ACTA ARITHMETICA
188.2 (2019)

On the parity of the Fourier coefficients of the Hauptmoduln $j_N(z)$ and $j_N^+(z)$

by

Moni Kumari (Mumbai) and Sujeet Kumar Singh (Khurda)

1. Introduction. Klein's j-function plays a central role in several areas of number theory. It is defined by

$$j(z) := \frac{E_4^3(z)}{\Delta(z)} = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n, \tag{1}$$

where $z \in \mathbb{H}$ and $q := e^{2\pi iz}$. Here $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ is Ramanujan's Delta function and $E_4(z)$ is the normalized Eisenstein series of weight 4.

In the theory of modular functions, the study of congruences for Fourier coefficients has a long history. In particular, congruences for the coefficients $c(n)$ of Klein's j-function have been studied by many mathematicians including O. Kolberg [3], J. Lehner [4], K. Ono [7] and J.-P. Serre [8]. In [1], C. Alfes says that "Surprisingly little is known about the behavior of the coefficients $c(n)$ modulo a prime". Regarding the parity of $c(n)$, it is easy to check that $c(n)$ is even whenever $n \not\equiv 7 \pmod 8$. This motivates the question of parity of $c(n)$ in the arithmetic progression $n \equiv 7 \pmod 8$. A large amount of work has recently been done in this direction, by a variety of methods (see [1, 6, 7]). In 2012, Ono and Ramsey [7] proved that there are infinitely many d such that $c(nd^2)$ is even whenever $n \equiv 7 \pmod 8$. Also, by using the mod p analogue of Atkin–Lehner's theorem and the generalized Borcherds product, they proved that for any $n \equiv 7 \pmod 8$, if there exists one odd integer d such that $c(nd^2)$ is odd, then there are infinitely many such integers. Recently, M. R. Murty and R. Thangadurai [6], in the spirit of Kolberg's [2] proof of the parity of the partition function, gave a range in which a suitable $n \equiv 7 \pmod 8$ can be chosen such that $c(n)$ is odd

2010 *Mathematics Subject Classification*: Primary 11F03; Secondary 11F30, 11F33.
Key words and phrases: modular functions, Fourier coefficients, congruences.
Received 27 October 2017; revised 14 April 2018.
Published online 15 February 2019.

DOI: 10.4064/aa171027-15-5

(respectively, even). According to [1], it is expected that the odd coefficients are supported on "one half" of the arithmetic progression $n \equiv 7 \pmod 8$.

Motivated by the work of Murty and Thangadurai [6], in this paper we obtain the parity of the Fourier coefficients of the Hauptmoduln $j_N(z)$ and $j_N^+(z)$, defined with respect to the congruence subgroup $\Gamma_0(N)$ and the Fricke group $\Gamma_0^+(N)$ respectively, for some particular values of N. Here the group $\Gamma_0^+(N)$ is generated by $\Gamma_0(N)$ and the Artin–Lehner involution $W_N = \left(\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right)$. For $N = 2, 3, 4, 5, 7, 13$ the groups $\Gamma_0(N)$ and $\Gamma_0^+(N)$ have genus zero and the corresponding Hauptmoduln $j_N(z)$ and $j_N^+(z)$ (see [5]) can be expressed by using the Dedekind η-function $\eta(z) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$ in the following way:

$$j_N(z) := \left(\frac{\eta(z)}{\eta(Nz)}\right)^{\frac{24}{N-1}} + \frac{24}{N-1}, \tag{2}$$

$$j_N^+(z) := \left(\frac{\eta(z)}{\eta(Nz)}\right)^{\frac{24}{N-1}} + \frac{24}{N-1} + N^{\frac{12}{N-1}} \left(\frac{\eta(Nz)}{\eta(z)}\right)^{\frac{24}{N-1}}. \tag{3}$$

For any positive integer N, we always denote the nth Fourier coefficients of $j_N(z)$ and $j_N^+(z)$ by $c_N(n)$ and $c_N^+(n)$ respectively. From the definitions (2) and (3), it is clear that $c_N(n)$ and $c_N^+(n)$ are integers. Also, note that for $N = 2, 3, 4, 5, 7, 13$,

$$j_N^+(z) = j_N(z) + N^{\frac{12}{N-1}} \left(\frac{\eta(Nz)}{\eta(z)}\right)^{\frac{24}{N-1}}.$$

