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Structural properties and formulae of the spectra of integral circulant graphs

by

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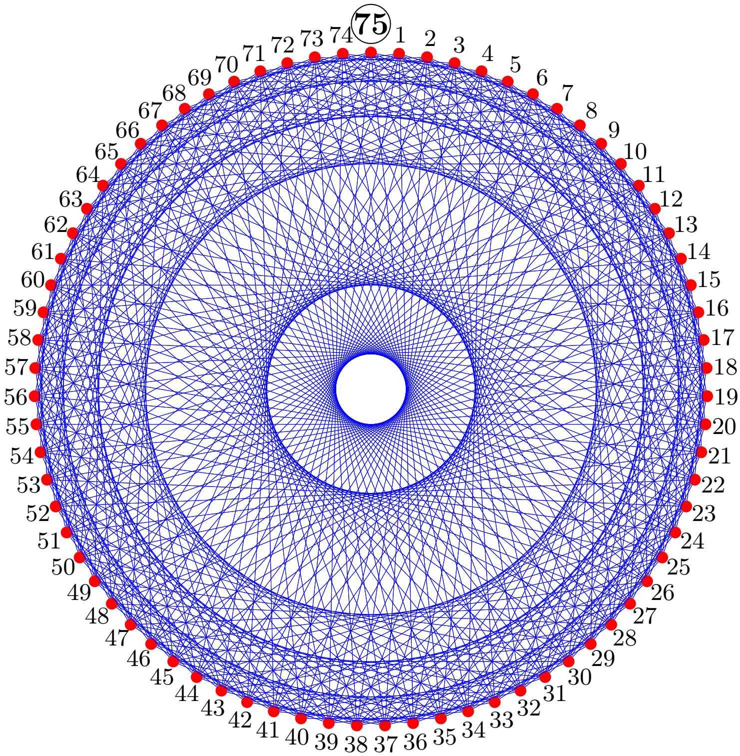

Rob's integral circulant birthday graph $\mathrm{ICG}(75, \{5, 15\})$
composed for and dedicated to my amicable colleague Rob Tijdeman on the occasion of his 75th birthday

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1. Introduction. *Integral circulant graphs*, i.e. graphs having a circulant adjacency matrix (see [7] for the general theory of circulant matrices) with integral eigenvalues, are objects exhibiting algebraic, arithmetic and combinatorial features at the same time. Since they are Cayley graphs on finite cyclic groups, algebraic aspects of these groups are reflected by properties of the corresponding graphs, as one would expect. The additional characteristic of having integral spectrum, which means that all eigenvalues of the adjacency matrices of these graphs are integers (1), is the basis for extra features of arithmetic nature.

"Can one hear the shape of a drum?", asked Kac [10] in 1966, which has become a synonym for the problem of deciding whether a given Riemannian manifold is uniquely determined by its spectrum. Fisher [8] formulated the discrete analogue of Kac's query and thus transferred it to the examination of spectra of linear graphs by use of their adjacency matrices. Since the mid-twentieth century one has tried to find out which graphs ensure an affirmative answer to Kac's question, and some results were given for different types of graphs (see [6] for a survey, and [5] for graph spectra in general). Very little and hardly anything of general nature is known about this problem for integral circulant graphs (see [16] for recent results), although So [18] conjectured more than a decade ago that these graphs are uniquely determined by their spectrum.

By the works of So [18] and Klotz and T. Sander [11] each integral circulant graph $\mathrm{ICG}(n,\mathcal{D})$ is characterised by its order n and a non-empty set $\mathcal{D} \subseteq D(n) := \{d > 0 : d \mid n\}$ of positive divisors of n in such a way that it has vertex set $\mathbb{Z}/n\mathbb{Z}$ and edge set $\{(a,b) : a,b \in \mathbb{Z}/n\mathbb{Z}, \gcd(a-b,n) \in \mathcal{D}\}$. An example, displayed on the frontispiece, is *Rob's integral circulant birthday graph* $\mathrm{ICG}(75,\{5,15\})$. By definition, each $\mathrm{ICG}(n,\mathcal{D})$ has a circulant adjacency matrix, and consequently each of its eigenvalues can explicitly be evaluated as a sum of roots of unity (see [7]). For integral circulant graphs this sum of complex numbers turns out to be a sum of Ramanujan sums (see (2.1) or [11]) and thus is an integer. Two of the arithmetic properties of the divisor set $\mathcal{D}$ reflected by graph-theoretical features of $\mathrm{ICG}(n,\mathcal{D})$ are the following: If $n \in \mathcal{D}$ then apparently $\mathrm{ICG}(n,\mathcal{D})$ has loops, which is the reason why one usually assumes that $n \notin \mathcal{D}$. Moreover, it is known that $\mathrm{ICG}(n,\mathcal{D})$ is connected if and only if the elements of $\mathcal{D}$ are coprime (see [4, Proposition 1]).

The characterisation of integral circulant graphs given in the preceding paragraph immediately implies So's conjecture for all $\mathrm{ICG}(p^k,\mathcal{D})$ of prime

(1) As a side note, we remind the reader of the still widely unanswered question "Which graphs have integral spectra?", posed by Harary and Schenk [9] in 1974. One of the few general results states that although several constructions of (classes of) such graphs are known, it is unlikely for a randomly chosen graph to have integral spectrum (see [1], also for references).

power order p^k. In fact, an integral circulant graph of prime power order is uniquely determined by its largest eigenvalue. In contrast, the situation is much more intricate for graphs $\mathrm{ICG}(n, \mathcal{D})$, where n is a composite integer. E.g. for $n = 216 = 2^3 \cdot 3^3$, $\mathcal{D} := \{6, 24\}$ and $\mathcal{E} := \{6, 18, 54\}$, the two non-isomorphic graphs $\mathrm{ICG}(216, \mathcal{D})$ and $\mathrm{ICG}(216, \mathcal{E})$ both have the same largest eigenvalue 18. Observe that incidentally both graphs also have the same eigenvalue of least absolute value, namely 0, their so-called *dominating* eigenvalue [16, Corollary 2.4]). This shows that in general more than one eigenvalue has to be considered in order to confirm So's conjecture—even in the special case of *multiplicative divisor sets* $\mathcal{D}$ (see beginning of Section 5 for a definition). Concerning arbitrary divisor sets there are even examples of non-isomorphic integral circulant graphs of the same order having the same spectrum (disregarding multiplicities of eigenvalues), e.g. $\mathrm{ICG}(48, \{2, 3, 4, 24\})$ and $\mathrm{ICG}(48, \{2, 3, 8, 16, 24\})$ both having spectrum $\{21, 5, 3, -1, -3, -13\}$ (see Fig. 1). Therefore, multiplicities of eigenvalues will certainly be crucial in a potential proof of So's conjecture in full generality.

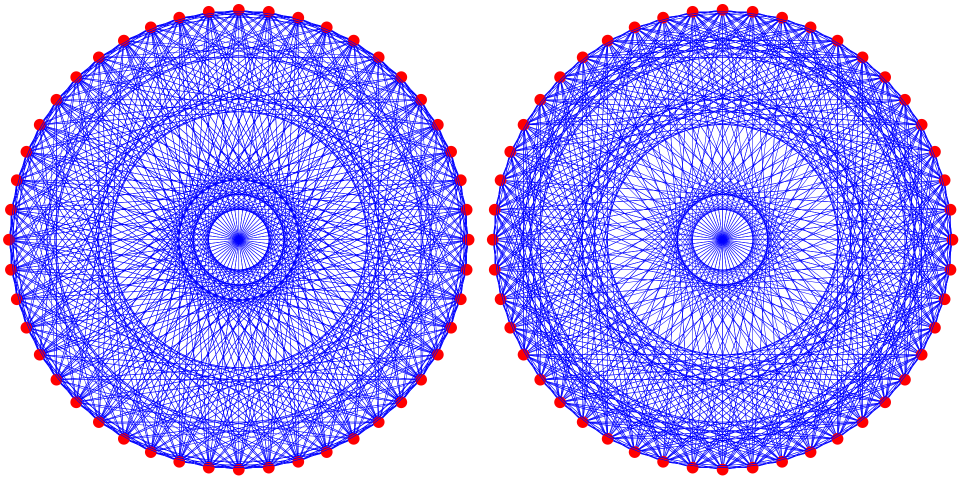

Fig. 1. $\mathrm{ICG}(48, \{2, 3, 4, 24\})$ and $\mathrm{ICG}(48, \{2, 3, 8, 16, 24\})$: non-isomorphic graphs with identical spectrum $\{21, 5, 3, -1, -3, -13\}$

For arbitrary n and multiplicative divisor sets $\mathcal{D}$ a slightly weaker form of So's conjecture was recently confirmed by T. Sander and the author [16]. In [15] the author examined the role of the eigenvalue 0 and clarified the interplay between the dimension of the kernel of $\mathrm{ICG}(n, \mathcal{D})$ and the graph itself for all positive integers n and multiplicative $\mathcal{D}$. In order to facilitate a deeper understanding of the dependencies between integral circulant graphs and their spectra, we extend the work dealing with the kernel to the entire spectrum. We deduce several structural spectral properties of $\mathrm{ICG}(p^k, \mathcal{D})$

with prime power order p^k and establish an explicit parameterisation of the spectrum of $\mathrm{ICG}(n, \mathcal{D})$ for multiplicative divisor sets $\mathcal{D}$. Our most important tool will be the new concept of the *leaping set* of $\mathcal{D}$.

