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A note on the distribution of normalized prime gaps

by

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Dedicated to the 75th birthday of Robert Tijdeman

1. Introduction. The Prime Number Theorem implies that the average value of

$$d_n = p_{n+1} - p_n \tag{1.1}$$

is $(1 + o(1)) \log N$ if $n \in [N, 2N]$, for example, where $\mathbb{P} = \{p_i\}_{i=1}^{\infty}$ is the set of primes. This motivates the investigation of the sequence $\{d_n/\log p_n\}_{n=1}^{\infty}$ or $\{d_n/\log n\}_{n=1}^{\infty}$ (which is asymptotically the same). Erdős formulated the conjecture that the set of its limit points is

$$J = \left\{ \frac{d_n}{\log n} \right\}' = [0, \infty]. \tag{1.2}$$

He writes in [Erd 1955]: "It seems certain that $d_n/\log n$ is everywhere dense in $(0, \infty)$" (after mentioning the conjecture $\liminf_{n\to\infty} d_n/\log n = 0$). The fact that $\infty \in J$ was proved already in 1931 by Westzynthius [Wes 1931].

In 2005 Goldston, Yıldırım and the author [GPY 2006], [GPY 2009] showed $0 \in J$, which is the hitherto only concrete known element of J. On the other hand, already 60 years ago Ricci [Ric 1954] and Erdős [Erd 1955] proved (simultaneously and independently) that J has a positive Lebesgue measure. A partial result towards the full conjecture (1.2) was shown by the author in [Pin 2016]: there exists an ineffective constant c such that

$$[0, c] \subset J. \tag{1.3}$$

W. Banks, T. Freiberg and J. Maynard [BFM 2016] have recently proved that for any sequence of $k = 9$ nonnegative real numbers $\beta_1 \le \cdots \le \beta_k$ we

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have

$$\{\beta_j - \beta_i : 1 \le i < j \le k\} \cap J \neq \emptyset. \tag{1.4}$$

As a corollary they found that if λ denotes the Lebesgue measure, then

$$\lambda([0,T] \cap J) \ge (1 + o(1))T/8. \tag{1.5}$$

2. Generalization and improvement. The purpose of this note is to generalize this result for the case when d_n is normalized by a rather general function $f(n) \to \infty$, that is, to consider instead of J the more general case of the set of limit points

$$J_f = \left\{ \frac{d_n}{f(n)} \right\}' \tag{2.1}$$

where we require f to belong to the class $\mathcal{F}$ below.

Definition. A function $f(n) \nearrow \infty$ belongs to $\mathcal{F}$ if for any $\varepsilon > 0$,

$$(1-\varepsilon)f(N) \le f(n) \le (1+\varepsilon)f(N) \quad \text{for } n \in [N, 2N],\ N > N_0, \tag{2.2}$$

and further

$$f(n) \ll \frac{\log n \log_2 n \log_4 n}{(\log_3 n)^2} \tag{2.3}$$

where $\log_\nu n$ denotes the ν-times iterated logarithm.

The first condition means that $f(n)$ is slowly oscillating, while the second one that it does not grow more quickly than the Erdős–Rankin function, which until the recent dramatic new developments by Maynard [May 2016], Ford–Green–Konyagin–Tao [FGKT 2016], and Ford–Green–Konyagin–Maynard–Tao [FG$^+$ 2018] described the largest known gap between consecutive primes. The improvement means that it is sufficient to work with $k = 5$ values of β_i in (1.4) instead of $k = 9$ values. As an immediate corollary we obtain a lower bound $(1 + o(1))T/4$ instead of (1.5) for the Lebesgue measure of the more general set $[0,T] \cap J_f$.

