APPLICATIONES MATHEMATICAE
Online First version

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ANALYSIS OF A DYNAMIC CONTACT PROBLEM WITH FRICTION, DAMAGE AND ADHESION

Abstract. We study a dynamic contact problem for viscoelastic materials with damage. The contact is modelled with Tresca's friction law and a first order differential equation which describes adhesion effect of contact surfaces; the damage of the material is described by a function whose evolution is governed by a parabolic inclusion. Under appropriate assumptions, we provide a variational formulation of the mechanical problem and establish the existence and uniqueness of a weak solution.

1. Introduction. Processes of adhesion are important in many industrial settings, especially when a glue is added to prevent relative motion of the surfaces. For this industrial interest adhesive contact problems have recently received increased attention in the mathematical literature. An early attempt to study models of contact with adhesion was made in [8]–[10]. Analysis and numerical simulations of frictionless contact problems with adhesion can be found in [4], [5], [15], [16]. In [4] the dynamic frictionless adhesive contact problem, with normal compliance condition, was modelled and analyzed; a fully discrete scheme was introduced and some numerical examples were included. The unilateral quasistatic contact problem with local friction and adhesion was studied in [5]; an existence result, for a friction coefficient small enough, was established. The main new idea in these papers is the introduction of the surface state variable β, the bonding field, which has values between 0 and 1 and measures the fractional density of

2010 *Mathematics Subject Classification*: 47J20, 49J40, 74M10, 74M15.
Key words and phrases: adhesion, dynamic process, viscoelastic materials, damage, Tresca's friction law.
Received 15 July 2016; revised 17 May 2018.
Published online 14 December 2018.

DOI: 10.4064/am2312-6-2018

active bonds. When $\beta = 0$ there are no active bonds; when $\beta = 1$ all the bonds are active; when $0 < \beta < 1$ partial adhesion takes place.

On the other hand, material damage, which may be caused by the growth of internal microcracks, appears in many applications of solids mechanics. Since it directly reduces the usefulness of the structures or components, the subject is important in design engineering. General models for damage were derived in [11], [12] from the virtual power principle. In these papers the evolution of the microscopic cracks which cause the damage is determined by a parabolic inclusion. In an isotropic and homogeneous elastic material, the damage function is defined by

$$\zeta = \mathcal{E}_{\mathrm{eff}}/\mathcal{E}_{\mathrm{Y}},$$

where $\mathcal{E}_{\mathrm{Y}}$ is the Young modulus of the original material and $\mathcal{E}_{\mathrm{eff}}$ is the current one. It follows that the damage function ζ has values between 0 and 1. Recent modeling, analysis and numerical simulations of contact problems which include the evolution of material damage can be found in [3], [13], [17], [18], [22], [23] and references therein.

This paper is a continuation and an extension of [25]. There, the constitutive law was assumed to be viscoelastic; the quasistatic adhesive contact problem with Tresca's friction law was investigated and the rate of the bonding field was assumed to be irreversible. Here, the novelty consists in dealing with a dynamic contact problem with Tresca's friction law in which both adhesion and damage are taken into account. Moreover, the adhesion field is described by a general function which may change sign and allows rebonding after debonding, and the process is assumed to be with memory so that it depends on the bonding history. Also, we assume that the mechanical properties of the body are described by a nonlinear viscoelastic constitutive law with damage, such that the damage does not affect the viscosity of the material, but only its elastic behaviour. We derive a variational formulation of the mechanical problem for which we prove the existence of a unique weak solution, and obtain regularity results for the solution. The proof is based on the regularization method (see e.g. [6]), nonlinear evolution equations with monotone operators, a version of the Cauchy–Lipschitz theorem and the Banach fixed point theorem.

The rest of this paper is organized as follows. In Section 2 we present the notation and some preliminaries. Section 3 is dedicated to describing the mechanical problem and deriving its variational formulation. The main existence and uniqueness theorem is established in Section 4.

2. Notation and preliminaries. Here we introduce the notation we shall use and some preliminary materials. Let Ω be a bounded domain, in the numerical space $\mathbb{R}^d$ (d=2, 3) of variables $x = (x_1, \dots, x_d)$, with a Lipschitz boundary Γ. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$u \cdot v = \sum_{i=1}^{d} u_i v_i, \qquad |u| = \sqrt[2]{u \cdot u}, \quad \forall u, v \in \mathbb{R}^d;$$
$$\sigma \cdot \xi = \sum_{1 \le i,j \le d} \sigma_{ij} \xi_{ij}, \quad |\sigma| = \sqrt[2]{\sigma \cdot \sigma}, \quad \forall \sigma, \xi \in \mathbb{S}^d.$$

We introduce the spaces

$$H = L^2(\Omega; \mathbb{R}^d), \qquad \mathcal{Q} = L^2(\Omega; \mathbb{S}^d),$$
$$H_1 = \{u \in H;\ \varepsilon(u) \in \mathcal{Q}\},$$
$$\mathcal{Q}_1 = \{\sigma \in \mathcal{Q};\ \mathrm{Div}\, \sigma \in H\}.$$

Here and below, $\varepsilon : H_1 \to \mathcal{Q}$ is the deformation operator, defined by

$$\varepsilon(u) = \tfrac{1}{2}(\nabla u + (\nabla u)^T), \qquad \forall u \in H_1,$$

where $(\nabla u)^T$ is the transpose of the matrix ∇u which is defined by

$$\nabla u = \left(\frac{\partial u_i}{\partial x_j} \right)_{1 \le i,j \le d},$$

and $\mathrm{Div} : \mathcal{Q}_1 \to H$ is the divergence operator, defined by

$$\mathrm{Div}\, \sigma = ((\mathrm{Div}\, \sigma)_i)_{1 \le i \le d} = \left(\sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j} \right)_{1 \le i \le d}, \qquad \forall \sigma \in \mathcal{Q}_1.$$

Note that H, $\mathcal{Q}$, H_1, $\mathcal{Q}_1$ are Hilbert spaces equipped with the respective canonical inner products

$$(u, v)_H = \int_\Omega u \cdot v \, dx, \qquad (\sigma, \tau)_{\mathcal{Q}} = \int_\Omega \sigma \cdot \tau \, dx,$$
$$(u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{Q}},$$
$$(\sigma, \tau)_{\mathcal{Q}_1} = (\mathrm{Div}\, \sigma, \mathrm{Div}\, \tau)_H + (\sigma, \tau)_{\mathcal{Q}},$$

and the associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{Q}}$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{Q}_1}$.

Let $H_\Gamma = H^{1/2}(\Gamma; \mathbb{R}^d)$ and let $\tilde{\gamma} : H_1 \to H_\Gamma$ be the trace map. For every $v \in H_1$ we also write v for the trace $\tilde{\gamma}(v)$ of v on Γ, and for all $v \in H_1$ we denote by v_ν and v_τ the normal and tangential components of v on the boundary Γ:

$$v_\nu = v \cdot \nu, \qquad v_\tau = v - v_\nu \nu \quad \text{on } \Gamma,$$

here and below ν represents the unit outward normal vector to Γ. In a similar manner, the normal and tangential components of a regular (say C^1) tensor field σ are defined by

$$\sigma_\nu = \sigma\nu \cdot \nu, \qquad \sigma_\tau = \sigma\nu - \sigma_\nu \nu \quad \text{on } \Gamma;$$

moreover the following Green's formula holds:

$$(\mathrm{Div}\,\sigma, v)_H + (\sigma, \varepsilon(v))_{\mathcal{Q}} = \int_\Gamma \sigma\nu \cdot v\, da, \qquad \forall v \in H_1, \tag{1}$$

where da is the surface measure element.

Next, for every real Banach space $(X, \|\cdot\|_X)$ and $T > 0$, we denote by $C([0,T];X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions from $[0,T]$ to X, and we use the standard notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $p \in [1,\infty]$ and $k \geq 1$. We need the following result (see, e.g., [24, p. 60]).

Proposition 1. *Let $(X, \|\cdot\|_X)$ be a real Banach space and let $F(t,\cdot): X \to X$ be an operator defined a.e. on $(0,T)$ which satisfies*

(i) *there exists $L_F > 0$ such that $\|F(t,z) - F(t,y)\|_X \leq L_F\|z - y\|_X$, $\forall z, y \in X$, a.e. $t \in (0,T)$;*
(ii) *there exists $p \in [1,\infty]$ such that the mapping $t \mapsto F(t,z)$ is in $L^p(0,T;X)$ for each $z \in X$.*

Then, for each $z_0 \in X$ there exists a unique function $z \in W^{1,p}(0,T;X)$ such that

$$\dot{z}(t) = F(t, z(t)), \qquad a.e.\ t \in (0,T),$$
$$z(0) = z_0.$$

Here and everywhere in this paper, the dot above a variable denotes its derivative with respect to time.

Let X and Y be real Hilbert spaces such that X is dense in Y and the injection map is continuous; the space Y is identified with its own dual and with a subspace of the dual X^* of X, i.e. $X \subset Y \subset X^*$ is a Gelfand triplet. Denote by $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_{X^*}$ and $\langle\cdot,\cdot\rangle_{X^*\times X}$ the norm on the spaces X, Y, X^* and the duality pairing between X and X^*, respectively.

An operator $A: X \to X^*$ is said to be *hemicontinuous* if the real function $t \mapsto \langle A(u + tv), w\rangle_{X^*\times X}$ is continuous on $[0,1]$ for all $u, v, w \in X$, and *monotone* if $\langle Av - Aw, v - w\rangle_{X^*\times X} \geq 0$ for all $v, w \in X$. The operator $A: X \to X^*$ is called *pseudomonotone* (see, e.g., [19]) if the following conditions are satisfied:

$$\begin{cases} \text{(i)} & A \text{ is bounded;} \\ \text{(ii)} & \text{if } (w_n) \subset X \text{ with } w_n \rightharpoonup w \text{ weakly in } X, \text{ and} \\ & \limsup \langle A(w_n), w_n - w\rangle_{X^*\times X} \leq 0, \text{ then} \\ & \liminf \langle A(w_n), w_n - v\rangle_{X^*\times X} \geq \langle A(w), w - v\rangle_{X^*\times X},\ \forall v \in X. \end{cases} \tag{2}$$

We have the following result which may be found in [1, p. 140].