2. Preliminaries. Let $\mathrm{ICG}(n, \mathcal{D})$ be an integral circulant graph with an arbitrary positive integer n and a divisor set $\mathcal{D} \subseteq D(n) = \{d > 0 : d \mid n\}$. As shown in [11, Theorem 16], the eigenvalues of (the adjacency matrix of) $\mathrm{ICG}(n, \mathcal{D})$, taking multiplicities into account, are integers given by

$$\lambda_j(n, \mathcal{D}) = \sum_{d \in \mathcal{D}} c(j, n/d) \qquad (1 \le j \le n), \tag{2.1}$$

where

$$c(j, n) := \sum_{\substack{\ell \bmod n \\ (\ell, n) = 1}} e(j\ell/n) = \sum_{d \mid \gcd(j,n)} d \cdot \mu(n/d)$$

denotes the Ramanujan sum (see [2, Chapters 8.3–8.4] or [17, Chapter I.3]) with the Möbius function μ and $e(x) := e^{2\pi i x}$ for real x. Then

$$\vec{\lambda}(n, \mathcal{D}) := (\lambda_1(n, \mathcal{D}), \ldots, \lambda_n(n, \mathcal{D}))$$

is called the *spectral vector* of $\mathrm{ICG}(n, \mathcal{D})$. Disregarding multiplicities we denote the set of all eigenvalues of $\mathrm{ICG}(n, \mathcal{D})$, i.e. its spectrum, by

$$\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})) := \{\lambda_j(n, \mathcal{D}) : 1 \le j \le n\}.$$

It is well known that integral circulant graphs are regular. Incidentally, $\lambda_n(n, \mathcal{D})$ is positive and the largest eigenvalue in absolute value (see Propositions 2.1 and 2.2). Moreover, this *spectral radius* $\lambda_n(n, \mathcal{D})$ equals the graph's degree of regularity (see [3] or [5, Proposition 3.1.2]).

In the following, let p^k always be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set. A typical example of the spectral vector of an integral circulant graph of prime power order looks like

$$\begin{aligned}
&\vec{\lambda}(3^5, \{1, 3, 3^3\}) = \\
&\quad (0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,-21,\\
&\quad\ 0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,6,0,0,-3,0,0,-3,0,0,\ 222).
\end{aligned}$$

The entries in $\vec{\lambda}(3^5, \{1, 3, 3^3\})$ reveal some kind of pseudo-periodicity, and the eigenvalues $0, -3, 6, -21, 222 \in \mathrm{Spec}(\mathrm{ICG}(3^5, \{1, 3, 3^3\}))$, arranged in order of their first occurrence in the spectral vector, have alternating signs and are growing in absolute value. The following well-known results about the spectral vector of $\mathrm{ICG}(p^k, \mathcal{D})$ are the basis for specifying these observations in general.

Proposition 2.1 ([16, Proposition 2.2(i)]).

$$\lambda_j(p^k, \mathcal{D}) = \lambda_{\gcd(j,p^k)}(p^k, \mathcal{D}) \quad \text{for } 1 \le j \le p^k.$$

The next statement will be refined by Proposition 3.3(v) below.

Proposition 2.2 ([16, Corollary 2.1(i)]).

$$|\lambda_{p^u}(p^k, \mathcal{D})| \le |\lambda_{p^v}(p^k, \mathcal{D})| \quad \text{for } 0 \le u \le v \le k.$$

Given a divisor set $\mathcal{D} = \{p^{k_1}, \ldots, p^{k_s}\}$, we call $\mathcal{K}_{\mathcal{D}} := \{k_1, \ldots, k_s\}$ its *exponent set.*

Proposition 2.3 ([16, Corollary 2.4(i)]). *The (dominating) eigenvalue $\lambda_1(p^k, \mathcal{D})$ of $\mathrm{ICG}(p^k, \mathcal{D})$ satisfies*

$$\lambda_1(p^k, \mathcal{D}) = \begin{cases} +1 & \text{if } k \in \mathcal{K}_{\mathcal{D}} \text{ and } k-1 \notin \mathcal{K}_{\mathcal{D}}, \\ -1 & \text{if } k \notin \mathcal{K}_{\mathcal{D}} \text{ and } k-1 \in \mathcal{K}_{\mathcal{D}}, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.4 ([15, Proposition 3.2]). *For $0 < \ell \le k$ we have*

$$\lambda_{p^\ell}(p^k, \mathcal{D}) - \lambda_{p^{\ell-1}}(p^k, \mathcal{D}) = \begin{cases} p^\ell & \text{if } k-\ell \in \mathcal{K}_{\mathcal{D}} \text{ and } k-\ell-1 \notin \mathcal{K}_{\mathcal{D}}, \\ -p^\ell & \text{if } k-\ell \notin \mathcal{K}_{\mathcal{D}} \text{ and } k-\ell-1 \in \mathcal{K}_{\mathcal{D}}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

In particular, (2.2) *includes the case $\ell = k$, where*

$$\lambda_{p^k}(p^k, \mathcal{D}) - \lambda_{p^{k-1}}(p^k, \mathcal{D}) = \begin{cases} p^k & \text{if } 0 \in \mathcal{K}_{\mathcal{D}}, \\ 0 & \text{if } 0 \notin \mathcal{K}_{\mathcal{D}}. \end{cases}$$

3. Structural properties of $\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$. Let p^k be a prime power. Given a divisor set $\mathcal{D} \subseteq D(p^k)$ with exponent set $\mathcal{K}_{\mathcal{D}}$, we define

$$\widetilde{\mathcal{K}}_{\mathcal{D}} := \mathcal{K}_{\mathcal{D}} \setminus \{k\}, \qquad \overline{\mathcal{K}}_{\mathcal{D}} := \{-1, 0, 1, \ldots, k-1\} \setminus \mathcal{K}_{\mathcal{D}}$$

and the *leaping set*

$$\mathcal{L}_{\mathcal{D}} := \{k - \ell : 0 \le \ell \le k-1,\ (\ell-1, \ell) \in (\widetilde{\mathcal{K}}_{\mathcal{D}} \times \overline{\mathcal{K}}_{\mathcal{D}}) \cup (\overline{\mathcal{K}}_{\mathcal{D}} \times \widetilde{\mathcal{K}}_{\mathcal{D}})\} \cup \{0\}.$$

It will become clear that the leaping set encodes essential information about the eigenvalues of the integral circulant graph and their multiplicities (see Theorem 4.2 and Example 4.3 as well as Theorem 5.1 and Example 5.2).

Lemma 3.1. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set with exponent set $\mathcal{K}_{\mathcal{D}}$ and leaping set $\mathcal{L}_{\mathcal{D}}$. Then for $1 \le \ell \le k$,*

$$\lambda_{p^\ell}(p^k,\mathcal{D}) - \lambda_{p^{\ell-1}}(p^k,\mathcal{D}) = \begin{cases} p^\ell & \text{if } \ell \in \mathcal{L}_{\mathcal{D}} \text{ with } (k-\ell-1, k-\ell) \in \overline{\mathcal{K}}_{\mathcal{D}} \times \widetilde{\mathcal{K}}_{\mathcal{D}}, \\ -p^\ell & \text{if } \ell \in \mathcal{L}_{\mathcal{D}} \text{ with } (k-\ell-1, k-\ell) \in \widetilde{\mathcal{K}}_{\mathcal{D}} \times \overline{\mathcal{K}}_{\mathcal{D}}, \\ 0 & \text{if } \ell \notin \mathcal{L}_{\mathcal{D}}. \end{cases}$$

Proof. The formula is a straightforward consequence of Proposition 2.4. In fact, setting $\ell' := k - \ell$, we have $0 \le \ell' < k$. First assume that $\ell \in \mathcal{L}_{\mathcal{D}}$, i.e. $(\ell'-1,\ell') \in (\widetilde{\mathcal{K}}_{\mathcal{D}} \times \overline{\mathcal{K}}_{\mathcal{D}}) \cup (\overline{\mathcal{K}}_{\mathcal{D}} \times \widetilde{\mathcal{K}}_{\mathcal{D}})$. In case $(\ell'-1,\ell') \in \widetilde{\mathcal{K}}_{\mathcal{D}} \times \overline{\mathcal{K}}_{\mathcal{D}}$, we have $k-\ell-1 = \ell'-1 \in \mathcal{K}_{\mathcal{D}}$ and $k-\ell = \ell' \notin \mathcal{K}_{\mathcal{D}}$, which implies the identity $\lambda_{p^\ell}(p^k,\mathcal{D}) - \lambda_{p^{\ell-1}}(p^k,\mathcal{D}) = -p^\ell$ by Proposition 2.4. In case $(\ell'-1,\ell') \in \overline{\mathcal{K}}_{\mathcal{D}} \times \widetilde{\mathcal{K}}_{\mathcal{D}}$, we similarly obtain $\lambda_{p^\ell}(p^k,\mathcal{D}) - \lambda_{p^{\ell-1}}(p^k,\mathcal{D}) = p^\ell$. We are left with the case $\ell \notin \mathcal{L}_{\mathcal{D}}$, where a comparable argument and Proposition 2.4 yield $\lambda_{p^\ell}(p^k,\mathcal{D}) - \lambda_{p^{\ell-1}}(p^k,\mathcal{D}) = 0$. ∎

