Hence and from $M \sim N$ we get for almost all n

$$N(\omega_n t_n)/N(t_n) < L'$$

with some constant $L' > 0$, which leads to a contradiction.

1.61. Concerning 1.6 let us notice that there exist non-measurable functions satisfying the conditions (*), 1.6, (a), (c). As an example we may take an arbitrary non-measurable function N satisfying the conditions $at \leqslant N(t) \leqslant bt$ for $t \geqslant 0$ where $0 < a < b$, $N(t) = N(-t)$ for $t < 0$. The non-measurable functions N, however, may always be eliminated in the investigation of linear spaces X^N provided that the condition (*) be satisfied. Indeed,

Let the function N satisfy the condition (*) and 1.6 (a), (b); then there exists a function M equivalent to N continuous and increasing for $t \geqslant 0$ and such that $M(t) = M(-t)$.

To prove this let us choose a positive integer m so that $N(t) \leqslant 1$ for $0 \leqslant t \leqslant 1/m$ and let ε_n be a decreasing sequence such that $0 < \varepsilon_n \leqslant \sup\limits_{0\leqslant s\leqslant 1/n} N(s)$ for $n > m$. This is possible, for $N(t)$ is continuous at 0 and $N(t) \neq 0$ for $t \neq 0$. Let us define first $M(t)$ for $t \geqslant 0$ as follows:

$$M(0) = 0,$$
$$M(1/n) = \sup_{0\leqslant s\leqslant 1/n} N(s)+\varepsilon_n,$$

$M(t)$ is equal to the linear function for $1/(n+1) \leqslant t \leqslant 1/n$, if $n \geqslant m$,

$M(t)$ is equal to an arbitrary continuous function increasing for $t > 1/m$ provided that it is chosen so that the continuity of M is preserved at the point $t = 1/m$.

For $t < 0$ we set $M(t) = M(-t)$. The equivalence of M and N results from the following inequalities, which result from 1.6 (c):

$$\sup_{0\leqslant s\leqslant 1/n} N(s) \leqslant DN(t), \qquad N\left(\frac{1}{n+1}\right) \geqslant \frac{N(t)}{D} \qquad \text{for} \qquad \frac{1}{n+1} < t < \frac{1}{n},$$

$$M(t) \leqslant M\left(\frac{1}{n}\right) \leqslant 2 \sup_{0\leqslant s\leqslant 1/n} N(s) \leqslant 2DN(t),$$

$$M(t) \geqslant M\left(\frac{1}{n+1}\right) \geqslant \sup_{0\leqslant s\leqslant 1/(n+1)} N(s) \geqslant N\left(\frac{1}{n+1}\right) \geqslant \frac{N(t)}{D}$$

for $1/(n+1) \leqslant t < 1/n$. In all these inequalities $n \geqslant m, n > 1/\delta$ and δ, D are constants involved in 1.6 (c) when $\varrho = 2$. We can suppose beforehand about N that $N(-t) = N(t)$ for $t > 0$, for every function N satisfying the condition 1.6 (b) is equivalent to $\overline{N}(t) = N(|t|)$; consequently $M \sim N$.

1.62. *Let N be non-decreasing for $t \geqslant 0$ and let $N(t) = N(-t)$ for all t; then the conditions* 1.6 (a), (b) *are equivalent to*

(+) $N(2t) \leqslant KN(t)$ *for some K in a neighbourhood of* 0.

Setting $t = s$ in 1.6 (a) we get (+). Conversely let the condition (+) be satisfied. The function N being monotone and even, we have for sufficiently small t, s

$$N(|t+s|) \leqslant N(|t|+|s|) \leqslant N(2\max(|t|, |s|)) \leqslant KN(\max(|t|, |s|)) \leqslant K[N(|t|)+N(|s|)].$$

If $|\omega| \leqslant 1$, then 1.6 (b) is satisfied with $D_\omega = 1$ and arbitrary ε_ω. Let $|\omega| > 1$; choose a positive integer n so that $2^{n-1} \leqslant |\omega| < 2^n$. It follows from (+) that in some neighbourhood of 0

$$N(|\omega|\,|t|) \leqslant N\left(2^n \frac{|\omega|}{2^n}\,|t|\right) \leqslant K^n N\left(\frac{|\omega|}{2^n}\,|t|\right) \leqslant K^n N(|t|).$$

1.7. THEOREM 1. *Let N satisfy the condition (*). The necessary and sufficient condition that X^N be a linear space is that the conditions* 1.6 (a), (c) *be satisfied. The function N may always be replaced by a continuous even function M increasing for $t \geqslant 0$ and equivalent to N, i. e. such that $X^M = X^N$.*

1.8. Now we shall discuss the possibility of introducing the norm in the linear spaces X^N. We remember that, in a linear space, the norm of type F is a functional $\| \ \|$ satisfying the conditions 1) $\|x\| \geqslant 0$, 2) $\|x\| = 0$ if and only if $x = 0$, 3) $\|x+y\| \leqslant \|x\|+\|y\|$, 4) $\|x\| = \|-x\|$, 5) the product ωx where ω is real is continuous with respect to the norm in both variables jointly. A linear space provided with an F-norm will be called the *F^*-space*; if, moreover, the axiom of completeness is satisfied it will be called the *F-space*.

Let the space X^N be linear; we shall say that *the sequence x_n of elements of X^N is ϱ-convergent to x_0* (in symbols $x_n \xrightarrow{\varrho} x_0$ as $n \to \infty$) if $\varrho(x_n - x_0) \to 0$ for $n \to \infty$. Suppose that an F-norm $\| \ \|$ is defined in X^N; the convergence with respect to the norm $\| \ \|$ will be called *equivalent to the convergence ϱ* if the relation $\|x_n - x_0\| \to 0$ as $n \to \infty$ implies $\varrho(x_n - x_0) \to 0$ for $n \to \infty$, and conversely. Let us observe that if the convergence with respect to the norm is equivalent to the convergence ϱ, then $\|x\| = 0$ if and only if $\varrho(x) = 0$.

1.81. *Let N satisfy the condition (*) and let the space X^N be linear. If:*

(a) $x \in X^N$, $\omega_n \to 0$, *then* $\omega_n x \xrightarrow{\varrho} 0$ *as* $n \to \infty$,

(b) $x_n \in X^N$, *the sequence ω_n is bounded and* $x_n \xrightarrow{\varrho} 0$ *as* $n \to \infty$, *then* $\omega_n x_n \xrightarrow{\varrho} 0$.

