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Exact Wiener–Ikehara theorems

by

WEN-BIN ZHANG* (Urbana, IL, and Guangzhou)

1. Introduction. The Wiener–Ikehara (henceforth, W-I) theorem is one of a few most famous theorems in tauberian theory [Bor]. It is the ultimate result of seeking a proof of the prime number theorem with as little assumed about the Riemann zeta function as possible [Ike31, Kor1]. In its long history the theorem has many different generalizations [Kor1]. Its most well-known form is as follows.

THEOREM (W-I). *Let $F(x)$ be a real-valued function with support in $[0, \infty)$ which is nondecreasing and continuous from the right. Suppose that the Laplace–Stieltjes transform*

$$\mathcal{F}(s) := \int_0^\infty e^{-sx}\, dF(x)$$

is convergent for $\Re s = \sigma > 1$. If, for some constant c, the analytic function

$$G(s) := \frac{\mathcal{F}(s+1)}{s+1} - \frac{c}{s}, \qquad \sigma > 0,$$

has an extension $G(it)$ such that $G(\sigma + it)$ converges to $G(it)$ as $\sigma \to 0+$ uniformly or in L^1 on every finite interval $[-T, T]$ then

$$\lim_{x\to\infty} e^{-x} F(x) = c.$$

Generalizations of the W-I theorem are usually devoted either to relaxing the "tauberian conditions" or to estimating the "error term" $F(x) - ce^x$; see, e.g. [GrVa, ReRo].

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* Cantonese: Cheung Man Ping.

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In [Zha14], it is shown that the tauberian condition that $G(\sigma+it)$ converges to $G(it)$ as $\sigma\to 0+$ uniformly or in L^1 on every finite interval $[-T,T]$ can be replaced by an "if and only if" condition that there exists $\lambda_0\ge 0$ such that, for every $\lambda>\lambda_0$,

$$\int_{-2\lambda}^{2\lambda}\frac{1}{2}\Big(1-\frac{|t|}{2\lambda}\Big)e^{ity}\big(G(\sigma+it)-G(\sigma'+it)\big)\,dt$$

approaches zero as σ, $\sigma'\to 0+$ uniformly for $y\ge y_0(\lambda)$.

We notice that, in applications, when $F(x)$ is a summatory function of a number-theoretic function, such as $\sum_{n\le x}\Lambda(n)$, the theorem is a kind of mean-value theorem like those, say, in probabilistic number theory. If the remaining tauberian condition that $F(x)$ is nondecreasing can be removed then the method of W-I theorem may also be used in the mean-value problem. This is the main motivation of this new paper.

Here we show further that the remaining tauberian condition can be replaced by an "if and only if" condition that $F(\log u)$ is a linearly slowly decreasing function of u (see Definition 1). A slowly decreasing function (hence, also a nondecreasing function) is linearly slowly decreasing. Also, summatory functions of many number-theoretic functions, such as the Möbius function $\mu(n)$, are linearly slowly decreasing but not nondecreasing. The new form of the W-I theorem makes it possible to apply the W-I theorem to, but not only, the mean-value problem for those functions. In fact, we prove an exact Wiener–Ikehara theorem which works for a much larger class of linearly slowly decreasing functions. Also, our theorem holds for poles of higher orders.

Let a real-valued function $f(x)$ be defined on $[a,\infty)$, $a>0$, and let α be a nonnegative real number.

Definition 1. $f(x)$ is said to be *linearly slowly decreasing* (henceforth, l.s.d.) with index α if

$$\liminf_{x\to\infty,\,y/x\to 1+}(x\log^\alpha x)^{-1}(f(y)-f(x))\ge 0.$$

In other words, for a l.s.d. function $f(x)$ with index α, given $\epsilon>0$, there exist a sufficiently large number $x(\epsilon)>a$ and a sufficiently small number $\eta(\epsilon)>0$ such that

$$(f(y)-f(x))/(x\log^\alpha x)>-\epsilon$$

for all (x,y) satisfying $x\ge x(\epsilon)$ and $x<y\le x(1+\eta(\epsilon))$.

We note that if $f(x)$ is l.s.d. with index α then it is also l.s.d. with any index $\beta>\alpha$.

For brevity, in the following discussion, if the index α is zero or the index is not emphasized, $f(x)$ is sometimes simply called a l.s.d. function.

Consider the W-I theorem in Laplace transform form. Let $F(x)$ be a real-valued Lebesgue measurable function with support in $[0,\infty)$. Assume that

$$\int_0^\infty e^{-\sigma x}|F(x)|\,dx<\infty \qquad \text{for all } \sigma>1.$$

Set

$$\int_0^\infty e^{-sx}F(x)\,dx=\frac{L}{(s-1)^{\alpha+1}}+G(s) \tag{1.1}$$

with real constants L and $\alpha\ge 0$ for $\sigma>1$.

Let

$$\Delta_\lambda(t):=\frac{1}{2}\left(1-\frac{|t|}{2\lambda}\right)^{+}, \qquad t\in\mathbb{R},$$

with $\lambda>0$. It has support $[-2\lambda,2\lambda]$. Also, let $\Delta_\lambda^{*m}(t)$ denote the m-fold convolution of $\Delta_\lambda(t)$ with itself for $m\in\mathbb{N}$. Note that $\Delta_\lambda^{*m}(t)$ has a compact support too.

THEOREM 1 (Exact Wiener–Ikehara Theorem). (1) *If* $F(\log u)$ *is a l.s.d. function of* u *with index* α *and there exist a constant* $\lambda_0\ge 0$ *and a positive integer* $m\ge 1+[\alpha]/2$ *such that the function* $G(s)$ *defined by* (1.1) *satisfies* (i) *the limit*

$$\lim_{\sigma\to 1+}\int_{-\infty}^{\infty}\Delta_\lambda^{*m}(t)e^{ity}G(\sigma+it)\,dt \tag{1.2}$$

exists for every $\lambda>\lambda_0$ *and every* $y\ge y_0(\lambda)$ *and* (ii) *we have*

$$\frac{1}{y^\alpha}\lim_{\sigma\to 1+}\int_{-\infty}^{\infty}\Delta_\lambda^{*m}(t)e^{ity}G(\sigma+it)\,dt=o_\lambda(1) \qquad \text{as } y\to\infty \tag{1.3}$$

then

$$F(x)=\frac{Le^x x^\alpha}{\Gamma(\alpha+1)}(1+o(1)) \qquad \text{as } x\to\infty, \tag{1.4}$$

where Γ *is the Euler gamma function.*

(2) *Conversely, if* (1.4) *holds as* $x\to\infty$ *then* (i) $F(\log u)$ *is a l.s.d. function of* u *with index* α *and* (ii) *for every* $\lambda>0$ *and each integer* $m\ge 1+[\alpha]/2$ *the limit* (1.2) *exists for every* $y>0$ *and* (1.3) *is satisfied as* $y\to\infty$.

REMARK 1. For $\lim_{x\to\infty}e^{-x}x^{-\alpha}F(x)=L/\Gamma(\alpha+1)$ to be true we must assume that (1.3) is satisfied for all $\lambda>\lambda_0$ as shown by an example with $\alpha=0$ in [Zha14].

Remark 2. If, for every $\lambda > \lambda_0$, there exists a sequence $\sigma_n(\lambda) \to 1+$ such that (1.2) exists and (1.3) holds with $\sigma = \sigma_n(\lambda)$ then the conclusion of the theorem is still true. This can be seen in the following proof.

The function $f(x) = x + Ax^{1-\epsilon}\cos x$ where $A > 0$, $0 < \epsilon < 1$ is not nondecreasing. It satisfies $\int_0^\infty e^{-sx} f(e^x)\,dx = (s-1)^{-1} + G(s)$ with $G(s) = A\int_1^\infty u^{-s-\epsilon}\cos u\,du$ and $\lim_{x\to\infty} e^{-x} f(e^x) = 1$. By Theorem 1, $f(x)$ is linearly slowly decreasing (with index $\alpha = 0$), as shown in a direct way in Section 2. Also, by the Riemann–Lebesgue lemma, $G(s)$ satisfies (1.2) and (1.3) with $m = 1$.

Combined with the Riemann–Lebesgue lemma, Theorem 1 yields an equivalent form.