Lemma 3.2. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set with exponent set $\mathcal{K}_{\mathcal{D}}$ and leaping set $\mathcal{L}_{\mathcal{D}} = \{\ell_0, \ell_1, \ldots, \ell_m\}$, say, for integers $0 = \ell_0 < \ell_1 < \cdots < \ell_m$. Setting $\ell_{m+1} := k+1$, we have, for $1 \le j \le m+1$,*

(i) $\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D}) = \lambda_{p^{\ell_{j-1}+1}}(p^k,\mathcal{D}) = \lambda_{p^{\ell_{j-1}+2}}(p^k,\mathcal{D}) = \cdots = \lambda_{p^{\ell_j-1}}(p^k,\mathcal{D})$,

(ii) *$k-\ell_{j-1}-1, k-\ell_{j-1}-2, \ldots, k-\ell_j+1, k-\ell_j$ either all lie in $\widetilde{\mathcal{K}}_{\mathcal{D}}$ or all in $\overline{\mathcal{K}}_{\mathcal{D}}$.*

Proof. First observe that by definition $\ell \notin \mathcal{L}_{\mathcal{D}}$ in the range $\ell_{j-1} < \ell < \ell_j$. So Lemma 3.1 yields $\lambda_{p^\ell}(p^k,\mathcal{D}) = \lambda_{p^{\ell-1}}(p^k,\mathcal{D})$ for $\ell_{j-1}+1 \le \ell \le \ell_j - 1$, which proves (i). The definition of the leaping set $\mathcal{L}_{\mathcal{D}}$ directly implies (ii). ∎

Proposition 3.3. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set with exponent set $\mathcal{K}_{\mathcal{D}}$ and leaping set $\mathcal{L}_{\mathcal{D}} = \{\ell_0, \ell_1, \ldots, \ell_m\}$, where we have $0 = \ell_0 < \ell_1 < \cdots < \ell_m$. Define*

$$i_0 = i_0(p^k,\mathcal{D}) := \begin{cases} 0 & \text{if } k-1 \notin \mathcal{K}_{\mathcal{D}}, \\ 1 & \text{if } k-1 \in \mathcal{K}_{\mathcal{D}}. \end{cases} \tag{3.1}$$

Then the following assertions hold for $1 \le j \le m$:

(i) $\lambda_{p^{\ell_j}}(p^k,\mathcal{D}) - \lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D}) = (-1)^{j-i_0} p^{\ell_j}$;

(ii) $\lambda_{p^{\ell_j}}(p^k,\mathcal{D}) = \lambda_1(p^k,\mathcal{D}) + \sum_{i=1}^{j} (-1)^{i-i_0} p^{\ell_i}$, *where*

$$\lambda_1(p^k,\mathcal{D}) = \begin{cases} +1 & \text{if } k \in \mathcal{K}_{\mathcal{D}} \text{ and } k-1 \notin \mathcal{K}_{\mathcal{D}}, \\ -1 & \text{if } k \notin \mathcal{K}_{\mathcal{D}} \text{ and } k-1 \in \mathcal{K}_{\mathcal{D}}, \\ 0 & \text{otherwise}; \end{cases} \tag{3.2}$$

(iii) $\operatorname{sign}(\lambda_{p^{\ell_j}}(p^k,\mathcal{D})) = (-1)^{j-i_0}$;

(iv) $p^{\ell_j} - p^{\ell_{j-1}} \le |\lambda_{p^{\ell_j}}(p^k, \mathcal{D})| \le p^{\ell_j}$; *more precisely,*

(iv$_1$) $|\lambda_{p^{\ell_1}}(p^k, \mathcal{D})| = p^{\ell_1} - |\lambda_1(p^k, \mathcal{D})|$,
(iv$_2$) $|\lambda_{p^{\ell_2}}(p^k, \mathcal{D})| = p^{\ell_2} - p^{\ell_1} + |\lambda_1(p^k, \mathcal{D})| < p^{\ell_2}$,
(iv$_3$) $p^{\ell_j} - p^{\ell_{j-1}} < |\lambda_{p^{\ell_j}}(p^k, \mathcal{D})| < p^{\ell_j}$ *for* $j \ge 3$.

(v) *If* $\lambda_{p^{\ell_j}}(p^k, \mathcal{D}) = -\lambda_{p^{\ell_{j-1}}}(p^k, \mathcal{D})$ *occurs in* $\mathrm{ICG}(p^k, \mathcal{D})$, *then* $p = 2$ *and either* $j = 1$, *in which case* $|\lambda_1(2^k, \mathcal{D})| = 1$ *and* $\ell_1 = 1$, *or* $j = 2$, *in which case* $\lambda_1(2^k, \mathcal{D}) = 0$ *and* $\ell_2 = \ell_1 + 1$.

REMARK 3.4. (i) Proposition 3.3(v) refines Propositon 2.2 by showing that an identity $|\lambda_{p^u}(p^k, \mathcal{D})| = |\lambda_{p^v}(p^k, \mathcal{D})|$ for some $0 \le u \le v \le k$ always implies $\lambda_{p^u}(p^k, \mathcal{D}) = \lambda_{p^v}(p^k, \mathcal{D})$ apart from very special types of integral circulant graphs of prime power order 2^k.

(ii) According to Proposition 3.3(v), $\lambda_{p^{\ell_j}}(p^k, \mathcal{D}) = -\lambda_{p^{\ell_{j-1}}}(p^k, \mathcal{D})$ cannot occur in an $\mathrm{ICG}(p^k, \mathcal{D})$ with $p \ge 3$. However, examples do exist for $p = 2$. In fact, for each $k \ge 2$ there are divisor sets $\mathcal{D} \subseteq D(2^k)$ such that $\mathrm{ICG}(2^k, \mathcal{D})$ has an eigenvalue $\lambda \ne 0$ with $-\lambda$ in $\mathrm{Spec}(\mathrm{ICG}(2^k, \mathcal{D}))$ as well, e.g.

(α) $\mathrm{ICG}(2^k, \{1\})$ has eigenvalues $\lambda_{2^{k-1}} = -2^{k-1}$, $\lambda_{2^k} = 2^{k-1}$ (and $\lambda_j = 0$ for all other j) for each $k \ge 1$;
(β) if $k \ge 2$, then $\mathrm{ICG}(2^k, \{1, 2, \ldots, 2^{k-2}, 2^k\})$ has

$$\lambda_j(2^k, \mathcal{D}) = \begin{cases} 1 & \text{for odd } j, \\ -1 & \text{for even } j < 2^k, \\ 2^k - 1 & \text{for } j = 2^k. \end{cases}$$

In Theorem 4.4 we shall identify all graphs of type $\mathrm{ICG}(2^k, \mathcal{D})$ having an eigenvalue $\lambda \ne 0$ such that $-\lambda$ is in $\mathrm{Spec}(\mathrm{ICG}(2^k, \mathcal{D}))$ as well.

Proof of Proposition 3.3. By Lemmas 3.2(i) and 3.1 we have

$$(3.3)\qquad \lambda_{p^{\ell_j}}(p^k, \mathcal{D}) - \lambda_{p^{\ell_{j-1}}}(p^k, \mathcal{D}) = \lambda_{p^{\ell_j}}(p^k, \mathcal{D}) - \lambda_{p^{\ell_j - 1}}(p^k, \mathcal{D}) = \begin{cases} p^{\ell_j} & \text{if } (k-\ell_j-1, k-\ell_j) \in \overline{\mathcal{K}}_{\mathcal{D}} \times \widetilde{\mathcal{K}}_{\mathcal{D}}, \\ -p^{\ell_j} & \text{if } (k-\ell_j-1, k-\ell_j) \in \widetilde{\mathcal{K}}_{\mathcal{D}} \times \overline{\mathcal{K}}_{\mathcal{D}}, \end{cases}$$

for $1 \le j \le m$. We prove (i) by induction on $j = 1, \ldots, m$. Let $j = 1$, and first assume that $p^{k-1} \notin \mathcal{D}$, i.e. $k - \ell_0 - 1 = k - 1 \in \overline{\mathcal{K}}_{\mathcal{D}}$ and $i_0 = 0$. Applying Lemma 3.2(ii) for $j = 1$ yields $k - \ell_1 \in \overline{\mathcal{K}}_{\mathcal{D}}$. Then $k - \ell_1 - 1 \in \widetilde{\mathcal{K}}_{\mathcal{D}}$ by definition of the leaping set $\mathcal{L}_{\mathcal{D}}$, and thus (3.3) implies that

$$\lambda_{p^{\ell_1}}(p^k, \mathcal{D}) - \lambda_{p^{\ell_0}}(p^k, \mathcal{D}) = -p^{\ell_1} = (-1)^{1-i_0} p^{\ell_1},$$

which proves (i) for $j = 1$ in this case. Analogously, (i) is shown for $j = 1$ if $p^{k-1} \in \mathcal{D}$. To complete the induction, assume as induction hypothesis that

