ACTA ARITHMETICA
184.2 (2018)

Sets generated by finite sets of algebraic numbers

by

C. L. STEWART (Waterloo)

For Professor Robert Tijdeman
on the occasion of his seventy-fifth birthday

1. Introduction. Let $S = \{p_1, \ldots, p_r\}$ be a finite set of prime numbers with $r \geq 2$ and let $(n_i)_{i=1}^{\infty}$ be the increasing sequence of positive integers composed of the primes from S. In 1973 [10] and 1974 [11] Tijdeman proved that there exist positive numbers c_1, c_2 and c_3, effectively computable in terms of S, such that for $n_i \geq c_3$,

$$\frac{n_i}{(\log n_i)^{c_1}} < n_{i+1} - n_i < \frac{n_i}{(\log n_i)^{c_2}}. \tag{1}$$

Tijdeman [10] also resolved a question of Wintner by proving that there exist infinite sets of primes S for which the associated sequence $(n_i)_{i=1}^{\infty}$ satisfies

$$\lim_{i\to\infty} (n_{i+1} - n_i) = \infty$$

(see also [5]).

In this note we shall study the distribution of the numbers formed when we take S to be a finite set of multiplicatively independent algebraic numbers of absolute value larger than 1 instead of a finite set of primes. Our first result corresponds to the lower bound in (1) and shows that such numbers are not close to each other.

THEOREM 1. *Let $\alpha_1, \ldots, \alpha_r$ be multiplicatively independent algebraic numbers with $|\alpha_i| > 1$ for $i = 1, \ldots, r$. Put*

$$T = \{\alpha_1^{h_1} \cdots \alpha_r^{h_r} \mid h_i \geq 0 \text{ for } i = 1, \ldots, r\}.$$

There exists a positive number c, which is effectively computable in terms of

2010 *Mathematics Subject Classification*: Primary 11N80; Secondary 11J86.
Key words and phrases: S-units, linear forms in logarithms.
Received 4 December 2017; revised 21 March 2018.
Published online 20 July 2018.

DOI: 10.4064/aa171204-9-4

$\alpha_1, \dots, \alpha_r$, *such that if t and t' are in T with $|t| \geq 3$ then*

$$|t - t'| > |t|/(\log |t|)^c.$$

Theorem 1 follows directly from lower bounds for linear forms in the logarithms of algebraic numbers [1, 2, 3, 6, 7].

We next obtain generalizations of the upper bound in (1). We consider two cases. For the first case we restrict our attention to sets of real algebraic numbers.

Theorem 2. *Let α_1 and α_2 be multiplicatively independent real algebraic numbers. Suppose $\alpha_i > 1$ for $i = 1, 2$ and put*

$$T = \{\alpha_1^{h_1} \alpha_2^{h_2} \mid h_i \geq 0 \text{ for } i = 1, 2\}.$$

There exists a positive number c_1, which is effectively computable in terms of α_1 and α_2, such that for any real number x with $x \geq 3$ there exists an element t of T with

$$|x - t| < x/(\log x)^{c_1}.$$

For the proof of Theorem 2 we modify the argument given by Tijdeman [11].

Finally we consider the case when the elements of T are not all real.

Theorem 3. *Let α_1, α_2 and α_3 be multiplicatively independent algebraic numbers with $|\alpha_i| > 1$ for $i = 1, 2, 3$. Suppose that α_1 and α_2 are positive real numbers and that $\alpha_3/|\alpha_3|$ is not a root of unity. Put*

$$T = \{\alpha_1^{h_1} \alpha_2^{h_2} \alpha_3^{h_3} \mid h_i \geq 0 \text{ for } i = 1, 2, 3\}.$$

There exists a positive number c_2, which is effectively computable in terms of α_1, α_2 and α_3, such that for any complex number z with $|z| \geq 3$ there exists an element t of T with

$$|z - t| < |z|/(\log |z|)^{c_2}.$$

Observe that if α_1 and α_2 are real numbers and $\alpha_3/|\alpha_3|$ is a root of unity then there is a positive number c_4 and complex numbers z of arbitrarily large modulus for which

$$|z - t| > c_4 |z|$$

for all elements t in T.

With Min Sha and Igor Shparlinski [8] we have applied both (1) and Theorem 3 in order to study the distribution of multiplicatively dependent vectors of algebraic numbers.