This suggests that the method used to study the Fourier coefficients $c_N^+(n)$ may also be applicable to get similar results for $c_N(n)$. In our setting, regarding the parity, this is exactly true. Therefore, we study the parity of the coefficients $c_N^+(n)$ for $N = 2, 3, 4, 7$ and give a detailed proof of our statements in Sections 2–4. By using the same techniques, we obtain the analogous results for $c_N(n)$, listed in the table in Section 5. Note that for $N = 5$ and 13, similar methods are applicable for the study of the parity of $c_N(n)$ and we include the result in the table. To the best of our knowledge, this is the first draft which deals with parity of coefficients of Hauptmoduln for congruence subgroups as well as for Fricke groups.

2. Parity of the Fourier coefficients of $j_2^+(z)$ and $j_4^+(z)$. In this section, we first prove that the Fourier coefficients of $j(z)$, $j_2^+(z)$ and $j_4^+(z)$ have the same parity. Then we use the earlier results of [6] for $j(z)$ to show the existence of infinitely many even and odd values for $c_2^+(n)$ (and also for $c_4^+(n)$) in the arithmetic progression $n \equiv 7 \pmod 8$.

Theorem 2.1. *Let $c(n)$, $c_2^+(n)$ and $c_4^+(n)$ denote the nth Fourier coefficients of $j(z)$, $j_2^+(z)$ and $j_4^+(z)$ respectively. Then*

$$c(n) \equiv c_2^+(n) \equiv c_4^+(n) \pmod 2. \tag{4}$$

Proof. From the definition of $j_2^+(z)$ in (3), we have

$$j_2^+(z) \equiv \left(\frac{\eta(z)}{\eta(2z)}\right)^{24} \pmod 2. \tag{5}$$

We know that

$$\left(\frac{\eta(z)}{\eta(2z)}\right)^{24} = \frac{1}{q}\prod_{n=1}^{\infty}\frac{(1-q^n)^{24}}{(1-q^{2n})^{24}} = \frac{1}{q}\prod_{n=1}^{\infty}\frac{1}{(1+q^n)^{24}}. \tag{6}$$

An easy calculation using the binomial theorem and the fact that $1+q^n \equiv 1-q^n \pmod 2$ yields

$$\frac{1}{q}\prod_{n=1}^{\infty}\frac{1}{(1+q^n)^{24}} \equiv \frac{1}{q}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{24}} \equiv \frac{1}{\Delta(z)} \pmod 2.$$

Combining (5) and (6) with the above congruence gives

$$j_2^+(z) \equiv \frac{1}{\Delta(z)} \pmod 2. \tag{7}$$

By using similar arguments, we can show that

$$j_4^+(z) \equiv \frac{1}{\Delta(z)} \pmod 2. \tag{8}$$

It is well-known that $E_4^3(z) \equiv 1 \pmod 2$, and using it in (1) we get

$$j(z) \equiv \frac{1}{\Delta(z)} \pmod 2. \tag{9}$$

Combining (7)–(9), we get

$$j(z) \equiv j_2^+(z) \equiv j_4^+(z) \pmod 2,$$

which completes the proof. ∎

In the literature there are many results about the parity of the coefficients $c(n)$ of Klein's j-function. From the above theorem the corresponding results hold for $c_2^+(n)$ and $c_4^+(n)$. In particular, a direct application of Theorem 2.1 using [6, Theorems 1.1 and 1.3] gives infinitely many intervals, each of which contains an integer $n \equiv 7 \pmod 8$ such that $c_2^+(n)$ (and $c_4^+(n)$) is even (respectively, odd). More precisely, we get the following results.

Corollary 2.2. *For $N = 2, 4$ and for every positive integer $t \geq 1$,*

(a) *the interval $I_{N,t} := [t, 4t(t+1)-1]$ contains an integer $n \equiv 7 \pmod 8$ with $c_N^+(n)$ odd,*
(b) *the interval $J_{N,t} := [16t-1, (4t+1)^2-1]$ contains an integer $n \equiv 7 \pmod 8$ with $c_N^+(n)$ even.*

In particular, $c_N^+(n)$ takes both even and odd values infinitely often in the arithmetic progression $n \equiv 7 \pmod 8$.