$$(3.4)\qquad (-1)^{(j-1)-i_0} p^{\ell_{j-1}} = \lambda_{p^{\ell_{j-1}}}(p^k, \mathcal{D}) - \lambda_{p^{\ell_{j-2}}}(p^k, \mathcal{D}) = \begin{cases} p^{\ell_{j-1}} & \text{if } (k - \ell_{j-1} - 1, k - \ell_{j-1}) \in \overline{\mathcal{K}}_{\mathcal{D}} \times \widetilde{\mathcal{K}}_{\mathcal{D}}, \\ -p^{\ell_{j-1}} & \text{if } (k - \ell_{j-1} - 1, k - \ell_{j-1}) \in \widetilde{\mathcal{K}}_{\mathcal{D}} \times \overline{\mathcal{K}}_{\mathcal{D}}, \end{cases}$$

where the last equality holds by (3.3). In case $k - \ell_{j-1} - 1 \in \overline{\mathcal{K}}_{\mathcal{D}}$, hence for $(-1)^{(j-1)-i_0} = 1$ according to (3.4), we also have $k - \ell_j \in \overline{\mathcal{K}}_{\mathcal{D}}$ by Lemma 3.2(ii). Then (3.3) implies that

$$\lambda_{p^{\ell_j}}(p^k, \mathcal{D}) - \lambda_{p^{\ell_{j-1}}}(p^k, \mathcal{D}) = -p^{\ell_j} = (-1)^{j-i_0} p^{\ell_j},$$

since $(-1)^{(j-1)-i_0} = 1$ in this case, and this proves (i) for those j satisfying $k - \ell_{j-1} - 1 \in \overline{\mathcal{K}}_{\mathcal{D}}$. The remaining case $k - \ell_{j-1} - 1 \in \widetilde{\mathcal{K}}_{\mathcal{D}}$ can be dealt with similarly, which completes the proof of (i).

From this we obtain

$$\begin{aligned} \lambda_{p^{\ell_j}}(p^k, \mathcal{D}) - \lambda_1(p^k, \mathcal{D}) &= \lambda_{p^{\ell_j}}(p^k, \mathcal{D}) - \lambda_{p^{\ell_0}}(p^k, \mathcal{D}) \\ &= \sum_{i=1}^{j} (\lambda_{p^{\ell_i}}(p^k, \mathcal{D}) - \lambda_{p^{\ell_{i-1}}}(p^k, \mathcal{D})) \\ &= \sum_{i=1}^{j} (-1)^{i-i_0} p^{\ell_i}, \end{aligned}$$

which proves (ii) by virtue of Proposition 2.3.

The alternating behaviour of $\operatorname{sign}(\lambda_{p^{\ell_j}}(p^k, \mathcal{D}))$ in (iii) is easily shown. In fact, we know by (i) that

$$(3.5)\qquad \lambda_{p^{\ell_j}}(p^k, \mathcal{D}) = \lambda_{p^{\ell_{j-1}}}(p^k, \mathcal{D}) + (-1)^{j-i_0} p^{\ell_j}.$$

Since (ii) implies that

$$(3.6)\qquad \begin{aligned} |\lambda_{p^{\ell_{j-1}}}(p^k, \mathcal{D}))| &\le 1 + \sum_{i=1}^{j-1} p^{\ell_i} \le \sum_{i=0}^{\ell_{j-1}} p^i = \frac{p^{\ell_{j-1}+1} - 1}{p - 1} \\ &< p^{\ell_{j-1}+1} \le p^{\ell_j}, \end{aligned}$$

the last summand in (3.5) dominates the right-hand side of that equation, and thus our assertion follows.

The formulae in (iv) sharpen the rough estimate used in (3.6) and follow by induction. It suffices to prove (iv$_1$), (iv$_2$) and (iv$_3$), since these obviously imply the leading inequality. Formula (iv$_1$) is a consequence of (ii), because the two identities there imply that

$$(3.7)\qquad \lambda_{p^{\ell_1}}(p^k, \mathcal{D}) = \lambda_1(p^k, \mathcal{D}) + (-1)^{1-i_0} p^{\ell_1},$$

where either $\lambda_1(p^k,\mathcal{D}) = 0$, thus $|\lambda_{p^{\ell_1}}(p^k,\mathcal{D})| = p^{\ell_1}$ by formula (3.7), or $\lambda_1(p^k,\mathcal{D}) = (-1)^{i_0}$, when (3.7) yields

$$|\lambda_{p^{\ell_1}}(p^k,\mathcal{D})| = |(-1)^{i_0} + (-1)^{1-i_0}p^{\ell_1}| = p^{\ell_1} - 1.$$

Similar reasoning shows that $|\lambda_{p^{\ell_2}}(p^k,\mathcal{D})| = p^{\ell_2} - p^{\ell_1}$ in case $\lambda_1(p^k,\mathcal{D}) = 0$ and $|\lambda_{p^{\ell_2}}(p^k,\mathcal{D})| = p^{\ell_2} - p^{\ell_1} + 1 < p^{\ell_2}$ in case $\lambda_1(p^k,\mathcal{D}) = (-1)^{i_0}$, and this verifies (iv$_2$). To complete the induction, it remains to prove (iv$_3$) for each $j \geq 3$. By induction hypothesis we may assume that

$$p^{\ell_{j-1}} - p^{\ell_{j-2}} \leq |\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D})| < p^{\ell_{j-1}}. \tag{3.8}$$

Then (i), (iii) and (3.8) together with the trivial bound $p^{\ell_{j-1}} < p^{\ell_j}$ imply

$$\begin{aligned}|\lambda_{p^{\ell_j}}(p^k,\mathcal{D})| &= |\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D}) + (-1)^{j-i_0}p^{\ell_j}|\\ &= \big|(-1)^{j-1-i_0}|\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D})| + (-1)^{j-i_0}p^{\ell_j}\big|\\ &= \big|p^{\ell_j} - |\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D})|\big| = p^{\ell_j} - |\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D})|.\end{aligned}$$

It follows from the upper bound in (3.8) that $|\lambda_{p^{\ell_j}}(p^k,\mathcal{D})| > p^{\ell_j} - p^{\ell_{j-1}}$, and we also obtain $|\lambda_{p^{\ell_j}}(p^k,\mathcal{D})| < p^{\ell_j}$, since $\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D}) \neq 0$ (e.g. by (iii)).

Finally, we prove (v). Assume that $\lambda_{p^{\ell_j}}(p^k,\mathcal{D}) = -\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D})$ for some $j \geq 1$. Then (i) implies $2\lambda_{p^{\ell_j}}(p^k,\mathcal{D}) = (-1)^{j-i_0}p^{\ell_j}$, which yields $p = 2$. Hence $\lambda_{2^{\ell_j}}(2^k,\mathcal{D}) = (-1)^{j-i_0}2^{\ell_j-1}$. Inserting this into (i) gives

$$\begin{aligned}\lambda_{2^{\ell_{j-1}}}(2^k,\mathcal{D}) &= \lambda_{2^{\ell_j}}(2^k,\mathcal{D}) - (-1)^{j-i_0}2^{\ell_j}\\ &= (-1)^{j-i_0}(2^{\ell_j-1} - 2^{\ell_j}) = (-1)^{j+1-i_0}2^{\ell_j-1},\end{aligned}$$

and therefore

$$2^{\ell_{j-1}} \leq 2^{\ell_j-1} = |\lambda_{2^{\ell_{j-1}}}(2^k,\mathcal{D})| \leq \begin{cases} 2^{\ell_{j-1}} & \text{for } j \leq 2,\\ 2^{\ell_{j-1}} - 1 & \text{for } j \geq 3,\end{cases} \tag{3.9}$$

by the leading inequality in (iv), and (iv$_2$) and (iv$_3$), respectively. The contradiction for $j \geq 3$ implies that $j = 1$ or $j = 2$. For these two values of j the inequalities in (3.9) turn into the identities

$$|\lambda_{2^{\ell_{j-1}}}(2^k,\mathcal{D})| = 2^{\ell_j-1} = 2^{\ell_{j-1}},$$

and consequently $\ell_j = \ell_{j-1}+1$. In case $j = 1$ this means that $|\lambda_1(2^k,\mathcal{D})| = 1$ and $\ell_1 = \ell_0+1 = 1$. For $j = 2$ we have $|\lambda_{2^{\ell_1}}(2^k,\mathcal{D})| = 2^{\ell_1}$, thus $\lambda_1(2^k,\mathcal{D}) = 0$ by (iv$_1$), and $\ell_2 = \ell_1 + 1$. This completes the proof of (v). ∎

4. Parameterisation of Spec(ICG($p^k,\mathcal{D}$)). Now we are able to show that the eigenvalues $\lambda_{p^\ell}(p^k,\mathcal{D})$ with $\ell \in \mathcal{L}_\mathcal{D}$ parameterise the set Spec(ICG($p^k,\mathcal{D}$)).