For the proof let us notice that if $x = \{t_n\} \in X^N$, then, by 1.6 (c),

$$\sum_{n=p}^{\infty} N(\omega t_n) \leqslant D \sum_{n=p}^{\infty} N(t_n) \quad \text{when} \quad \sum_{n=p}^{\infty} N(t_n) \leqslant \delta,\ |\omega| < r,$$

$$\varrho(\omega x) \leqslant D\varrho(x) \quad \text{when} \quad \varrho(x) < \delta,\ |\omega| < r,$$

where D, δ are constants of 1.6 (c) and the constant ϱ appearing in 1.6 (c) is denoted by r. The first of these inequalities and the continuity of N at 0 imply (a), the second inequality implies (b).

1.811. *Let $M \sim N$; then $\varrho_M(x_n) \to 0$ as $n \to \infty$ implies $\varrho_N(x_n) \to 0$ as $n \to \infty$, and conversely.*

This follows immediately from the definition of equivalent functions.

1.82. *Let the function N be non-decreasing for $t \geqslant 0$, $N(t) = N(-t)$ for all t and let it satisfy the condition (*); let the space X^N be linear. One can define in X^N an F-norm so that convergence with respect to the norm is equivalent to the convergence ϱ. With this definition of norm X^N is an F-space.*

By 1.6 (c) and 1.81 (a) we have $\varrho(x/\varepsilon) \to 0$ as $0 < \varepsilon \to \infty$, whence there exist $\varepsilon > 0$ satisfying the inequality

(+) $$\varrho(x/\varepsilon) < \varepsilon.$$

Let us define the norm $\|x\| = \inf \varepsilon$, the infimum being extended over the set of the $\varepsilon > 0$ satisfying (+). We shall verify that $\| \ \|$ satisfies all the axioms of F-norms. Let us observe first that $\varrho(x) \leqslant \varrho(x/\varepsilon) \leqslant \varepsilon$ when $\varrho(x/\varepsilon) \leqslant \varepsilon$, $0 < \varepsilon \leqslant 1$. We shall prove that

(++) $\|x_n\| \to 0$ *as* $n \to \infty$ *implies* $\varrho(x_n) \to 0$ *as* $n \to \infty$, *and conversely.*

Indeed, if $\|x_n\| \to 0$ as $n \to \infty$, then $\varrho(x_n/\varepsilon) < \varepsilon$ for $0 < \varepsilon < 1$ and large n, whence $\varrho(x_n) < \varepsilon$. Conversely let $\varrho(x_n) \to 0$ as $n \to \infty$: then in virtue of 1.81 (b) we have $\varrho(x_n/\varepsilon) < \varepsilon$ for large n, whence $\|x_n\| < \varepsilon$ for large n. The symmetry $\|x\| = \|-x\|$ follows directly from the fact that N is even; $\|0\| = 0$ is obvious. If $\|x\| = 0$, then applying (++) to the sequence $x, x, \ldots$ we get $\varrho(x) = 0$, *i. e.* $x = 0$. To prove the triangle inequality we may suppose that $\|x\| > 0$ $\|y\| > 0$. Given $\delta > 0$, there exist $\varepsilon > 0, \eta > 0$, satisfying (+) and $\varrho(y/\eta) < \eta$ respectively, such that $\varepsilon < \|x\|+\delta$, $\eta < \|y\|+\delta$. Then

$$N\left(\frac{t_n+s_n}{\varepsilon+\eta}\right) = N\left(\frac{|t_n+s_n|}{\varepsilon+\eta}\right) \leqslant N\left(\frac{|t_n|+|s_n|}{\varepsilon+\eta}\right) \leqslant N\left(\frac{|t_n|}{\varepsilon}\,\frac{\varepsilon}{\varepsilon+\eta}+\frac{|s_n|}{\eta}\,\frac{\eta}{\varepsilon+\eta}\right)$$

$$\leqslant \sup\left(N\left(\frac{|t_n|}{\varepsilon}\right), N\left(\frac{|s_n|}{\eta}\right)\right) \leqslant N\left(\frac{|t_n|}{\varepsilon}\right)+N\left(\frac{|s_n|}{\eta}\right),$$

whence

$$\varrho\left(\frac{x+y}{\varepsilon+\eta}\right) \leqslant \varrho\left(\frac{x}{\varepsilon}\right)+\varrho\left(\frac{y}{\eta}\right) < \varepsilon+\eta < \|x\|+\|y\|+2\delta,$$

which implies $\|x+y\| \leqslant \|x\|+\|y\|+2\delta$.

The continuity of ωx with respect to the variables ω, x follows by 1.81 and (++). Finally, (++) implies the equivalence of the ϱ-convergence to the convergence with respect to the norm. To prove the axiom of completeness let us choose a continuous function M satisfying (*) equivalent to N (this is possible in virtue of 1.61). Let $\|x_p-x_q\| \to 0$ as $p, q \to \infty$, whence because of 1.811 $\varrho_M(x_p-x_q) \to 0$ as $p, q \to \infty$. It follows $t_n^{(r)} \to t_n^{(0)}$ as $r \to \infty$ for $n = 1, 2, \ldots$ (we have set here $x_r = \{t_n^{(r)}\}$). For sufficiently large p, q we have $\varrho_M(x_p-x_q) < \varepsilon$, whence by the continuity of M

$$\varrho_M(x_p-x_0) \leqslant \varepsilon,$$

where $x_0 = \{t_n^{(0)}\}$, *i. e.* $\varrho_M(x_p-x_0) \to 0$ as $p \to \infty$ and consequently $\|x_p-x_0\| \to 0$ as $p \to \infty$. Since $x_p - x_0 \in X^N$ for large p, $x_p \in X^N$, whence $x_0 \in X^N$. Putting together 1.82 and 1.61 and taking into account 1.811 we get

THEOREM 2. *Each X^N linear sequence space forms an F-space if N satisfies the condition (*); the F-norm may be defined so that the ϱ-convergence is equivalent to the convergence with respect to the norm.*

1.9. *If X^N is a locally convex F-space, N satisfies the condition (*) and norm-convergence implies ϱ-convergence, then N is equivalent to a continuous even convex function satisfying the condition (*).*

By a known theorem of Banach the norm in X^N is equivalent to the norm of theorem 2, whence the convergence with respect to the norm is equivalent to the convergence ϱ. We may suppose by 1.61 that N is continuous, even and increasing for $t \geqslant 0$. The local convexity of X^N implies the existence of $\varepsilon > 0$ such that $\varrho(x_k) < \varepsilon$ for $k = 1, 2, \ldots, n$ implies $\varrho((x_1+x_2+\ldots+x_n)/n) < 1$. Let us denote by e_m the sequence $\{t_k^m\}$, $t_k^m = 1$ for $k = m$, $= 0$ for $k \neq m$. Writing for any positive integer p, $x_k = t(e_k+e_{k+n}+\ldots+e_{k+(p-1)n})$, we get $\varrho(x_k) = pN(t)$,

$$\varrho\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) = pnN\left(\frac{t}{n}\right),$$

whence for $N(t) \leqslant \varepsilon/p$ we have $N(t/n) \leqslant 1/pn$. If $0 < N(t) \leqslant \varepsilon$ and p is chosen so that $\varepsilon/(p+1) < N(t) \leqslant \varepsilon/p$, then $N(t/n) \leqslant 1/pn \leqslant (2/\varepsilon n)N(t)$. Given $\varrho > 0$ let us choose $q \geqslant 1$ so that $N(t/q) < \varepsilon$ for $|t| \leqslant \varrho$. We have

