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On a problem of Nathanson

by

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1. Introduction. Let $\mathbb{N}$ denote the set of all nonnegative integers and let h be an integer with $h \geq 2$. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, let

$$r_h(A,n) = \sharp\{(a_1,\ldots,a_h) \in A^h : a_1 + \cdots + a_h = n\}.$$

A set A is called an *asymptotic basis of order* h if $r_h(A,n) \geq 1$ for all sufficiently large integers n. In 1955, Stöhr [13] introduced the concept of minimal asymptotic basis. An asymptotic basis A of order h is *minimal* if no proper subset of A is an asymptotic basis of order h. This means that, for any $a \in A$, the set $E_a = hA \setminus h(A \setminus \{a\})$ is infinite.

In 1956, Härtter [5] showed that for every $h \geq 2$, there exists a minimal asymptotic basis of order h. Nathanson [10] presented an explicit construction of a minimal asymptotic basis of order 2 by using binary representations. For every $h \geq 2$, Jia and Nathanson [7] gave an explicit construction of a minimal asymptotic basis of order h. Chen and Chen [1] answered some problems of Nathanson on minimal asymptotic bases. For related problems concerning minimal asymptotic bases, see [2]–[4], [6], [8]–[9] and [12].

For any nonempty subset W of $\mathbb{N}$, denote by $\mathcal{F}^*(W)$ the set of all finite, nonempty subsets of W. Let $A(W)$ be the set of all numbers of the form $\sum_{f\in F} 2^f$, where $F \in \mathcal{F}^*(W)$.

In 1988, Nathanson [11] posed the following problem (see also Jia and Nathanson [7]).

PROBLEM 1.1. *Let* $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ *be a partition such that* $w \in W_i$ *implies either* $w-1 \in W_i$ *or* $w+1 \in W_i$. *Is*

$$A = A(W_0) \cup \cdots \cup A(W_{h-1})$$

a minimal asymptotic basis of order h?

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In 2011, Chen and Chen [1] obtained the following result.

THEOREM A. *Let $h \geq 2$ and t be the least integer with $t > \log h/\log 2$. Let $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ be a partition such that each W_i is infinite and contains t consecutive integers for $i = 1, \ldots, h$. Then*

$$A = A(W_0) \cup \cdots \cup A(W_{h-1})$$

is a minimal asymptotic basis of order h.

By Theorem A, the answer to Problem 1.1 is affirmative for $2 \leq h < 4$. We prove the following result, which shows that the answer to Problem 1.1 is negative for $h \geq 4$.

THEOREM 1.2. *Let h and t be integers with $2 \leq t \leq \log h/\log 2$. Then there exists a partition $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ such that each W_i is a union of infinitely many intervals of at least t consecutive integers and*

$$A = A(W_0) \cup \cdots \cup A(W_{h-1})$$

is not a minimal asymptotic basis of order h.

REMARK 1.3. For $2 \leq t < \log h/\log 2$, the following stronger result is proved: there exists a partition $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$ such that each set W_i is a union of infinitely many intervals of at least t consecutive integers and $n \in hA(W_0)$ for all sufficiently large integers n.

2. Proof of the theorem. Since $t \geq 2$, it follows that $h \geq 2^t \geq 4$. For any subset X of $\mathbb{N}$, let $2^X = \{2^x : x \in X\}$. Let $\{m_i\}_{i=1}^{\infty}$ be a sequence of integers with $m_1 > 2^{h+4}$ and $m_{i+1} - m_i > 2^{h+4}$ $(i \geq 1)$. For $a < b$, let $[a, b]$ denote the set of all integers x with $a \leq x \leq b$. Let

$$W_0 = [0, m_1] \cup \bigcup_{i=1}^{\infty} [m_i + t + 1, m_{i+1}],$$

$$W_j = \bigcup_{\substack{i=1 \\ i \equiv j \pmod{h-1}}}^{\infty} [m_i + 1, m_i + t], \qquad j = 1, \ldots, h-1.$$

It is clear that $\mathbb{N} = W_0 \cup \cdots \cup W_{h-1}$. If $w \in \mathbb{N} \setminus W_0$, then, by the definition of W_i, we have $w > m_1 > 2^{h+4}$ and $w - t \in W_0$. Write

$$A = A(W_0) \cup \cdots \cup A(W_{h-1}).$$

For any positive integer, let its binary expansion be

$$n = \sum_{f \in F_n} 2^f. \tag{2.1}$$

We will distinguish two cases: $h > 2^t$ and $h = 2^t$.