THEOREM 4.1. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set with corresponding exponent set $\mathcal{K}_{\mathcal{D}}$ and leaping set $\mathcal{L}_{\mathcal{D}}$. Then the eigenvalues $\lambda_{p^\ell}(p^k,\mathcal{D}) \in \mathrm{Spec}(\mathrm{ICG}(p^k,\mathcal{D}))$ with $\ell \in \mathcal{L}_{\mathcal{D}}$ are pairwise distinct and satisfy $\{\lambda_{p^\ell}(p^k,\mathcal{D}) : \ell \in \mathcal{L}_{\mathcal{D}}\} = \mathrm{Spec}(\mathrm{ICG}(p^k,\mathcal{D}))$.*

Proof. By Proposition 2.1 and Lemma 3.2(i), we have

$$\begin{aligned}\mathrm{Spec}(\mathrm{ICG}(p^k,\mathcal{D})) &= \{\lambda_j(p^k,\mathcal{D}) : 1 \le j \le p^k\}\\ &= \{\lambda_{p^\ell}(p^k,\mathcal{D}) : 0 \le \ell \le k\}\\ &= \{\lambda_{p^\ell}(p^k,\mathcal{D}) : \ell \in \mathcal{L}_{\mathcal{D}}\}.\end{aligned}$$

Hence it remains to show that the $\lambda_{p^\ell}(p^k,\mathcal{D})$ with $\ell \in \mathcal{L}_{\mathcal{D}}$ are pairwise distinct.

Let $\mathcal{L}_{\mathcal{D}} = \{\ell_0,\ell_1,\ldots,\ell_m\}$ with $0 = \ell_0 < \ell_1 < \cdots < \ell_m$. According to Proposition 2.2 we have

$$|\lambda_{p^{\ell_0}}(p^k,\mathcal{D})| \le |\lambda_{p^{\ell_1}}(p^k,\mathcal{D})| \le \cdots \le |\lambda_{p^{\ell_m}}(p^k,\mathcal{D})|.$$

By definition, $\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D}) \ne \lambda_{p^{\ell_j}}(p^k,\mathcal{D})$ for $1 \le j \le m$, and by Proposition 3.3(v) we know that $\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D}) \ne -\lambda_{p^{\ell_j}}(p^k,\mathcal{D})$ for $1 \le j \le m$ in case $p \ne 2$ and $\lambda_{2^{\ell_{j-1}}}(2^k,\mathcal{D}) \ne -\lambda_{2^{\ell_j}}(2^k,\mathcal{D})$ for $3 \le j \le m$. Combining all these facts we conclude that

$$|\lambda_{p^{\ell_0}}(p^k,\mathcal{D})| < |\lambda_{p^{\ell_1}}(p^k,\mathcal{D})| < \cdots < |\lambda_{p^{\ell_m}}(p^k,\mathcal{D})|$$

for $p \ne 2$, which already shows in this case that the $\lambda_{p^\ell}(p^k,\mathcal{D})$ with $\ell \in \mathcal{L}_{\mathcal{D}}$ are pairwise distinct, and

$$\begin{aligned}|\lambda_{2^{\ell_0}}(2^k,\mathcal{D})| &\le |\lambda_{2^{\ell_1}}(2^k,\mathcal{D})| \le |\lambda_{2^{\ell_2}}(2^k,\mathcal{D})|\\ &< |\lambda_{2^{\ell_3}}(2^k,\mathcal{D})| < \cdots < |\lambda_{2^{\ell_m}}(2^k,\mathcal{D})|\end{aligned} \tag{4.1}$$

if $p = 2$. Since $\lambda_{2^{\ell_0}}(2^k,\mathcal{D}) \ne \lambda_{2^{\ell_1}}(2^k,\mathcal{D})$ and $\lambda_{2^{\ell_1}}(2^k,\mathcal{D}) \ne \lambda_{2^{\ell_2}}(2^k,\mathcal{D})$ by definition, the only remaining obstacle could be $\lambda_{2^{\ell_0}}(2^k,\mathcal{D}) = \lambda_{2^{\ell_2}}(2^k,\mathcal{D})$, hence by (4.1),

$$\lambda_{2^{\ell_0}}(2^k,\mathcal{D}) = -\lambda_{2^{\ell_1}}(2^k,\mathcal{D}) = \lambda_{2^{\ell_2}}(2^k,\mathcal{D}).$$

However, according to Proposition 3.3(v), the first of these equalities implies $|\lambda_1(2^k,\mathcal{D})| = 1$, while the second one implies $\lambda_1(2^k,\mathcal{D}) = 0$. This contradiction completes our proof. ∎

The following result lists the eigenvalues of any $\mathrm{ICG}(p^k,\mathcal{D})$ in increasing order and provides their multiplicities.

THEOREM 4.2. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set with exponent set $\mathcal{K}_{\mathcal{D}}$ and leaping set $\mathcal{L}_{\mathcal{D}} = \{\ell_0,\ell_1,\ldots,\ell_m\}$, where we have $0 = \ell_0 < \ell_1 < \cdots < \ell_m$. Set $\lambda_{p^{\ell_j}} = \lambda_{p^{\ell_j}}(p^k,\mathcal{D})$ for $0 \le j \le m$. Then:*

(i) *If $k-1 \notin \mathcal{K}_{\mathcal{D}}$, then m is even and the increasing sequence*

$$\lambda_{p^{\ell_{m-1}}} < \lambda_{p^{\ell_{m-3}}} < \cdots < \lambda_{p^{\ell_3}} < \lambda_{p^{\ell_1}} < \lambda_{p^{\ell_0}} < \lambda_{p^{\ell_2}} < \lambda_{p^{\ell_4}} < \cdots < \lambda_{p^{\ell_{m-2}}} < \lambda_{p^{\ell_m}}$$

contains all eigenvalues of $\mathrm{ICG}(p^k, \mathcal{D})$*, where*

$$\lambda_{p^{\ell_0}} = \lambda_1 = \begin{cases} 0 & \text{for } k \notin \mathcal{K}_{\mathcal{D}}, \\ 1 & \text{for } k \in \mathcal{K}_{\mathcal{D}}. \end{cases}$$

(ii) *If $k-1 \in \mathcal{K}_{\mathcal{D}}$, then m is odd and the increasing sequence*

$$\lambda_{p^{\ell_{m-1}}} < \lambda_{p^{\ell_{m-3}}} < \cdots < \lambda_{p^{\ell_4}} < \lambda_{p^{\ell_2}} < \lambda_{p^{\ell_0}} < \lambda_{p^{\ell_1}} < \lambda_{p^{\ell_3}} < \cdots < \lambda_{p^{\ell_{m-2}}} < \lambda_{p^{\ell_m}}$$

contains all eigenvalues of $\mathrm{ICG}(p^k, \mathcal{D})$*, where*

$$\lambda_{p^{\ell_0}} = \lambda_1 = \begin{cases} 0 & \text{for } k \in \mathcal{K}_{\mathcal{D}}, \\ -1 & \text{for } k \notin \mathcal{K}_{\mathcal{D}}. \end{cases}$$

The multiplicity $\mu_{\mathrm{ICG}(p^k,\mathcal{D})}(\lambda_{p^{\ell_j}}(p^k, \mathcal{D}))$ of the eigenvalue $\lambda_{p^{\ell_j}}(p^k, \mathcal{D})$ satisfies

$$\mu_{\mathrm{ICG}(p^k,\mathcal{D})}(\lambda_{p^{\ell_j}}(p^k, \mathcal{D})) = \begin{cases} p^{k-\ell_j} - p^{k-\ell_{j+1}} & \text{for } 0 \le j < m, \\ p^{k-\ell_m} & \text{for } j = m. \end{cases} \tag{4.2}$$

Before we prove the theorem above, let us look at some examples.

Example 4.3. (i) First consider $\mathrm{ICG}(3^5, \{1, 3, 3^3\})$, which is our sample graph from Section 2 with $p^k = 3^5$. We have the exponent set $\mathcal{K}_{\mathcal{D}} = \{0, 1, 3\}$ and leaping set $\mathcal{L}_{\mathcal{D}} = \{0, 1, 2, 3, 5\}$, thus $m = 4$. Observe that neither $k - 1 = 4$ nor $k = 5$ belongs to $\mathcal{K}_{\mathcal{D}}$. By Theorem 4.2(i) and formula (2.1), we obtain the complete list

$$\lambda_{3^3} = -21 < \lambda_{3^1} = -3 < \lambda_{3^0} = 0 < \lambda_{3^2} = 6 < \lambda_{3^5} = 222$$

of eigenvalues in increasing order, having respective multiplicities

$$3^2 - 3^0 = 8, \quad 3^4 - 3^3 = 54, \quad 3^5 - 3^4 = 162, \quad 3^3 - 3^2 = 18, \quad 3^0 = 1$$

by (4.2). This is confirmed by taking a look at the spectral vector $\vec{\lambda}(3^5, \{1, 3, 3^3\})$ in Section 2.

(ii) Secondly, let us look at $\mathrm{ICG}(2^6, \{1, 2^2, 2^5\})$. We have $\mathcal{K}_{\mathcal{D}} = \{0, 2, 5\}$ and $\mathcal{L}_{\mathcal{D}} = \{0, 1, 3, 4, 5, 6\}$, thus $m = 5$. Observe that $k - 1 = 5 \in \mathcal{K}_{\mathcal{D}}$, while $k = 6 \notin \mathcal{K}_{\mathcal{D}}$. Now Theorem 4.2(ii) and formula (2.1) provide the list

$$\lambda_{2^5} = -23 < \lambda_{2^3} = -7 < \lambda_{2^0} = -1 < \lambda_{2^1} = 1 < \lambda_{2^4} = 9 < \lambda_{2^6} = 41$$

of all eigenvalues, with respective multiplicities

$$2^1 - 2^0 = 1, \quad 2^3 - 2^2 = 4, \quad 2^6 - 2^5 = 32, \quad 2^5 - 2^3 = 24, \quad 2^2 - 2^1 = 2, \quad 2^0 = 1$$

according to (4.2).