2. Linear forms in the logarithms of algebraic numbers. For any algebraic number α the *height* of α is the maximum of the absolute values of the relatively prime integer coefficients of the minimal polynomial of α. Let $\alpha_1, \dots, \alpha_n$ be algebraic numbers of heights at most $A_1, \dots, A_n$ respectively.

Let $b_1, \ldots, b_n$ be non-zero integers of absolute value at most B with $B \geq 2$. Put

$$\Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \quad \text{and} \quad d = [\mathbb{Q}(\alpha_1, \ldots, \alpha_n) : \mathbb{Q}].$$

Baker [1, 2] and Feldman [3] proved the following result.

Lemma 4. *There is a positive number c, which depends on $A_1, \ldots, A_n$, n and d, such that if $\Lambda \neq 0$ then*

$$|\Lambda| > B^{-c}.$$

For a sharp explicit dependence of c in Lemma 4 on the parameters $A_1, \ldots, A_n$, n and d, see Matveev [6, 7].

3. Proof of Theorem 1. Let $c_1, c_2, \ldots$ be positive numbers which are effectively computable in terms of $\alpha_1, \ldots, \alpha_r$. Let t be in T with $|t| \geq 3$. Then

$$t = \alpha_1^{h_1} \cdots \alpha_r^{h_r}$$

with $h_i \geq 0$ for $i = 1, \ldots, r$. Suppose t' is in T with $t' \neq t$. We have

$$t' = \alpha_1^{j_1} \cdots \alpha_r^{j_r}$$

with $j_i \geq 0$ for $i = 1, \ldots, r$. Then

$$|t - t'| = |t|\,|\alpha_1^{j_1 - h_1} \cdots \alpha_r^{j_r - h_r} - 1|.$$

Since $t \neq t'$, we may apply Lemma 4, as in [9, Theorem A], to obtain

$$|t - t'| > |t| B^{-c_1}, \tag{2}$$

where

$$B = \max(4, |j_1 - h_1|, \ldots, |j_r - h_r|).$$

We may suppose that $|t'| \leq 2|t|$, since otherwise the result holds, and thus

$$B < c_2 \log |t|. \tag{3}$$

Our result now follows from (2) and (3) since $|t| \geq 3$.

4. A preliminary result for the proof of Theorem 2. Let α_1 and α_2 be multiplicatively independent real algebraic numbers with $\alpha_i > 1$ for $i = 1, 2$. Let $\ell_0/k_0, \ell_1/k_1, \ldots$ be the sequence of convergents to $\log \alpha_1 / \log \alpha_2$. Our next result gives a bound on the growth of the k_i's. The proof depends upon Lemma 4 and is due to Tijdeman [11] when α_1 and α_2 are distinct primes.

Lemma 5. *There exists a positive number c, which is effectively computable in terms of α_1 and α_2, such that*

$$k_{j+1} < k_j^c \quad \text{for } j = 2, 3, \ldots.$$

Proof. Replacing $\log p/\log q$ by $\log\alpha_1/\log\alpha_2$ in [11, proof of the Lemma] and noting that $k_j \ge 2$ for $j \ge 2$ we obtain the result. ■

5. Proof of Theorem 2. Let $c_1, c_2, \ldots$ denote positive numbers which are effectively computable in terms of α_1 and α_2. Let x be a real number with $x \ge 3$ and let t be the largest element of T with $t \le x$. Then

$$\frac{x}{\max(\alpha_1,\alpha_2)} \le t$$

and so

$$\tfrac{1}{2}\log x < \log t \tag{4}$$

for $x > c_1$.

We have

$$t = \alpha_1^{h_1}\alpha_2^{h_2}$$

with h_1 and h_2 non-negative integers. We may assume, without loss of generality, that

$$\alpha_1^{h_1} \ge t^{1/2}$$

and so

$$h_1 \ge \frac{1}{2\log\alpha_1}\log t. \tag{5}$$

Let $\ell_0/k_0, \ell_1/k_1, \ldots$ be the sequence of convergents from the continued fraction expansion of $\log\alpha_1/\log\alpha_2$. Recall that the convergents with even index are smaller that $\log\alpha_1/\log\alpha_2$ and those with odd index are larger. Choose j to be the largest odd integer for which

$$k_j \le h_1; \tag{6}$$

certainly $k_1 \le h_1$ for $x > c_2$. Then k_{j+2} exceeds h_1 and by (4) and (5),