Theorem 2 (Exact Wiener–Ikehara Theorem). (1.4) *holds if* $F(\log u)$ *is a l.s.d. function of* u *with index* α *and there exist a constant* $\lambda_0 \ge 0$ *and a positive integer* $m \ge 1 + [\alpha]/2$ *such that, for each* $\lambda > \lambda_0$,

$$\frac{1}{y^\alpha}\int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}(G(\sigma+it) - G(\sigma'+it))\,dt \tag{1.5}$$

approaches zero as σ, $\sigma' \to 1+$ *uniformly for* $y \ge y_0(\lambda)$ (>0).

Conversely, if (1.4) *holds then* (i) $F(\log u)$ *is l.s.d. with index* α *and* (ii) *for all* $\lambda > 0$ *and integers* $m \ge 1 + [\alpha]/2$, (1.5) *approaches zero as* $\sigma, \sigma' \to 1+$ *uniformly for* $y > 0$.

Remark 3. In particular, $\lim_{x\to\infty} e^{-x}F(x)\,dx = L$ if and only if $F(\log u)$ is linearly slowly decreasing with index 0 and

$$\int_{-\infty}^{\infty} \Delta_\lambda(t)e^{ity}(G(\sigma+it) - G(\sigma'+it))\,dt$$

approaches zero as σ, $\sigma' \to 1+$ uniformly for $y \ge y_0(\lambda)$ (>0).

Corollary 1. *If* $F(\log u)$ *is l.s.d. with index* α *and there exists* $\lambda_0 \ge 0$ *such that for all* $\lambda > \lambda_0$,

$$\lim_{\sigma,\sigma'\to 1+} \int_{-2\lambda}^{2\lambda} |G(\sigma+it) - G(\sigma'+it)|\,dt = 0$$

then (1.4) *holds.*

The classical W-I theorem is a special case of Corollary 1 with index 0.

Corollary 2. *If* $F(\log u)$ *is linearly slowly decreasing with index* $\gamma + n - 1$, *where* $0 \le \gamma < 1$, $n \in \mathbb{N}$, *and*

$$\int_0^\infty e^{-sx}F(x)\,dx = \frac{\phi(s)}{(s-1)^{\gamma+n}} + \psi(s) \quad \textit{for } \Re s = \sigma > 1,$$

where $\phi(s)$ *and* $\psi(s)$ *are analytic on* $\{\sigma \geq 1\}$ *and* $\phi(1) \neq 0$, *then*

$$F(x) = \frac{\phi(1)e^x x^{\gamma+n-1}}{\Gamma(\gamma+n)}(1+o(1)) \quad \text{as } x \to \infty.$$

This is an extension of Theorem 7.7 of [BaDi] dealing with nondecreasing functions.

We also prove Wiener–Ikehara upper and lower bound theorems with looser conditions.

Definition 2. $f(x)$ is said to be *linearly bounded below decreasing* with index α if

$$\liminf_{x\to\infty,\, y/x\to 1+} (x\log^\alpha x)^{-1}(f(y)-f(x)) > -\infty. \tag{1.6}$$

In other words, for a linearly bounded below decreasing function $f(x)$ with index α, there exist positive constants A, x_0 and η such that

$$(x\log^\alpha x)^{-1}(f(y)-f(x)) > -A$$

for all (x,y) satisfying $x \geq x_0$ and $x < y \leq x(1+\eta)$.

Recall that $\Delta_\lambda^{*m}(t)$ is the m-fold convolution of $\Delta_\lambda(t)$ with itself. The Fourier transform of $\Delta_\lambda^{*m}(t)$ is the mth power of the Fourier transform $k_\lambda(x)$ of $\Delta_\lambda(t)$, i.e.,

$$\int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{itx}\,dt = k_\lambda^m(x) = \left(\lambda\left(\frac{\sin\lambda x}{\lambda x}\right)^2\right)^m. \tag{1.7}$$

Moreover,

$$\int_{-\infty}^{\infty} k_\lambda^m(x)\,dx = \lambda^{m-1}C_m$$

with constant

$$C_m = \int_{-\infty}^{\infty}\left(\frac{\sin x}{x}\right)^{2m} dx;$$

in particular, $C_1 = \pi$. Also,

$$\int_{|x|\geq\delta} k_\lambda^m(x)\,dx \leq \frac{2}{(2m-1)\lambda^m\delta^{2m-1}} \quad \text{for } \delta > 0. \tag{1.8}$$

Theorem 3 (W-I Upper Bound). *If* $F(\log u)$ *is linearly bounded below decreasing with index* α *and if there exist a positive integer* $m \geq 1 + [\alpha]/2$ *and positive constants* λ, y_0, K, C_m *such that*

$$\limsup_{\sigma\to 1+} \int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}G(\sigma+it)\,dt < K\lambda^{m-1}C_m y^\alpha \tag{1.9}$$

for each $y \geq y_0$ *then* $F(x)(e^x x^\alpha)^{-1} \leq C$ *for some constant* $C > 0$.

Conversely, if $F(x)(e^x x^\alpha)^{-1} \le C$ *for some constant* $C > 0$ *then, for every* $\lambda > 0$ *and every* $m \ge 1 + [\alpha]/2$, (1.9) *holds with some constants* K *and* C_m *uniformly for* $y \ge y_0(\lambda)$ (> 0).

Remark 4. Since $F(x)$ is real-valued, from (1.1) we see that $G(\sigma)$ is real-valued for real $\sigma > 1$. By the reflection principle, $G(\bar{s}) = \overline{G(s)}$ for complex s. Moreover, $\Delta_\lambda^{*m}(t)$ is an even function of real t. Hence the left-hand side of (1.9) (and also (1.11)) is real-valued.

We now investigate the existence of a positive lower bound for $F(x)/(e^x x^\alpha)$. Let $F(x)$ be nonnegative and satisfy the convention

$$\lim_{\sigma\to 1+} (\sigma - 1)^{\alpha+1} \int_0^\infty e^{-\sigma x} F(x)\, dx = L \tag{1.10}$$

with the constant L in (1.1).

Theorem 4 (W-I Lower Bound). *If* $F(x)(e^x x^\alpha)^{-1} \ge c$ *for all* $x \ge x_0$ (> 0) *and some constant* $c > 0$ *then, for every* $\lambda > 0$ *and every integer* $m \ge 1 + [\alpha]/2$, *there exists a constant* $\gamma > 0$ *such that*

$$\liminf_{\sigma\to 1+} \int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t) e^{ity} G(\sigma + it)\, dt \ge \left(\frac{-L}{\Gamma(\alpha + 1)} + \gamma \right) \lambda^{m-1} C_m y^\alpha \tag{1.11}$$

for $y \ge y_0(\lambda)$.

Conversely, assume that there exist positive constants λ, γ, y_0 *and an integer* $m \ge 1 + [\alpha]/2$ *such that* (1.11) *holds for all* $y \ge y_0$. *If* $F(\log u)$ *is nondecreasing or linearly bounded below decreasing with index* $\beta < \alpha$ *and if* $F(x)(e^x x^\alpha)^{-1} \le C$ *for some constant* $C > 0$ *then* $F(x)(e^x x^\alpha)^{-1} \ge c$ *for all* $x \ge x_0$ (> 0) *with some constant* $c > 0$.

Remark 5. The condition (1.10) alone implies neither an upper bound nor a positive lower bound for $F(x)(e^x x^\alpha)^{-1}$. In case $\alpha = 0$, this is shown by the example given in [DiZh].

Remark 6. Note that, for a positive lower bound, we assume that $F(\log u)$ is nondecreasing or linearly bounded below decreasing with index $\beta < \alpha$. This is much stronger than being linearly bounded below decreasing with index α.

As a preliminary application of the exact Wiener–Ikehara theorem, in Section 5 we sketch a direct proof of the proposition $M(x) := \sum_{n\le x} \mu(n) = o(x)$, where μ is the Möbius function, without appealing to $N(x) + M(x)$ as usual. Also, we show the nonequivalence of $M(x) = o(x)$ and $\psi(x) \ll x$, where $\psi(x)$ is the Chebyshev function.

2. Linearly slowly decreasing functions. In this section, we compare the definition of l.s.d. functions with the well-known definition of slowly decreasing functions and set up some basic properties of l.s.d. functions. Readers who want to look quickly at the proofs of the theorems may go directly to Lemma 1 and leave the rest of the section for a second read.

Note that the sum of two l.s.d. functions with the same index on the same interval $[a,\infty)$ is l.s.d. with that index. Also, the product of a l.s.d. function with a nonnegative constant is l.s.d. with the same index again.