(iii) Finally, we study the disconnected graph $\mathrm{ICG}(5^2, \{5, 5^2\})$ with loops (observe that $\gcd(5, 5^2) \neq 1$ and $5^2 \in \mathcal{D}$). We clearly have $\mathcal{K}_{\mathcal{D}} = \{1, 2\}$ and $\mathcal{L}_{\mathcal{D}} = \{0, 1\}$. By Theorem 4.2(ii), (2.1) and (4.2), the spectrum consists of the two eigenvalues $\lambda_{5^0} = 0$ with multiplicity $5^2 - 5^1 = 20$ and $\lambda_{5^1} = 5$ with multiplicity $5^1 = 5$.

For later use the following table provides the essential information of (i), (ii) and (iii), in particular the eigenvalues $\lambda_{p^{\ell_j}}(p^k, \mathcal{D})$ and their multiplicities $\mu(\lambda_{p^{\ell_j}}(p^k, \mathcal{D})) := \mu_{\mathrm{ICG}(p^k,\mathcal{D})}(\lambda_{p^{\ell_j}}(p^k, \mathcal{D}))$.

Table 1. Eigenvalues and multiplicities of some integral circulant graphs

p^k	$\mathcal{D}$	$\mathcal{K}_{\mathcal{D}}$	$\mathcal{L}_{\mathcal{D}} = \{\ell_0, \ell_1, \dots, \ell_m\}$	$\lambda_{p^{\ell_j}}(p^k, \mathcal{D})$ / $\mu(\lambda_{p^{\ell_j}}(p^k, \mathcal{D}))$ $(j = 0, 1, \dots, m)$					
2^6	$\{1, 2^2, 2^5\}$	$\{0, 2, 5\}$	$\{0, 1, 3, 4, 5, 6\}$	**−1**	**1**	**−7**	**9**	**−23**	**41**
				32	24	4	2	1	1
3^5	$\{1, 3, 3^3\}$	$\{0, 1, 3\}$	$\{0, 1, 2, 3, 5\}$	**0**	**−3**	**6**	**−21**	**222**	
				162	54	18	8	1	
5^2	$\{5, 5^2\}$	$\{1, 2\}$	$\{0, 1\}$	**0**	**5**				
				20	5				

Proof of Theorem 4.2. The values of $\lambda_{p^{\ell_0}} = \lambda_1$ were determined in (3.2) as stated. Proposition 3.3(iii) tells us that $\mathrm{sign}(\lambda_{p^{\ell_1}}) = (-1)^{1-i_0}$, which yields $\lambda_{p^{\ell_1}} < \lambda_{p^{\ell_0}}$ for $k - 1 \notin \mathcal{K}_{\mathcal{D}}$ and $\lambda_{p^{\ell_1}} > \lambda_{p^{\ell_0}}$ for $k - 1 \in \mathcal{K}_{\mathcal{D}}$ according to (3.2) and (3.1). By Propositions 3.3(iii) and 2.2, the sequence $\lambda_{p^{\ell_j}}$ for $j = 1, \dots, m$ is alternating and non-decreasing in absolute value, where strict inequalities are a consequence of Proposition 3.3(i) or Theorem 4.1. This implies the order of the eigenvalues as given in (i) and (ii), respectively. Observe that $\lambda_{p^{\ell_m}} = \lambda_{p^k}$ by Lemma 3.2(i), and λ_{p^k} as the spectral radius of $\mathrm{ICG}(p^k, \mathcal{D})$ is always positive, which yields the parity of m.

It remains to determine the multiplicity of the eigenvalue $\lambda := \lambda_{p^{\ell_j}}(p^k, \mathcal{D})$ for any fixed j with $0 \leq j \leq m$ (recall that we set $\ell_{m+1} = k + 1$). By Lemma 3.2(i) and Theorem 4.1 we find that $\lambda_{p^\ell} = \lambda$ for some $1 \leq \ell \leq k$ if and only if $\ell \in \{\ell_j, \ell_j + 1, \dots, \ell_{j+1} - 1\}$. Therefore Proposition 2.1 implies that

$$\{1 \leq i \leq p^k : \lambda_i = \lambda\} = \big\{1 \leq i \leq p^k : \gcd(i, p^k) \in \{p^{\ell_j}, p^{\ell_j+1}, \dots, p^{\ell_{j+1}-1}\}\big\}.$$

Consequently,

$$\begin{aligned} \mu_{\mathrm{ICG}(p^k,\mathcal{D})}(\lambda) &= \big|\big\{1 \leq i \leq p^k : \gcd(i, p^k) \in \{p^{\ell_j}, p^{\ell_j+1}, \dots, p^{\ell_{j+1}-1}\}\big\}\big| \\ &= \sum_{t=\ell_j}^{\ell_{j+1}-1} |\{1 \leq i \leq p^k : \gcd(i, p^k) = p^t\}| \end{aligned}$$

$$= \sum_{t=\ell_j}^{\ell_{j+1}-1} |\{1 \le i \le p^{k-t} : \gcd(i, p^{k-t}) = 1\}|$$

$$= \sum_{t=\ell_j}^{\ell_{j+1}-1} \varphi(p^{k-t}) = \begin{cases} p^{k-\ell_j} - p^{k-\ell_{j+1}} & \text{for } 0 \le j < m, \\ p^{k-\ell_m} & \text{for } j = m, \end{cases}$$

where φ denotes Euler's totient function. ∎

Proposition 3.3(v) shows that $\mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$ may contain a non-zero eigenvalue together with its negative only in case $p = 2$; Remark 3.4(ii) and Example 4.3(ii) provide some examples. The following result characterises all corresponding divisor sets $\mathcal{D}$ and those pairs of eigenvalues. We remind the reader of the definition $i_0(p^k, \mathcal{D})$ in (3.1) and set

$$\kappa(p^k, \mathcal{D}) := \begin{cases} \max\{\ell \ge 0 : \{k-\ell, k-\ell+1, \ldots, k\} \subseteq \mathcal{K}_{\mathcal{D}}\} & \text{if } k \in \mathcal{K}_{\mathcal{D}}, \\ \max\{\ell \ge 0 : \{k-\ell, k-\ell+1, \ldots, k\} \cap \mathcal{K}_{\mathcal{D}} = \emptyset\} & \text{if } k \notin \mathcal{K}_{\mathcal{D}}. \end{cases} \tag{4.3}$$

THEOREM 4.4. *Let p^k be a prime power and $\mathcal{D} \subseteq D(p^k)$ a divisor set with exponent set $\mathcal{K}_{\mathcal{D}}$ and leaping set $\mathcal{L}_{\mathcal{D}} = \{\ell_0, \ell_1, \ldots, \ell_m\}$, where we have $0 = \ell_0 < \ell_1 < \cdots < \ell_m$. Then the following two statements are equivalent:*

(i) *There is a $\lambda \ne 0$ such that $\pm\lambda \in \mathrm{Spec}(\mathrm{ICG}(p^k, \mathcal{D}))$.*

(ii) *$p = 2$ and $|\lambda| = 2^{\kappa(2^k, \mathcal{D})}$ for a divisor set $\mathcal{D} \subseteq D(2^k)$ satisfying one of the following two conditions:*

(α) *$k - \kappa - 2 \in \mathcal{K}_{\mathcal{D}}$ with $\kappa = \kappa(2^k, \mathcal{D})$ for some $0 \le \kappa \le k-2$ in case $k \in \mathcal{K}_{\mathcal{D}}$,*

(β) *$k - \kappa - 2 \notin \mathcal{K}_{\mathcal{D}}$ with $\kappa = \kappa(2^k, \mathcal{D})$ for some $0 \le \kappa \le k-1$ in case $k \notin \mathcal{K}_{\mathcal{D}}$,*

and, setting $i_0 = i_0(2^k, \mathcal{D})$, we have

$$\lambda_{2^\kappa}(2^k, \mathcal{D}) = (-1)^{i_0} 2^\kappa = (-1)^{i_0}|\lambda|,$$
$$\lambda_{2^{\kappa+1}}(2^k, \mathcal{D}) = (-1)^{i_0+1} 2^\kappa = -\lambda_{2^\kappa}(2^k, \mathcal{D}) = (-1)^{i_0+1}|\lambda|,$$

where $\kappa = \ell_0 = 0$ and $\kappa + 1 = \ell_1 = 1$ in case $\kappa = 0$, and $\kappa = \ell_1$ and $\kappa + 1 = \ell_2$ in case $\kappa > 0$ (when $\lambda_{2^{\ell_0}}(2^k, \mathcal{D}) = \lambda_1(2^k, \mathcal{D}) = 0$).