Proposition 2. *Let $X \subset Y \subset X^*$ be a Gelfand triplet. Assume that $A: X \to X^*$ is a hemicontinuous and monotone operator and there are real* *constants* $\mathcal{C}_1 > 0$, $\mathcal{C}_2$ *and* $\mathcal{C}_3 > 0$ *such that*

$$\langle Av, v\rangle_{X^*\times X} \geq \mathcal{C}_1\|v\|_X^2 + \mathcal{C}_2, \qquad \forall v \in X, \tag{3}$$

$$\|Av\|_{X^*} \leq \mathcal{C}_3(\|v\|_X + 1), \qquad \forall v \in X. \tag{4}$$

Then, given $w_0 \in Y$ *and* $f \in L^2(0,T;X^*)$, *there exists a unique function* w *which satisfies*

$$\begin{gathered} w \in L^2(0,T;X) \cap C([0,T];Y), \qquad \dot{w} \in L^2(0,T;X^*), \\ \dot{w}(t) + Aw(t) = f(t) \quad \text{in } X^*, \text{ a.e. } t \in (0,T), \\ w(0) = w_0. \end{gathered}$$

The following abstract result can be found in [2, p. 124].

Proposition 3. *Let* $X \subset Y \subset X^*$ *be a Gelfand triplet and let* K *be a nonempty, closed and convex set of* X. *Assume that* $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ *is a continuous and symmetric bilinear form and there are real constants* $c_2 > 0$ *and* c_1 *such that*

$$a(v,v) + c_1\|v\|_Y^2 \geq c_2\|v\|_X^2, \qquad \forall v \in X. \tag{5}$$

Then, for each $w_0 \in K$ *and each* $f \in L^2(0,T;Y)$, *there exists a unique function* $w \in W^{1,2}(0,T;Y) \cap L^2(0,T;X)$ *such that*

$$\begin{aligned} & w(t) \in K, \qquad \forall t \in [0,T], \\ & \langle \dot{w}(t), v - w(t)\rangle_{X^*\times X} + a\big(w(t), v - w(t)\big) \geq \big(f(t), v - w(t)\big)_Y, \\ & \qquad\qquad\qquad\qquad\qquad\qquad \forall v \in K, \text{ a.e. } t \in (0,T), \\ & w(0) = w_0. \end{aligned}$$

We end this section with the following Gronwall type inequality.

Proposition 4. *Assume that* $f, g : [a,b] \to \mathbb{R}$ *are continuous functions which satisfy*

$$f(t) \leq g(t) + c\int_a^t f(s)\,ds, \qquad \forall t \in [a,b],$$

where $c > 0$ *is a constant. Then*

$$f(t) \leq g(t) + c\int_a^t g(s)\exp(c(t-s))\,ds, \qquad \forall t \in [a,b].$$

The proof can be found in [14, p. 162].

3. Problem statement. Assumptions. Variational formulation. The physical setting is as follows. A deformable body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ (with $d = 2,3$). The body is described by a nonlinear viscoelastic constitutive law with damage and the process is dynamic in the time interval of interest $[0,T]$. We assume that the boundary Γ of the domain Ω is Lipschitz continuous, and it is partitioned into three disjoint measurable parts $\Gamma_1, \Gamma_2, \Gamma_3$ such that meas $(\Gamma_1) > 0$. The body is clamped on Γ_1 and therefore the displacement field vanishes there, while volume forces of density f_0 act in Ω, and surface tractions of density f_2 act on Γ_2. The contact is supposed to be bilateral, adhesive and governed by Tresca's friction law. To simplify the notation, we do not indicate explicitly the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$. Under the above assumptions, the classical formulation of our problem is the following.

Problem 5. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to \mathbb{S}^d$, a damage field $\zeta : \Omega \times [0, T] \to \mathbb{R}$ and a bonding field $\beta : \Gamma_3 \times [0, T] \to \mathbb{R}$ such that

$$\sigma = \mathcal{A}\varepsilon(\dot{u}) + \mathcal{B}(\varepsilon(u), \zeta) \quad \text{in } \Omega \times (0, T), \tag{6}$$

$$\dot{\zeta} - \kappa\Delta\zeta + \partial I_{[0,1]}(\zeta) \ni \mathcal{G}(\varepsilon(u), \zeta) \quad \text{in } \Omega \times (0, T), \tag{7}$$

$$\rho\ddot{u} = \operatorname{Div}\sigma + f_0 \quad \text{in } \Omega \times (0, T), \tag{8}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0, T), \tag{9}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{10}$$

$$\frac{\partial\zeta}{\partial\nu} = 0 \quad \text{on } \Gamma \times (0, T), \tag{11}$$

$$u_\nu = 0 \quad \text{on } \Gamma_3 \times (0, T), \tag{12}$$

$$\begin{cases} |\sigma_\tau + p_\tau(\beta, u_\tau)| \le g_b, \\ |\sigma_\tau + p_\tau(\beta, u_\tau)| < g_b \ \Rightarrow \ \dot{u}_\tau = 0, \\ |\sigma_\tau + p_\tau(\beta, u_\tau)| = g_b \ \Rightarrow \\ \quad \exists\lambda \ge 0 : \sigma_\tau + p_\tau(\beta, u_\tau) = -\lambda\dot{u}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0, T), \tag{13}$$

$$\dot{\beta} = H_{\mathrm{ad}}(\beta, \theta_\beta, |R_\tau(u_\tau)|) \quad \text{on } \Gamma_3 \times (0, T), \tag{14}$$

$$\beta(0) = \beta_0 \quad \text{on } \Gamma_3, \tag{15}$$

$$\zeta(0) = \zeta_0 \quad \text{in } \Omega, \tag{16}$$

$$\text{(i) } u(0) = u_0, \ \text{(ii) } \dot{u}(0) = v_0 \quad \text{in } \Omega. \tag{17}$$

Equation (6) represents the viscoelastic constitutive law with damage in which ε denotes the linearized strain tensor, $\mathcal{A}$ is the viscosity operator and $\mathcal{B}$ is the elasticity operator. Here we assume that the damage affects only the elastic behaviour of the material, and therefore $\mathcal{B}$ is a function of the strain and the damage field. (7) is a parabolic differential inclusion which describes the evolution of the damage field, where Δ denotes the Laplace operator, $\kappa > 0$ is the microcrack diffusion constant and $\mathcal{G}$ is the damage source function. The indicator function $I_{[0,1]} : \mathbb{R} \to (-\infty, \infty]$ is given by

$$I_{[0,1]}(s) = \begin{cases} 0 & \text{if } s \in [0,1], \\ \infty & \text{otherwise.} \end{cases}$$

The subdifferential of $I_{[0,1]}$ at s is the set

$$\partial I_{[0,1]}(s) = \{r \in \mathbb{R};\, I_{[0,1]}(z) - I_{[0,1]}(s) \geq r(z-s),\, \forall z \in \mathbb{R}\}, \qquad \forall s \in \mathbb{R},$$

i.e.

$$\partial I_{[0,1]}(s) = \begin{cases} (-\infty, 0] & \text{if } s = 0, \\ 0 & \text{if } s \in (0,1), \\ [0,\infty) & \text{if } s = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore the subdifferential term $\partial I_{[0,1]}(\zeta)$ in (7) guarantees that ζ is restricted to values between 0 and 1; when $\zeta = 1$ the material is damage-free and has its full capacity; when $\zeta = 0$ the material is completely damaged; when $0 < \zeta < 1$ there is partial additional damage. Equation (8) is the dynamic equation of motion where ρ is the mass density. Equations (9)–(10) are the displacement-traction boundary conditions where $\sigma\nu$ represents the Cauchy stress vector. Equation (11) means that the normal derivative of ζ, denoted by $\partial\zeta/\partial\nu$, vanishes on Γ. Therefore, there is no influx of microcracks across the boundary. Conditions (12)–(13) represent the bilateral contact with Tresca's friction law in which adhesion is taken into account and g_b is a friction bound. Here, p_τ is a general prescribed function. In particular, we may consider the case

$$p_\tau(\beta, v) = \begin{cases} q_\tau(\beta) v & \text{if } 0 \leq |v| \leq L, \\ q_\tau(\beta) L v / |v| & \text{if } |v| > L, \end{cases}$$

where $L > 0$ is a limit bound constant and q_τ is nonnegative tangential stiffness function (see, e.g., [21]). In [25] the following form of q_τ has been employed:

$$q_\tau(\beta) = c_\tau \beta^2 \qquad \text{on } \Gamma_3 \times (0,T),$$

where $c_\tau$is a given positive material parameter. Equation (14) represents the evolution of the bonding field described by a general function H_{ad} which may change sign. This condition implies that cycles of rebonding after debonding may take place (see [16], [22] for details). Moreover, the process depends on the bonding history, which we denote by

$$\theta_\beta(t,x) = \int_0^t \beta(s,x)\, ds.$$

Here and below, $R_\tau : \mathbb{R}^d \to \mathbb{R}^d$ is a truncation operator defined by

$$R_\tau(v) = \begin{cases} v & \text{if } 0 \le |v| \le L, \\ Lv/|v| & \text{if } |v| > L. \end{cases}$$

The introduction of the operator R_τ is motivated by the mathematical arguments where $L > 0$ is a characteristic length of the bond, beyond which there is no any additional traction. Clearly, R_τ satisfies

$$\begin{cases} |R_\tau(v)| \le L, & \forall v \in \mathbb{R}^d, \\ \big||R_\tau(w)| - |R_\tau(v)|\big| \le |w - v|, & \forall w, v \in \mathbb{R}^d. \end{cases} \tag{18}$$

An example of the adhesion rate function H_{ad} is

$$H_{\mathrm{ad}}(\beta, r) = -(c_\tau \beta r^2 - \epsilon_a)_+ \quad \text{on } \Gamma_3 \times (0, T),$$

where c_τ, ϵ_a are given positive material parameters and $(\alpha)_+$ denotes the positive part of α, that is, $(\alpha)_+ = \max\{\alpha, 0\}$. Since $\dot{\beta} \le 0$, the process is irreversible and once debonding occurs, bonding cannot be reestablished (see, e.g., [4], [25]). Another example, in which H_{ad} depends on all three variables, is

$$H_{\mathrm{ad}}(\beta, \theta_\beta, r) = -\gamma_1 \beta r^2 + \gamma_2 \frac{\beta_+(1-\beta)_+}{1 + d_*(\theta_\beta)^2},$$

where γ_1, γ_2 are given positive material parameters and $d_* > 0$ is the history weight factor (see, e.g., [7], [16], [22]). Finally, (15)–(17) are the initial conditions.