3. Parity of the Fourier coefficients of $j_3^+(z)$. From the definition (3), we have

$$j_3^+(z) = \left(\frac{\eta(z)}{\eta(3z)}\right)^{12} + 12 + 3^6\left(\frac{\eta(3z)}{\eta(z)}\right)^{12} \tag{10}$$
$$\equiv \left(\frac{\eta(z)}{\eta(3z)}\right)^{12} + \left(\frac{\eta(3z)}{\eta(z)}\right)^{12} \pmod 2.$$

In view of the right hand side of the above congruence, we introduce two functions:

$$f_3^+(z) = \left(\frac{\eta(z)}{\eta(3z)}\right)^{12} = \sum_{n\geq -1} a_3^+(n)q^n, \tag{11}$$

$$g_3^+(z) = \left(\frac{\eta(3z)}{\eta(z)}\right)^{12} = \sum_{n\geq 1} b_3^+(n)q^n. \tag{12}$$

Here $a_3^+(n)$ and $b_3^+(n)$ are the nth Fourier coefficients of f_3^+ and g_3^+ respectively. By using the product formula for the η-function and simplifying, we get

$$f_3^+(z) = \frac{1}{q}\prod_{n=1}^{\infty}\frac{(1-q^n)^{12}}{(1-q^{3n})^{12}} \equiv \sum_{n=-1}^{\infty} a_3^+(4n+3)q^{4n+3} \pmod 2. \tag{13}$$

Similarly

$$g_3^+(z) \equiv \sum_{n=1}^{\infty} b_3^+(4n+1)q^{4n+1} \pmod 2. \tag{14}$$

We note that $a_3^+(n)$ and $b_3^+(n)$ are even whenever $n \not\equiv 3 \pmod 4$ and $n \not\equiv 1 \pmod 4$ respectively. Therefore, from (10) we have

$$c_3^+(n) \equiv a_3^+(n) + b_3^+(n) \pmod 2$$

for all $n \geq 1$. More precisely,

$$c_3^+(n) \equiv \begin{cases} a_3^+(n) \pmod 2 & \text{if } n \equiv 3 \pmod 4, \\ b_3^+(n) \pmod 2 & \text{if } n \equiv 1 \pmod 4, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

For $n \equiv 0, 2 \pmod 4$, $c_3^+(n)$ is even, and therefore it is natural to ask about the parity of $c_3^+(n)$ in the arithmetic progressions $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$. Furthermore, the above congruence shows that it is sufficient to study the parity of $a_3^+(n)$ and $b_3^+(n)$. The following theorems give the answer.

Theorem 3.1. *Let t be a positive integer.*

(a) *Assume $3t(t+1)$ is not of the form $l(l+1)$ for any $l \in \mathbb{N}$. Then the interval $I_{3,t} := [12t-1, 6t(t+1)-1]$ contains an integer $n \equiv 3 \pmod 4$ with $a_3^+(n)$ odd.*

(b) *Assume* $6t(2t+1)$ *is not of the form* $l(l+1)$ *for any* $l \in \mathbb{N}$. *Then the interval* $J_{3,t} := [24t-1, 12t(2t+1)-1]$ *contains an integer* $n \equiv 3 \pmod 4$ *with* $a_3^+(n)$ *even.*

In particular, there are infinitely many integers $n \equiv 3 \pmod 4$ *for which* $c_3^+(n)$ *is even (respectively, odd).*

The next statement is an analogous result for the Fourier coefficients $b_3^+(n)$.

THEOREM 3.2. *Let* t *be a positive integer.*

(a) *If* $t(t+1)$ *is not of the form* $3l(l+1)$ *for any* $l \in \mathbb{N}$, *then the interval* $I'_{3,t} := [4t+1, 2t(t+1)+1]$ *contains an integer* $n \equiv 1 \pmod 4$ *such that* $b_3^+(n)$ *is odd.*

(b) *If* $2t(2t+1)$ *is not of the form* $3l(l+1)$ *for any* $l \in \mathbb{N}$, *then the interval* $J'_{3,t} := [8t+1, 4t(2t+1)+1]$ *contains an integer* $n \equiv 1 \pmod 4$ *such that* $b_3^+(n)$ *is even.*

In particular, there are infinitely many integers $n \equiv 1 \pmod 4$ *for which* $c_3^+(n)$ *is even (respectively, odd).*

To obtain these results, we prove the following lemmas related to the parity of $a_3^+(n)$ and $b_3^+(n)$.