$$k_{j+2} > \frac{1}{4\log\alpha_1}\log x$$

for $x > c_3$. By Lemma 5, $k_{j+2} < k_{j+1}^{c_4}$ and so

$$k_{j+1} > \left(\frac{1}{4\log\alpha_1}\log x\right)^{1/c_4},$$

hence

$$k_{j+1} > (\log x)^{c_5} \qquad \text{for } x > c_6. \tag{7}$$

Put

$$t' = \alpha_1^{h_1-k_j}\alpha_2^{h_2+\ell_j}$$

and note that by (6) t' is in T. Further since $t < t'$ we have $x < t'$. By [4, Theorems 167 and 171],

$$0 < \frac{\ell_j}{k_j} - \frac{\log \alpha_1}{\log \alpha_2} < \frac{1}{k_j k_{j+1}}$$

and so

$$\log(t'/t) < \frac{\log \alpha_2}{k_{j+1}}. \tag{8}$$

It follows from (7) and (8) that $\log(t'/t) < \frac{1}{4}$, hence

$$\log(t'/t) = \log(1 + (t'-t)/t) > \frac{t'-t}{2t}, \tag{9}$$

and thus, by (8) and (9),

$$t' - t < \frac{(2\log \alpha_2)t}{k_{j+1}} \quad \text{for } x > c_6. \tag{10}$$

Recall (7) and that $t \le x < t'$. We see from (10) that

$$x - t < c_7 \frac{x}{(\log x)^{c_8}} < \frac{x}{(\log x)^{c_9}} \quad \text{for } x > c_{10}.$$

Note that this suffices to prove Theorem 2 since

$$x - t \le x - 1 \le \frac{x}{1 + \frac{1}{x-1}} \le \frac{x}{(\log x)^{c_{11}}} \quad \text{for } 3 \le x \le c_{10}.$$

6. Preliminaries for the proof of Theorem 3. For any non-zero complex number z let $\mathrm{Im}(z)$ denote the imaginary part of z, let $\mathrm{Arg}(z)$ denote the argument of z chosen so that $0 \le \mathrm{Arg}(z) < 2\pi$, and let $\log z$ denote the principal branch of the logarithm so that $0 \le \mathrm{Im}(\log z) < 2\pi$.

Let ν be a real number with $0 \le \nu < 2\pi$ and let α be an algebraic number with $|\alpha| > 1$ for which $\alpha/|\alpha|$ is not a root of unity. For each positive integer k let b_k be the smallest positive integer for which

$$|\mathrm{Arg}(\alpha^{b_k}) - \nu| \le 2\pi/k.$$

LEMMA 6. *There exists a positive number c, which is effectively computable in terms of α, such that*

$$b_k < k^c \quad \textit{for } k \ge 2.$$

Proof. Suppose $k \ge 2$. By Dirichlet's box principle there exists an integer r_k with $1 \le r_k \le k$ and an integer m_k such that

$$\left| r_k \log \frac{\alpha}{|\alpha|} - m_k 2\pi i \right| \le \frac{2\pi}{k}.$$

Notice that $0 \le m_k \le r_k$.

Let $c_1, c_2, \ldots$ denote positive numbers which are effectively computable in terms of α. By Lemma 4,

$$\left| r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \right| \geq \frac{1}{(2r_k)^{c_1}} \geq \frac{1}{k^{c_2}}.$$

If

$$0 < \frac{1}{i}\left(r_k \log \frac{\alpha}{|\alpha|} - m_k 2\pi i \right) \leq \frac{2\pi}{k}$$

then there exists an integer q_k with $1 \leq q_k \leq 2\pi k^{c_2}$ for which

$$\left| q_k \left(r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \right) - \nu i \right| \leq \frac{2\pi}{k}.$$

On the other hand, if $r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1)$ is of the form yi with $-2\pi/k \leq y < 0$ then there exists an integer q_k with $1 \leq q_k \leq 2\pi k^{c_2}$ for which

$$\left| 2\pi i + q_k \left(r_k \log \frac{\alpha}{|\alpha|} - 2m_k \log(-1) \right) - \nu i \right| \leq \frac{2\pi}{k}.$$