Definition (Schmidt [Sch25a]). A function $f(x)$ is said to be *slowly decreasing* if

$$\liminf_{x\to\infty,\, y/x\to 1+} (f(y)-f(x)) \ge 0.$$

Since Schmidt's significant works [Sch25a, Sch25b], slowly decreasing functions have been used extensively in tauberian theory; see, e.g., Wiener [Wie32] and Hardy [Har49], and also [Kor1].

Nondecreasing functions are slowly decreasing and slowly decreasing functions are linearly slowly decreasing with index 0. The function $f(x) = x + Ax^{1-\epsilon}\cos x$ for $x>0$, where $A>0$, $0<\epsilon<1$, is l.s.d. with index 0 but neither nondecreasing nor slowly decreasing. Actually,

$$f(y)-f(x) \ge -A(\lambda^{1-\epsilon}+1)x^{1-\epsilon}$$

for $x<y\le \lambda x$ with $\lambda>1$ and hence

$$\liminf_{x\to\infty,\, y/x\to 1+} x^{-1}(f(y)-f(x)) \ge 0.$$

Therefore $f(x)$ is l.s.d. On the other hand, for $x_n = 2n\pi$, $y_n = x_n+\pi$,

$$f(y_n)-f(x_n) = \pi - A(x_n+\pi)^{1-\epsilon} - Ax_n^{1-\epsilon}$$

and hence

$$\liminf_{x\to\infty,\, y/x\to 1+} (f(y)-f(x)) = -\infty.$$

Therefore, $f(x)$ is not slowly decreasing.

As a less elementary example of l.s.d. functions, consider a real-valued function $f(n_i)$ defined on Beurling numbers $\mathcal{N}=\{n_i\}$ [BaDi, Beu] (see also Section 5). If $|f(n_i)|\le 1$ for all $n_i\in\mathcal{N}$ and $\mathcal{N}$ has a density $A>0$, i.e., $N(x) := \sum_{n_i\le x} 1 \sim Ax$, then the summatory function $F(x) = \sum_{n_i\le x} f(n_i)$ is l.s.d. with index 0. In this case, since $|f(n_i)|\le 1$ and $N(x)\sim Ax$, we see that

$$F(y)-F(x) = \sum_{x<n_i\le y} f(n_i) \ge -(N(y)-N(x)) = -A(y-x)+o(y)$$

and hence $\liminf_{x\to\infty,\, y/x\to 1+} x^{-1}(F(y)-F(x)) \ge 0$. In particular, the summatory function $M(x) := \sum_{n_i\le x}\mu(n_i)$ of the Möbius function $\mu(n_i)$ on $\mathcal{N}$ [Zha87] is l.s.d. On the other hand, in the particular case of rational integers $\mathbb{N}$, if p is a rational prime then $M(p)-M(p-) = -1$ and

$\liminf_{x\to\infty,\, y/x\to 1+}(M(y)-M(x)) \le -1$, i.e., $M(x)$ is not slowly decreasing.

We note that a function $f(x)$ is l.s.d. with index α if and only if

$$\liminf_{x\to\infty,\, y-x\to 0+} e^{-x}x^{-\alpha}(f(e^y)-f(e^x)) \ge 0. \tag{2.1}$$

In applications of the Wiener–Ikehara theorem in the form of Laplace transforms, given $f(x)$, we investigate $F(x) := f(e^x)$, i.e. $F(\log x) = f(x)$. For instance, in the well-known Wiener–Ikehara proof of the prime number theorem, we investigate $\psi(e^x)$.

Proposition 1. (1) *$f(u) := F(\log u)$ is a l.s.d. function of u with index α if and only if*

$$\liminf_{x\to\infty,\, y-x\to 0+} e^{-x}x^{-\alpha}(F(y)-F(x)) \ge 0.$$

Hence if $F(x)$ is l.s.d. then $f(u) := F(\log u)$ is l.s.d. with the same index.

(2) *$f(u) := F(\log u)$ is a slowly decreasing function of u if and only if*

$$\liminf_{x\to\infty,\, y-x\to 0+} (F(y)-F(x)) \ge 0.$$

Hence if $F(x)$ is slowly decreasing then $f(u) := F(\log u)$ is slowly decreasing (hence, l.s.d.).

The proof is straightforward.

The converses of the "hence" parts of Proposition 1 are usually not true. The function $F(x) := x(1+\cos x)$ is not l.s.d. with index 0 while $f(u) = F(\log u)$ is. Actually, for $x_n = 2n\pi$, $y_n = x_n + \pi$,

$$x_n^{-1}(F(y_n)-F(x_n)) = -2$$

and hence $\liminf x^{-1}(F(y)-F(x)) \le -2$. On the other hand,

$$\begin{aligned} f(u)-f(v) &= (\log u)(1+\cos(\log u)) - (\log v)(1+\cos(\log v)) \\ &= \left(\log\frac{u}{v}\right)(1+\cos(\log u)) - 2(\log v)\sin\frac{\log uv}{2}\sin\frac{\log u/v}{2} \end{aligned}$$

and so $v^{-1}(f(u)-f(v)) \to 0$ as $v\to\infty$, $u/v\to 1+$.

We then set up some basic properties of linearly bounded below decreasing functions. A function $f(x)$ is linearly bounded below decreasing with index α if and only if

$$\liminf_{x\to\infty,\, y-x\to 0+} (e^x x^\alpha)^{-1}(f(e^y)-f(e^x)) > -\infty. \tag{2.2}$$

If $F(x)$ is linearly bounded below decreasing with index α then $f(u) := F(\log u)$ is linearly bounded below decreasing with the same index.

Also, a l.s.d. function with index α is linearly bounded below decreasing with the same index.

LEMMA 1. *If $F(\log u)$ is linearly bounded below decreasing with index α then*

$$\liminf_{x\to\infty} F(x)(e^x x^\alpha)^{-1} \ge -c \tag{2.3}$$

for some positive constant c.

Proof. We have

$$(e^x x^\alpha)^{-1}(F(y) - F(x)) \ge -A$$

for $x \ge x_0$ and $0 < y - x < 2\delta$ with some $\delta > 0$. In particular, fixing $\delta > 0$, we get

$$e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}\big(F(x_0+(n+1)\delta) - F(x_0+n\delta)\big) \ge -A, \qquad n = 0, 1, \ldots.$$

Hence

$$\begin{aligned}
&e^{-(x_0+(n+1)\delta)}(x_0+(n+1)\delta)^{-\alpha}F(x_0+(n+1)\delta)\\
&\quad\ge \big(e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}F(x_0+n\delta) - A\big)e^{-\delta}\left(\frac{x_0+n\delta}{x_0+(n+1)\delta}\right)^{\alpha}\\
&\quad= e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}F(x_0+n\delta)e^{-\delta}\left(\frac{x_0+n\delta}{x_0+(n+1)\delta}\right)^{\alpha}\\
&\qquad - Ae^{-\delta}\left(\frac{x_0+n\delta}{x_0+(n+1)\delta}\right)^{\alpha}.
\end{aligned}$$

Then, by induction,

$$\begin{aligned}
&e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}F(x_0+n\delta)\\
&\qquad\ge e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}F(x_0) - A\sum_{k=1}^{n} e^{-k\delta}\left(\frac{x_0+(n-k)\delta}{x_0+n\delta}\right)^{\alpha}.
\end{aligned}$$

Therefore

$$\liminf_{n\to\infty} e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}F(x_0+n\delta) \ge -Ae^{-\delta}/(1-e^{-\delta}).$$

In general, if $x_0 + n\delta \le x < x_0 + (n+1)\delta$ then

$$e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}\big(F(x) - F(x_0+n\delta)\big) \ge -A$$

and hence

$$\begin{aligned}
&e^{-x}x^{-\alpha}F(x)\\
&\ge \big(e^{-(x_0+n\delta)}(x_0+n\delta)^{-\alpha}F(x_0+n\delta) - A\big)e^{-(x-(x_0+n\delta))}\left(\frac{x_0+n\delta}{x}\right)^{\alpha}.
\end{aligned}$$

It follows that

$$\liminf_{x\to\infty} e^{-x}x^{-\alpha}F(x) \ge -Ae^{-\delta}/(1-e^{-\delta}) - A = -A(1-e^{-\delta})^{-1}. \ \blacksquare$$

3. Proof of Theorem 1. For $\alpha > 0$ and $\sigma > 1$, we first have

$$\frac{1}{(s-1)^\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-(s-1)x} x^{\alpha-1}\,dx. \tag{3.1}$$

This equality can be seen by contour integration.