LEMMA 3.3. *For any integer* n, *we have*

$$\sum_{k=0}^{\infty} a_3^+(n-6k(k+1)-1) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = 2l(l+1) \text{ for some } l \in \mathbb{N}, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Proof. We know that $(x+y)^{2^m} \equiv x^{2^m} + y^{2^m} \pmod 2$, where x and y are integers. Hence

$$\eta^{12}(z) = q^{1/2} \prod_{n=1}^{\infty} (1-q^n)^{12} \equiv q^{1/2} \prod_{n=1}^{\infty} (1-q^{4n})^3 \pmod 2. \tag{15}$$

Now, using the well-known Jacobi identity

$$\prod_{n=1}^{\infty} (1-q^n)^3 = \sum_{k=1}^{\infty} (-1)^k (2k+1) q^{k(k+1)/2}$$

in the last expression of (15), we obtain

$$\eta^{12}(z) \equiv q^{1/2} \sum_{n=0}^{\infty} q^{2n(n+1)} \pmod 2. \tag{16}$$

By similar arguments,

$$\eta^{12}(3z) \equiv q^{3/2} \sum_{n=0}^{\infty} q^{6n(n+1)} \pmod 2. \tag{17}$$

By using (16) and (17) in (11), we get

$$\sum_{n=-1}^{\infty} a_3^+(n)q^n \equiv \frac{1}{q}\frac{\sum_{n=0}^{\infty} q^{2n(n+1)}}{\sum_{n=0}^{\infty} q^{6n(n+1)}} \pmod 2.$$

Simplifying the above yields

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty} a_3^+(n-6k(k+1)-1)\Big)q^n \equiv \sum_{n=0}^{\infty} q^{2n(n+1)} \pmod 2.$$

Comparing the corresponding coefficients of q^n on both sides of the above congruence completes the proof. ∎

Lemma 3.4. *For any integer $n \geq 1$, we have*

$$\sum_{k=0}^{\infty} b_3^+(n-2k(k+1)+1) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = 6l(l+1) \text{ for some } l \in \mathbb{N}, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Proof. This follows by the same method as in the proof of Lemma 3.3. ∎

Now we are ready to prove our theorems.

Proof of Theorem 3.1(a). Let $n = 6t(t+1)$ and so from the assumption n is not of the form $2l(l+1)$ for any l. Substituting this value of n in Lemma 3.3, we get

$$\sum_{k=0}^{\infty} a_3^+\big(6t(t+1)-6k(k+1)-1\big) \equiv 0 \pmod 2.$$

Note that the above sum is finite since $a_3^+(n) = 0$ if $n < -1$. Thus the above congruence is actually

$$\sum_{k=0}^{t-1} a_3^+\big(6t(t+1)-6k(k+1)-1\big) + a_3^+(-1) \equiv 0 \pmod 2.$$

Equivalently, by using $a_3^+(-1) = 1$, we can write

$$\sum_{k=0}^{t-1} a_3^+(\alpha_{3,t}(k)) \equiv 1 \pmod 2,$$

where $\alpha_{3,t}(k) := 6t(t+1) - 6k(k+1) - 1$ which is $\equiv 3 \pmod 4$. Hence there exists k such that $0 \leq k \leq t-1$ and $a_3^+(\alpha_{3,t}(k))$ is odd. Now to complete the proof, it is sufficient to show that $\alpha_{3,t}(k) \in I_{3,t}$ for each $0 \leq k \leq t-1$. This follows from the fact that $\alpha_{3,t}(k)$ is a decreasing function of k and $\alpha_{3,t}(0), \alpha_{3,t}(t-1) \in I_{3,t}$. ∎

Proof of Theorem 3.1(b). Towards a contradiction, suppose that $a_3^+(m)$ is odd whenever $m \equiv 3 \pmod 4$ and $m \in J_{3,t}$. Put $n = 12t(2t+1)$ in

Lemma 3.3 to get

$$\sum_{k=0}^{\infty} a_3^+\big(12t(2t+1)-6k(k+1)-1\big) \equiv 0 \pmod 2.$$

By similar arguments,

$$\sum_{k=0}^{2t-1} a_3^+(\beta_{3,t}(k)) \equiv 1 \pmod 2, \tag{18}$$

where $\beta_{3,t}(k) := 12t(2t+1)-6k(k+1)-1$ which is $\equiv 3 \pmod 4$. Again, one can easily see that $\beta_{3,t}(k) \in J_{3,t}$ for all $0 \le k \le 2t-1$.