Theorem 1. *If $f \in \mathcal{F}$, then for any sequence of $k = 5$ nonnegative real numbers $\beta_1 \le \dots \le \beta_k$ we have*

$$\{\beta_j - \beta_i : 1 \le i < j \le k\} \cap J_f \neq \emptyset. \tag{2.4}$$

Corollary 2. *If $f \in \mathcal{F}$, then*

$$\lambda([0,T] \cap J_f) \ge (1 + o(1))T/4. \tag{2.5}$$

Remark. The author is aware that the same assertion appeared in a preprint of Jacques Benatar (arXiv:1505.03104v1) about five months earlier than the arXiv preprint of the author. However his work, containing several important true assertions, seems to be incorrect at this point, namely in the proof of his Proposition 1.2. In his crucial inequality (5.3), in the lower estimate of the quantity on the LHS there seems to be an incorrect extra condition $h \neq h'$ for the summation in the second factor on the RHS which appears with the negative exponent -1. If this condition is removed, then the corresponding factor will be larger than the quantity in the second line of his (5.2), its reciprocal smaller and therefore the lower bound for the LHS smaller, i.e. weaker. This unfortunately destroys the proof of Proposition 5.2 which should prove Proposition 1.2 about the measure of the limit points of the normalized prime gaps dealt with in the present work.

In an earlier work [Pin 2016] we showed that for any $f \in \mathcal{F}$ there exists an ineffective constant c_f such that $[0, c_f] \subset J_f$. We further remark that since β_i can be arbitrarily large, Theorem 1 includes the improvement of the Erdős–Rankin function given in (2.3) proved recently in [May 2016] and [FGKT 2016]. (We note that the proof uses some refinement of the argument of [May 2016], so it does not represent an independent new proof.)

In connection with the original Erdős conjecture for general $f \in \mathcal{F}$ we remark that it was proved in [Pin 2014ar] that the conjecture is in some sense valid for almost all functions $f \in \mathcal{F}$. More precisely, it was shown in [Pin 2014ar] that if $\{f_n\}_{n=1}^{\infty} \subset \mathcal{F}$ with $\lim_{x\to\infty} f_{n+1}(x)/f_n(x) = \infty$, then

$$J_{f_n} = [0, \infty] \tag{2.6}$$

apart from at most 98 exceptional functions f_n.

3. Proof. The generalization to $f \in \mathcal{F}$ instead of the single $f = \log n$ runs completely analogously to the proofs in [Pin 2014ar]. Therefore we will only describe how to improve $k = 9$ to $k = 5$ in Theorem 1.

This result will follow from the following improvement of Theorem 4.3 of their work. Let $\mathcal{Z}$ be given by (4.8) of [BFM 2016].

Theorem 3. *Let m, k and $\varepsilon = \varepsilon(k)$ be fixed. If k is a sufficiently large multiple of $4m+1$ and ε is sufficiently small, then there is some $N(m,k,\varepsilon)$ such that the following holds for $N \geq N(m,k,\varepsilon)$ with*

$$w = \varepsilon \log N, \qquad W = \prod_{p \leq w,\, p \nmid \mathcal{Z}} p. \tag{3.1}$$

Let $\mathcal{H} = \{h_1, \ldots, h_k\}$ be an admissible k-tuple (that is, it does not cover all residue classes modulo p for any prime p) such that

$$0 \leq h_1 < \cdots < h_k \leq N \tag{3.2}$$

and

$$p \,\Big|\, \prod_{1 \leq i < j \leq k} (h_j - h_i) \ \Rightarrow\ p \leq w. \tag{3.3}$$

Let $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_{4m+1}$ *be a partition of* $\mathcal{H}$ *into* $4m+1$ *sets of equal size and let* b *be an integer with*

$$\Big(\prod_{i=1}^{k}(b+h_i), W\Big) = 1. \tag{3.4}$$

Then there is some $n \in (N, 2N]$ *with* $n \equiv b \pmod{W}$ *and some set of distinct indices* $\{i_1, \dots, i_{m+1}\} \subseteq \{1, \dots, 4m+1\}$ *such that*

$$\big|\mathcal{H}_i(n) \cap \mathbb{P}\big| \ge 1 \quad \textit{for all } i \in \{i_1, \dots, i_{m+1}\}. \tag{3.5}$$

Remark. The original analogous statement (4.20) of [BFM 2016] should have been stated with ≥ 1 instead of $= 1$ (oral communication of James Maynard). This form is enough to imply their Corollary 1.2 or our Corollary 2.