LEMMA 2. *For $\alpha \ge 0, m \in \mathbb{N}$ with $m \ge 1 + [\alpha]/2$,*

$$y^{-\alpha} \int_0^\infty x^\alpha k_\lambda^m(y-x)\,dx = \lambda^{m-1} C_m (1 + o_\lambda(1)), \tag{3.2}$$

where $o_\lambda(1) \to 0$ as $y \to \infty$.

Proof. Since $2m - \alpha > 1$ the integral on the left-hand side is convergent.

Let $\alpha = \gamma + n - 1$ with $0 \le \gamma < 1$ and $n \in \mathbb{N}$. To prove (3.2), we apply induction on n. Let $f(y; m, \gamma + n - 1)$ denote the integral on the left-hand side. By changing variable,

$$f(y; m, \gamma + n - 1) = \int_{-\infty}^{y} (y-u)^{\gamma+n-1} k_\lambda^m(u)\,du.$$

Then, by l'Hôpital's rule,

$$\lim_{y\to\infty} \frac{f(y; m, \gamma+n-1)}{y^{\gamma+n-1}} = \lim_{y\to\infty} \frac{f(y; m, \gamma+n-2)}{y^{\gamma+n-2}}.$$

Hence it suffices to show that, for $0 \le \gamma < 1$,

$$\lim_{y\to\infty} \frac{f(y; m, \gamma)}{y^\gamma} = \lambda^{m-1} C_m, \tag{3.3}$$

i.e.,

$$\lim_{y\to\infty} y^{-\gamma} \int_0^\infty x^\gamma k_\lambda^m(y-x)\,dx = \lambda^{m-1} C_m.$$

The equality is plain for $\gamma = 0$. For $0 < \gamma < 1$, we first have

$$\begin{aligned}\liminf_{y\to\infty} \frac{f(y; m, \gamma)}{y^\gamma} &\ge \liminf_{y\to\infty} \frac{(y-\delta)^\gamma \int_{y-\delta}^{y+\delta} k_\lambda^m(y-x)\,dx}{y^\gamma} \\ &= \int_{-\delta}^{\delta} k_\lambda^m(u)\,du\end{aligned}$$

for all $\delta > 0$ and hence

$$\liminf_{y\to\infty} \frac{f(y; m, \gamma)}{y^\gamma} \ge \lambda^{m-1} C_m.$$

Then, with $\delta = y^{\gamma/(2m-1)}$,

$$\int_0^{y+\delta} x^\gamma k_\lambda^m(y-x)\,dx \le (y+\delta)^\gamma \lambda^{m-1} C_m,$$

$$\int_{y+\delta}^{2y} x^\gamma k_\lambda^m(y-x)\,dx \le (2y)^\gamma \int_\delta^y k_\lambda^m(u)\,du \le \frac{2^\gamma}{(2m-1)\lambda^m},$$

and

$$\int_{2y}^{\infty} x^\gamma k_\lambda^m(y-x)\,dx \le \int_y^\infty 2^\gamma u^\gamma k_\lambda^m(u)\,du \le \frac{2^\gamma}{(2m-1-\gamma)\lambda^m y^{2m-1-\gamma}}.$$

Hence

$$\limsup_{y\to\infty} \frac{f(y;m,\gamma)}{y^\gamma} \le \lambda^{m-1} C_m$$

for $2m-1 \ge 1$. ∎

Applying (3.1), we now write (1.1) in the form

$$\int_0^\infty e^{-sx} F(x)\,dx = \frac{L}{\Gamma(\alpha+1)} \int_0^\infty e^{-(s-1)x} x^\alpha\,dx + G(s).$$

Multiplying both sides by $\Delta_\lambda^{*m}(t)e^{ity}$, integrating with respect to t on $(-\infty,\infty)$, then exchanging the integration order on the left-hand side and in the first term on the right-hand side and applying (1.7), we obtain the equality

$$\int_0^\infty e^{-\sigma x} F(x) k_\lambda^m(y-x)\,dx = \frac{L}{\Gamma(\alpha+1)} \int_0^\infty e^{-(\sigma-1)x} x^\alpha k_\lambda^m(y-x)\,dx + \int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t) e^{ity} G(\sigma+it)\,dt. \tag{3.4}$$

3.1. Proof of necessity. Assume (1.4), i.e., $F(x) \sim Le^x x^\alpha/\Gamma(\alpha+1)$. First, for $v = e^y > u = e^x$,

$$\begin{aligned}(u\log^\alpha u)^{-1}(F(\log v) - F(\log u)) &= (e^x x^\alpha)^{-1}(F(y) - F(x)) \\ &= \frac{L}{\Gamma(\alpha+1)}\left(e^{y-x}\left(\frac{y}{x}\right)^\alpha (1+o(1)) - 1 + o(1)\right)\end{aligned}$$

as $u\to\infty$ and $u/v \to 1+$, i.e., $x\to\infty$ and $y-x\to 0+$. Therefore,

$$\liminf\,(u\log^\alpha u)^{-1}(F(\log v) - F(\log u)) \ge 0,$$

i.e., $F(\log u)$ is l.s.d. with index α.

Then, with $m \ge 1 + [\alpha]/2$, by the dominated convergence theorem, the left-hand side and the first term on the right-hand side of (3.4) have finite limits as $\sigma \to 1+$ and the limitation and integration are exchangeable there. Hence, the second term on the right-hand side also has a limit. Therefore the limit (1.2) exists for every $\lambda > 0$ and $y > 0$ and it equals

$$\int_0^\infty \left(e^{-x}F(x) - \frac{Lx^\alpha}{\Gamma(\alpha+1)} \right) k_\lambda^m(y-x)\,dx = \int_0^\infty o(x^\alpha) k_\lambda^m(y-x)\,dx.$$

Then (1.3) follows from Lemma 2.

3.2. Proof of sufficiency. Assume that $F(\log u)$ is l.s.d. with index α and that (1.2) and (1.3) are satisfied.

By Lemma 1, we may further assume that $\liminf_{x\to\infty}(e^x x^\alpha)^{-1}F(x) > 0$. Otherwise, we replace $F(x)$ by $F_1(x) = ae^x x^\alpha + F(x)$ with any constant a greater than the constant c given in Lemma 1. The latter satisfies the conditions of Theorem 1 with L replaced by $L_1 = L + a\Gamma(\alpha+1)$. Then $\lim\,(e^x x^\alpha)^{-1}F(x) = L/\Gamma(\alpha+1)$ follows from $\lim\,(e^x x^\alpha)^{-1}F_1(x) = L_1/\Gamma(\alpha+1)$. Hence we may further assume $F(x) \ge 0$ for $x \ge x_0$.

We begin with the equality (3.4). Letting $\sigma \to 1+$, by the dominated convergence theorem, the partial integral of the left-hand side on the interval $0 \le x \le x_0$ has a finite limit and, by the monotone convergence theorem, the partial integral on the interval $x_0 \le x < \infty$ has a limit. Thus the left-hand side has a limit too and the limitation and integration are interchangeable. Also, by the monotone convergence theorem, the first term on the right-hand side has a finite limit

$$\frac{L}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha k_\lambda^m(y-x)\,dx$$

for $2m > 1 + \alpha$. Therefore, by (1.2) and (1.3), and Lemma 2, we arrive at

$$\begin{aligned} \int_0^\infty e^{-x}F(x)k_\lambda^m(y-x)\,dx &= \frac{L}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha k_\lambda^m(y-x)\,dx + o_\lambda(1)y^\alpha \\ &= \frac{L\lambda^{m-1}C_m y^\alpha}{\Gamma(\alpha+1)}(1+o_\lambda(1)) \end{aligned} \tag{3.5}$$

for $\lambda > \lambda_0$ and $y \ge y_0(\lambda)$, where $o_\lambda(1) \to 0$ as $y \to \infty$. It follows, since

$$\int_0^{x_0} e^{-x}F(x)k_\lambda^m(y-x)\,dx = o_\lambda(1),$$

that

$$\int_{y-\delta_0}^{y+\delta_0} e^{-x}F(x)k_\lambda^m(y-x)\,dx \le \frac{L\lambda^{m-1}C_m y^\alpha}{\Gamma(\alpha+1)}(1+o_\lambda(1)) \tag{3.6}$$

with $\delta_0 > 0$ for $y \geq \max\{y_0(\lambda), x_0 + \delta_0\}$. The left-hand side of (3.6) equals

$$e^{-(y-\delta_0)} F(y-\delta_0) \int_{y-\delta_0}^{y+\delta_0} k_\lambda^m(y-x)\, dx + I_1(\delta_0) + I_2(\delta_0), \tag{3.7}$$

where

$$I_1(\delta_0) := \int_{y-\delta_0}^{y+\delta_0} (e^{-x} - e^{-(y-\delta_0)}) F(x) k_\lambda^m(y-x)\, dx,$$
$$I_2(\delta_0) := \int_{y-\delta_0}^{y+\delta_0} e^{-(y-\delta_0)} (F(x) - F(y-\delta_0)) k_\lambda^m(y-x)\, dx.$$