To obtain a variational formulation of the mechanical problem, we introduce the space V and the convex set $\mathcal{K}$ defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1,\ v_\nu = 0 \text{ on } \Gamma_3\},$$
$$\mathcal{K} = \{\zeta \in H^1(\Omega);\ 0 \le \zeta \le 1, \text{ a.e. } x \in \Omega\}.$$

Since $\mathrm{meas}(\Gamma_1) > 0$, Korn's inequality holds:

$$C_K \|v\|_{H_1} \le \|\varepsilon(v)\|_{\mathcal{Q}}, \qquad \forall v \in V, \tag{19}$$

where $C_K > 0$ is a positive constant depending only on Ω and Γ_1. A proof of Korn's inequality can be found, for instance, in [20, p. 79]. Over the space V, we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{Q}}, \qquad \forall u, v \in V,$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (19) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Therefore $(V, (\cdot,\cdot)_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a positive constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3;\mathbb{R}^d)} \le c_0 \|v\|_V, \qquad \forall v \in V. \tag{20}$$

In the study of the mechanical problem (6)–(17) we consider the following assumptions. We assume that the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\text{(21)} \quad \begin{cases} \text{(i) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2, \\ \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(ii) there exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \quad |\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)| \leq L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|, \\ \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega; \\ \text{(iii) the mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega \\ \quad \text{for any } \varepsilon \in \mathbb{S}^d; \\ \text{(iv) the mapping } x \mapsto \mathcal{A}(x, 0_{\mathbb{S}^d}) \text{ belongs to } \mathcal{Q}. \end{cases}$$

We assume that the operator $\mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$ satisfies

$$\text{(22)} \quad \begin{cases} \text{(i) there exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \quad |\mathcal{B}(x,\varepsilon_1,\xi_1) - \mathcal{B}(x,\varepsilon_2,\xi_1)| \leq L_{\mathcal{B}} (|\varepsilon_1 - \varepsilon_2| + |\xi_1 - \xi_2|), \\ \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \forall \xi_1, \xi_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ \text{(ii) the mapping } x \mapsto \mathcal{B}(x,\varepsilon,\xi) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall \varepsilon \in \mathbb{S}^d, \ \forall \xi \in \mathbb{R}; \\ \text{(iii) the mapping } x \mapsto \mathcal{B}(x, 0_{\mathbb{S}^d}, 0_{\mathbb{R}}) \text{ belongs to } \mathcal{Q}. \end{cases}$$

We assume that the damage source function $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$ satisfies

$$\text{(23)} \quad \begin{cases} \text{(i) there exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad |\mathcal{G}(x,\varepsilon_1,\xi_1) - \mathcal{G}(x,\varepsilon_2,\xi_1)| \leq L_{\mathcal{G}} (|\varepsilon_1 - \varepsilon_2| + |\xi_1 - \xi_2|), \\ \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \forall \xi_1, \xi_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ \text{(ii) the mapping } x \mapsto \mathcal{G}(x,\varepsilon,\xi) \text{ is Lebesgue measurable on } \Omega, \\ \quad \forall \varepsilon \in \mathbb{S}^d, \ \forall \xi \in \mathbb{R}; \\ \text{(iii) the mapping } x \mapsto \mathcal{G}(x, 0_{\mathbb{S}^d}, 0_{\mathbb{R}}) \text{ belongs to } L^2(\Omega). \end{cases}$$

We assume that the tangential contact function $p_\tau : \Gamma_3 \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$(24)\quad \begin{cases} \text{(i) there exists } L_\tau > 0 \text{ such that} \\ \quad |p_\tau(x,\beta_1,r_1) - p_\tau(x,\beta_2,r_2)| \le L_\tau(|\beta_1-\beta_2| + |r_1-r_2|), \\ \quad \forall \beta_1,\beta_2 \in [0,1],\ \forall r_1,r_2 \in \mathbb{R}^d,\ \text{ a.e. } x \in \Gamma_3; \\ \text{(ii) } r\cdot\nu(x) = 0 \ \Rightarrow\ p_\tau(x,\beta,r)\cdot\nu(x) = 0, \\ \quad \forall \beta \in \mathbb{R},\ \forall r \in \mathbb{R}^d,\ \text{ a.e } x \in \Gamma_3; \\ \text{(iii) the mapping } x \mapsto p_\tau(x,\beta,r) \text{ is Lebesgue measurable on } \Gamma_3 \\ \quad \forall \beta \in \mathbb{R},\ \forall r \in \mathbb{R}^d; \\ \text{(iv) the mapping } x \mapsto p_\tau(x,0_{\mathbb{R}},0_{\mathbb{R}^d}) \text{ belongs to } L^2(\Gamma_3;\mathbb{R}^d). \end{cases}$$

As in [22], the adhesion rate function $H_{\mathrm{ad}} : \Gamma_3 \times \mathbb{R} \times \mathbb{R} \times [0,L] \to \mathbb{R}$ is assumed to satisfy

$$(25)\quad \begin{cases} \text{(i) there exists } L_{H_{\mathrm{ad}}} > 0 \text{ such that} \\ \quad |H_{\mathrm{ad}}(x,\beta_1,z,r) - H_{\mathrm{ad}}(x,\beta_2,z,r)| \le L_{H_{\mathrm{ad}}}|\beta_1-\beta_2|, \\ \quad \text{a.e. } x \in \Gamma_3,\ \forall \beta_1,\beta_2,z \in \mathbb{R},\ \forall r \in [0,L]; \\ \text{(ii) } |H_{\mathrm{ad}}(x,\beta_1,z_1,r_1) - H_{\mathrm{ad}}(x,\beta_2,z_2,r_2)| \\ \qquad\qquad \le L_{H_{\mathrm{ad}}}(|\beta_1-\beta_2| + |z_1-z_2| + |r_1-r_2|), \\ \quad \forall \beta_1,\beta_2 \in [0,1],\ \forall z_1,z_2 \in \mathbb{R},\ \forall r_1,r_2 \in [0,L],\ \text{ a.e. } x \in \Gamma_3; \\ \text{(iii) the mapping } x \to H_{\mathrm{ad}}(x,\beta,z,r) \text{ is measurable on } \Gamma_3, \\ \quad \forall \beta,z \in \mathbb{R},\ \forall r \in [0,L]; \\ \text{(iv) the mapping } (\beta,z,r) \to H_{\mathrm{ad}}(x,\beta,z,r) \text{ is continuous on} \\ \quad \mathbb{R}\times\mathbb{R}\times[0,L],\ \text{ a.e. } x \in \Gamma_3; \\ \text{(v) } H_{\mathrm{ad}}(x,0,z,r) = 0,\ \forall z \in \mathbb{R},\ \forall r \in [0,L],\ \text{ a.e. } x \in \Gamma_3; \\ \text{(vi) } H_{\mathrm{ad}}(x,\beta,z,r) \ge 0,\ \forall \beta \le 0,\ \forall z \in \mathbb{R},\ \forall r \in [0,L],\ \text{ a.e. } x \in \Gamma_3, \\ \quad H_{\mathrm{ad}}(x,\beta,z,r) \le 0,\ \forall \beta \ge 1,\ \forall z \in \mathbb{R},\ \forall r \in [0,L],\ \text{ a.e. } x \in \Gamma_3. \end{cases}$$

We suppose that the mass density satisfies

(26) $\rho \in L^\infty(\Omega)$ and there exists $\rho_* > 0$ such that $\rho(x) \ge \rho_*$ a.e. $x \in \Omega$.

Also, we suppose that the friction bound function $g_b : \Gamma_3 \to \mathbb{R}^+$ satisfies

$$(27)\qquad g_b \in L^\infty(\Gamma_3).$$

The body forces and surface tractions have the regularity:

$$(28)\qquad f_0 \in L^2(0,T;H), \qquad f_2 \in L^2(0,T;L^2(\Gamma_2;\mathbb{R}^d)).$$

Finally, we assume that the initial data satisfy

$$(29)\qquad \beta_0 \in L^\infty(\Gamma_3), \qquad 0 \le \beta_0(x) \le 1, \qquad \text{a.e. } x \in \Gamma_3,$$

$$(30)\qquad \zeta_0 \in \mathcal{K},$$

$$(31)\qquad \text{(i) } u_0 \in V, \qquad \text{(ii) } v_0 \in H.$$

Next, we define a modified inner product on the Hilbert space H by

$$((u, w))_H = (\rho u, w)_H, \qquad \forall u, w \in H, \tag{32}$$

and let $\|\!|\cdot|\!\|_H$ be the associated norm, i.e.

$$\|\!|w|\!\|_H = [(\rho w, w)_H]^{1/2}, \qquad \forall w \in H. \tag{33}$$

It follows from (26) and (33) that $\|\!|\cdot|\!\|_H$ and $\|\cdot\|_H$ are equivalent norms on H. Moreover, the inclusion mapping of $(V, \|\cdot\|_V)$ into $(H, \|\!|\cdot|\!\|_H)$ is continuous and dense. Identifying H with its own dual, we can write the Gelfand triplet

$$V \subset H \subset V^*.$$

We use $\langle\, ,\rangle_{V^*\times V}$ to represent the duality pairing between V^* and V. Then

$$\langle v, w\rangle_{V^*\times V} = ((v, w))_H, \qquad \forall v \in H, \forall w \in V. \tag{34}$$

Using Riesz's theorem, from (28) we can define $f \in L^2(0, T; V)$ by

$$(f(t), w)_V = \int_\Omega f_0(t)\cdot w\, dx + \int_{\Gamma_2} f_2(t)\cdot w\, da, \qquad \forall w \in V, \text{ a.e. } t \in (0, T). \tag{35}$$

Below, we use the functionals $j_\tau : V \to \mathbb{R}$ and $j_{\mathrm{ad}} : L^\infty(\Gamma_3)\times V\times V \to \mathbb{R}$ defined by

$$j_\tau(w) = \int_{\Gamma_3} g_b|w_\tau|\, da, \qquad \forall w \in V, \tag{36}$$

$$j_{\mathrm{ad}}(\beta, v, w) = \int_{\Gamma_3} p_\tau(\beta, v_\tau)\cdot w_\tau\, da, \qquad \forall(\beta, v, w) \in L^\infty(\Gamma_3)\times V\times V. \tag{37}$$

Thanks to (24) and (20), there exists $L_{\mathrm{ad}} > 0$ such that

$$\begin{aligned} &|j_{\mathrm{ad}}(\beta_1, u_1, w) - j_{\mathrm{ad}}(\beta_2, u_2, w)| \\ &\le L_{\mathrm{ad}}(\|\beta_1-\beta_2\|_{L^2(\Gamma_3)} + \|u_1-u_2\|_V)\|w\|_V, \qquad \forall u_1, u_2, w \in V, \\ &\qquad\qquad \forall \beta_1, \beta_2 \in L^\infty(\Gamma_3),\ 0 \le \beta_1, \beta_2 \le 1, \text{ a.e. } x \in \Gamma_3. \end{aligned} \tag{38}$$