Proof. (i)⇒(ii): According to Theorem 4.1 there exist $i \ne j$ such that $\lambda = \lambda_{p^{\ell_j}}(p^k, \mathcal{D})$ and $-\lambda = \lambda_{p^{\ell_i}}(p^k, \mathcal{D})$, where we may assume that $i < j$, after interchanging λ and $-\lambda$ if necessary. Hence $\ell_i < \ell_j$, and by Proposition 2.2 it follows that $|\lambda_{p^\ell}(p^k, \mathcal{D})| = |\lambda_{p^{\ell_j}}(p^k, \mathcal{D})|$ for all $\ell_i \le \ell \le \ell_j$, in particular

$$|\lambda_{p^{\ell_j - 1}}(p^k, \mathcal{D})| = |\lambda_{p^{\ell_j}}(p^k, \mathcal{D})| = |\lambda| \ne 0. \tag{4.4}$$

Since $\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D}) \neq \lambda_{p^{\ell_j}}(p^k,\mathcal{D})$ by Proposition 3.3(i), we conclude that $\lambda_{p^{\ell_j}}(p^k,\mathcal{D}) = -\lambda_{p^{\ell_{j-1}}}(p^k,\mathcal{D})$, which implies $p=2$, and moreover $j=1$ with $\ell_1 = 1$ or $j=2$ with $\ell_2=\ell_1+1$ by Proposition 3.3(v). We consider the two cases $j=1$ and $j=2$ separately.

Case 1: $j = 1$ and $\ell_1 = \ell_0 + 1 = 1$. By (4.4) in this case we have $\lambda_1(2^k,\mathcal{D}) = \lambda_{2^{\ell_0}}(2^k,\mathcal{D}) \neq 0$. This implies that $k-1 \notin \mathcal{K}_{\mathcal{D}}$ if and only if $k \in \mathcal{K}_{\mathcal{D}}$ due to (3.2), and more precisely we obtain

$$\lambda_{2^{\ell_0}}(2^k,\mathcal{D}) = \lambda_1(2^k,\mathcal{D}) = (-1)^{i_0} \tag{4.5}$$

by (3.2) and (3.1). Therefore

$$\begin{aligned}\lambda_2(2^k,\mathcal{D}) = \lambda_{2^{\ell_1}}(2^k,\mathcal{D}) &= \lambda_{2^{\ell_0}}(2^k,\mathcal{D}) + (-1)^{1-i_0}2^{\ell_1}\\ &= (-1)^{i_0} + 2\cdot(-1)^{1-i_0} = (-1)^{i_0+1}\end{aligned}$$

according to Proposition 3.3(i). Moreover, $\ell_1 = 1 \in \mathcal{L}_{\mathcal{D}}$ implies $k-2 \in \mathcal{K}_{\mathcal{D}}$ if and only if $k-1 \notin \mathcal{K}_{\mathcal{D}}$ if and only if $k \in \mathcal{K}_{\mathcal{D}}$. Consequently, either $k-2, k \in \mathcal{K}_{\mathcal{D}}$ and $k-1 \notin \mathcal{K}_{\mathcal{D}}$, or $k-2, k \notin \mathcal{K}_{\mathcal{D}}$ and $k-1 \in \mathcal{K}_{\mathcal{D}}$. These two situations are covered by the case $\kappa(2^k,\mathcal{D}) = 0$ of (α) and (β), respectively, and by (4.4) and (4.5) we have $|\lambda| = |\lambda_{2^{\ell_0}}(2^k,\mathcal{D})| = 1 = 2^{\kappa}$.

Case 2: $j = 2$ and $\ell_2 = \ell_1 + 1$. Here by (4.4) and the overall inequality in Proposition 3.3(iv) we have

$$|\lambda_{2^{\ell_1}}(2^k,\mathcal{D})| = |\lambda_{2^{\ell_2}}(2^k,\mathcal{D})| \geq 2^{\ell_2} - 2^{\ell_1} = 2^{\ell_1+1} - 2^{\ell_1} = 2^{\ell_1}. \tag{4.6}$$

Proposition 3.3(iv$_1$) implies that we have $\lambda_1(2^k,\mathcal{D})$=0 and $|\lambda_{2^{\ell_1}}(2^k,\mathcal{D})|$=$2^{\ell_1}$. From $\lambda_1(2^k,\mathcal{D}) = 0$ we deduce by (3.2) that either $k-1$ and k both lie in $\mathcal{K}_{\mathcal{D}}$, or none of them does. In both cases $\kappa(2^k,\mathcal{D}) \geq 1$. Now assume that $\kappa(2^k,\mathcal{D}) = \kappa$ for some $\kappa \geq 1$, i.e. $k-\kappa-1 \notin \mathcal{K}_{\mathcal{D}}$ if and only if $k-\kappa \in \mathcal{K}_{\mathcal{D}}$ (if and only if $k \in \mathcal{K}_{\mathcal{D}}$), thus $\ell_1 = \kappa \in \mathcal{L}_{\mathcal{D}}$. Since $\ell_2 = \ell_1+1 = \kappa+1 \in \mathcal{L}_{\mathcal{D}}$, we also have $k-\kappa-2 \in \mathcal{K}_{\mathcal{D}}$ if and only if $k-\kappa-1 \notin \mathcal{K}_{\mathcal{D}}$ if and only if $k \in \mathcal{K}_{\mathcal{D}}$. These situations are exactly the ones covered by the cases $\kappa \geq 1$ in (α) and (β), respectively. By (4.4) and (4.6) we finally obtain $|\lambda| = |\lambda_{2^{\ell_1}}(2^k,\mathcal{D})| = 2^{\ell_1} = 2^{\kappa}$.

(ii)⇒(i): This implication obviously holds by the facts gathered in part (ii). ∎

5. The spectrum of ICG($n,\mathcal{D}$) for arbitrary n and multiplicative $\mathcal{D}$. The idea to consider integral circulant graphs ICG($n,\mathcal{D}$) with *multiplicative divisor sets* $\mathcal{D}$ was introduced and applied by Le and the author in [12] and [13], and again used by the author in [14]. For a positive integer d and a number p in the set $\mathbb{P}$ of all primes, we denote by $e_p(d)$ the order of p in d. A non-empty finite set $\mathcal{D}$ of positive integers is called *multiplicative* if $\mathcal{D} = \prod_{p\in\mathbb{P}} \mathcal{D}_p$, where $\mathcal{D}_p := \{p^{e_p(d)} : d \in \mathcal{D}\}$ for each prime p, and the product of sets $D_1, \dots, D_t$ of positive integers is defined as

$$\prod_{i=1}^{t} D_i := \{d_1 \cdot \ldots \cdot d_t : d_i \in D_i\}.$$

Observe that $\mathcal{D}_p \neq \{1\}$ only for those finitely many primes dividing at least one of the $d \in \mathcal{D}$. Hence $\prod_{p \in \mathbb{P}} \mathcal{D}_p$ can be regarded as a finite product for any finite set $\mathcal{D}$.

THEOREM 5.1. *Given a positive integer n, let $n = p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$ be its prime factorisation. For $1 \leq i \leq r$ let $\mathcal{D}_i \subseteq D(p_i^{k_i})$ be a divisor set with exponent set $\mathcal{K}_i := \mathcal{K}_{\mathcal{D}_i}$ and leaping set $\mathcal{L}_i := \mathcal{L}_{\mathcal{D}_i}$, and define the multiplicative divisor set $\mathcal{D} := \mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_r \subseteq D(n)$.*

(i) *We have*

$$\begin{aligned} \text{Spec}(\text{ICG}(n, \mathcal{D})) &= \prod_{i=1}^{r} \text{Spec}(\text{ICG}(p_i^{k_i}, \mathcal{D}_i)) \\ &= \prod_{i=1}^{r} \{\lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_i) : \ell_i \in \mathcal{L}_i\}. \end{aligned} \tag{5.1}$$

(ii) *For each $\lambda \in \text{Spec}(\text{ICG}(n, \mathcal{D}))$, let*

$$L(\lambda) := \Big\{(\ell_1, \dots, \ell_r) \in \mathcal{L}_1 \times \cdots \times \mathcal{L}_r : \prod_{i=1}^{r} \lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_i) = \lambda\Big\}.$$

Then the multiplicity $\mu_{\text{ICG}(n,\mathcal{D})}(\lambda)$ of the eigenvalue λ satisfies

$$\mu_{\text{ICG}(n,\mathcal{D})}(\lambda) = \sum_{(\ell_1,\dots,\ell_r) \in L(\lambda)} \prod_{i=1}^{r} \mu_{\text{ICG}(p_i^{k_i}, \mathcal{D}_i)}(\lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_i)),$$

which can be evaluated by (4.2).