CASE 1: $h > 2^t$. In this case, we will prove that all integers n with $n \geq h2^{h(2t+1)}$ are in $hA(W_0)$. Thus A is not a minimal asymptotic basis of order h.

Let $n \geq h2^{h(2t+1)}$. Now we split some terms of the sum in (2.1) into sums. First, we split all 2^f with $f \in F_n \setminus W_0$ into 2^t terms 2^{f-t}. Then all terms are in 2^{W_0} and each term repeats at most $2^t + 1$ times in the summation. We continue to split terms in the summation. For any term 2^w in the summation, if $w > 2t+1$ and none of 2^{w-i} $(1 \leq i \leq 2t+1)$ appears in the summation, we split 2^w (split one of 2^w if there are several such terms) as follows:

(a) $2^w = 2^{w-1} + 2^{w-1}$ if $w - 1 \in W_0$;
(b) $2^w = (2^t+1)2^{w-t-1} + \cdots + (2^t+1)2^{w-2t+1} + (2^t+1)2^{w-2t} + 2 \times 2^{w-2t-1}$ if $w - 1 \notin W_0$.

In case (b), by the definition of W_0 and $w \in W_0$, we know that the integers $w - t - i$ $(1 \leq i \leq t+1)$ are all in W_0.

Since each split increases the number of terms by at least 1, the splitting procedure must terminate in finitely many steps. In the final summation, all terms are in 2^{W_0} and each term repeats at most $2^t + 1$ times. If 2^w $(w > 2t + 1)$ appears, then at least one of 2^{w-i} $(1 \leq i \leq 2t + 1)$ appears. Let the final summation be

$$n = \sum_{j=1}^{s} 2^{w_j}$$

with $0 \leq w_1 \leq \cdots \leq w_s$. Let $w_0 = 0$. Thus

$$0 \leq w_{i+1} - w_i \leq w_{i+1} - (w_{i+1} - 2t - 1) = 2t + 1, \qquad i = 0, 1, \ldots, s - 1.$$

Since

$$h2^{h(2t+1)} \leq n = \sum_{j=1}^{s} 2^{w_j} \leq (2^t + 1) \sum_{w=0}^{w_s} 2^w = (2^t + 1)(2^{w_s+1} - 1) < h2^{w_s+1},$$

it follows that $w_s \geq h(2t + 1)$. On the other hand,

$$w_s = \sum_{i=0}^{s-1} (w_{i+1} - w_i) \leq s(2t + 1).$$

Hence $s \geq h$. Noting that $2^t + 1 \leq h$ and $s \geq h$, we can split the final summation into h nonempty sums such that all terms in each sum are distinct. So $n \in hA(W_0)$.

CASE 2: $h = 2^t$. It is clear that $4 \in A(W_0)$. Now we prove that $E_4 = hA \setminus h(A \setminus \{4\})$ is a finite set. Thus A is not a minimal asymptotic basis of order h.

Let $n > m_2$. We will show that $n \in h(A \setminus \{4\})$, that is, $n \notin E_4$. Consider the following subcases:

SUBCASE 2.1: $F_n \cap W_0 \neq \{2\}$.

SUBCASE 2.1.1: $F_n \cap W_0 \neq \emptyset$ and $|F_n \setminus W_0| \geq h-1$. Then $F_n \setminus W_0$ has a partition

$$F_n \setminus W_0 = L_1 \cup \cdots \cup L_{h-1},$$

where $L_i \neq \emptyset$ $(1 \leq i \leq h-1)$ and for every L_i there exists a W_j $(j \geq 1)$ with $L_i \subseteq W_j$. Let $L_0 = F_n \cap W_0$ and

$$a_i = \sum_{l \in L_i} 2^l, \qquad 0 \leq i \leq h-1.$$

Then

$$a_i \in A \setminus \{4\}, \quad 0 \leq i \leq h-1, \quad \text{and} \quad n = a_0 + \cdots + a_{h-1}.$$

Hence $n \in h(A \setminus \{4\})$.