We turn now to derive a variational formulation of the mechanical problem (6)–(17). To this end, assume that $(u, \sigma, \zeta, \beta)$ are smooth functions satisfying (6)–(17) and let $w \in V$ and $t \in [0, T]$. Using (8), (32), (34) and Green's formula (1), we obtain

$$\langle \ddot u(t), w\rangle_{V^*\times V} + (\sigma(t), \varepsilon(w))_{\mathcal{Q}} - \int_\Omega f_0(t)\cdot w\, dx = \int_\Gamma \sigma(t)\nu\cdot w\, da, \tag{39}$$

and by (10) we find

$$\int_\Gamma \sigma(t)\nu\cdot w\, da = \int_{\Gamma_2} f_2(t)\cdot w\, da + \int_{\Gamma_3} \sigma_\tau(t)\cdot w_\tau\, da. \tag{40}$$

Then (39), (40) and (35) lead to

$$(41)\qquad \langle \ddot{u}(t), w - \dot{u}(t)\rangle_{V^*\times V} + \big(\sigma(t), \varepsilon(w) - \varepsilon(\dot{u}(t))\big)_{\mathcal{Q}} - \int_{\Gamma_3} \sigma_\tau(t)\cdot(w_\tau - \dot{u}_\tau(t))\,da = (f(t), w - \dot{u}(t))_V.$$

On the other hand,

$$-\int_{\Gamma_3} \big(\sigma_\tau(t) + p_\tau(\beta(t), u_\tau(t))\big)\cdot w_\tau\,da \le \int_{\Gamma_3} |\sigma_\tau(t) + p_\tau(\beta(t), u_\tau(t))|\,|w_\tau|\,da,$$

and taking into account the boundary condition (13), we deduce that

$$(42)\qquad -\int_{\Gamma_3} \sigma_\tau(t)\cdot w_\tau\,da \le \int_{\Gamma_3} |\sigma_\tau(t) + p_\tau(\beta(t), u_\tau(t))|\,|w_\tau|\,da + \int_{\Gamma_3} p_\tau(\beta(t), u_\tau(t))\cdot w_\tau\,da \le \int_{\Gamma_3} g_b|w_\tau|\,da + \int_{\Gamma_3} p_\tau(\beta(t), u_\tau(t))\cdot w_\tau\,da.$$

Also, using (13) we obtain

$$\int_{\Gamma_3} \big(\sigma_\tau(t) + p_\tau(\beta(t), u_\tau(t))\big)\cdot \dot{u}_\tau(t)\,da = -\int_{\Gamma_3} |\sigma_\tau(t) + p_\tau(\beta(t), u_\tau(t))|\,|\dot{u}_\tau(t)|\,da = -\int_{\Gamma_3} g_b|\dot{u}_\tau(t)|\,da,$$

which gives

$$\int_{\Gamma_3} \sigma_\tau(t)\cdot \dot{u}_\tau(t)\,da = -\int_{\Gamma_3} g_b|\dot{u}_\tau(t)|\,da - \int_{\Gamma_3} p_\tau(\beta(t), u_\tau(t))\cdot \dot{u}_\tau(t)\,da,$$

and keeping in mind (42) and (36)–(37), we find

$$(43)\qquad -\int_{\Gamma_3} \sigma_\tau(t)\cdot(w_\tau - \dot{u}_\tau(t))\,da \le j_\tau(w) - j_\tau(\dot{u}(t)) + j_{\mathrm{ad}}(\beta(t), u(t), w) - j_{\mathrm{ad}}(\beta(t), u(t), \dot{u}(t)).$$

Now, using integration by parts and applying condition (11) we get

$$(44)\qquad -(\Delta\zeta(t), \psi)_{L^2(\Omega)} = (\nabla\zeta(t), \nabla\psi)_{L^2(\Omega,\mathbb{R}^d)}, \qquad \forall\psi\in H^1(\Omega).$$

Moreover, from (7) one has

$$\big(\dot{\zeta}(t) - \kappa\Delta\zeta(t)\big)(\psi - \zeta(t)) + I_{[0,1]}(\psi) - I_{[0,1]}(\zeta(t)) \ge \mathcal{G}\big(\varepsilon(u(t)), \zeta(t)\big)(\psi - \zeta(t)), \qquad \forall\psi\in H^1(\Omega),\ \text{a.e. } x\in\Omega,$$

and since $0 \le \zeta(t) \le 1$, we obtain

$$(45)\qquad \big(\dot\zeta(t), \psi - \zeta(t)\big)_{L^2(\Omega)} - \kappa(\Delta\zeta(t), \psi - \zeta(t))_{L^2(\Omega)} \ge \big(\mathcal{G}(\varepsilon(u(t)), \zeta(t)), \psi - \zeta(t)\big)_{L^2(\Omega)}, \quad \forall \psi \in \mathcal{K}.$$

Finally, using (41), (43)–(45) and keeping in mind (14)–(17), we obtain the following variational formulation of problem (6)–(17) in terms of displacement, damage and adhesion fields.

PROBLEM 6. Find a displacement field $u : [0,T] \to V$, a damage field $\zeta : [0,T] \to H^1(\Omega)$ and a bonding field $\beta : [0,T] \to L^\infty(\Gamma_3)$ such that

$$(46)\qquad \langle \ddot u(t), w - \dot u(t)\rangle_{V^*\times V} + \big(\mathcal{A}(\varepsilon(\dot u(t))), \varepsilon(w - \dot u(t))\big)_{\mathcal{Q}} + \big(\mathcal{B}(\varepsilon(u(t)), \zeta(t)), \varepsilon(w - \dot u(t))\big)_{\mathcal{Q}} + j_{\rm ad}\big(\beta(t), u(t), w - \dot u(t)\big) + j_\tau(w) - j_\tau(\dot u(t)) \ge (f(t), w - \dot u(t))_V, \quad \forall w \in V, \text{ a.e. } t \in (0,T),$$

$\zeta(t) \in \mathcal{K}$,

$$(47)\qquad \big(\dot\zeta(t), \psi - \zeta(t)\big)_{L^2(\Omega)} + \kappa(\nabla\zeta(t), \nabla\psi - \nabla\zeta(t))_H \ge \big(\mathcal{G}(\varepsilon(u(t)), \zeta(t)), \psi - \zeta(t)\big)_{L^2(\Omega)}, \quad \forall \psi \in \mathcal{K}, \text{ a.e. } t \in (0,T),$$

$$(48)\qquad \dot\beta(t) = H_{\rm ad}\big(\beta(t), \theta_\beta(t), |R_\tau(u_\tau(t))|\big), \quad \text{a.e. } t \in (0,T),$$

$$(49)\qquad \beta(0) = \beta_0,$$

$$(50)\qquad \zeta(0) = \zeta_0,$$

$$(51)\qquad \text{(i) } u(0) = u_0, \quad \text{(ii) } \dot u(0) = v_0.$$

4. Existence and uniqueness of a weak solution. The following theorem is the main result of this paper.

THEOREM 7. *Assume* (21)–(31) *are fulfilled. Then problem* (46)–(51) *has a unique solution* $\{u, \beta, \zeta\}$. *Moreover, the solution satisfies*

$$(52)\qquad u \in W^{1,2}(0,T;V) \cap C^1([0,T];H), \quad \ddot u \in L^2(0,T;V^*),$$

$$(53)\qquad \beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)), \quad 0 \le \beta(t) \le 1, \ \textit{a.e. } x \in \Gamma_3, \ \forall t \in [0,T],$$

$$(54)\qquad \zeta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

The proof will be carried out in several steps. In the rest of this paper, the same letter c will be used to denote different positive constants independent of $t \in (0,T)$.

STEP 1. Consider the following problem.

PROBLEM 8. Let $\eta \in L^2(0,T;V)$. Find a function $\zeta_\eta : [0,T] \to H^1(\Omega)$ such that

$$(55)\quad \begin{cases} \zeta_\eta(t) \in \mathcal{K}, \\ (\dot{\zeta}_\eta(t), \psi - \zeta_\eta(t))_{L^2(\Omega)} + \kappa(\nabla\zeta_\eta(t), \nabla\psi - \nabla\zeta_\eta(t))_H \\ \qquad \geq \big(\mathcal{G}(\varepsilon(\eta(t)), \zeta_\eta(t)), \psi - \zeta_\eta(t)\big)_{L^2(\Omega)}, \ \forall\psi \in \mathcal{K}, \ \text{a.e. } t \in (0,T), \end{cases}$$

$$(56)\quad \zeta_\eta(0) = \zeta_0.$$

Lemma 9. *Assume that* (23) *and* (30) *hold. Then problem* (55)–(56) *has a unique solution* ζ_η *which satisfies*

$$(57)\qquad \zeta_\eta \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

Furthermore, there is a constant $c > 0$ *such that for all* $\eta_1, \eta_2 \in L^2(0,T;V)$,

$$(58)\qquad \|\zeta_{\eta_1}(t) - \zeta_{\eta_2}(t)\|^2_{L^2(\Omega)} \leq c\int_0^t \|\eta_1(s) - \eta_2(s)\|^2_V\,ds, \qquad \forall t \in [0,T],$$

where ζ_{η_i} $(i = 1,2)$ *is the solution corresponding to* η_i.