Therefore, since

$$\log(\alpha/|\alpha|)^{q_k r_k} = i \operatorname{Arg}(\alpha^{q_k r_k})$$

and since $k \geq 2$, we have

$$b_k \leq q_k r_k \leq 2\pi k^{1+c_2} < k^{c_3}$$

as required. ∎

7. Proof of Theorem 3. Let z be a complex number with $|z| \geq 3$ and put $\nu = \operatorname{Arg}(z)$. Let $b_1, b_2, \ldots$ be defined as in §6 with α replaced by α_3. Let $c_1, c_2, \ldots$ denote positive numbers which are effectively computable in terms of α_1, α_2 and α_3. It suffices, as in the proof of Theorem 2, to establish our result for $|z| > c_1$. In particular we may suppose that $|z|$ exceeds $\max(9, |\alpha_3|^{2b_2})$. We now choose k so that

(11) $$|\alpha_3|^{2b_k} \leq |z| < |\alpha_3|^{2b_{k+1}};$$

since $|z|$ exceeds $|\alpha_3|^{2b_2}$ and since the sequence $(b_i)_{i=1}^{\infty}$ is non-decreasing, k is well defined. By the definition of b_k we have

$$|\operatorname{Arg}(\alpha_3^{b_k}) - \operatorname{Arg}(z)| \leq 2\pi/k.$$

Further

(12) $$3 \leq |z|^{1/2} \leq |z|/|\alpha_3|^{b_k} < |z|.$$

We now choose non-negative integers j_1 and j_2 such that

(13) $$\left| \alpha_1^{j_1} \alpha_2^{j_2} - \frac{|z|}{|\alpha_3|^{b_k}} \right| < \frac{|z|/|\alpha_3|^{b_k}}{(\log|z|)^{c_2}},$$

which is possible by (12) and Theorem 2. Put

$$t = \alpha_1^{j_1}\alpha_2^{j_2}\alpha_3^{b_k}.$$

Notice that by (13),

$$\big||z| - |t|\big| < \frac{|z|}{(\log|z|)^{c_2}}. \tag{14}$$

In addition, since $\mathrm{Arg}(t) = \mathrm{Arg}(\alpha_3^{b_k})$,

$$|\mathrm{Arg}(z) - \mathrm{Arg}(t)| \le 2\pi/k. \tag{15}$$

On the other hand, by (11),

$$\log|z| < 2b_{k+1}\log|\alpha_3|$$

and so by Lemma 6,

$$\log|z| < k^{c_3} \tag{16}$$

since $k \ge 2$. Thus by (15) and (16),

$$|\mathrm{Arg}(z) - \mathrm{Arg}(t)| < \frac{2\pi}{(\log|z|)^{c_4}}. \tag{17}$$

Our result now follows from (14) and (17).

Acknowledgements. This research was supported in part by the Canada Research Chairs Program and by grant A3528 from the Natural Sciences and Engineering Research Council of Canada.

References

[1] A. Baker, *A sharpening of the bounds for linear forms in logarithms*, Acta Arith. 21 (1972), 117–129.

[2] A. Baker, *The theory of linear forms in logarithms*, in: Transcendence Theory: Advances and Applications, Academic Press, London, 1977, 1–27.

[3] N. I. Feldman, *Improved estimate for a linear form of the logarithms of algebraic numbers*, Mat. Sb. (N.S.) 77 (1968), 432–436 (in Russian); English transl.: Math. USSR-Sbornik 6 (1968), 393–406.

[4] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 5th ed., Oxford Univ. Press, 1979.

[5] J. Kim and C. L. Stewart, *Well spaced integers generated by an infinite set of primes*, Proc. Amer. Math. Soc. 143 (2015), 915–923.

[6] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers*, Izv. Ross. Akad. Nauk Ser. Mat. 62 (1998), 81–136 (in Russian); English transl.: Izv. Math. 62 (1998), 723–772.

[7] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in logarithms of algebraic numbers. II*, Izv. Ross. Akad. Nauk Ser. Mat. 64 (2000), 125–180 (in Russian); English transl.: Izv. Math. 64 (2000), 1217–1269.

[8] M. Sha, I. Shparlinski and C. L. Stewart, *On the distribution of multiplicatively dependent vectors*, to appear.

[9] T. N. Shorey, A. J. van der Poorten, R. Tijdeman and A. Schinzel, *Applications of the Gel'fond–Baker method to Diophantine equations*, in: Transcendence Theory: Advances and Applications, Academic Press, London, 1977, 59–77.
[10] R. Tijdeman, *On integers with many small prime factors*, Compos. Math. 26 (1973), 319–330.
[11] R. Tijdeman, *On the maximal distance between integers composed of small primes*, Compos. Math. 28 (1974), 159–162.

C. L. Stewart
Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
E-mail: cstewart@uwaterloo.ca