(+) $$N(qt) \leqslant D_q N(t) \quad \text{for} \quad |t| \leqslant \varrho.$$

Indeed, this inequality is true with some constant for $|t| < \delta_q$ in virtue of 1.6 (c), and since $\inf N(t) > 0$, $\sup N(qt) < \infty$ in $\langle \delta_q, \varrho \rangle$, we may choose $D_q > 0$ so that (+) is satisfied in the whole of the interval $|t| \leqslant \varrho$. By the inequality proved previously

$$N\left(\frac{t}{n}\right) = N\left(q\frac{|t|}{qn}\right) \leqslant D_q N\left(\frac{|t|}{qn}\right) \leqslant \frac{1}{n}\,\frac{2}{\varepsilon}\,D_q N(t) \quad \text{as} \quad |t| \leqslant \varrho.$$

If $0 < \omega \leqslant 1$ let us choose n so that $1/(n+1) < \omega \leqslant 1/n$. As $1/n \leqslant 2\omega$, setting $K_\varrho = 4D_q/\varepsilon$ we get

(++) $$N(\omega t) \leqslant K_\varrho \omega N(t) \quad \text{for} \quad 0 \leqslant \omega \leqslant 1,\ |t| \leqslant \varrho.$$

Let us write

$$P(t) = \sup_{0<\omega\leqslant 1} \frac{N(\omega t)}{\omega}.$$

By (++) it follows that $0 \leqslant P(t) < \infty$ for each t; moreover, $P(at) \leqslant aP(t)$ for $0 \leqslant a \leqslant 1$. By (++) $P(t) \leqslant K_1 N(t)$ for $|t| \leqslant 1$; moreover $P(t) \geqslant N(t)$, whence $P \sim N$. Let us set $Q(t) = P(t)/t$ for $t > 0$; this function is non-decreasing on the half axis $t > 0$, for we have for $0 < t' < t''$

$$Q(t') = \frac{P(t')}{t'} = \frac{P\left(\frac{t'}{t''}t''\right)}{t'} \leqslant \frac{\frac{t'}{t''}P(t'')}{t'} = \frac{P(t'')}{t''} = Q(t'').$$

Let us now define

$$M(t) = \int_0^t Q(s)\,ds \quad \text{for} \quad t \geqslant 0, \qquad M(t) = M(-t) \quad \text{for} \quad t < 0.$$

The function M is evidently convex, moreover $N \sim M$. The equivalence follows by the inequalities

$$M(t) \leqslant tQ(t) = P(t), \qquad M(t) \geqslant \int_{t/2}^{t} Q(s)\,ds = \frac{t}{2}\,Q\left(\frac{t}{2}\right) = P\left(\frac{t}{2}\right) \quad \text{for } t \geqslant 0,$$

$$P\left(\frac{t}{2}\right) \geqslant N\left(\frac{t}{2}\right) \geqslant \frac{1}{D}N(t) \quad \text{in a neighbourhood of } 0.$$

1.91. *Let the space X^N be linear. If the function N satisfies the condition $(*)$ and N is equivalent to a convex function, then X^N is a B-space with norm-convergence equivalent to ϱ-convergence.*

Let the function M be convex and equivalent to N. The continuity of N at 0 implies the same for the function M, whence M is continuous everywhere. We may suppose that $M(t) = M(-t)$ for $t \geqslant 0$; moreover $M(0) = 0$ for $N(0) = 0$ and $M \sim N$. The function M satisfies the condition $(*)$. Indeed, if $t_n \to 0$ as $n \to \infty$, then $M(t_n) \to 0$; if $M(t_n) \to 0$, then $t_n \to 0$ as $n \to \infty$. In the contrary case we would infer from the continuity and convexity of M that $M(t) = 0$ in some neighbourhood of 0, whence also $N(t)$ in a neighbourhood of 0, which is impossible, for N satisfies the condition $(*)$. Let W denote the set of the elements x satisfying the condition $\varrho_M(x) \leqslant 1$. From the properties of M it follows that the set W is convex and symmetric with respect to 0. By 1.82 there exists in $X^N = X^M$ an F-norm, say $\| \|^*$, such that ϱ_M-convergence is equivalent to the convergence with respect to $\| \|^*$. Thus choosing a sufficiently small $r > 0$ we deduce from $\|x\|^* \leqslant r$ that $\varrho_M(x) \leqslant 1$, *i. e.* 0 is an inner point of W in the $\| \|^*$-normed space X^N. The set W is also bounded; in fact $M(\omega t) = M(|\omega|\,|t|) \leqslant |\omega| M(|t|)$ as $|\omega| \leqslant 1$, whence $\varrho_M(\omega x) \leqslant |\omega|\, \varrho_M(x)$ for $|\omega| \leqslant 1$. From the above properties of W it follows by a known theorem that the norm $\| \|^*$ is equivalent to a B-norm. Let us observe that the corresponding B-norm may be obtained as follows. Let us set $\|x\| = \inf k$, the infimum being taken over the set of the $k > 0$ satisfying the inequality $\varrho_M(x/k) \leqslant 1$[1]. The axioms of the norm are easily verified. One can also prove directly the equivalence of the convergence ϱ_M and the convergence with respect to $\| \|$, which implies in turn the axiom of completeness.

1.92. As a corollary to 1.9 and 1.91 and taking into account that for even functions M non-decreasing for $M \geqslant 0$ the condition 1.6 (a) is equivalent to 1.62 (+), we deduce

THEOREM 3. *Let the function N satisfy the condition $(*)$, then*

(a) *X^N is a Banach space if and only if the function N is equivalent to a continuous convex even function M vanishing only at 0 and satisfying the inequality 1.62 (+),*

(b) *if X^N is a B_0-space, X^N is a B-space.*

Here norm-convergence is always to understand as equivalent to ϱ-convergence.

[1] This method of the introduction of the norm in X^N is known (see [2]).