The needed change in the deduction of Theorem 4.3 is the following. First, using $4m+1 \mid k$ we write

$$\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_{4m+1} \tag{3.6}$$

as a partition of $\mathcal{H}$ into $4m+1$ sets each of size $k/(4m+1)$. Instead of the quantity S in [BFM 2016] we introduce, with a new parameter $\alpha = \alpha(m)$, the new quantity $S(\alpha)$, where α will be chosen relatively small (we will see that $\alpha(m) = 1/(5m)$ is a good choice, for example). Thus, with a further parameter β, let

$$S(\alpha,\beta) = \sum_{N<n\le 2N} \Big(\sum_{i=1}^{k} 1_{\mathbb{P}}(n+h_i) - \beta m - \alpha \sum_{j=1}^{4m+1} \sum_{\substack{h,h'\in\mathcal{H}_j\\ h\ne h'}} 1_{\mathbb{P}}(n+h)1_{\mathbb{P}}(n+h')\Big) \times \Big(\sum_{\substack{d_1,\dots,d_k\\ d_i\mid n+h_i\,\forall i}} \lambda_{d_1,\dots,d_k}\Big)^2 \tag{3.7}$$

where under the summation sign we consider unordered pairs $h, h' \in \mathcal{H}_j$. Let

$$\beta = \beta(\alpha) = \max_{\ell\in\mathbb{Z}^+}\bigg(\ell - \alpha\binom{\ell}{2}\bigg). \tag{3.8}$$

Then for any given n the contribution of any set $\mathcal{H}_j$ to $S(\alpha,\beta) = S(\alpha)$ is at most β, so if we have for every $n \in (N, 2N]$ at most m sets of the form $\mathcal{H}_j$ with

$$\sum_{h\in\mathcal{H}_j} 1_{\mathbb{P}}(n+h) > 0, \tag{3.9}$$

then

$$S(\alpha) \le 0. \tag{3.10}$$

In contrast to the choice $\varrho \in (0,1)$ and $\delta\varrho \log k = 2m$ in [BFM 2016], we will now choose $\delta\varrho\log k$ much larger:

$$\delta\varrho\log k = u := \frac{4m+1}{4\alpha}, \qquad \alpha = \frac{1}{5m}, \qquad \varrho \in (0,1). \tag{3.11}$$

This implies, by an easy calculation,

$$\beta = \frac{5m+1}{2}. \tag{3.12}$$

Using the same argument for the estimation of the negative double sum as in [BFM 2016] we obtain a choice of a function F such that

$$\begin{aligned} S(\alpha) &\geq \frac{N}{W} B^{-k} I_k(F) \Bigg(\sum_{i=1}^{k} \frac{u}{k}(1+O(\gamma)) - \beta m \\ &\qquad - 4\alpha \sum_{j=1}^{4m+1} \sum_{\substack{h,h'\in\mathcal{H}_j \\ h\neq h'}} \frac{u^2}{k^2}(1+O(\delta+\gamma)) \Bigg) \\ &= \frac{N}{W} B^{-k} I_k(F) \Bigg(u(1+O(\gamma)) - \frac{(5m+1)m}{2} \\ &\qquad - \frac{4(4m+1)}{5m} \binom{k/(4m+1)}{2} \frac{u^2}{k^2}(1+O(\delta+\gamma)) \Bigg). \end{aligned} \tag{3.13}$$

By the above choice of the parameters in (3.11) we will deduce from (3.13) with $\gamma = (\log k)^{-1/2}$ that

$$\begin{aligned} \frac{S(\alpha) W B^k}{N I_k(F)} &\geq \frac{5m(4m+1)(1+O(\gamma))}{4} - \frac{(5m+1)m}{2} \\ &\qquad - \frac{5m(4m+1)(1+O(\delta+\gamma))}{8} \\ &= \frac{m(1+O(m(\delta+\gamma)))}{8} > 0, \end{aligned} \tag{3.14}$$

which contradicts (3.10).

In order to see the validity of the last inequality in (3.14) we can choose

$$m < (\log k)^{1/4} \ \Leftrightarrow\ \delta \asymp \frac{m^2}{\log k} \ll \frac{1}{\sqrt{\log k}}, \tag{3.15}$$

which implies that the quantities $m\gamma$ and $m\delta$ can be chosen arbitrarily small if k is sufficiently large.

This contradiction proves Theorem 3, and consequently our Theorem 1. Corollary 2 follows from it in the same way as Corollary 1.2 from Theorem 1.1 in [BFM 2016].

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