Proof. Let $(\eta, \phi) \in L^2(0,T;V) \times L^2(0,T;L^2(\Omega))$. From (23) we can define $f_{\eta\phi} \in L^2(0,T;L^2(\Omega))$ by

$$f_{\eta\phi}(t) = \mathcal{G}(\varepsilon(\eta(t)), \phi(t)), \qquad \text{a.e. } t \in (0,T).$$

On the other hand, let $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ be the bilinear form defined by

$$a(\varphi, \psi) = \kappa(\nabla\varphi, \nabla\psi)_H, \qquad \forall\varphi, \psi \in H^1(\Omega).$$

After some algebraic computations, we find that a satisfies (5) and it is continuous and symmetric on $H^1(\Omega)$. Therefore, applying Proposition 3 with $X = H^1(\Omega), Y = L^2(\Omega)$ and $K = \mathcal{K}$, we can see that, for each $(\eta, \phi) \in L^2(0,T;V) \times L^2(0,T;L^2(\Omega))$, the system

$$(59)\quad \begin{cases} \zeta_{\eta\phi}(t) \in \mathcal{K}, \\ (\dot{\zeta}_{\eta\phi}(t), \psi - \zeta_{\eta\phi}(t))_{L^2(\Omega)} + \kappa(\nabla\zeta_{\eta\phi}(t), \nabla\psi - \nabla\zeta_{\eta\phi}(t))_H \\ \qquad \geq \big(\mathcal{G}(\varepsilon(\eta(t)), \phi(t)), \psi - \zeta_{\eta\phi}(t)\big)_{L^2(\Omega)}, \quad \forall\psi \in \mathcal{K}, \ \text{a.e. } t \in (0,T), \end{cases}$$

$$(60)\quad \zeta_{\eta\phi}(0) = \zeta_0$$

has a unique solution $\zeta_{\eta\phi}$ which satisfies

$$(61)\qquad \zeta_{\eta\phi} \in W^{1,2}(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)).$$

To continue, assume $\zeta_{\eta\phi_1}$, $\zeta_{\eta\phi_2}$ are two solutions to system (59)–(60), corresponding to $(\eta, \phi) = (\eta, \phi_1)$ and $(\eta, \phi) = (\eta, \phi_2)$, respectively. Then

$$\int_0^t \big(\dot\zeta_{\eta\phi_1}(s) - \dot\zeta_{\eta\phi_2}(s), \zeta_{\eta\phi_1}(s) - \zeta_{\eta\phi_2}(s)\big)_{L^2(\Omega)}\,ds \\ + \kappa\int_0^t a\big(\zeta_{\eta\phi_1}(s) - \zeta_{\eta\phi_2}(s), \zeta_{\eta\phi_1}(s) - \zeta_{\eta\phi_2}(s)\big)\,ds \\ \le \int_0^t \big(\mathcal{G}(\varepsilon(\eta(s)),\phi_1(s)) - \mathcal{G}(\varepsilon(\eta(s)),\phi_2(s)), \zeta_{\eta\phi_1}(s) - \zeta_{\eta\phi_2}(s)\big)_{L^2(\Omega)}\,ds,$$

and using the inequality

$$\lambda\delta \le \tfrac{1}{2}\lambda^2 + \tfrac{1}{2}\delta^2, \qquad \forall \lambda, \delta \in \mathbb{R},$$

we deduce that

$$\|\zeta_{\eta\phi_1}(t) - \zeta_{\eta\phi_2}(t)\|^2_{L^2(\Omega)} \le c\int_0^t \|\phi_1(s) - \phi_2(s)\|^2_{L^2(\Omega)}\,ds \\ + c\int_0^t \|\zeta_{\eta\phi_1}(s) - \zeta_{\eta\phi_2}(s)\|^2_{L^2(\Omega)}\,ds.$$

From Gronwall's inequality, we obtain

$$(62)\qquad \|\zeta_{\eta\phi_1}(t) - \zeta_{\eta\phi_2}(t)\|^2_{L^2(\Omega)} \le c\int_0^t \|\phi_1(s) - \phi_2(s)\|^2_{L^2(\Omega)}\,ds, \qquad \forall t \in [0,T].$$

Now, for each $\eta \in L^2(0,T;V)$, consider the operator $\Phi_\eta : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ defined by

$$(63)\qquad \Phi_\eta\phi = \zeta_{\eta\phi}, \qquad \forall \phi \in L^2(0,T;L^2(\Omega)),$$

where $\zeta_{\eta\phi}$ is the unique solution satisfying (59)–(60). Using (62)–(63) we get

$$\|\Phi_\eta\phi_1(t) - \Phi_\eta\phi_2(t)\|^2_{L^2(\Omega)} \le c\int_0^t \|\phi_1(s) - \phi_2(s)\|^2_{L^2(\Omega)}\,ds, \qquad \forall t \in [0,T].$$

Reiterating the last inequality n times, we infer that

$$\|\Phi^n_\eta\phi_1 - \Phi^n_\eta\phi_2\|^2_{L^2(0,T;L^2(\Omega))} \le \frac{(cT)^n}{n!}\|\phi_1 - \phi_2\|^2_{L^2(0,T;L^2(\Omega))},$$

which implies that, for n sufficiently large, a power Φ^n_η of Φ_η is a contraction in the Banach space $L^2(0,T;L^2(\Omega))$. Therefore, Φ_η has a unique fixed point $\phi^*_\eta \in L^2(0,T;L^2(\Omega))$. Now, let $\eta \in L^2(0,T;V)$, let ϕ^*_η be the fixed point of Φ_η, let $\zeta_\eta = \zeta_{\eta\phi^*_\eta}$ and keeping in mind (59)–(61), it is straightforward to see that ζ_η is a unique solution to problem (55)–(56) such that (57) holds. Finally, assume ζ_{η_1}, ζ_{η_2} are two solutions to problem (55)–(56), corresponding to η_1

and η_2, respectively. Then we get

$$\|\zeta_{\eta_1}(t)-\zeta_{\eta_2}(t)\|^2_{L^2(\Omega)} \le c\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_V\,ds + c\int_0^t \|\zeta_{\eta_1}(s)-\zeta_{\eta_2}(s)\|^2_{L^2(\Omega)}\,ds, \qquad \forall t\in[0,T].$$

Thus, using Gronwall's inequality, we obtain (58). ∎

Now, consider the following problem.

PROBLEM 10. Let $\eta\in L^2(0,T;V)$. Find a function $\beta_\eta:[0,T]\to L^\infty(\Gamma_3)$ such that

$$\dot\beta_\eta(t)=H_{\mathrm{ad}}\big(\beta_\eta(t),\theta_{\beta_\eta}(t),|R_\tau(\eta_\tau(t))|\big), \qquad \text{a.e. } t\in(0,T), \tag{64}$$

$$\beta_\eta(0)=\beta_0, \tag{65}$$

where

$$\theta_{\beta_\eta}(t)=\int_0^t \beta_\eta(s)\,ds, \qquad \forall t\in[0,T]. \tag{66}$$

LEMMA 11. *Assume* (25) *and* (29) *are fulfilled. Then problem* (64)–(66) *has a unique solution which satisfies*

$$\begin{cases} \text{(i)}\ \ \beta_\eta\in W^{1,\infty}(0,T;L^\infty(\Gamma_3)),\\ \text{(ii)}\ \ 0\le\beta_\eta(t)\le 1,\ \ \forall t\in[0,T],\ \ a.e.\ x\in\Gamma_3. \end{cases} \tag{67}$$

Moreover, there exists a constant $c>0$ *such that for all* $\eta_1,\eta_2\in L^2(0,T;V)$,

$$\|\beta_{\eta_1}(t)-\beta_{\eta_2}(t)\|^2_{L^2(\Gamma_3)}\le c\int_0^t \|\eta_1(s)-\eta_2(s)\|^2_V\,ds, \qquad \forall t\in[0,T], \tag{68}$$

where β_{η_i} $(i=1,2)$ *is the solution corresponding to* η_i.

Proof. Fix $(\eta,\theta)\in L^2(0,T;V)\times L^\infty(0,T;L^\infty(\Gamma_3))$ and consider $F:[0,T]\times L^\infty(\Gamma_3)\to L^\infty(\Gamma_3)$ defined by

$$F(t,\beta)=H_{\mathrm{ad}}(\beta,\theta(t),|R_\tau(\eta_\tau(t))|), \quad \forall\beta\in L^\infty(\Gamma_3),\ \text{a.e. } t\in(0,T). \tag{69}$$

Let $\beta_1,\beta_2\in L^\infty(\Gamma_3)$. From (69) and (25), we obtain

$$\begin{aligned} |F(t,\beta_1)-F(t,\beta_2)| &\le \big|H_{\mathrm{ad}}\big(\beta_1,\theta(t),|R_\tau(\eta_\tau(t))|\big)-H_{\mathrm{ad}}\big(\beta_2,\theta(t),|R_\tau(\eta_\tau(t))|\big)\big|\\ &\le L_{H_{\mathrm{ad}}}|\beta_1-\beta_2|, \qquad \text{a.e. } t\in(0,T), \end{aligned}$$

which implies that

$$\|F(t,\beta_1)-F(t,\beta_2)\|_{L^\infty(\Gamma_3)}\le L_{H_{\mathrm{ad}}}\|\beta_1-\beta_2\|_{L^\infty(\Gamma_3)}, \qquad \text{a.e. } t\in(0,T).$$

Hence, F is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, $t\mapsto F(t,\beta)$ belongs to $L^\infty(0,T;L^\infty(\Gamma_3))$,

$\forall\beta \in L^\infty(\Gamma_3)$. Thus, using Proposition 1, we deduce that there exists a unique $\beta_{\eta\theta} \in W^{1,\infty}(0,T;L^\infty(\Gamma_3))$ such that

$$\dot\beta_{\eta\theta}(t) = H_{\mathrm{ad}}(\beta_{\eta\theta}(t), \theta(t), |R_\tau(\eta_\tau(t))|), \qquad \text{a.e. } t \in (0,T), \tag{70}$$

$$\beta_{\eta\theta}(0) = \beta_0. \tag{71}$$

Moreover, from (70)–(71), (25), (29) and using arguments similar to those in [16], [22], we deduce that

$$0 \le \beta_{\eta\theta}(t) \le 1, \qquad \forall t \in [0,T], \qquad \text{a.e. } x \in \Gamma_3. \tag{72}$$

Now, let $\beta_{\eta\theta_1}$, $\beta_{\eta\theta_2}$ be two solutions satisfying (70)–(71). It follows from (25) that

$$\begin{aligned}\|\beta_{\eta\theta_1}(t) - \beta_{\eta\theta_2}(t)\|_{L^\infty(\Gamma_3)} \le c\int_0^t \|\beta_{\eta\theta_1}(s) - \beta_{\eta\theta_2}(s)\|_{L^\infty(\Gamma_3)}\,ds \\ + c\int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^\infty(\Gamma_3)}\,ds.\end{aligned}$$

Using Gronwall's inequality we get

$$\|\beta_{\eta\theta_1}(t) - \beta_{\eta\theta_2}(t)\|_{L^\infty(\Gamma_3)} \le c\int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^\infty(\Gamma_3)}\,ds. \tag{73}$$

To continue, for each $\eta \in L^2(0,T;V)$, consider the operator $\Theta_\eta : L^\infty(0,T;L^\infty(\Gamma_3)) \to L^\infty(0,T;L^\infty(\Gamma_3))$ defined by

$$\Theta_\eta\theta(t) = \int_0^t \beta_{\eta\theta}(s)\,ds, \qquad \forall t \in [0,T],\ \forall\theta \in L^\infty(0,T;L^\infty(\Gamma_3)), \tag{74}$$

where $\beta_{\eta\theta}$ is the unique solution of (70)–(71). Using (73)–(74) we have

$$\begin{aligned}\|\Theta_\eta\theta_1(t) - \Theta_\eta\theta_2(t)\|_{L^\infty(\Gamma_3)} &\le \int_0^t \|\beta_{\eta\theta_1}(s) - \beta_{\eta\theta_2}(s)\|_{L^\infty(\Gamma_3)}\,ds \\ &\le c\int_0^t\int_0^s \|\theta_1(r) - \theta_2(r)\|_{L^\infty(\Gamma_3)}\,dr\,ds \\ &\le c\int_0^t \|\theta_1(r) - \theta_2(r)\|_{L^\infty(\Gamma_3)}\,dr, \qquad \forall t \in [0,T].\end{aligned}$$