2.1. Throughout this section we suppose that N and M are Baire functions; this hypothesis will not be mentioned in the subsequent considerations. We shall use the notation

$$\varrho_N(x) = \int_0^1 N(x(t))\,dt;$$

moreover, the subscript N will be omitted when there is no doubt about the considered function N. Since N is a Baire function, the function $N(x(t))$ is measurable when $x(t)$ is measurable. The set of those x for which $\varrho_N(x) < \infty$ will be denoted by X^N. Saying "N satisfies the condition $(*)$" we shall always mean the condition 1.1 $(*)$. We shall also often need the following condition:

(o) $N(t_n) \to \infty$ *as* $n \to \infty$ *if and only if* $|t_n| \to \infty$ *as* $n \to \infty$.

The condition (o) obviously implies that N is bounded in every finite interval.

The functions N and M will be said to be *equivalent* if they have the following properties:

There exist constants $A > 0$, $B > 0$, $K > 0$, $r > 0$ such that the following inequalities are satisfied:

(+) $N(t) \leqslant BM(t)$ for $M(t) \geqslant r$, $M(t) \leqslant AN(t)$ for $N(t) \geqslant r$,

(++) $N(t) \leqslant K$ for $M(t) < r$, $M(t) \leqslant K$ for $N(t) < r$.

For this definition of equivalence we shall use the same notation as in section 1 to denote the equivalent functions: $N \sim M$.

2.12. *Let N and M satisfy the condition* (o); *then $N \sim M$ if and only if simultaneously $N(t) \leqslant BM(t)$, $M(t) \leqslant AN(t)$ for sufficiently large $|t|$.*

2.13. *Let $M(t,s)$ and $N(t,s)$ be two non-negative Baire functions of the variables t, s, defined for all real t, s. A necessary and sufficient condition that for arbitrary functions x, y the inequality*

$$\int_0^1 M(x(t), y(t))\,dt < \infty$$

imply

$$\int_0^1 N(x(t), y(t))\,dt < \infty$$

is that for certain constants $C > 0$, $K > 0$, $r > 0$ the following inequalities be satisfied:

(+) $N(t,s) \leqslant CM(t,s)$ *as* $M(t,s) \geqslant r$,

(++) $N(t,s) \leqslant K$ *as* $M(t,s) < r$.

Sufficiency is evident.

Necessity. Let us supppose that (+) is not satisfied. Then there exist t_n, s_n such that

$$\sum_{n=1}^{\infty} \frac{1}{M(t_n, s_n)} \leqslant 1, \quad N(t_n, s_n) \geqslant n M(t_n, s_n) \quad \text{for} \quad n = 1, 2, \ldots$$

Let $\delta_1, \delta_2, \ldots$ denote disjoint intervals in (0,1) with lengths

$$\frac{1}{1^2 M(t_1, s_1)}, \frac{1}{2^2 M(t_2, s_2)}, \ldots$$

Let

$$(\substack{++\\+}) \quad x(s) = \begin{cases} t_n & \text{for} \quad s \in \delta_n, n = 1, 2, \ldots, \\ 0 & \text{for} \quad s \in (0,1) - \bigcup_1^{\infty} \delta_n, \end{cases} \qquad y(s) = \begin{cases} s_n & \text{for} \quad s \in \delta_n, n = 1, 2, \ldots, \\ 0 & \text{for} \quad s \in (0,1) - \bigcup_1^{\infty} \delta_n. \end{cases}$$

Then

$$\int_0^1 M\big(x(s), y(s)\big) ds = \sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \int_0^1 N\big(x(s), y(s)\big) ds \geqslant \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

contrary to hypothesis. Let us now suppose that (++) is not satisfied. There exist t_n, s_n such that $N(t_n, s_n) \to \infty$ for $n \to \infty$, $M(t_n, s_n) \leqslant r$. Let us choose positive a_n so that

$$\sum_{n=1}^{\infty} a_n \leqslant 1, \quad \sum_{n=1}^{\infty} a_n N(t_n, s_n) = \infty,$$

and then disjoint intervals δ_n in (0,1) with lengths a_n; next we define the functions x, y by aid of $(\substack{++\\+})$. We obtain

$$\int_0^1 M\big(x(s), y(s)\big) ds < \infty, \qquad \int_0^1 N\big(x(s), y(s)\big) ds = \infty,$$

which leads to a contradiction.

Setting $M(t, s) = M(t)$, $N(t, s) = N(t)$ for arbitrary t, s we deduce from 2.13 that

2.14. *A necessary and sufficient condition that the inequality $\varrho_M(x) < \infty$ imply $\varrho_N(x) < \infty$ is the existence of constants $B > 0$, $K > 0$, $r > 0$ such that*

$$N(t) \leqslant BM(t) \quad \textit{for} \quad M(t) \geqslant r,$$
$$N(t) \leqslant K \quad \textit{for} \quad M(t) < r.$$

2.2. (a) *$X^N = X^M$ if and only if $N \sim M$;*

(b) *X^N is identical with the space of all measurable functions if and only if M is bounded in $(-\infty, \infty)$.*

Ad (a). It immediately follows from 2.14 and the definition of equivalence.

Ad (b). Sufficiency is trivial.

Necessity. If N is unbounded, then there exist t_n such that

$$N(t_n) \to \infty \quad \text{as} \quad n \to \infty, \qquad \sum_{n=1}^{\infty} \frac{1}{N(t_n)} \leqslant 1.$$

Let us choose disjoint intervals δ_n in (0,1) with lengths $1/N(t_n)$ and let us set

$$x(s) = \begin{cases} t_n & \text{for} \quad s \in \delta_n, \ n = 1, 2, \ldots, \\ 0 & \text{for} \quad s \in (0,1) - \bigcup_{n=1}^{\infty} \delta_n. \end{cases}$$

Obviously $\varrho_N(x) = \infty$ and X^N does not contain all measurable functions.

2.3. *The following conditions are necessary and sufficient that X^N be a linear space:*

(a) *There exist constants $C > 0$, $K > 0$, $r > 0$ such that*

$$N(t+s) \leqslant C\big(N(t) + N(s)\big) \quad \textit{for} \quad N(t) + N(s) \geqslant r;$$

(a′) *$N(t+s) \leqslant K$ for $N(t) + N(s) < r$;*

(b) *for each ω there exist constants $D_\omega > 0$, $K_\omega > 0$, $r_\omega > 0$ such that*

$$N(\omega t) \leqslant D_\omega N(t) \quad \textit{for} \quad N(t) \geqslant r_\omega;$$

(b′) *$N(\omega t) \leqslant K_\omega$ for $N(t) < r_\omega$.*

To prove (a), (a′) let us write $M(t, s) = N(t) + N(s)$, $N(t, s) = N(t+s)$. As $\varrho_N(x) < \infty$, $\varrho_N(y) < \infty$ implies $\varrho_N(x+y) < \infty$, it is sufficient to apply 2.13 to $M(t, s)$, $N(t, s)$. To prove (b), (b′) let us make use of 2.14 replacing $N(t)$ by $N(\omega t)$ and $M(t)$ by $N(t)$. Then we take into account that $\varrho_N(x) < \infty$ implies $\varrho_N(\omega x) < \infty$ for each ω.