Now, the number of terms on the left hand side of (18) is even, namely $2t$, and by the assumption on $a_3^+(m)$ each term is odd. Therefore, the left hand side of (18) is $\equiv 0 \pmod 2$, a contradiction. ■

Proof of Theorem 3.2. Putting $n = 2t(t+1)$ (respectively, $= 4t(2t+1)$) in Lemma 3.4 and following the method of proof of Theorem 3.1(a) (respectively, Theorem 3.1(b)), we get the result. ■

4. Parity of the Fourier coefficients of $j_7^+(z)$. In this section we investigate the parity of the coefficients $c_7^+(n)$. From the definition (3), we have

$$j_7^+(z) \equiv \left(\frac{\eta(z)}{\eta(7z)}\right)^4 + \left(\frac{\eta(7z)}{\eta(z)}\right)^4 \pmod 2. \tag{19}$$

To make our calculations easy, we introduce two functions f_7^+ and g_7^+ with Fourier coefficients $a_7^+(n)$ and $b_7^+(n)$ respectively (as in the case of $j_3^+(z)$),

$$f_7^+(z) := \left(\frac{\eta(z)}{\eta(7z)}\right)^4 \quad \text{and} \quad g_7^+(z) := \left(\frac{\eta(7z)}{\eta(z)}\right)^4. \tag{20}$$

From (19) and (20), it is clear that to determine the parity of $c_7^+(n)$, it is sufficient to know the parity of $a_7^+(n)$ and $b_7^+(n)$. More precisely, for all $n \ge 1$, we have

$$c_7^+(n) \equiv a_7^+(n) + b_7^+(n) \pmod 2.$$

Now simplifying the eta quotients, we get

$$f_7^+(z) = \frac{1}{q}\prod_{n=1}^{\infty}\frac{(1-q^n)^4}{(1-q^{7n})^4} \equiv \sum_{n=-1}^{\infty} a_7^+(4n+3)q^{4n+3} \pmod 2,$$

and similarly

$$g_7^+(z) \equiv \sum_{n=1}^{\infty} b_7^+(4n+1)q^{4n+1} \pmod 2.$$

Note that $a_7^+(n)$ and $b_7^+(n)$ are even whenever $n \not\equiv 3 \pmod 4$ and $n \not\equiv 1 \pmod 4$ respectively. Combining the above, we get

$$c_7^+(n) \equiv \begin{cases} a_7^+(n) \pmod 2 & \text{if } n \equiv 3 \pmod 4, \\ b_7^+(n) \pmod 2 & \text{if } n \equiv 1 \pmod 4, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

For $n \equiv 0, 2 \pmod 4$, $c_7^+(n)$ is even. Hence it is natural to investigate the parity of $c_7^+(n)$ in the arithmetic progressions $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$. The following theorems answer these questions.

THEOREM 4.1. *Let t be a positive integer.*

(a) *Assume that $7t(3t-1)$ is neither of the form $l(3l-1)$ nor $l(3l+1)$ for any $l \in \mathbb{N}$. Then the interval $I_{7,t} := [56t-29, 14t(3t-1)-1]$ contains an integer $n \equiv 3 \pmod 4$ with $a_7^+(n)$ odd.*

(b) *Assume that $14t(6t-1)$ is neither of the form $l(3l-1)$ nor $l(3l+1)$ for any $l \in \mathbb{N}$. Then the interval $J_{7,t} := [112t-29, 28t(6t-1)-1]$ contains an integer $n \equiv 3 \pmod 4$ with $a_7^+(n)$ even.*

In particular, $c_7^+(n)$ takes both even and odd values infinitely often in the arithmetic progression $n \equiv 3 \pmod 4$.