Proof. [12, Proposition 4.1] states that

$$\lambda_j(n, \mathcal{D}) = \prod_{i=1}^{r} \lambda_{j \bmod p_i^{k_i}}(p_i^{k_i}, \mathcal{D}_i).$$

By Proposition 2.1, Lemma 3.2(i) and Theorem 4.1, for $1 \leq j \leq n$,

$$\begin{aligned} \lambda_j(n, \mathcal{D}) &= \prod_{i=1}^{r} \lambda_{j \bmod p_i^{k_i}}(p_i^{k_i}, \mathcal{D}_i) \\ &= \prod_{i=1}^{r} \lambda_{\gcd(j, p_i^{k_i})}(p_i^{k_i}, \mathcal{D}_i) = \prod_{i=1}^{r} \lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_i) \end{aligned} \tag{5.2}$$

with suitable $\ell_i \in \mathcal{L}_i$ for $1 \leq i \leq r$. This shows in particular that

$$\text{Spec}(\text{ICG}(n, \mathcal{D})) \subseteq \prod_{i=1}^{r} \text{Spec}(\text{ICG}(p_i^{k_i}, \mathcal{D}_i)). \tag{5.3}$$

By the Chinese remainder theorem there exists a ring isomorphism

$$\psi_n : \mathbb{Z}/p_1^{k_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_r^{k_r}\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}, \qquad (j_1, \ldots, j_r) \mapsto j,$$

such that $j \equiv j_i \bmod p_i^{k_i}$ for $1 \le i \le r$. By the first equality of (5.2) this implies that

$$\prod_{i=1}^{r} \lambda_{j_i}(p_i^{k_i}, \mathcal{D}_i) = \lambda_{\psi_n(j_1,\ldots,j_r)}(n, \mathcal{D}) \tag{5.4}$$

for any given $(j_1, \ldots, j_r) \in \bigtimes_{i=1}^{r}\{1, \ldots, p_i^{k_i}\}$. This confirms that

$$\prod_{i=1}^{r} \mathrm{Spec}(\mathrm{ICG}(p_i^{k_i}, \mathcal{D}_i)) \subseteq \mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})),$$

and together with (5.3) the first identity of (5.1) is proven. The second equality in (5.1) is an immediate consequence of Theorem 4.1; this completes the proof of (i).

In fact, (5.4) not only implies the product formula (5.1) for sets of spectra, but it even allows one to determine the multiplicity of each eigenvalue λ of $\mathrm{ICG}(n, \mathcal{D})$. According to (5.4) and the definition of $L(\lambda)$ we have

$$\begin{aligned}
\mu_{\mathrm{ICG}(n,\mathcal{D})}(\lambda) &= \Big|\Big\{(j_1, \ldots, j_r) \in \bigtimes_{i=1}^{r}\{1, \ldots, p_i^{k_i}\} : \lambda_{\psi_n(j_1,\ldots,j_r)}(n, \mathcal{D}) = \lambda\Big\}\Big| \\
&= \Big|\Big\{(j_1, \ldots, j_r) \in \bigtimes_{i=1}^{r}\{1, \ldots, p_i^{k_i}\} : \prod_{i=1}^{r} \lambda_{j_i}(p_i^{k_i}, \mathcal{D}_i) = \lambda\Big\}\Big| \\
&= \sum_{(\ell_1,\ldots,\ell_r)\in L(\lambda)} \prod_{i=1}^{r} |\{1 \le j_i \le p_i^{k_i} : \lambda_{j_i}(p_i^{k_i}, \mathcal{D}_i) = \lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_i)\}| \\
&= \sum_{(\ell_1,\ldots,\ell_r)\in L(\lambda)} \prod_{i=1}^{r} \mu_{\mathrm{ICG}(p_i^{k_i},\mathcal{D}_i)}(\lambda_{p_i^{\ell_i}}(p_i^{k_i}, \mathcal{D}_i)),
\end{aligned}$$

which completes the proof of the theorem. ■

Example 5.2. (i) Let $n = 2^6 \cdot 3^5$ and $\mathcal{D} = \{1, 3, 4, 12, 27, 32, 96, 108, 864\}$. Apparently $\mathcal{D} = \{1, 4, 32\} \cdot \{1, 3, 27\}$, hence $\mathcal{D} \subseteq D(n)$ is a multiplicative divisor set. By Theorem 5.1(i) and Example 4.3(i),(ii) (see Table 1) we obtain

$$\begin{aligned}
&\mathrm{Spec}(\mathrm{ICG}(n, \mathcal{D})) \\
&\quad = \{-5106, -1554, -861, -222, -189, -138, -123, -42, -27, \\
&\qquad -21, -6, -3, 0, 3, 6, 21, 54, 69, 147, 222, 246, 483, 1998, 9102\}.
\end{aligned}$$

Theorem 5.1(ii) and Table 1 at the end of Example 4.3 yield Table 2, which exhibits all eigenvalues (in bold print) of

$$S_{2^6} := \mathrm{Spec}(\mathrm{ICG}(2^6, \{1,4,32\})),$$
$$S_{3^5} := \mathrm{Spec}(\mathrm{ICG}(3^5, \{1,3,27\}))$$

and Spec(ICG($n, \mathcal{D}$)) with the corresponding multiplicities (in brackets).

Table 2. Eigenvalues and multiplicities of $\mathrm{ICG}(2^6 \cdot 3^5, \{1,4,32\} \cdot \{1,3,27\})$

S_{2^6} \ S_{3^5}	**0** (162)	**−3** (54)	**6** (18)	**−21** (8)	**222** (1)
−1 (32)	**0** (5184)	**3** (1728)	**−6** (576)	**21** (256)	**−222** (32)
1 (24)	**0** (3888)	**−3** (1296)	**6** (432)	**−21** (192)	**222** (24)
−7 (4)	**0** (648)	**21** (216)	**−42** (72)	**147** (32)	**−1554** (4)
9 (2)	**0** (324)	**−27** (108)	**54** (36)	**−189** (16)	**1998** (2)
−23 (1)	**0** (162)	**69** (54)	**−138** (18)	**483** (8)	**−5106** (1)
41 (1)	**0** (162)	**−123** (54)	**246** (18)	**−861** (8)	**9102** (1)

In particular, the multiplicity of the eigenvalue 0 in ICG($n, \mathcal{D}$) equals

$$\mu_{\mathrm{ICG}(n,\mathcal{D})}(0) = \sum_{(\ell,\ell')\in L(0)} \mu_{\mathrm{ICG}(2^6,\{1,4,32\})}(\lambda_{2^\ell}(2^6,\{1,4,32\})) \times \mu_{\mathrm{ICG}(3^5,\{1,3,27\})}(\lambda_{3^{\ell'}}(3^5,\{1,3,27\}))$$
$$= (32+24+4+2+1+1)\cdot 162 = 10368.$$

By the definition given in (4.3), we have $\kappa(2^6,\{1,4,32\}) = 0$ and $\kappa(3^5,\{1,3,27\}) = 1$. According to [15, Theorem 4.1], $\mu_{\mathrm{ICG}(n,\mathcal{D})}(0)$ could be determined directly by

$$\mu_{\mathrm{ICG}(n,\mathcal{D})}(0) = 2^6\cdot 3^5\left(1 - \frac{1}{2^{\kappa(2^6,\{1,4,32\})}\cdot 3^{\kappa(3^5,\{1,3,27\})}}\right)$$
$$= 2^6\cdot 3^5\left(1-\frac{1}{2^0\cdot 3^1}\right) = 10368,$$

and this is the dimension of the kernel of ICG($n, \mathcal{D}$).

Observe that $\mathrm{ICG}(2^6 \cdot 3^5, \{1,4,32\} \cdot \{1,3,27\})$ has a non-zero eigenvalue obtained as a product of eigenvalues of S_{2^6} and S_{3^5} in two different ways, namely $21 = (-1)\cdot(-21) = (-7)\cdot(-3)$, hence

$$\mu_{\mathrm{ICG}(n,\mathcal{D})}(21) = 32\cdot 8 + 4\cdot 54 = 472.$$

(ii) Let $n = 2^6\cdot 3^5\cdot 5^2$ and $\mathcal{D} = \{1,4,32\}\cdot\{1,3,27\}\cdot\{5,25\}$. By Example 4.3(iii), Spec(ICG($5^2, \{5,25\}$)) consists of the two eigenvalues 0 with multiplicity 20 and 5 with multiplicity 5. Then Theorem 5.1(i) implies that

$$\mathrm{Spec}(\mathrm{ICG}(n,\mathcal{D})) = \{5\cdot\lambda : \lambda \in \mathrm{Spec}(\mathrm{ICG}(2^6\cdot 3^5, \{1,4,32\}\cdot\{1,3,27\}))\},$$

and for $5\cdot\lambda$, $\lambda\neq 0$, by Theorem 5.1(ii) we have

$$\mu_{\mathrm{ICG}(n,\mathcal{D})}(5\cdot\lambda) = 5\cdot\mu_{\mathrm{ICG}(2^6\cdot 3^5,\{1,4,32\}\cdot\{1,3,27\})}(\lambda),$$

where the multiplicities on the right-hand side have already been calculated in part (i) of this example.

To complete the picture, we determine the multiplicity of the eigenvalue 0. Since

$$\kappa(2^6,\{1,4,32\}) = 0, \qquad \kappa(3^5,\{1,3,27\}) = \kappa(5^2,\{5,25\}) = 1,$$

according to [15, Theorem 4.1] we obtain

$$\begin{aligned}\mu_{\mathrm{ICG}(n,\mathcal{D})}(0) &= 2^6\cdot 3^5\cdot 5^2\left(1-\frac{1}{2^{\kappa(2^6,\{1,4,32\})}\cdot 3^{\kappa(3^5,\{1,3,27\})}\cdot 5^{\kappa(5^2,\{5,25\})}}\right)\\ &= 2^6\cdot 3^5\cdot 5^2\left(1-\frac{1}{2^0\cdot 3^1\cdot 5^1}\right) = 362880.\end{aligned}$$

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