2.31. *Let N satisfy the conditions 2.3 (b) and (b′) and let the sequence $N(t_n)$ be bounded; then for every ω the sequence $N(\omega t_n)$ is also bounded.*

Suppose that $N(t_n) \leqslant L$ for $n = 1, 2, \ldots$ If $N(t_n) \geqslant r_\omega$ then $N(\omega t_n) \leqslant D_\omega L$; if $N(t_n) < r_\omega$, then $N(\omega t_n) \leqslant K_\omega$.

2.4. *Let N satisfy the conditions* (a), (a′), (b), (b′) *of* 2.3; *then for every $\varrho > 0$ there exist positive constants D, F such that*

$$N(\omega t) \leqslant DN(t) \quad \textit{for} \quad N(t) \geqslant r_\varrho, \ |\omega| < \varrho,$$
$$N(\omega t) \leqslant F \quad \textit{for} \quad N(t) < r_\varrho, \ |\omega| < \varrho.$$

Let us suppose that $|\omega_n| < \varrho$, $N(t_n) \to \infty$ as $n \to \infty$ where $N(t_n) > 0$, and that $N(\omega_n t_n)/N(t_n) \to \infty$ as $n \to \infty$. As in 1.6 we define functions for $n = 1, 2, \ldots$ and $|\omega| < \varrho$

$$\varphi_n(\omega) = \frac{N(\omega 4 t_n)}{N(t_n)}.$$

It follows from 2.3 (b) that for every ω in $(-\varrho, \varrho)$ the sequence $\varphi_n(\omega)$ is bounded. Since $N(t_n) \to \infty$, it follows from 2.31 for $\omega \neq 0$ that $N(\omega t_n) \to \infty$ and, the functions $N(\omega t_n)$ being measurable, there exists in $(-\varrho, \varrho)$ a measurable set E of measure $> \frac{7}{4}\varrho$ such that $N(\omega 4 t_n) \geqslant r$ for $\omega \in E$ and sufficiently large n. Thus by 2.3 (a) it follows that for $\omega_1, \omega_2 \in E$ and sufficiently large n

$$\varphi_n(\omega_1 + \omega_2) \leqslant C[\varphi_n(\omega_1) + \varphi_n(\omega_2)],$$

whence by 1.2 B there is a constant L such that

$$\varphi_n(\omega t_n) \leqslant L \quad \text{for} \quad |\omega| < \varrho/4 \text{ and large } n;$$

therefore

$$\frac{N(\omega t_n)}{N(t_n)} \leqslant L \quad \text{for} \quad |\omega| < \varrho \text{ and large } n,$$

in contradiction to the relation $N(\omega_n t_n)/N(t_n) \to \infty$ as $n \to \infty$.

To prove the second inequality of our proposition let us suppose that for certain t_n, ω_n such that $|\omega_n| < \varrho$, $N(t_n) < r_\varrho$ we have $N(\omega_n t_n) \to \infty$. Let us define measurable functions for $n = 1, 2, \ldots$

$$\varphi_n(\omega) = N(\omega 4 t_n).$$

By 2.31 the sequence $\varphi_n(\omega)$ is bounded as $\omega \in (-\varrho, \varrho)$. The same argument as in the proof of 1.2, B, with the same notation except that δ is replaced by ϱ, leads to the representation 1.2 (++), whence by 2.3 (a) there follows 1.2 $(^{++}_{+})$ (where K is to be replaced by C) if $N(\omega_1 4 t_n) + N(-\omega_2 4 t_n) \geqslant r$. On the other hand, if $N(\omega_1 4 t_n) + N(-\omega_2 4 t_n) < r$, then from 2.3 (a′) we deduce

$$\varphi_n(\omega) = N(\omega 4 t_n) \leqslant K \quad \text{for} \quad |\omega| < \varrho/4;$$

consequently the sequence $N(\omega 4 t_n)$ is uniformly bounded in $(-\varrho/4, \varrho/4)$, whence also the sequence $N(\omega t_n)$ in $(-\varrho, \varrho)$, which is contrary to the fact that $N(\omega_n t_n) \to \infty$ as $n \to \infty$ where $|\omega_n| < \varrho$.

2.5. *Let X^N be a linear space different from the space of all measurable functions; then N satisfies the condition* (o).

By 2.3 and 2.4 $N(\omega)$ is bounded in $(-\varrho, \varrho)$, whence in every finite interval. Therefore if $N(t_n) \to \infty$ as $n \to \infty$, then $|t_n| \to \infty$ as $n \to \infty$. Conversely, suppose that $|t_n| \to \infty$ as $n \to \infty$, and let $r > 0$ be given. If we had $N(t_n) < r$ for $n = 1, 2, \ldots$, then, by 2.4, $N(\omega t_n) \leqslant \max(D_{r_1} r, F)$ for $|\omega| < 1$, $n = 1, 2, \ldots$, whence $N(t)$ would be bounded in $(-\infty, \infty)$, which is contrary to 2.2 (b).

2.6. *Let N satisfy the condition* (o) *and let X^N be a linear space. Then there exists a function M equivalent to N, continuous increasing for $t \geqslant 0$ such that $M(t) = M(-t)$ and satisfying the conditions* (o), (*).

Let us observe first that we may suppose that $N(t) = N(-t)$ for arbitrary t. Indeed, the condition (o) being satisfied by N, it follows by 2.3 (b), (b′) that $N \sim \overline{N}$ where $\overline{N}(t) = N(|t|)$.

Let m be the smallest positive integer such that $\sup\limits_{0 \leqslant s \leqslant m} N(s) > 0$. Let us set $M(n) = \varepsilon_n \sup\limits_{0 \leqslant s \leqslant n} N(s)$ for $n \geqslant m$ where $0 < \varepsilon_n < 1$ are chosen to form an increasing sequence. We define $M(t)$ as a linear function taking on the value 0 at $t = 0$, in the interval $0 \leqslant t \leqslant m$; for $n \leqslant t \leqslant n+1$ with $n \geqslant m$, the function $M(t)$ is equal to the linear function, $M(t) = M(-t)$ for $t < 0$. M obviously satisfies the condition (*); the condition (o) being satisfied by N, it follows that $\sup\limits_{0 \leqslant s \leqslant n} N(s) \to \infty$ as $n \to \infty$, which implies that M satisfies the condition (o) too. For $n \leqslant t \leqslant n+1$ where $n \geqslant m$ we have

$$M(t) \leqslant M(n+1) = \varepsilon_{n+1} \sup_{0 \leqslant s \leqslant n+1} N(s) = \varepsilon_{n+1} \sup_{0 \leqslant \omega \leqslant (n+1)/t} N(\omega t) \leqslant$$
$$\leqslant \varepsilon_{n+1} \sup_{0 \leqslant \omega \leqslant 2} N(\omega t),$$
$$M(t) \geqslant M(n) \geqslant \varepsilon_n N(n) \geqslant \varepsilon_m N(n).$$