We have, for sufficiently small $\delta_0 > 0$,

$$\begin{aligned} I_1(\delta_0) &\geq -3\delta_0 \int_{y-\delta_0}^{y+\delta_0} e^{-x} F(x) k_\lambda^m(y-x)\, dx \\ &\geq -3\delta_0 \frac{L\lambda^{m-1} C_m y^\alpha}{\Gamma(\alpha+1)} (1 + o_\lambda(1)) \end{aligned} \tag{3.8}$$

by (3.6). To estimate $I_2(\delta_0)$, we note that

$$f(u) := F(\log u)$$

is l.s.d. with index α. Hence

$$\liminf_{u\to\infty,\, v-u\to 0+} (e^u u^\alpha)^{-1} (F(v) - F(u)) \geq 0.$$

Then, given $\epsilon > 0$,

$$(e^u u^\alpha)^{-1} (F(v) - F(u)) > -\epsilon \tag{3.9}$$

for $u \geq u_0$, $0 < v - u < 2\delta_1(\epsilon)$. Thus, for δ_0 satisfying $0 < \delta_0 \leq \min\{\delta_1, \epsilon/3\}$,

$$I_2(\delta_0) \geq -\epsilon (y-\delta_0)^\alpha \int_{y-\delta_0}^{y+\delta_0} k_\lambda^m(y-x)\, dx. \tag{3.10}$$

From (3.6)–(3.8) and (3.10), we obtain

$$\begin{aligned} \big(e^{-(y-\delta_0)} F(y-\delta_0) - \epsilon (y-\delta_0)^\alpha\big) &\int_{y-\delta_0}^{y+\delta_0} k_\lambda^m(y-x)\, dx \\ &\leq (1+\epsilon) \frac{L\lambda^{m-1} C_m y^\alpha (1 + o_\lambda(1))}{\Gamma(\alpha+1)} \end{aligned}$$

for $\lambda > \lambda_0$ and $y \geq \max\{x_0 + u_0 + \delta_0, y_0(\lambda)\}$. Therefore

$$e^{-(y-\delta_0)}F(y-\delta_0) \le \epsilon(y-\delta_0)^\alpha + \frac{(1+\epsilon)L\lambda^{m-1}C_m y^\alpha(1+o_\lambda(1))}{\Gamma(\alpha+1)\int_{y-\delta_0}^{y+\delta_0} k_\lambda^m(y-x)\,dx} \le \epsilon(y-\delta_0)^\alpha + \frac{(1+\epsilon)^2 L y^\alpha(1+o_\lambda(1))}{\Gamma(\alpha+1)} \tag{3.11}$$

for

$$\int_{y-\delta_0}^{y+\delta_0} k_\lambda^m(y-x)\,dx \ge \lambda^{m-1}C_m - \frac{2}{(2m-1)\lambda^m\delta_0^{2m-1}} \ge \frac{\lambda^{m-1}C_m}{1+\epsilon}$$

with $\lambda > \lambda_1(\epsilon)$ ($\ge \lambda_0$) (δ_0 is independent of λ). Dividing both sides of (3.11) by $(y-\delta_0)^\alpha$ and letting $y \to \infty$, we conclude that

$$\limsup_{x\to\infty}\,(e^x x^\alpha)^{-1}F(x) \le \frac{L}{\Gamma(\alpha+1)} \tag{3.12}$$

since ϵ may be arbitrarily small.

It remains to show that

$$\liminf_{x\to\infty}\,(e^x x^\alpha)^{-1}F(x) \ge \frac{L}{\Gamma(\alpha+1)}. \tag{3.13}$$

By (3.12), $F(x)e^{-x} \le Kx^\alpha$ for $x \ge x_1$. Then, for $y > \max\{y_0(\lambda), x_0, x_1+\delta\}$ with $\delta > 0$, the left-hand side of (3.5) is at most

$$\int_{y-\delta}^{y+\delta} e^{-x}F(x)k_\lambda^m(y-x)\,dx + \int_0^{x_1} e^{-x}F(x)k_\lambda^m(y-x)\,dx + K\Big(\int_{x_1}^{y-\delta} x^\alpha k_\lambda^m(y-x)\,dx + \int_{y+\delta}^{\infty} x^\alpha k_\lambda^m(y-x)\,dx\Big). \tag{3.14}$$

We have

$$\int_{x_1}^{y-\delta} x^\alpha k_\lambda^m(y-x)\,dx = \int_\delta^{y-x_1} (y-u)^\alpha k_\lambda^m(u)\,du \le \frac{y^\alpha}{\lambda^m\delta^{2m-1}(2m-1)},$$

and

$$\int_{y+\delta}^{\infty} x^\alpha k_\lambda^m(y-x)\,dx = \int_\delta^\infty (y+u)^\alpha k_\lambda^m(u)\,du \le (2y)^\alpha\int_\delta^\infty \frac{du}{\lambda^m u^{2m}} + \int_y^\infty (2u)^\alpha\frac{du}{\lambda^m u^{2m}} = \frac{(2y)^\alpha}{(2m-1)\lambda^m\delta^{2m-1}} + \frac{2^\alpha}{(2m-1-\alpha)\lambda^m y^{2m-1-\alpha}}$$

since $2m > 1 + \alpha$. Therefore, by (3.14),

$$\begin{aligned}(3.15)\qquad & e^{-(y+\delta)}F(y+\delta)\int_{y-\delta}^{y+\delta} k_\lambda^m(y-x)\,dx + I_3(\delta) + I_4(\delta) \\ &\geq \frac{L\lambda^{m-1}C_m y^\alpha}{\Gamma(\alpha+1)}(1+o_\lambda(1)) - \frac{(1+2^\alpha)y^\alpha}{(2m-1)\lambda^m\delta^{2m-1}} + \frac{O(1)}{\lambda^m}\end{aligned}$$

where

$$I_3(\delta) := \int_{y-\delta}^{y+\delta} (e^{-x} - e^{-(y+\delta)})F(y+\delta)k_\lambda^m(y-x)\,dx,$$

$$I_4(\delta) := \int_{y-\delta}^{y+\delta} e^{-x}(F(x) - F(y+\delta))k_\lambda^m(y-x)\,dx.$$

We have

$$\begin{aligned}(3.16)\qquad I_3(\delta) &= \int_{y-\delta}^{y+\delta} (e^{y+\delta-x} - 1)e^{-(y+\delta)}F(y+\delta)k_\lambda^m(y-x)\,dx \\ &< (e^{2\delta}-1)K(y+\delta)^\alpha\lambda^{m-1}C_m \leq 3\delta K y^\alpha \lambda^{m-1}C_m\end{aligned}$$

for sufficiently small δ. Also, for $y \geq u_0 + \delta_1$ and $y - \delta \leq x \leq y + \delta$ with $0 < \delta \leq \delta_1$, by (3.9),

$$e^{-x}(F(x) - F(y+\delta)) < \epsilon x^\alpha.$$

Hence, by (3.2),

$$(3.17)\qquad I_4(\delta) < \epsilon \int_{y-\delta}^{y+\delta} x^\alpha k_\lambda^m(y-x)\,dx < \epsilon\lambda^{m-1}C_m y^\alpha(1+o_\lambda(1)).$$