SUBCASE 2.1.2: $F_n \cap W_0 \neq \emptyset$ and $1 \leq |F_n \setminus W_0| \leq h-2$. Let

$$F_n \setminus W_0 = \{f_0, \ldots, f_{l-1}\}$$

with $f_0 > \cdots > f_{l-1}$. Then $f_0 \geq m_1 + 1 > 2^{h+4}$. Let

$$f_i = f_0 - (i - l + 1), \qquad l \leq i \leq h-2,$$

and $f_{h-1} = f_{h-2}$. Set

$$a_0 = \sum_{f \in F_n \cap W_0} 2^f, \qquad a_i = 2^{f_i}, \quad 1 \leq i \leq h-1.$$

Since

$$f_l > f_{l+1} > \cdots > f_{h-2} = f_{h-1} > 2^{h+4} - (h-2-l+1) \geq 2^{h+4} - (h-2) > 2,$$

it follows that

$$a_i \in A \setminus \{4\}, \quad 0 \leq i \leq h-1, \quad \text{and} \quad n = a_0 + \cdots + a_{h-1}.$$

Hence $n \in h(A \setminus \{4\})$.

SUBCASE 2.1.3: $F_n \cap W_0 \neq \emptyset$ and $F_n \setminus W_0 = \emptyset$. That is, $F_n \subseteq W_0$. Let

$$F_n = \{g_0, \ldots, g_{k-1}\}$$

with $g_0 > \cdots > g_{k-1}$. Since

$$n > m_2 > 2^{h+4} > 1 + 2 + 2^2 + \cdots + 2^{h+3},$$

we have $g_0 \geq h+4$.

If $k = 1$, then $F_n = \{g_0\}$. Let

$$a_i = 2^{g_0 - i - 1}, \qquad 0 \leq i \leq h-2,$$

and $a_{h-1} = a_{h-2}$. Since

$$a_0 > a_1 > \cdots > a_{h-2} = a_{h-1} = 2^{g_0 - h + 1} > 4,$$

it follows that

$$a_i \in A \setminus \{4\}, \quad 0 \le i \le h-1, \quad \text{and} \quad n = a_0 + \cdots + a_{h-1}.$$

Hence $n \in h(A \setminus \{4\})$.

If $k \ge 2$ and $2^{g_1} + \cdots + 2^{g_{k-1}} \ne 4$, then we take

$$a_0 = 2^{g_1} + \cdots + 2^{g_{k-1}}, \qquad a_i = 2^{g_0 - i}, \quad 1 \le i \le h-2,$$

and $a_{h-1} = a_{h-2}$. Since

$$a_1 > \cdots > a_{h-2} = a_{h-1} = 2^{g_0 - h + 2} > 4,$$

it follows that

$$a_i \in A \setminus \{4\}, \quad 0 \le i \le h-1, \quad \text{and} \quad n = a_0 + \cdots + a_{h-1}.$$

Hence $n \in h(A \setminus \{4\})$.

If $k \ge 2$ and $2^{g_1} + \cdots + 2^{g_{k-1}} = 4$, then we take $a_0 = a_1 = 2$,

$$a_i = 2^{g_0 - i + 1}, \qquad 2 \le i \le h-2,$$

and $a_{h-1} = a_{h-2}$. Since

$$a_2 > \cdots > a_{h-2} = a_{h-1} = 2^{g_0 - h + 3} > 4,$$

it follows that

$$a_i \in A \setminus \{4\}, \quad 0 \le i \le h-1, \quad \text{and} \quad n = a_0 + \cdots + a_{h-1}.$$

Hence $n \in h(A \setminus \{4\})$.

Subcase 2.1.4: $F_n \cap W_0 = \emptyset$. If $|F_n| \ge h$, then as in Subcase 2.1.1 we have $n \in h(A \setminus \{4\})$. If $|F_n| \le h-1$, then as in Subcase 2.1.2 we have $n \in h(A \setminus \{4\})$.

Subcase 2.2: $F_n \cap W_0 = \{2\}$. As $n > m_2$, we have $F_n \setminus W_0 \ne \emptyset$. If $f \in F \setminus W_0$, then $f > m_1 > 2^{h+4}$ and $f - t \in W_0$ (see the arguments before Case 1). Let

$$a_0 = 2^2 + \sum_{f \in F_n \setminus \{2\}} 2^{f-t}, \qquad a_1 = \cdots = a_{h-1} = \sum_{f \in F_n \setminus \{2\}} 2^{f-t}.$$

Then

$$a_i \in A(W_0) \setminus \{4\}, \quad 0 \le i \le h-1, \quad \text{and} \quad n = a_0 + \cdots + a_{h-1}$$

as $h = 2^t$. Hence $n \in h(A(W_0) \setminus \{4\})$.

This completes the proof of Theorem 1.2.

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