And since by 2.4 and the condition (o) for N the inequalities

$$\sup_{0 \leqslant \omega \leqslant 2} N(\omega t) \leqslant DN(t) \quad \text{for} \quad t \text{ sufficiently large},$$

$$N(t) = N\left(\frac{t}{n} n\right) \leqslant DN(n) \quad \text{for } n \text{ sufficiently large}, \ n \leqslant t \leqslant n+1,$$

are satisfied, we can choose r so great that for $N(t) \geqslant r$, $M(t) \geqslant r$

$$DN(t) \geqslant M(t) \geqslant \frac{\varepsilon_m}{D} N(t).$$

The functions N, M satisfy the condition (o), whence for $N(t) < r$ or $M(t) < r$ we have $M(t) \leqslant K$ or $N(t) \leqslant K$, K being a constant, which gives $N \sim M$.

2.7. Theorem 4. A. *Suppose that N is not bounded. Then X^N forms a linear space if and only if:*

(a) *N satisfies the condition* (o);

(b) *there are constants $C > 0$, $r > 0$ such that $N(t+s) \leqslant C(N(t)+N(s))$ for $|t|+|s| > r$,*

(c) *for each $\varrho > 0$ there exist constants $D > 0$, $r_0 > 0$ such that $N(\omega t) \leqslant DN(t)$ for $|\omega| < \varrho$, $|t| > r_0$.*

B. *If N is not bounded and X^N is a linear space, we may replace N by a continuous even function M increasing for $t \geqslant 0$, satisfying the condition* (∗) *and equivalent to N, i. e. such that $X^M = X^N$.*

Ad A. Necessity. (a) follows by 2.2 (b) and (2.5); (b), (c) follow from 2.3, 2.4, and (a).

Sufficiency. The set X^N is non-empty, *e. g.* $x(t) = 0$ for every t belongs to X^N, it is sufficient to apply the condition (o) and 2.3.

Ad B. Since N must satisfy the condition (o), we may apply 2.6.

2.71. *If N is non-decreasing for $t \geqslant 0$, $N(t) = N(-t)$ for all t, and N satisfies the condition* (o), *then the conditions* 2.7 (b), (c) *are equivalent to the following one:*

$$(+) \qquad N(2t) \leqslant KN(t)$$

for sufficiently large $|t|$, K being a constant.

The proof is analogous to that in 1.62.

2.8. Let X^N be a linear space; as in 1.8, we introduce the notion of the ϱ-convergence. The sequence $x_n \in X$ is called *ϱ-convergent to the element* $x_0 \in X^N$ if $\varrho(x_n - x_0) \to 0$ as $n \to \infty$; we shall write this: $x_n \overset{\varrho}{\to} x_0$ as $n \to \infty$.

2.81. *Let X^N be a linear space and let N not vanish identically. Then a necessary condition that the product ωx be continuous with respect to the ϱ-convergence is that the condition* (∗) *be satisfied.*

Under the supplementary hypothesis that N satisfies the condition (o), (∗) *is a sufficient condition for the continuity of ωx.*

Necessity. Let $t_n \to 0$ as $n \to \infty$, $x_n(s) = 1$ for $s \in (0,1)$, $n = 0, 1, 2, \ldots$ obviously $x_n \overset{\varrho}{\to} x_0$ as $n \to \infty$, whence $t_n x_n \overset{\varrho}{\to} 0$, *i. e.* $\varrho(t_n x_n) = \varrho(t_n x_0) = N(t_n) \to 0$ as $n \to \infty$. Let $N(t_n) \to 0$ as $n \to \infty$; we may suppose that $t_n \to t_0$ or $|t_n| \to \infty$. Let us suppose also, for example, that $N(1) \neq 0$. Assuming $t_0 \neq 0$ we may suppose that all $t_n \neq 0$. Setting $x_n(s) = t_n$ for $s \in (0,1)$ $n = 1, 2, \ldots$ we get $x_n/t_n \overset{\varrho}{\to} 0$ as $n \to \infty$, whence $\varrho(x_n/t_n) = N(1) \to 0$ which is contradictory. Similarly we are led to a contradiction in the case $|t_n| \to \infty$; thus $t_n \to 0$. We omit here the proof of sufficiency, for it follows immediately from 2.84.

Theorem 2.81 justifies the necessity of supposing the condition (∗) when introducing in X^N a norm such that the convergence with respect to the norm is equivalent to the ϱ-convergence (this equivalence is understood in the same sense as in 1.8).

2.82. *Let N satisfy the conditions* (∗), (o), *and let X^N be a linear space. Under these hypotheses the statements* 1.81 (a), (b) *are satisfied.*

Let us observe at first that $N(0) = 0$ and that N is continuous at 0. It follows by 2.7, A, that for $x \in X^N$ the inequality

$$(+) \qquad \int_E N(\omega x(t))\,dt \leqslant D \int_E N(x(t))\,dt \quad \text{when} \quad |\omega| < \varrho$$

is satisfied where $E = \{t: |x(t)| > r_0, t \in (0,1)\}$, D, r_0, ϱ are constants as in 2.7, A. Let $x_n \overset{\varrho}{\to} 0$; then $N(x_n(t))$ converges in measure to 0; thus it follows from (∗) that $x_n(t)$ tends in measure to 0. Let φ_n denote the characteristic function of the set $E_n = \{t: |x_n(t)| > r_0, t \in (0,1)\}$. As $(1-\varphi_n(t))x_n(t)$ tends in measure to 0 and $|(1-\varphi_n(t))x_n(t)| \leqslant r_0$ for $n = 1, 2, \ldots$, we have $\int_0^1 N(\omega_n(1-\varphi_n(t))x_n(t))\,dt \to 0$ as $n \to \infty$ $|\omega_n| < \varrho$. By (+)

$$\int_0^1 N(\varphi_n(t)\,\omega_n x_n(t))\,dt \leqslant D \int_0^1 N(\varphi_n(t) x_n(t))\,dt \leqslant D\varrho(x_n)$$

as $|\omega_n| < \varrho$ for $n = 1, 2, \ldots$, whence

$$\varrho(\omega_n x_n) = \int_0^1 N((1-\varphi_n(t))\,\omega_n x_n(t))\,dt + \int_0^1 N(\varphi_n(t)\,\omega_n x_n(t))\,dt \to 0 \quad \text{as} \quad n \to \infty.$$

The proof of 1.81, (a) is analogous.