THEOREM 4.2. *Let t be a positive integer.*

(a) *Assume that $t(3t-1)$ is neither of the form $7l(3l-1)$ nor $7l(3l+1)$ for any $l \in \mathbb{N}$. Then the interval $I'_{7,t} := [8t-3, 2t(3t-1)+1]$ contains an integer $n \equiv 1 \pmod 4$ with $b_7^+(n)$ odd.*

(b) *Assume that $2t(3t-1)$ is neither of the form $7l(3l-1)$ nor $7l(3l+1)$ for any $l \in \mathbb{N}$. Then the interval $J'_{7,t} := [16t-3, 4t(6t-1)+1]$ contains an integer $n \equiv 1 \pmod 4$ with $b_7^+(n)$ even.*

In particular, $c_7^+(n)$ takes both even and odd values infinitely often in the arithmetic progression $n \equiv 1 \pmod 4$.

We need the following lemmas to prove Theorems 4.1 and 4.2.

LEMMA 4.3. *For any integer n, we have*

$$\begin{aligned} &\sum_{k=0}^{\infty} a_7^+\bigl(n - 14k(3k-1) - 1\bigr) + \sum_{k=0}^{\infty} a_7^+\bigl(n - 14k(3k+1) - 1\bigr) \\ &\equiv \begin{cases} 1 \pmod 2 & \text{if } n = 2l(3l-1) \text{ or } n = 2l(3l+1) \text{ for some } l \in \mathbb{N}, \\ 0 \pmod 2 & \text{otherwise.} \end{cases} \end{aligned} \tag{21}$$

Proof. Using the product identity

$$\prod_{n=1}^{\infty} (1-q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2},$$

we obtain

$$\prod_{n=1}^{\infty}(1-q^{4n}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{2n(3n-1)} \equiv \sum_{n=-\infty}^{\infty} q^{2n(3n-1)} \pmod 2 \tag{22}$$

and

$$\prod_{n=1}^{\infty}(1-q^{28n}) = \sum_{n=-\infty}^{\infty}(-1)^n q^{14n(3n-1)} \equiv \sum_{n=-\infty}^{\infty} q^{14n(3n-1)} \pmod 2. \tag{23}$$

Also from (20), we have

$$f_7^+(z) = \frac{\eta^4(z)}{\eta^4(7z)} = \frac{1}{q}\prod_{n=1}^{\infty}\frac{(1-q^n)^4}{(1-q^{7n})^4} \equiv \frac{1}{q}\prod_{n=1}^{\infty}\frac{1-q^{4n}}{1-q^{28n}} \pmod 2. \tag{24}$$

Substituting the congruences (22) and (23) in (24), we get

$$\sum_{n=-1}^{\infty} a_7^+(n)q^n \equiv \frac{\sum_{n=0}^{\infty} q^{2n(3n-1)} + \sum_{n=1}^{\infty} q^{2n(3n+1)}}{\sum_{n=0}^{\infty} q^{14n(3n-1)+1} + \sum_{n=1}^{\infty} q^{14n(3n+1)+1}} \pmod 2.$$

Multiplication of the series in the above congruence gives

$$\sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\infty} a_7^+(n-14k(3k-1)-1) + \sum_{k=0}^{\infty} a_7^+(n-14k(3k+1)-1)\Big)q^n$$
$$\equiv \Big(\sum_{n=0}^{\infty} q^{2n(3n-1)} + \sum_{n=1}^{\infty} q^{2n(3n+1)}\Big) \pmod 2.$$

Now comparing the corresponding coefficients of q^n on both sides of the above congruence gives the required result. ■

LEMMA 4.4. *For any integer $n \geq 1$, we have*

$$\sum_{k=0}^{\infty} b_7^+(n-2k(3k-1)) + \sum_{k=0}^{\infty} b_7^+(n-2k(3k+1)) \tag{25}$$
$$\equiv \begin{cases} 1 \pmod 2 & \text{if } n = 14l(3l-1)+1 \text{ or } n = 14l(3l+1)+1 \text{ for some } l, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

Proof. The proof, using similar arguments to the proof of Lemma 4.3, is left to the reader. ■