Fixing sufficiently small δ satisfying $0 < \delta \leq \min\{\delta_1, \epsilon/(3K)\}$, by (3.15)–(3.17) we have

$$\begin{aligned}&\lambda^{m-1}C_m e^{-(y+\delta)}F(y+\delta) + 2\epsilon\lambda^{m-1}C_m y^\alpha(1+o_\lambda(1)) \\ &\qquad\geq \frac{L\lambda^{m-1}C_m y^\alpha}{\Gamma(\alpha+1)}(1+o_\lambda(1)) - \frac{(1+2^\alpha)y^\alpha}{(2m-1)\lambda^m\delta^{2m-1}} + \frac{O(1)}{\lambda^m}\end{aligned}$$

for $y \geq \max\{y_0(\lambda), x_0, u_0 + x_1 + \delta_1\}$ and $\lambda > \lambda_0$. Dividing both sides by y^α and taking limits as $y \to \infty$, we see that

$$\begin{aligned}&\lambda^{m-1}C_m \liminf_{x\to\infty} e^{-x}x^{-\alpha}F(x) + 2\epsilon\lambda^{m-1}C_m \\ &\qquad\geq \frac{L\lambda^{m-1}C_m}{\Gamma(\alpha+1)} - \frac{(1+2^\alpha)}{(2m-1)\lambda^m\delta^{2m-1}}.\end{aligned}$$

Finally, dividing both sides by $\lambda^{m-1}C_m$ and taking limits as $\lambda \to \infty$, we see that

$$\liminf_{x\to\infty} e^{-x}x^{-\alpha}F(x) + 2\epsilon \geq \frac{L}{\Gamma(\alpha+1)}.$$

for every $\epsilon > 0$. Thus (3.13) follows.

This completes the proof of the sufficiency of the conditions of Theorem 1, and also the proof of the theorem.

4. Proofs of Theorems 2–4

Proof of Theorem 2. Assume that (1.5) approaches zero as σ, $\sigma' \to 1+$ uniformly for $y \geq y_0(\lambda)$. By Cauchy's criterion, limit (1.2) exists for every $\lambda > \lambda_0$ and every $y \geq y_0(\lambda)$. Let $\ell(\lambda, y)$ denote the limit function of (1.2). Then

$$y^{-\alpha} \int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}G(\sigma+it)\,dt$$

converges to $y^{-\alpha}\ell(\lambda, y)$ as $\sigma \to 1+$ uniformly for $y \geq y_0(\lambda)$. Given $\epsilon > 0$, by (1.5),

$$\left|y^{-\alpha} \int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}G(\sigma+it)\,dt - y^{-\alpha}\ell(\lambda, y)\right| < \epsilon \tag{4.1}$$

for $1 < \sigma < 1 + \delta(\lambda)$. Fixing $\sigma = \sigma_0 < 1 + \delta(\lambda)$ and noting that $G(\sigma_0 + it)$ is a continuous function of t, by the Riemann–Lebesgue lemma we have

$$\lim_{y\to\infty} \int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}G(\sigma_0+it)\,dt = 0.$$

It follows from (4.1) that

$$\limsup_{y\to\infty} y^{-\alpha}\ell(\lambda, y) \leq \lim_{y\to\infty} y^{-\alpha} \int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}G(\sigma_0+it)\,dt + \epsilon = \epsilon.$$

Similarly, $\liminf_{y\to\infty} y^{-\alpha}\ell(\lambda, y) \geq -\epsilon$, whence $\lim_{y\to\infty} y^{-\alpha}\ell(\lambda, y) = 0$, i.e., (1.3) holds for every $\lambda > \lambda_0$. Since $F(\log u)$ is l.s.d. with index α, by Theorem 1, $\lim_{x\to\infty} e^{-x}x^{-\alpha}F(x) = L/\Gamma(\alpha+1)$.

Conversely, by (3.4),

$$\int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}\big(G(\sigma+it) - G(\sigma'+it)\big)\,dt$$
$$= \int_{0}^{\infty} (e^{-(\sigma-1)x} - e^{-(\sigma'-1)x})\left(e^{-x}F(x) - \frac{Lx^\alpha}{\Gamma(\alpha+1)}\right)k_\lambda^m(y-x)\,dx.$$

If $\lim_{x\to\infty} e^{-x}x^{-\alpha}F(x) = L/\Gamma(\alpha+1)$ then

$$(e^{-(\sigma-1)x} - e^{-(\sigma'-1)x})\left(e^{-x}F(x) - \frac{Lx^\alpha}{\Gamma(\alpha+1)}\right) \le \epsilon x^\alpha$$

uniformly for $0 \le x < \infty$ when $1 < \sigma \le \sigma' < 1+\eta$. Hence, with $m \ge 1 + [\alpha]/2$, by Lemma 2,

$$\frac{1}{y^\alpha}\Big|\int_{-\infty}^{\infty} \Delta_\lambda^{*m}(t)e^{ity}\big(G(\sigma+it) - G(\sigma'+it)\big)\,dt\Big|$$
$$\le \epsilon y^{-\alpha}\int_0^\infty x^\alpha k_\lambda^m(y-x)\,dx = \epsilon\lambda^{m-1}C_m(1+o_\lambda(1)) < 2\epsilon\lambda^{m-1}C_m$$

for $y \ge y_1$. This shows that (1.5) approaches zero as $\sigma, \sigma' \to 1+$ uniformly for $y > 0$.

Moreover, by Theorem 1, $F(\log u)$ is l.s.d. with index α.

Proof of Theorem 3. As in the proof of Theorem 1, we may further assume that $F(x) \ge 0$ for $x \ge x_0$.

We begin with (3.4). As $\sigma \to 1+$, the left-hand side has a limit $\int_0^\infty e^{-x}F(x)k_\lambda^m(y-x)\,dx$ by the dominated convergence theorem and the monotone convergence theorem. Moreover, the first term on the right-hand side has a limit $\ll y^\alpha$ by Lemma 2. Hence the second term on the right-hand side has a limit, by (1.9), which is at most $K\lambda^{m-1}C_m y^\alpha$. Therefore

$$(0 \le) \int_y^{y+\delta} e^{-x}F(x)k_\lambda^m(y-x)\,dx \le \int_{x_0}^\infty e^{-x}F(x)k_\lambda^m(y-x)\,dx \le By^\alpha \tag{4.2}$$

for $y \ge \max\{y_0, x_0\}$ and every $\delta > 0$ with some constant $B > 0$. The left-hand side equals

$$e^{-y}F(y)\int_{-\delta}^0 k_\lambda^m(u)\,du + I_5 + I_6, \tag{4.3}$$

where

$$I_5 := \int_{-\delta}^0 (e^{-(y-u)} - e^{-y})F(y-u)k_\lambda^m(u)\,du,$$
$$I_6 := \int_{-\delta}^0 e^{-y}(F(y-u) - F(y))k_\lambda^m(u)\,du.$$

Since $F(\log u)$ is linearly bounded below decreasing,

$$e^{-y}(F(x) - F(y)) \ge -Ay^\alpha$$

for $y \ge y_1$, $y < x < y + 2\delta_1$ with some constants $A > 0$ and $(1 \ge)\ \delta_1 > 0$.

Hence,

$$I_6 \ge -Ay^{\alpha} \int_{-\delta_1}^{0} k_{\lambda}^{m}(u)\,du \ge -Ey^{\alpha}. \tag{4.4}$$

Also, by (4.2),

$$I_5 \ge -2 \int_{-\delta_1}^{0} e^{-(y-u)} F(y-u) k_{\lambda}^{m}(u)\,du \ge -2By^{\alpha}. \tag{4.5}$$

From (4.2)–(4.5), we see that

$$e^{-y}F(y) \int_{0}^{\delta_1} k_{\lambda}^{m}(u)\,du \le (3B+E)y^{\alpha},$$

and so $e^{-y}F(y) \le Cy^{\alpha}$.

Conversely, if $F(x) \le Ce^{x}x^{\alpha}$ is true then by (3.4),

$$\int_{-\infty}^{\infty} \Delta_{\lambda}^{*m}(t)e^{ity}G(\sigma+it)\,dt \le \left(C + \frac{|L|}{\Gamma(\alpha+1)}\right) \int_{0}^{\infty} e^{-(\sigma-1)x} x^{\alpha} k_{\lambda}^{m}(y-x)\,dx$$

and hence the left-hand side of (1.9) is at most

$$\left(C + \frac{|L|}{\Gamma(\alpha+1)}\right) \int_{0}^{\infty} x^{\alpha} k_{\lambda}^{m}(y-x)\,dx < K\lambda^{m-1}C_m y^{\alpha}$$

for $y \ge y_0(\lambda)$ by Lemma 2.