2.83. *Let N and M satisfy the condition* (∗) *and let $N \sim M$, i. e. $X^N = X^M$. Then $\varrho_N(x_n) \to 0$ as $n \to \infty$ implies $\varrho_M(x_n) \to 0$ as $n \to \infty$ and conversely.*

It is sufficient to prove that $\varrho_N(x_n) \to 0$, $x_n \in X^N$ implies $\varrho_M(x_n) \to 0$. According to 2.1 (+) we have $M(t) \leqslant BN(t)$ for $N(t) \geqslant r$ and $M(t) \leqslant K$ for $N(t) < r$. Let $E_n = \{t: N(x_n(t)) \geqslant r, t \in (0,1)\}$ for $n = 1, 2, \ldots$ and let φ_n denote the characteristic function of the set E_n. We have

$$\int_0^1 M((x_n(t))\varphi_n(t)\,dt \leqslant B \int_0^1 N(x_n(t))\varphi_n(t)\,dt \leqslant B\varrho_N(x_n) \to 0.$$

Now $M(x_n(t))(1-\varphi_n(t)) \leqslant K$, $M(x_n(t))$ tends in measure to 0 in $(0,1)$, for $N(x_n(t))$ tends in measure to 0 and the condition (∗) is satisfied.

Thus

$$\int_0^1 M(x_n(t))(1-\varphi_n(t))\,dt \to 0 \quad \text{as} \quad n\to\infty,$$

whence

$$\varrho_M(x_n) = \int_0^1 M(x_n(t))\,dt \to 0 \quad \text{as} \quad n\to\infty.$$

2.84. *Let N satisfy the conditions $(*)$, (o), let N be non-decreasing for $t\geqslant 0$, $N(t)=N(-t)$ for all t, and let X^N be a linear space. One can define in X^N an F-norm such that the convergence with respect to the norm is equivalent to the ϱ-convergence. With this norm, X^N is an F-space.*

The set of the $\varepsilon>0$ satisfying the inequality 1.82 (+) is non-empty, as follows by 2.82. Let us define the norm as in 1.82, *i. e.* $\|x\|=\inf\varepsilon$, the infimum being taken over the set of those $\varepsilon>0$ which satisfy 1.82 (+). Arguing as in 1.82 and taking into account the lemmata 2.82 we can verify that $\|\ \|$ satisfies the axioms of F-norm and that the convergence with respect to the norm is equivalent to the ϱ-convergence. Let us call attention here to the role of the condition $(*)$, considering, for example, the axiom: $\|x\|=0$ if and only $x=0$. If $\|x\|=0$, then $\varrho(x)=0$, for the convergence with respect to the norm implies the ϱ-convergence, and since $(*)$ implies $N(t)\neq 0$ for $t\neq 0$ we have $x(t)=0$ almost everywhere, *i. e.* $x=0$. Conversely, if $x(t)=0$ almost everywhere, then $N(|x(t)|/\varepsilon)=0$ almost everywhere, $\varrho(x/\varepsilon)=0$, *i. e.* $\|x\|=0$. To prove the axiom of completeness[2]) let M be a continuous even function satisfying the condition $(*)$, equivalent to N; such a function exists in virtue of 2.6. Let $\|x_p-x_q\|\to 0$ as $p,q\to\infty$; by 2.83 $\varrho_M(x_p-x_q)\to 0$ as $p,q\to\infty$. Let us choose $\varepsilon_n>0$ so that $\sum_n \varepsilon_n[M(1/n^2)]^{-1}<\infty$ and then an increasing sequence p_n of indices such that $\varrho_M(x_{p_n}-x_{p_{n+1}})\leqslant\varepsilon_n$ for $n=1,2,\ldots$ Let $E_n=\{t\colon |x_{p_n}(t)-x_{p_{n+1}}(t)|\geqslant 1/n^2,\ t\in(0,1)\}$, $F_n=(0,1)-E_n$ for $n=1,2,\ldots$ The series $\sum_{n=1}^{\infty}|E_n|$ converges, for $\varrho_M(x_{p_n}-x_{p_{n+1}})\geqslant |E_n|M(1/n^2)$, therefore $|\overline{\lim_n} E_n|=0$, $|\underline{\lim_n} F_n|=1$, whence it follows that the series $\sum_n |x_{p_n}(t)-x_{p_{n+1}}(t)|$ converges almost everywhere, *i. e.* $x_{p_n}(t)\to x_0(t)$ almost everywhere. Let $\varepsilon>0$ be fixed; for sufficiently large q and almost all n we have $\int_0^1 M(x_{p_n}(t)-x_q(t))\,dt\leqslant\varepsilon$, which implies together with the continuity of M that

$$\underline{\lim_n}\int_0^1 M(x_{p_n}(t)-x_q(t))\,dt \geqslant \int_0^1 M(x_0(t)-x_q(t))\,dt,$$

[2]) For the quoted proof of the completeness of X^M compare [1].

i. e. $\varrho_M(x_0-x_q)\leqslant\varepsilon$ for large q. It follows that $x_0\in X^N$, $\varrho_N(x_n-x_0)\to 0$, whence $\|x_n-x_0\|\to 0$ as $n\to\infty$.

Taking into account 2.6, 2.84, and 2.83 we deduce

THEOREM 5. *Every linear space of functions X^N forms an F-space if N satisfies the conditions* (o) *and* $(*)$; *the F-norm may be defined so that the ϱ-convergence is equivalent to the convergence with respect to the norm.*

2.9. *If X^N is a locally convex F-space, N satisfies the conditions* (o), $(*)$ *and norm-convergence implies ϱ-convergence, then N is equivalent to continuous convex even function satisfying the conditions* $(*)$, (o).