Proof of Theorem 4.1(a). Suppose, contrary to our claim, that $a_7^+(m)$ is even for all $m \in I_{7,t}$. Putting $n = 14t(3t-1)$ in Lemma 4.3 and using the hypothesis of the theorem, we obtain

$$\sum_{k=0}^{\infty} a_7^+\big(14t(3t-1)-14k(3k-1)-1\big) + \sum_{k=0}^{\infty} a_7^+\big(14t(3t-1)-14k(3k+1)-1\big)$$
$$\equiv 0 \pmod 2.$$

The above congruence can be written as

$$\sum_{k=0}^{\infty} a_7^+(\alpha_{7,t}(k)) + \sum_{k=0}^{\infty} a_7^+(\beta_{7,t}(k)) \equiv 0 \pmod 2, \tag{26}$$

where $\alpha_{7,t}(k) = 14t(3t-1) - 14k(3k-1) - 1$ and $\beta_{7,t}(k) = 14t(3t-1) - 14k(3k+1) - 1$.

Since $a_7^+(m) = 0$ for $m < -1$, the sums on the left hand side of (26) are finite and the maximum value of k in the sums will be t. Therefore, we can write the congruence (26) as

$$\sum_{k=0}^{t-1}\{a_7^+(\alpha_{7,t}(k)) + a_7^+(\beta_{7,t}(k))\} + a_7^+(\alpha_{7,t}(t)) + a_7^+(\beta_{7,t}(t)) \equiv 0 \pmod 2.$$

But $a_7^+(\alpha_{7,t}^+(t)) = a_7^+(-1) = 1$ and $a_7^+(\beta_{7,t}(t)) = a_7^+(-28t-1) = 0$ for any $t \in \mathbb{N}$. Thus,

$$\sum_{k=0}^{t-1}\{a_7^+(\alpha_{7,t}(k)) + a_7^+(\beta_{7,t}(k))\} \equiv 1 \pmod 2. \tag{27}$$

Clearly, for a fixed t, $\alpha_{7,t}$ and $\beta_{7,t}$ are decreasing functions of k. In view of $\alpha_{7,t}(0), \alpha_{7,t}(t-1), \beta_{7,t}(0), \beta_{7,t}(t-1) \in I_{7,t}$, it follows that

$$\alpha_{7,t}(k), \beta_{7,t}(k) \in I_{7,t} \quad \text{for all } 0 \le k \le t-1.$$

Now the assumption on $a_7^+(m)$ implies that the left hand side of (27) is $\equiv 0 \pmod 2$, a contradiction. ■

Proof of Theorem 4.1(b). Assume on the contrary that $a_7^+(m)$ is odd for every $m \in J_{7,t}$ and $m \equiv 3 \pmod 4$. As before, putting $n = 28t(6t-1)$ in Lemma 4.3 and using the hypothesis of the theorem, we obtain

$$\sum_{k=0}^{\infty} a_7^+\big(28t(6t-1) - 14k(3k-1) - 1\big) + \sum_{k=0}^{\infty} a_7^+\big(28t(6t-1) - 14k(3k+1) - 1\big) \equiv 0 \pmod 2.$$

For $k \in \mathbb{N} \cup \{0\}$, we denote $\gamma_{7,t}(k) = 28t(6t-1) - 14k(3k-1) - 1$ and $\delta_{7,t}(k) = 28t(6t-1) - 14k(3k+1) - 1$. Using these notations, we write the above congruence as

$$\sum_{k=0}^{\infty} a_7^+(\gamma_{7,t}(k)) + \sum_{k=0}^{\infty} a_7^+(\delta_{7,t}(k)) \equiv 0 \pmod 2.$$

The maximum k in the above sums will be $2t$ because subsequent summands are zero. Therefore the above congruence becomes

$$\sum_{k=0}^{2t-1}\{a_7^+(\gamma_{7,t}(k)) + a_7^+(\delta_{7,t}(k))\} + a_7^+(\gamma_{7,t}(2t)) + a_7^+(\delta_{7,t}(2t)) \equiv 0 \pmod 2. \tag{28}$$

Note that $a_7^+(\gamma_7^+(2t)) = a_7^+(-1) = 1$ and $a_7^+(\delta_{7,t}(2t)) = a_7^+(-56t-1) = 0$ for any $t \in \mathbb{N}$. Substituting these values in (28), we get

$$\sum_{k=0}^{2t-1} \{a_7^+(\gamma_{7,t}(k)) + a_7^+(\delta_{7,t}(k))\} \equiv 1 \pmod 2. \tag{29}$$

Arguing as before, we find that $\gamma_{7,t}(k), \delta_{7,t}(k) \in J_{7,t}$ for all $0 \le k \le 2t-1$.