Proof of Theorem 4. Suppose first that $F(x)(e^{x}x^{\alpha})^{-1} \ge c$ for all $x \ge x_0$ with some constant $c > 0$. By (3.4),

$$\int_{-\infty}^{\infty} \Delta_{\lambda}^{*m}(t)e^{ity}G(\sigma+it)\,dt = \int_{0}^{\infty} e^{-\sigma x}F(x)k_{\lambda}^{m}(y-x)\,dx - \frac{L}{\Gamma(\alpha+1)} \int_{0}^{\infty} e^{-(\sigma-1)x} x^{\alpha} k_{\lambda}^{m}(y-x)\,dx.$$

By the monotone convergence theorem, the first term on the right-hand side has a limit as $\sigma \to 1+$ and the limitation and integration are interchangeable. Thus

$$\liminf_{\sigma\to 1+} \int_{-\infty}^{\infty} \Delta_{\lambda}^{*m}(t)e^{ity}G(\sigma+it)\,dt \ge \int_{0}^{\infty} e^{-x}F(x)k_{\lambda}^{m}(y-x)\,dx - \frac{L}{\Gamma(\alpha+1)} \int_{0}^{\infty} x^{\alpha}k_{\lambda}^{m}(y-x)\,dx \ge \left(c - \frac{L}{\Gamma(\alpha+1)}\right)\lambda^{m-1}C_m y^{\alpha}(1+o_{\lambda}(1))$$

by Lemma 2, where $o_\lambda(1) \to 0$ as $y \to \infty$. Thus (1.11) holds with $\gamma = c/2$ for $y \ge y_0(\lambda)$.

Conversely, assume that (1.11) holds with positive constants λ, γ, y_0 and an integer $m \ge 1 + [\alpha]/2$. By (3.4) and (1.11), for $y \ge y_0$, $1 < \sigma < 1 + \eta(y)$,

$$\int_0^\infty e^{-\sigma x} F(x) k_\lambda^m(y-x)\,dx - \frac{L}{\Gamma(\alpha+1)} \int_0^\infty e^{-(\sigma-1)x} x^\alpha k_\lambda^m(y-x)\,dx > \left(-\frac{L}{\Gamma(\alpha+1)} + \frac{\gamma}{2}\right)\lambda^{m-1} C_m y^\alpha.$$

Taking limits on the left-hand side as $\sigma \to 1+$ and then applying Lemma 2, we arrive at

$$\int_0^\infty e^{-x} F(x) k_\lambda^m(y-x)\,dx > \frac{\gamma'}{3} y^\alpha \tag{4.6}$$

for $y > \max\{y_0, y_1\}$, where $\gamma' = \gamma\lambda^{m-1}C_m$. If we assume that $F(x)/(e^x x^\alpha) \le C$, then the left-hand side is at most

$$C \int_{|x-y|\ge\delta} x^\alpha k_\lambda^m(y-x)\,dx + \int_{y-\delta}^{y+\delta} e^{-x} F(x) k_\lambda^m(y-x)\,dx.$$

The first term is bounded above by

$$C\left(\frac{(1+2^\alpha)y^\alpha}{(2m-1)\lambda^m \delta^{2m-1}} + \frac{2^\alpha}{(2m-1-\alpha)\lambda^m y^{2m-1-\alpha}}\right)$$

as shown in the proof of the sufficiency part of Theorem 1. For sufficiently large $\delta > 0$ (independent of y), this is not larger than $\gamma' y^\alpha/6$. Fixing $\delta = \delta_0$, we see that

$$\int_{y-\delta_0}^{y+\delta_0} e^{-x} F(x) k_\lambda^m(y-x)\,dx \ge \frac{\gamma'}{6} y^\alpha \tag{4.7}$$

for $y \ge \max\{y_0, y_1 + \delta_0\}$. If $F(x)$ is nondecreasing, then

$$e^{-y+\delta_0} F(y+\delta_0)\lambda^{m-1} C_m \ge \frac{\gamma'}{6} y^\alpha$$

and $F(x)(e^x x^\alpha)^{-1} \ge c$ for $x \ge x_0$ with some small constant $c > 0$ follows.

If, instead, $F(\log u)$ is linearly bounded below decreasing with index $\beta < \alpha$ then there exist positive constants A, δ_1 and x_1 such that

$$(e^x x^\beta)^{-1}(F(y) - F(x)) \ge -A$$

for $x \ge x_1$, $0 \le y - x < 2\delta_1$, i.e.,

$$(e^x x^\beta)^{-1}(F(x) - F(y)) \le A. \tag{4.8}$$

We claim that there exists a constant $B > 0$ such that

$$(e^x x^\beta)^{-1}(F(x) - F(y+\delta_0)) \le AB \tag{4.9}$$

for $y \ge \max\{y_0, y_1, x_1\} + \delta_0$, $y - \delta_0 \le x \le y + \delta_0$. Then, in this case, the left-hand side of (4.7) equals

$$\begin{aligned}
\int_{y-\delta_0}^{y+\delta_0} e^{-x}F(y+\delta_0)k_\lambda^m(y-x)\,dx &+ \int_{y-\delta_0}^{y+\delta_0} e^{-x}(F(x) - F(y+\delta_0))k_\lambda^m(y-x)\,dx \\
&\le e^{-y+\delta_0}F(y+\delta_0)\lambda^{m-1}C_m + AB\int_{y+\delta_0}^{y+\delta_0} x^\beta k_\lambda^m(y-x)\,dx \\
&= e^{-y+\delta_0}F(y+\delta_0)\lambda^{m-1}C_m + \lambda^{m-1}C_m O(y^\beta)
\end{aligned}$$

by Lemma 2. Therefore,

$$e^{-y+\delta_0}F(y+\delta_0)\lambda^{m-1}C_m + \lambda^{m-1}C_m O(y^\beta) \ge \frac{\gamma'}{6}y^\alpha$$

and so $F(x)(e^x x^\alpha)^{-1} \ge c$ for $x \ge x_0$ with some constant $c > 0$.

It remains to show (4.9). Let $k = [2\delta_0/\delta_1]$ and let Y denote $y + \delta_0$. By (4.8), we have

$$(e^{Y-n\delta_1}(Y-n\delta_1)^\beta)^{-1}\big(F(Y-n\delta_1) - F(Y-(n-1)\delta_1)\big) \le A \tag{4.10}$$

for $n = 1, \ldots, k$. We first show, by induction, that

$$\begin{aligned}
&(e^{Y-n\delta_1}(Y-n\delta_1)^\beta)^{-1}(F(Y-n\delta_1) - F(Y)) \\
&\qquad\le A + A\sum_{\ell=1}^{n-1} e^{\ell\delta_1}\left(\frac{Y-(n-\ell)\delta_1}{Y-n\delta_1}\right)^\beta
\end{aligned} \tag{4.11}$$

for $n = 1, \ldots, k$. Hence (4.9) holds for $x = Y - n\delta_1$ ($\ge y - \delta_0$) for $n = 1, \ldots, k$ with

$$B = 2 + 2^\beta \sum_{\ell=1}^{k-1} e^{\ell\delta_1}$$

since

$$\frac{Y-(n-\ell)\delta_1}{Y-n\delta_1} \le 2.$$

To show (4.11), for $n = 1$, it is merely (4.10). Then, we have

$$\begin{aligned}
&\big(e^{Y-(n+1)\delta_1}(Y-(n+1)\delta_1)^\beta\big)^{-1}\big(F(Y-(n+1)\delta_1) - F(Y)\big) \\
&\quad= (e^{Y-n\delta_1}(Y-n\delta_1)^\beta)^{-1}(F(Y-n\delta_1) - F(Y))e^{\delta_1}\left(\frac{Y-n\delta_1}{Y-(n+1)\delta_1}\right)^\beta \\
&\qquad+ \big(e^{Y-(n+1)\delta_1}(Y-(n+1)\delta_1)^\beta\big)^{-1}\big(F(Y-(n+1)\delta_1) - F(Y-n\delta_1)\big)
\end{aligned}$$

$$\le \Big(A + A\sum_{\ell=1}^{n-1} e^{\ell\delta_1}\Big(\frac{Y-(n-\ell)\delta_1}{Y-n\delta_1}\Big)^{\beta}\Big)e^{\delta_1}\Big(\frac{Y-n\delta_1}{Y-(n+1)\delta_1}\Big)^{\beta} + A$$
$$= A + A\sum_{\ell=1}^{n} e^{\ell\delta_1}\Big(\frac{Y-(n+1-\ell)\delta_1}{Y-(n+1)\delta_1}\Big)^{\beta}.$$

Hence (4.11) holds for $n = 1, \dots, k$.