Reiterating this inequality n times, we obtain

$$\|\Theta_\eta\theta_1 - \Theta_\eta\theta_2\|_{L^\infty(0,T;L^\infty(\Gamma_3))} \le \frac{(cT)^n}{n!}\|\theta_1 - \theta_2\|_{L^\infty(0,T;L^\infty(\Gamma_3))},$$

which implies that, for n sufficiently large, a power Θ_η^n of Θ_η is a contraction in the Banach space $L^\infty(0,T;L^\infty(\Gamma_3))$. Therefore, Θ_η has a unique fixed point $\theta_\eta^* \in L^\infty(0,T;L^\infty(\Gamma_3))$ given by

$$\theta_\eta^*(t) = \int_0^t \beta_{\eta\theta_\eta^*}(s)\,ds, \qquad \forall t \in [0,T].$$

Now, let $\eta \in L^2(0,T;V)$, let θ_η^* be the fixed point of Θ_η, let $\beta_\eta = \beta_{\eta\theta_\eta^*}$, let $\theta_{\beta_\eta} = \theta_\eta^*$ and keeping in mind (70)–(72), it is straightforward to see that β_η is a unique solution to problem (64)–(66) such that (67) holds.

Finally, assume β_{η_1}, β_{η_2} are two solutions to problem (64)–(66), corresponding to η_1 and η_2, respectively. Then from (25) and (18) we obtain

$$\begin{aligned}
&\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|^2_{L^2(\Gamma_3)} \\
&\quad \le c\int_0^t \|\beta_{\eta_1}(s) - \beta_{\eta_2}(s)\|^2_{L^2(\Gamma_3)}\,ds + c\int_0^t \|\theta^*_{\eta_1}(s) - \theta^*_{\eta_2}(s)\|^2_{L^2(\Gamma_3)}\,ds \\
&\qquad + c\int_0^t \|\eta_1(s) - \eta_2(s)\|^2_{L^2(\Gamma_3;\mathbb{R}^d)}\,ds \\
&\quad \le c\int_0^t \|\beta_{\eta_1}(s) - \beta_{\eta_2}(s)\|^2_{L^2(\Gamma_3)}\,ds + c\int_0^t\int_0^s \|\beta_{\eta_1}(r) - \beta_{\eta_2}(r)\|^2_{L^2(\Gamma_3)}\,dr\,ds \\
&\qquad + c\int_0^t \|\eta_1(s) - \eta_2(s)\|^2_{L^2(\Gamma_3;\mathbb{R}^d)}\,ds,
\end{aligned}$$

which leads to

$$\begin{aligned}
\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|^2_{L^2(\Gamma_3)} \le {}& c\int_0^t \|\beta_{\eta_1}(r) - \beta_{\eta_2}(r)\|^2_{L^2(\Gamma_3)}\,dr \\
&+ c\int_0^t \|\eta_1(s) - \eta_2(s)\|^2_{L^2(\Gamma_3,\mathbb{R}^d)}\,ds,
\end{aligned}$$

which, together with Gronwall's inequality (20), gives (68). ■

STEP 2. Let $\alpha > 0$. We define a regularized functional $j_\alpha : V \to \mathbb{R}$ by

$$j_\alpha(w) = \int_{\Gamma_3} g_b\sqrt{|w_\tau|^2 + \alpha^2}\,da, \qquad \forall w \in V,$$

which represents an approximation of j_τ. More precisely,

$$|j_\alpha(w) - j_\tau(w)| \le \alpha\|g_b\|_{L^1(\Gamma_3)}, \qquad \forall w \in V. \tag{75}$$

Moreover, j_α has a Gâteaux derivative $j'_\alpha : V \to V^*$ given by

$$\langle j'_\alpha(y), w\rangle_{V^*\times V} = \int_{\Gamma_3} \frac{g_b}{\sqrt{|y_\tau|^2 + \alpha^2}}\,y_\tau\cdot w_\tau\,da, \qquad \forall w, y \in V. \tag{76}$$

Since j_α is a convex function, it follows (see, e.g., [19]) that j'_α is a hemicontinuous operator and satisfies

$$(77)\qquad \langle j'_\alpha(y), w - y\rangle_{V^*\times V} \le j_\alpha(w) - j_\alpha(y), \quad \forall w, y \in V,$$

which leads to

$$(78)\qquad \langle j'_\alpha(y) - j'_\alpha(w), y - w\rangle_{V^*\times V} \ge 0, \quad \forall w, y \in V.$$

To continue, let $\alpha > 0$, let $\eta \in L^2(0,T;V)$, let β_η be the unique solution of problem (64)–(66), let ζ_η be the unique solution of problem (55)–(56). We can define $f_\eta \in L^2(0,T;V^*)$ by

$$(79)\qquad \langle f_\eta(t), w\rangle_{V^*\times V} = (f(t), w)_V - \big(\mathcal{B}(\varepsilon(\eta(t)), \zeta_\eta(t)), \varepsilon(w)\big)_{\mathcal{Q}} - j_{\mathrm{ad}}(\beta_\eta(t), \eta(t), w), \quad \text{a.e. } t \in (0,T),\ \forall w \in V,$$

and we consider the following regularized problem.

Problem 12. Let $\alpha > 0$ and let $\eta \in L^2(0,T;V)$. Find a function $v^\alpha_\eta : [0,T] \to V$ such that

$$(80)\qquad \langle \dot v^\alpha_\eta(t), w\rangle_{V^*\times V} + (\mathcal{A}(\varepsilon(v^\alpha_\eta(t))), \varepsilon(w))_{\mathcal{Q}} + \langle j'_\alpha(v^\alpha_\eta(t)), w\rangle_{V^*\times V} = \langle f_\eta(t), w\rangle_{V^*\times V}, \quad \forall w \in V,\ \text{a.e. } t \in (0,T),$$

$$(81)\qquad v^\alpha_\eta(0) = v_0.$$

Lemma 13. *Assume that* (21)–(30) *and* (31)(ii) *hold. Then problem* (80)–(81) *has a unique solution* v^α_η *which satisfies*

$$(82)\qquad v^\alpha_\eta \in L^2(0,T;V) \cap C([0,T];H), \quad \dot v^\alpha_\eta \in L^2(0,T;V^*).$$

Proof. Let $A : V \to V^*$ be defined by

$$(83)\qquad \langle Aw, z\rangle_{V^*\times V} = (\mathcal{A}(\varepsilon(w)), \varepsilon(z))_{\mathcal{Q}} + \langle j'_\alpha(w), z\rangle_{V^*\times V}, \quad \forall w, z \in V.$$

From (21) and (78), we deduce that A is a monotone operator. Moreover, using (83), (21) one has, for all $\lambda, \lambda_0 \in \mathbb{R}$,

$$|\langle A(w+\lambda v) - A(w+\lambda_0 v), z\rangle_{V^*\times V}| \le L_{\mathcal{A}}|\lambda - \lambda_0|\,\|v\|_V\|z\|_V + |\langle j'_\alpha(w+\lambda v) - j'_\alpha(w+\lambda_0 v), z\rangle_{V^*\times V}|, \quad \forall w, v, z \in V,$$

and since j'_α is hemicontinuous, it follows that A is hemicontinuous from V to V^*. To continue, since $\langle j'_\alpha(v), v\rangle_{V^*\times V} \ge 0$, using (21) and (83) we get

$$\begin{aligned} m_{\mathcal{A}}\|v\|_V^2 &\le \big(\mathcal{A}(\varepsilon(v)) - \mathcal{A}(\varepsilon(0)), \varepsilon(v) - \varepsilon(0)\big)_{\mathcal{Q}} \\ &\le (\mathcal{A}(\varepsilon(v)), \varepsilon(v))_{\mathcal{Q}} - (\mathcal{A}(\varepsilon(0)), \varepsilon(v))_{\mathcal{Q}} \\ &\le (\mathcal{A}(\varepsilon(v)), \varepsilon(v))_{\mathcal{Q}} + \langle j'_\alpha(v), v\rangle_{V^*\times V} - (\mathcal{A}(0), \varepsilon(v))_{\mathcal{Q}} \\ &\le \langle Av, v\rangle_{V^*\times V} + \|\mathcal{A}(0)\|_{\mathcal{Q}}\|v\|_V, \end{aligned}$$

which leads to

$$m_{\mathcal{A}}\|v\|_V^2 \le \langle Av, v\rangle_{V^*\times V} + \frac{1}{2m_{\mathcal{A}}}\|\mathcal{A}(0)\|_{\mathcal{Q}}^2 + \frac{m_{\mathcal{A}}}{2}\|v\|_V^2,$$

which implies that

$$\frac{m_{\mathcal{A}}}{2}\|v\|_V^2 - \frac{1}{2m_{\mathcal{A}}}\|\mathcal{A}(0)\|_{\mathcal{Q}}^2 \le \langle Av, v\rangle_{V^*\times V}.$$

On the other hand, from (83), (20), (21) and (76), one has

$$\begin{aligned}
&|\langle Av, w\rangle_{V^*\times V}|\\
&\quad\le |(\mathcal{A}(\varepsilon(v)), \varepsilon(w))_{\mathcal{Q}}| + |\langle j'_\alpha(v), w\rangle_{V^*\times V}|\\
&\quad\le \big|\big(\mathcal{A}(\varepsilon(v)) - \mathcal{A}(\varepsilon(0)), \varepsilon(w)\big)_{\mathcal{Q}}\big| + |(\mathcal{A}(\varepsilon(0)), \varepsilon(w))_{\mathcal{Q}}| + |\langle j'_\alpha(v), w\rangle_{V^*\times V}|\\
&\quad\le L_{\mathcal{A}}\|v\|_V\|w\|_V + \|\mathcal{A}(0)\|_{\mathcal{Q}}\|w\|_V + c_0\|g_b\|_{L^2(\Gamma_3)}\|w\|_V,
\end{aligned}$$

which implies that

$$\|Av\|_{V^*} \le L_{\mathcal{A}}\|v\|_V + \|\mathcal{A}(0)\|_{\mathcal{Q}} + c_0\|g_b\|_{L^2(\Gamma_3)}. \tag{84}$$

Thus, A satisfies conditions (3)–(4) with $\mathcal{C}_1 = m_{\mathcal{A}}/2$, $\mathcal{C}_2 = -\frac{1}{2m_{\mathcal{A}}}\|\mathcal{A}(0)\|_{\mathcal{Q}}^2$ and $\mathcal{C}_3 = \max(L_{\mathcal{A}}, \|\mathcal{A}(0)\|_{\mathcal{Q}} + c_0\|g_b\|_{L^2(\Gamma_3)})$. Now, keeping in mind (31)(ii), it follows from Proposition 2.2 that there exists a unique function v_η^α satisfying

$$\dot v_\eta^\alpha(t) + Av_\eta^\alpha(t) = f_\eta(t) \quad \text{in } V^*, \text{ a.e. } t \in (0,T), \tag{85}$$

$$v_\eta^\alpha(0) = v_0, \tag{86}$$

such that (82) holds. Thus problem (80)–(81) has a unique solution v_η^α which satisfies (82). ■

Step 3. Next, we introduce the space $\mathcal{W} = \{v \in \mathcal{V};\ \dot v \in \mathcal{V}^*\}$ which is a separable and reflexive Banach space equipped with the norm

$$\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|\dot v\|_{\mathcal{V}^*},$$

where $\mathcal{V} = L^2(0,T;V)$ and $\mathcal{V}^* = L^2(0,T;V^*)$. We have the following result.