Now, the assumption on $a_7^+(m)$ and the fact that the sum on the left hand side of (29) has an even number of terms gives a contradiction. ∎

Proof of Theorem 4.2. Putting $n = 2t(3t-1)+1$ (respectively, $= 4t(6t-1)+1$) in Lemma 4.4 and following the method of the proof of Theorem 4.1(a) (respectively, Theorem 4.1(b)), we get the results. ∎

5. Parity of the Fourier coefficients of $j_N(z)$. In the following table, we give analoguous results about the parity of the Fourier coefficients of $j_N(z)$ for $N = 2,3,4,5,7,13$. The proofs of these statements use the same techniques as in Sections 2–4. We remark that $c_N(n)$ is even if n is not in the arithmetic progression mentioned in the second column.

Coefficients of $j_N(z)$	Arithmetic progressions	Interval which contains an n with $c_N(n)$ odd	Interval which contains an n with $c_N(n)$ even
$c_2(n)$	$n \equiv 7 \pmod 8$	$[t, 4t(t+1)-1]$	$[16t-1, (4t+1)^2-1]$
$c_3(n)$	$n \equiv 3 \pmod 4$	$[12t-1, 6t(t+1)-1]$ $3t(t+1) \neq l(l+1)$ for any l	$[24t-1, 12t(2t+1)-1]$, $6t(2t+1) \neq l(l+1)$ for any l
$c_4(n)$	$n \equiv 7 \pmod 8$	$[t, 4t(t+1)-1]$	$[16t-1, (4t+1)^2-1]$
$c_5(n)$	$n \equiv 1 \pmod 2$	$[10t-1, 5t(t+1)-1]$, $5t(t+1) \neq l(l+1)$ for any l	$[20t-1, 10t(2t+1)-1]$, $10t(2t+1) \neq l(l+1)$ for any l
$c_7(n)$	$n \equiv 3 \pmod 4$	$[56t-29, 14t(3t-1)-1]$, $7t(3t-1) \neq l(3l-1) \neq l(3l+1)$ for any l	$[112t-29, 28t(6t-1)-1]$, $14t(6t-1) \neq l(3l-1) \neq l(3l+1)$ for any l
$c_{13}(n)$	$n \equiv 1 \pmod 2$	$[52t-27, 13t(3t-1)-1]$, $13t(3t-1) \neq l(3l-1) \neq l(3l+1)$for any l	$[104t-27, 26t(6t-1)-1]$, $26t(6t-1) \neq l(3l-1) \neq l(3l+1)$ for any l

6. Further remarks. For the proof of our results, we rely on the fact that for $N = 2,3,4,5,7,13$ the corresponding Hauptmoduln $j_N(z)$ and $j_N^+(z)$ have an explicit expression in terms of an η-function. By using the method of this paper, we could not conclude any result about the parity of $c_N^+(n)$

for $N = 5, 13$. It would be interesting to extend these results to congruence subgroups and Fricke groups of higher level. Moreover, one may try to get some shorter intervals having the same properties as in the statements of this paper.

Similar to Klein's j-function, we expect that the Fourier coefficients of $j_N(z)$ and $j_N^+(z)$ in some arithmetic progression are both even and odd with density $\frac{1}{2}$.

Acknowledgments. The first author is grateful to her advisor Dr. Brundaban Sahu for his continuous support and making some important corrections in an earlier version of this paper. The authors thank Dr. Jaban Meher for some comments.

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Moni Kumari
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Mumbai 400 005, India
E-mail: moni@math.tifr.res.in

Sujeet Kumar Singh
National Institute of
Science Education and Research
HBNI, Bhubaneswar
Via-Jatni
Khurda, Odisha, 752050, India
E-mail: sujeet.singh@niser.ac.in