Suppose that X^N is a locally convex F-space; by the theorem of Banach and theorem 5 the convergence with respect to the norm is equivalent to the ϱ-convergence; by 2.6, 2.83 we may suppose that N is continuous and increasing for $t\geqslant 0$, $N(t)=N(-t)$ for all t. The local convexity of X^N implies the existence of $\varepsilon>0$ such that from the inequality $\varrho(x_k)<\varepsilon$ for $k=1,2,\ldots,n$ there follows $\varrho((x_1+x_2+\ldots+x_n)/n)<1$. For positive integers q,p,n such that $p\leqslant n$ and real t let us define the functions x_k for $k=1,2,\ldots,n$ as follows:

$$x_k(s)=\begin{cases} t\varepsilon_{r(k-1+i)} & \text{for} \quad i/nq<s<(i+1)/nq, \quad i=0,1,2,\ldots,n-1,\\ 0 & \text{elsewhere in} \quad (0,1),\end{cases}$$

where $\varepsilon_m=1$ for $m=0,1,\ldots,p-1$, $\varepsilon_m=0$ for $m\geqslant p$ and $r(m)$ denotes the residue of m modulo n. Now

$$\varrho(x_k)=\frac{1}{q}\frac{p}{n}N(t), \qquad \varrho\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)=\frac{1}{q}N\left(\frac{p}{n}t\right),$$

whence

$$\frac{1}{q}N\left(\frac{p}{n}t\right)<1 \quad \text{for} \quad \frac{1}{q}\frac{p}{n}N(t)<\varepsilon.$$

The continuity of N, p, n being arbitrary, implies that $q^{-1}N(\omega t)\leqslant 1$ for $0\leqslant\omega\leqslant 1$ if $q^{-1}\omega N(t)\leqslant\varepsilon$. If $\omega N(t)\geqslant\varepsilon$ and $\varepsilon q\leqslant\omega N(t)\leqslant\varepsilon(q+1)$ for positive integer q, then

$$\frac{1}{q+1}\omega N(t)\leqslant\varepsilon,$$

whence

$$\frac{1}{q+1}N(\omega t)\leqslant 1, \qquad N(\omega t)\leqslant q+1=\frac{q+1}{\varepsilon q}\varepsilon q\leqslant\frac{2}{\varepsilon}\omega N(t).$$

Setting $C=2/\varepsilon$ we have

$$(+) \qquad N(\omega t)\leqslant C\omega N(t) \quad \text{for} \quad \omega N(t)\geqslant\varepsilon.$$

We shall prove that

$$\varliminf_{t\to\infty}\frac{N(t)}{t} > 0.$$

In the contrary case there exist t_n such that $t_n\to\infty$, $N(t_n)/t_n\to 0$ as $n\to\infty$. Now $N(t_n)\to\infty$, since N satisfies the condition (o) and we may suppose that $\omega_n=\varepsilon/N(t_n)\leqslant 1$. As $\omega_n N(t_n)=\varepsilon$ we infer from (+) that

$$N\left(\varepsilon\frac{t_n}{N(t_n)}\right)\leqslant C\varepsilon,$$

which contradicts the condition (o), for $\varepsilon t_n/N(t_n)\to\infty$ as $n\to\infty$. Let us choose $\delta>0$ and $T>0$ so that

$$N(t)/t\geqslant\delta \quad \text{for} \quad t\geqslant T \quad \text{and} \quad \delta T\geqslant\varepsilon.$$

Then for $\omega t\geqslant T$, $0<\omega\leqslant 1$, $t>0$

$$\omega N(t)=\omega t\frac{N(t)}{t}\geqslant T\delta\geqslant\varepsilon,$$

for $\omega t\geqslant T$ implies $t\geqslant T$; therefore, by (+),

(++) $$N(\omega t)\leqslant C\omega N(t) \quad \text{for} \quad \omega t\geqslant T,\ 0<\omega\leqslant 1.$$

Let us set

$$P(t)=\begin{cases}\sup\limits_{\substack{0<\omega\leqslant 1\\ \omega t\geqslant T}}\dfrac{N(\omega t)}{\omega} & \text{for} \quad t\geqslant T,\\ N(t) & \text{for} \quad 0\leqslant t<T.\end{cases}$$

From (++) and the continuity of N it follows that $P(t)<\infty$ for every t. The following inequality is satisfied:

$$P(at)\leqslant aP(t) \quad \text{as} \quad 0<a\leqslant 1,\ at\geqslant T$$

for

$$P(at)=\sup_{\substack{0<\omega\leqslant 1\\ \omega at\geqslant T}}\frac{N(\omega at)}{\omega}=a\sup_{\substack{0<\omega\leqslant 1\\ \omega at\geqslant T}}\frac{N(\omega at)}{\omega a}\leqslant a\sup_{\substack{0<\omega\leqslant 1\\ \omega t\geqslant T}}\frac{N(\omega t)}{\omega}=aP(t);$$

it follows that the function $P(t)/t$ is non-decreasing as $t\geqslant T$, since for $T\leqslant t'<t''$

$$\frac{P(t')}{t'}=\frac{P\left(\frac{t'}{t''}t''\right)}{t'}\leqslant\frac{\frac{t'}{t''}P(t'')}{t'}=\frac{P(t'')}{t''}, \quad \text{for} \quad \frac{t'}{t''}t''=t'\geqslant T.$$

Let us now write

$$Q(t)=\begin{cases}P(t)/t & \text{for} \quad t\geqslant T,\\ tP(T)/T^2 & \text{for} \quad 0\leqslant t\leqslant T,\end{cases}$$

$$M(t)=\int_0^t Q(s)\,ds \quad \text{for} \quad t\geqslant 0, \qquad M(t)=M(-t) \quad \text{for} \quad t\leqslant 0.$$

The function M is convex and, as in 1.91, one can prove that $N\sim M$.

2.91. *If X^N is a linear space, N satisfies the conditions* (o), (*) *and N is equivalent to a convex function, then X^N is a B-space with norm-convergence equivalent to ϱ-convergence.*

Let the function M be convex and equivalent to N; this, together with (o), implies that M is continuous. We may assume that M is even; the condition (o) is satisfied by M, for it is satisfied by N. It is easily seen that modifying suitably M in a neighbourhood of 0 we may obtain a convex function equivalent to M satifying the condition (*); thus we can suppose that M satisfies (*). Using 2.84 and arguing as in 1.91[3]) we can prove that the norm described in lemma 2.84 is equivalent to a B-norm.

2.92. As a corollary to 2.9, 2.91 and applying 2.71 we get

THEOREM 6. (a) *Let N satisfy the conditions* (o) and (*). *Then X^N is a Banach space if and only if N is equivalent to a continuous even convex function vanishing only at* 0 *and satifying the inequality* 2.71 (+).

(b) *If X^N is a B_0-space, then X^N is a B-space.*

Here norm-convergence is always to understand as equivalent to ϱ-convergence.

References

[1] S. Kaczmarz et L. Nikliborc, *Sur les suites des fonctions convergentes en moyenne*, Fundamenta Mathem. 11 (1928), p. 151-168.

[2] W. A. J. Luxembourg, *Banach function spaces*, thesis, Delft 1955.

[3] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Ac. Pol. (1932), p. 207-220.

[4] — *Über Räume* (L^M), Bull. Ac. Pol. (1936), p. 93-107.

[5] F. Riesz, *Untersuchungen über Systeme integrierbarer Funktionen*, Math. Ann. 69 (1910), p. 449-497.

[6] — *Les systèmes d'équations linéaires à une infinité d'inconnues*, Collection Borel, Paris 1913.

Reçu par la Rédaction le 21. 9. 1957

[3]) Compare the footnote on p. 105.