Lemma 14. *There exists a function $v_\eta \in \mathcal{W}$ and a subsequence of $\{v_\eta^\alpha\}$, again denoted by $\{v_\eta^\alpha\}$, such that as $\alpha \to 0$, the following convergences hold:*

$$v_\eta^\alpha \rightharpoonup v_\eta \quad \text{weakly in } L^2(0,T;V), \tag{87}$$

$$\dot v_\eta^\alpha \rightharpoonup \dot v_\eta \quad \text{weakly in } L^2(0,T;V^*), \tag{88}$$

$$v_\eta^\alpha(t) \rightharpoonup v_\eta(t) \quad \text{weakly in } H,\ \forall t \in [0,T]. \tag{89}$$

Proof. It follows from (80) that

$$\begin{aligned}
\langle \dot v_\eta^\alpha(s), v_\eta^\alpha(s)\rangle_{V^*\times V} &+ \big(\mathcal{A}(\varepsilon(v_\eta^\alpha(s))), \varepsilon(v_\eta^\alpha(s))\big)_{\mathcal{Q}} + \langle j'_\alpha(v_\eta^\alpha(s)), v_\eta^\alpha(s)\rangle_{V^*\times V}\\
&= \langle f_\eta(s), v_\eta^\alpha(s)\rangle_{V^*\times V}, \quad s \in (0,T),
\end{aligned}$$

and since $\langle j'_\alpha(v_\eta^\alpha(s)), v_\eta^\alpha(s)\rangle_{V^*\times V} \ge 0$, we get

$$\begin{aligned}
\langle \dot v_\eta^\alpha(s), v_\eta^\alpha(s)\rangle_{V^*\times V} &+ \big(\mathcal{A}(\varepsilon(v_\eta^\alpha(s))) - \mathcal{A}(\varepsilon(0)), \varepsilon(v_\eta^\alpha(s))\big)_{\mathcal{Q}}\\
&\le \langle f_\eta(s), v_\eta^\alpha(s)\rangle_{V^*\times V} - \big(\mathcal{A}(\varepsilon(0)), \varepsilon(v_\eta^\alpha(s))\big)_{\mathcal{Q}}, \quad s \in (0,T),
\end{aligned}$$

and using (21), we deduce that

$$\langle \dot{v}_\eta^\alpha(s), v_\eta^\alpha(s)\rangle_{V^*\times V} + m_{\mathcal{A}}\|v_\eta^\alpha(s)\|_V^2 \le \frac{1}{m_{\mathcal{A}}}\|f_\eta(s)\|_{V^*}^2 + \frac{m_{\mathcal{A}}}{4}\|v_\eta^\alpha(s)\|_V^2 + \frac{1}{m_{\mathcal{A}}}\|\mathcal{A}(0)\|_{\mathcal{Q}}^2 + \frac{m_{\mathcal{A}}}{4}\|v_\eta^\alpha(s)\|_V^2.$$

Integrating both sides, one has

$$\|v_\eta^\alpha(t)\|_H^2 + \int_0^t \|v_\eta^\alpha(s)\|_V^2\, ds \le c\int_0^t \|f_\eta(s)\|_{V^*}^2\, ds + \|v_0\|_H^2 + c, \qquad \forall t\in[0,T].$$

From this, we get

$$\|v_\eta^\alpha\|_{L^\infty(0,T;H)} + \|v_\eta^\alpha\|_{L^2(0,T;V)} \le c. \tag{90}$$

To continue, using (84)–(85), we obtain

$$\|\dot{v}_\eta^\alpha(s)\|_{V^*} \le L_{\mathcal{A}}\|v_\eta^\alpha(s)\|_V + \|\mathcal{A}(0)\|_{\mathcal{Q}} + c_0\|g_b\|_{L^2(\Gamma_3)} + \|f_\eta(s)\|_{V^*},\ \ s\in(0,T),$$

which gives

$$\|\dot{v}_\eta^\alpha(s)\|_{V^*}^2 \le c\|v_\eta^\alpha(s)\|_V^2 + c\|f_\eta(s)\|_{V^*}^2 + c, \qquad \text{a.e. } s\in(0,T);$$

integrating both sides on $(0,t)$, we get

$$\int_0^t \|\dot{v}_\eta^\alpha(s)\|_{V^*}^2\, ds \le c\int_0^t \|v_\eta^\alpha(s)\|_V^2\, ds + c\int_0^t \|f_\eta(s)\|_{V^*}^2\, ds + c,$$

which, together with (90), implies that

$$\|\dot{v}_\eta^\alpha\|_{L^2(0,T;V^*)} \le c. \tag{91}$$

In (90)–(91), c is a positive constant independent of α. Thus, from standard compactness arguments, there exists a function $v_\eta \in L^2(0,T;V)\cap W^{1,2}(0,T;V^*)$ and a subsequence of $\{v_\eta^\alpha\}$, still denoted by $\{v_\eta^\alpha\}$, such that the convergences (87)–(88) hold. Now, since the inclusion map $\mathcal{W}\subset C([0,T];H)$ is continuous, the convergence (89) follows from (87)–(88). ■

Lemma 15. *For any $z\in L^2(0,T;V)$, the following properties hold:*

$$\liminf_{\alpha\to 0}\int_0^T \langle \dot{v}_\eta^\alpha(s), v_\eta^\alpha(s) - z(s)\rangle_{V^*\times V}\, ds \ge \int_0^T \langle \dot{v}_\eta(s), v_\eta(s) - z(s)\rangle_{V^*\times V}\, ds, \tag{92}$$

$$\liminf_{\alpha\to 0}\int_0^T [j_\alpha(v_\eta^\alpha(s)) - j_\alpha(z(s))]\, ds \ge \int_0^T [j_\tau(v_\eta(s)) - j_\tau(z(s))]\, ds, \tag{93}$$

$$\liminf_{\alpha\to 0}\int_0^T (\mathcal{A}(\varepsilon(v_\eta^\alpha(s))),\varepsilon(v_\eta^\alpha(s)-z(s)))_{\mathcal{Q}}\,ds \geq \int_0^T(\mathcal{A}(\varepsilon(v_\eta(s))),\varepsilon(v_\eta(s)-z(s)))_{\mathcal{Q}}\,ds. \tag{94}$$

Proof. Using (89) and (81) we find that $v_\eta^\alpha(0)=v_\eta(0)$. Moreover, we obtain

$$\liminf_{\alpha\to 0}\int_0^T\langle \dot v_\eta^\alpha(s), v_\eta^\alpha(s)\rangle_{V^*\times V}\,ds = \liminf_{\alpha\to 0}\left(\tfrac12 |||v_\eta^\alpha(T)|||_H^2-\tfrac12|||v_\eta(0)|||_H^2\right) \geq \tfrac12|||v_\eta(T)|||_H^2-\tfrac12|||v_\eta(0)|||_H^2 \geq \int_0^T\langle \dot v_\eta(s), v_\eta(s)\rangle_{V^*\times V}\,ds,$$

which together with (88) implies (92).

To establish (93), we write

$$\int_0^T[j_\alpha(v_\eta^\alpha(s))-j_\alpha(z(s))]\,ds = \int_0^T[j_\alpha(v_\eta^\alpha(s))-j_\tau(v_\eta^\alpha(s))]\,ds+\int_0^T[j_\tau(z(s))-j_\alpha(z(s))]\,ds + \int_0^T[j_\tau(v_\eta^\alpha(s))-j_\tau(z(s))]\,ds.$$

Keeping in mind (75), (87) and using standard lower semicontinuity arguments, in the last equality, we get

$$\liminf_{\alpha\to 0}\int_0^T[j_\alpha(v_\eta^\alpha(s))-j_\alpha(z(s))]\,ds = \liminf_{\alpha\to 0}\int_0^T[j_\tau(v_\eta^\alpha(s))-j_\tau(z(s))]\,ds \geq \int_0^T[j_\tau(v_\eta(s))-j_\tau(z(s))]\,ds.$$

To continue, let $\mathcal{F}:\mathcal{V}\to\mathcal{V}^*$ be the operator defined by

$$\langle \mathcal{F}w,z\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T\big(\mathcal{A}(\varepsilon(w(s))),\varepsilon(z(s))\big)_{\mathcal{Q}}\,ds,\qquad \forall w,z\in\mathcal{V}. \tag{95}$$

From (80), (77) and (95), we get

$$\langle \mathcal{F}v_\eta^\alpha, v_\eta^\alpha-v_\eta\rangle_{\mathcal{V}^*\times\mathcal{V}} \leq \int_0^T\langle \dot v_\eta^\alpha(s), v_\eta(s)-v_\eta^\alpha(s)\rangle_{V^*\times V}\,ds + \int_0^T[j_\alpha(v_\eta(s))-j_\alpha(v_\eta^\alpha(s))]\,ds+\int_0^T\langle f_\eta(s), v_\eta^\alpha(s)-v_\eta(s)\rangle_{V^*\times V}\,ds.$$

Passing to lim sup as $\alpha \to 0$ in the last inequality by using (92)–(93) and (87), we obtain

$$\limsup_{\alpha\to 0} \langle \mathcal{F}(v_\eta^\alpha), v_\eta^\alpha - v_\eta\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0,$$

and since the operator $\mathcal{F}$ is bounded, hemicontinuous and monotone, we deduce that $\mathcal{F}$ is pseudomonotone (see, e.g., [19]), and keeping in mind (2), (87) and (95) we deduce the convergence (94). ∎

STEP 4. In this step we consider the following variational problem.