Finally, for $Y - \delta_1 \le x < Y$, (4.9) follows directly from (4.8). For $Y - n\delta_1 \le x < Y - (n-1)\delta_1$ with $2 \le n \le k+1$, by a similar argument, we have

$$(e^x x^\beta)^{-1}(F(x) - F(Y)) \le A + Ae^{\delta_1}2^\beta\Big(1 + 2^\beta\sum_{\ell=1}^{n-2} e^{\ell\delta_1}\Big).$$

This completes the proof of (4.9) as well as the proof of Theorem 4.

5. Beurling generalized primes. As a preliminary application of the exact Wiener–Ikehara theorem we study the Beurling generalized primes.

A *Beurling generalized prime number system* (henceforth, a *g-prime system*) $\mathcal{P} = \{p_i\}$ is a sequence of real numbers satisfying

$$1 < p_1 \le p_2 \le \cdots, \quad p_i \to \infty.$$

Associated with $\mathcal{P}$ we have the multiplicative semigroup of *g-integers* it generates,

$$\mathcal{N} = \mathcal{N}_{\mathcal{P}} : 1 = n_1 < n_2 \le n_3 \le \cdots.$$

The g-prime counting functions are defined as

$$\pi_{\mathcal{P}}(x) := \#\{i \ge 1 : p_i \le x\},$$
$$\Pi_{\mathcal{P}}(x) := \pi_{\mathcal{P}}(x) + \tfrac{1}{2}\pi_{\mathcal{P}}(x^{1/2}) + \tfrac{1}{3}\pi_{\mathcal{P}}(x^{1/3}) + \cdots,$$
$$\psi_{\mathcal{P}}(x) := \sum_{p_i^{\alpha_i} \le x} \log p_i = \int_1^x \log x \, d\Pi_{\mathcal{P}}(x),$$

and the g-integer counting function and associated g-zeta function as

$$N_{\mathcal{P}}(x) := \sum_{n_i \le x} 1, \qquad \zeta(s) = \zeta_{\mathcal{P}}(s) := \sum_{n_i \le x} n_i^{-s} = \int_{1-}^{\infty} x^{-s}\, dN_{\mathcal{P}}(x).$$

For convenience, the subscript $\mathcal{P}$ is dropped if there is no need to emphasize a particular $\mathcal{P}$.

The central scheme of the theory of Beurling generalized primes is whether one of $\mathcal{P}$ or $\mathcal{N}_{\mathcal{P}}$ being "reasonably near" its classical counterpart implies the same for the other one.

THEOREM 5. *Assume that $\mathcal{N}$ is a Beurling generalized number system with density $A > 0$ and zeta function $\zeta(s)$. If $1/\zeta(\sigma+it)$ converges in L_1 on every fixed interval $-T \le t \le T$ as $\sigma \to 1+$ then $M(x) := \sum_{n\le x}\mu(n) = o(x)$.*

Proof. On every fixed interval $-T \le t \le T$, $\int_{-T}^{T} dt/|\zeta(\sigma+it)|$ is bounded for $\sigma > 1$ and

$$\int_{-T}^{T}\left|\frac{1}{\zeta(\sigma_2+it)} - \frac{1}{\zeta(\sigma_1+it)}\right| dt \to 0$$

as $1 < \sigma_1 < \sigma_2$, $\sigma_2 \to 1+$. Then

$$\begin{aligned}\int_{-T}^{T}&\left|\frac{1}{(\sigma_2+it)\zeta(\sigma_2+it)} - \frac{1}{(\sigma_1+it)\zeta(\sigma_1+it)}\right| dt\\ &\le \int_{-T}^{T}\left|\frac{1}{\sigma_1+it}\left(\frac{1}{\zeta(\sigma_2+it)} - \frac{1}{\zeta(\sigma_1+it)}\right)\right| dt\\ &\quad + \int_{-T}^{T}\left|\frac{\sigma_1-\sigma_2}{(\sigma_2+it)(\sigma_1+it)}\frac{1}{\zeta(\sigma_2+it)}\right| dt\\ &\to 0\end{aligned}$$

as $\sigma_2 \to 1+$; that is, $1/((\sigma+it)\zeta(\sigma+it))$ converges in L_1. The function

$$G(s) := \int_1^\infty x^{-s-1}M(x)\,dx = \frac{1}{s\zeta(s)}, \qquad \sigma > 1,$$

satisfies the conditions of Corollary 1. Thus $M(e^x) = o(e^x)$. ∎

In the case of the rational integers $\mathbb{N}$, the exact Wiener–Ikehara theorem and Theorem 5 show that $M(x) = o(x)$ if and only if the Riemann zeta function $\zeta(s)$ has no zeros on the line $\sigma = 1$.

Also, in the case of the Beurling–Diamond primes $\mathcal{P}$ [Beu, BaDi], the associated zeta function $\zeta_{\mathcal{P}}$ has a representation [Dia]

$$\zeta_{\mathcal{P}}(s) = \frac{(s-1-i)^{1/2}(s-1+i)^{1/2}}{s-1}\exp\left\{\int_1^\infty x^{-s}\,d(\Pi_c(x) - \Pi_{\mathcal{P}}(x))\right\},$$

where $\Pi_c(x) - \Pi_{\mathcal{P}}(x) = O(x^{1/2})$. The system $\mathcal{N}_{\mathcal{P}}$ satisfies the condition of Theorem 5. We conclude that $M_{\mathcal{P}}(x) = o(x)$. Note that the Beurling–Diamond primes $\mathcal{P}$ do not satisfy the PNT [Zha87]. This indicates that the "equivalence" of $M(x) = o(x)$ and the PNT in $\mathbb{N}$ does not hold in the Beurling generalized numbers. This leaves the conjecture that $M(x) = o(x)$ and the Chebyshev bound $\psi(x) \ll x$ are equivalent there.

To attack the conjecture, it is shown in [DeDiVi] and [Zha18] that, in a g-prime system $\mathcal{N}$ with $N(x)$ satisfying $N(x) \sim Ax$ and

$$\int_1^\infty x^{-\sigma-1}|N(x) - Ax|\,dx = O((\sigma-1)^{-\beta}), \qquad \sigma > 1,$$

with some constant $\beta \in [0, 1/2)$, if $\psi(x) \ll x$ then $M(x) = o(x)$, i.e., the Chebyshev bound implies $M(x) = o(x)$. This proves the conjecture in one direction in this case.

However, we now know that the conjecture is false in the opposite direction. Take the case of the generalized number system $\mathcal{N}_B$ given in [DiZh]. The system has a density $A > 0$ since $N_B(x) - Ax = xE(x)$ with

$$E(x) = O(\log\log\log\log x/\log x).$$

Also $N_B(x)$ satisfies

$$\int_1^\infty x^{-2}|N_B(x) - Ax|\,dx < \infty.$$

Since

$$\frac{1}{\zeta_B(s)} = \frac{s-1}{(s-1)\zeta_B(s)}$$

has a continuous extension to $\{\sigma \geq 1\}$, $\mathcal{N}_B$ satisfies the conditions of Theorem 5. We conclude that $M_B(x) = o(x)$. However, in this system, the Chebyshev bounds do not hold for

$$\limsup_{x\to\infty} \frac{\pi_B(x)}{x/\log x} = \infty \quad \text{and} \quad \liminf_{x\to\infty} \frac{\pi_B(x)}{x/\log x} = 0.$$

Therefore, for this system $M_B(x) = o(x)$ does not imply the Chebyshev upper bound though $N_B(x)$ satisfies the conditions of Theorem 3.1 of [DeDiVi] or Corollary 1 in [Zha18], as just shown. This shows that the conjecture is false.

6. Concluding remarks. New significant applications of the exact Wiener–Ikehara theorems are possible. Also the condition that $F(\log u)$ is l.s.d. may replace the nondecreasing condition on $F(x)$ in many tauberian theorems. These will be the subjects of our next paper.

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Wen-Bin Zhang
Department of Mathematics
University of Illinois at Urbana-Champaign
Urbana, IL 61801, U.S.A.
and
Department of Mathematics
South China University of Technology
Guangzhou, People's Republic of China
E-mail: w-zhang1@math.uiuc.edu