PROBLEM 16. Let $\eta \in L^2(0,T;V)$. Find a function $v_\eta \in \mathcal{W}$ such that

$$\begin{aligned}(96)\qquad & \langle \dot v_\eta(s), w - v_\eta(s)\rangle_{V^*\times V} + \big(\mathcal{A}(\varepsilon(v_\eta(s))), \varepsilon(w - v_\eta(s))\big)_{\mathcal{Q}} \\ &\quad + j_\tau(w) - j_\tau(v_\eta(s)) \ge \langle f_\eta(s), w - v_\eta(s)\rangle_{V^*\times V}, \\ &\quad \forall w\in V,\ \text{a.e. } s\in(0,T),\end{aligned}$$

$$(97)\qquad v_\eta(0) = v_0.$$

We have the following existence and uniqueness result.

LEMMA 17. *Assume that* (21)–(30) *and* (31)(ii) *hold. Then problem* (96)–(97) *has a unique solution* v_η *which satisfies*

$$(98)\qquad v_\eta \in L^2(0,T;V)\cap C([0,T];H), \qquad \dot v_\eta \in L^2(0,T;V^*).$$

Proof. In view of (80), (77) we deduce that the function v_η^α satisfies

$$\begin{aligned}(99)\qquad & \int_0^T \langle \dot v_\eta^\alpha(s), z(s) - v_\eta^\alpha(s)\rangle_{V^*\times V}\, ds \\ &+ \int_0^T \big(\mathcal{A}(\varepsilon(v_\eta^\alpha(s))), \varepsilon(z(s) - v_\eta^\alpha(s))\big)_{\mathcal{Q}}\, ds + \int_0^T [j_\alpha(z(s)) - j_\alpha(v_\eta^\alpha(s))]\, ds \\ &\ge \int_0^T \langle f_\eta(s), z(s) - v_\eta^\alpha(s)\rangle_{V^*\times V}\, ds, \qquad \forall z\in L^2(0,T;V).\end{aligned}$$

Let $t\in(0,T)$ and $r>0$ be such that $t+r\in(0,T)$, and let $w\in V$. Then in (99) we put

$$z(s) = \begin{cases} w & \text{for } s\in(t,t+r),\\ v_\eta(s) & \text{elsewhere,}\end{cases}$$

and pass to the lim sup as $\alpha\to 0$, to obtain

$$\begin{aligned}&\frac{1}{r}\int_t^{t+r} \langle \dot v_\eta(s), w - v_\eta(s)\rangle_{V^*\times V}\, ds + \frac{1}{r}\int_t^{t+r} \big(\mathcal{A}(\varepsilon(v_\eta(s))), \varepsilon(w - v_\eta(s))\big)_{\mathcal{Q}}\, ds \\ &\quad + \frac{1}{r}\int_t^{t+r} [j_\tau(w) - j_\tau(v_\eta(s))]\, ds \ge \frac{1}{r}\int_t^{t+r} \langle f_\eta(s), w - v_\eta(s)\rangle_{V^*\times V}\, ds, \qquad \forall w\in V.\end{aligned}$$

Let v_η be the function obtained in Lemma 14. Since $v_\eta^\alpha(0) = v_\eta(0)$, using (86) and passing to the limit as $r \to 0$ in the last inequality we deduce that the function v_η is a solution of problem (96)–(97) such that (98) holds.

For the uniqueness, let $v_1, v_2 \in \mathcal{W}$ be two functions satisfying (96)–(97). Setting, in (96), $(v_\eta, w) = (v_1, v_2)$ and $(v_\eta, w) = (v_2, v_1)$, and adding the two inequalities we get

$$\int_0^T \langle \dot v_1(s) - \dot v_2(s), v_1(s) - v_2(s)\rangle_{V^*\times V}\, ds + \int_0^T \big(\mathcal{A}(\varepsilon(v_1(s))) - \mathcal{A}(\varepsilon(v_2(s))), \varepsilon(v_1(s) - v_2(s))\big)_{\mathcal{Q}}\, ds \le 0.$$

Keeping in mind (97) and (21), we deduce that

$$\tfrac{1}{2}|||v_1(T) - v_2(T)|||_H^2 + m_{\mathcal{A}}\|v_1 - v_2\|_{L^2(0,T;V)}^2 \le 0,$$

which shows that $v_1 = v_2$. ∎

STEP 5. Let $\eta \in L^2(0,T;V)$, let v_η be the unique solution of problem (96)–(97), let β_η be the unique solution of problem (64)–(66), let ζ_η be the unique solution of problem (55)–(56), and define $\Lambda : L^2(0,T;V) \to L^2(0,T;V)$ by

$$\Lambda\eta(t) = \int_0^t v_\eta(s)\, ds + u_0, \qquad \forall \eta \in L^2(0,T;V),\ \forall t \in [0,T]. \tag{100}$$

We have the following result.

LEMMA 18. *The operator Λ has a unique fixed point $\eta^* \in L^2(0,T;V)$.*

Proof. Let $\eta_1, \eta_2 \in L^2(0,T;V)$, then by (96) and (79),

$$\begin{aligned}
&\int_0^t \langle \dot v_{\eta_1}(s) - \dot v_{\eta_2}(s), v_{\eta_1}(s) - v_{\eta_2}(s)\rangle_{V^*\times V}\, ds \\
&\quad + \int_0^t \big(\mathcal{A}(\varepsilon(v_{\eta_1}(s))) - \mathcal{A}(\varepsilon(v_{\eta_2}(s))), \varepsilon(v_{\eta_1}(s) - v_{\eta_2}(s))\big)_{\mathcal{Q}}\, ds \\
&\le \int_0^t j_{\mathrm{ad}}\big(\beta_{\eta_1}(s), \eta_1(s), v_{\eta_2}(s) - v_{\eta_1}(s)\big)\, ds \\
&\quad - \int_0^t j_{\mathrm{ad}}\big(\beta_{\eta_2}(s), \eta_2(s), v_{\eta_2}(s) - v_{\eta_1}(s)\big) \\
&\quad + \int_0^t \big(\mathcal{B}(\varepsilon(\eta_1(s)), \zeta_{\eta_1}(s)) - \mathcal{B}(\varepsilon(\eta_2(s)), \zeta_{\eta_2}(s)), \varepsilon(v_{\eta_2}(s) - v_{\eta_1}(s))\big)_{\mathcal{Q}}\, ds.
\end{aligned}$$

Using (97), (21)–(22), (38) and (67)(ii), we obtain

$$\begin{aligned}\|\!|v_{\eta_1}(t)-v_{\eta_2}(t)|\!\|_H^2 &+ \int_0^t \|v_{\eta_1}(s)-v_{\eta_2}(s)\|_V^2\,ds\\ &\le c\int_0^t \|\beta_{\eta_1}(s)-\beta_{\eta_2}(s)\|_{L^2(\Gamma_3)}\|v_{\eta_1}(s)-v_{\eta_2}(s)\|_V\,ds\\ &\quad + c\int_0^t \|\eta_1(s)-\eta_2(s)\|_V\|v_{\eta_1}(s)-v_{\eta_2}(s)\|_V\,ds\\ &\quad + c\int_0^t \|\zeta_{\eta_1}(s)-\zeta_{\eta_2}(s)\|_V\|v_{\eta_1}(s)-v_{\eta_2}(s)\|_V\,ds.\end{aligned}$$

We then use the inequality

$$\lambda\delta \le \lambda^2 + \tfrac{1}{4}\delta^2, \qquad \forall \lambda,\delta\in\mathbb{R}.$$

We deduce that

$$\begin{aligned}\int_0^t \|v_{\eta_1}(s)-v_{\eta_2}(s)\|_V^2\,ds &\le c\int_0^t \|\beta_{\eta_1}(s)-\beta_{\eta_2}(s)\|_{L^2(\Gamma_3)}^2\,ds\\ &\quad + c\int_0^t \|\eta_1(s)-\eta_2(s)\|_V^2\,ds + c\int_0^t \|\zeta_{\eta_1}(s)-\zeta_{\eta_2}(s)\|_{L^2(\Omega)}^2\,ds.\end{aligned}$$

Therefore, by (68) and (58),

$$\int_0^t \|v_{\eta_1}(s)-v_{\eta_2}(s)\|_V^2\,ds \le c\int_0^t \|\eta_1(s)-\eta_2(s)\|_V^2\,ds. \tag{101}$$

Now, using (100), we obtain

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_V^2 \le c\int_0^t \|v_{\eta_1}(s)-v_{\eta_2}(s)\|_V^2\,ds,$$

which, together with (101), implies that

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_V^2 \le c\int_0^t \|\eta_1(s)-\eta_2(s)\|_V^2\,ds, \qquad \forall t\in[0,T].$$

Reiterating the last inequality n times, we infer that

$$\|\Lambda\eta_1-\Lambda\eta_2\|_{L^2(0,T;V)}^2 \le \frac{(cT)^n}{n!}\|\eta_1-\eta_2\|_{L^2(0,T;V)}^2,$$

which implies that, for n sufficiently large, a power Λ^n of Λ is a contraction in the Banach space $L^2(0,T;V)$. Hence Λ has a unique fixed point $\eta^*\in L^2(0,T;V)$. ■

Now, we have all the ingredients to prove Theorem 7. Let Λ be the operator defined by (100), let η^* be the fixed point of Λ, let $\beta = \beta_{\eta^*}$ be the unique solution of problem (64)–(66) corresponding to η^*, let $\zeta = \zeta_{\eta^*}$ be the unique solution of problem (55)–(56), and let $u : [0,T] \to V$ be the displacement field defined by

$$u(t) = \eta^*(t) = \int_0^t v_{\eta^*}(s)\,ds + u_0, \qquad \forall t \in [0,T], \tag{102}$$

where v_{η^*} is the unique solution of problem (96)–(97) corresponding to η^*. We conclude by (96)–(97), (64)–(66), (55)–(56) and (102) that $\{u, \beta, \zeta\}$ is a solution of problem (46)–(51). Moreover, the regularity (52)–(54) follows from (67), (57), (98) and (102). The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ and of the uniqueness of the solution of problems (64)–(66), (55)–(56) and (96)–(97). Finally, it is easy to see, in this case, that the function σ defined by (6) satisfies

$$\sigma \in L^2(0,T;\mathcal{Q}), \qquad \operatorname{Div}\sigma \in L^2(0,T;V^*).$$

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