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CONVERGENCE FOR VARIANTS OF CHEBYSHEV–HALLEY METHODS USING RESTRICTED CONVERGENCE DOMAINS

Abstract. We present a local convergence analysis for some variants of Chebyshev–Halley methods of approximating a locally unique solution of a nonlinear equation in a Banach space setting. We only use hypotheses reaching up to the second Fréchet derivative of the operator involved in contrast to earlier studies using Lipschitz hypotheses on the second Fréchet derivative and other more restrictive conditions. This way the applicability of these methods is expanded. We also show how to improve the semilocal convergence in the earlier studies under the same conditions using our new idea of restricted convergence domains leading to: weaker sufficient convergence criteria, tighter error bounds on the distances involved and an at least as precise information on the location of the solution. Numerical examples where earlier results cannot be applied but our results can, are also provided.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution x^* of a nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a twice Fréchet differentiable operator defined on an open and convex subset D of a Banach space X with values in a Banach space Y. Numerous problems in computational sciences can be written in the form of equation (1.1), using mathematical modelling [1–27]. The solutions of these equations can rarely be found in explicit form. This explains why

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most solution methods for these equations are iterative. Newton's method defined for $n = 0, 1, \dots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.2}$$

where $x_0 \in D$ is an initial point, is undoubtedly the most popular quadratically convergent method for generating a sequence $\{x_n\}$ approximating x^* [3, 18]. Gutiérrez and Hernández [11] responding to the need for higher convergence order methods studied the semilocal convergence of third order methods defined by

$$x_{n+1} = x_n - \left[I + \tfrac{1}{2}L_F(x_n)(I - \delta L_F(x_n))^{-1}\right]F'(x_n)^{-1}F(x_n), \tag{1.3}$$

where $\delta \in [0, 1]$ and the operator L_F is defined by

$$L_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n). \tag{1.4}$$

The computation of the inverse linear operator $I - \delta L_F(x_n)$ is very expensive in general or impossible. That is why in the study by Kou and Wang [21] the following alternative method to (1.3) was studied (for $\alpha = 1$, $\delta \in [0, 1]$ and $\theta \in [-2, 2]$):

$$\begin{aligned} z_n &= x_n - F'(x_n)^{-1}F(x_n) - A_nF'(x_n)^{-1}F(x_n),\\ x_{n+1} &= z_n - F'(x_n)^{-1}F(z_n) - B_nF'(x_n)^{-1}F(z_n), \end{aligned} \tag{1.5}$$

where

$$\begin{aligned} A_n &= \frac{\alpha}{2}L_F(x_n) + \frac{\delta}{2}L_F(x_n)^2 + \frac{\delta^2}{2}L_F(x_n)^3,\\ B_n &= L_F(x_n) + \theta F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(z_n). \end{aligned}$$

Here $\alpha, \theta, \delta \in S$ where $S = \mathbb{R}$ or $S = \mathbb{C}$. Notice that $I + \delta L_F(x_n) + \delta^2 L_F(x_n)^2$ is used as an approximation to $(I - \delta L_F(x_n))^{-1}$. The semilocal convergence analysis was presented in [21] under the conditions $(\mathcal{C})$ (specified in Section 3). As already noted in [21] these conditions generalize the conditions in [11].

In the present study, we first study the local convergence (not given in [11] or [21]) under weaker conditions. Notice that in [21] one of the conditions is given by

$$\|F'(x^*)^{-1}(F''(x) - F''(y))\| \le w(\|x - y\|) \tag{1.6}$$

where $w : [0, +\infty) \to [0, +\infty)$ is a continuous and nondecreasing function. This hypothesis limits the applicability of methods (1.3)–(1.5). As a motivational example let $X = Y = \mathbb{R}$, $D = [-1/2, 5/2]$ and define a function F on D by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0,\\ 0, & x = 0. \end{cases}$$

Notice that $x^* = 1$. We have

$$\begin{aligned} F'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ F''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x, \\ F'''(x) &= 6 \ln x^2 + 60x^2 - 12x + 22. \end{aligned}$$

Since $F'''(x)$ is unbounded on D, condition (1.6) cannot be satisfied. Hence the results in [11, 21] cannot be used to solve the simple scalar equation $F(x) = 0$.

In the present study, we show convergence without the use of (1.6). Moreover, the convergence order is computed using the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [3]. In the semilocal convergence case, using our new idea of restricted convergence domains, we present a new convergence analysis with the following advantages denoted by $(\mathcal{A})$: larger convergence domain; tighter error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ and an at least as precise information on the location of the solution x^*. The advantages are obtained since the new majorizing functions and parameters are tighter than the old ones. Moreover, these advantages are obtained under the same computational cost as in the earlier studies [11, 21].

The rest of the paper is organized as follows: In Sections 2 and 3, we analyze the local and semilocal convergence of method (1.5), respectively. Numerical examples are given in Section 4.

2. Local convergence. The local convergence analysis of method (1.5) is based on some scalar functions and parameters. Let $L_0 > 0, L > 0, M \geq 1$ and $\alpha, \delta, \theta \in S$ be such that $|\alpha| < 2/(ML)$. Define functions g_1, h_1, g_2, h_2 on the interval $[0, 1/(L_0))$ by

$$\begin{aligned} g_1(t) &= \frac{1}{2(1-L_0 t)}\left[Lt + \frac{ML}{1-L_0 t}\left(|\alpha| + \frac{|\delta| MLt}{(1-L_0 t)^2} + \frac{\delta^2 M^2 L^2 t^2}{(1-L_0 t)^4}\right)\right], \\ h_1(t) &= g_1(t) - 1, \\ g_2(t) &= \frac{L g_1^2(t) t}{2(1-L_0 t)} + \frac{ML_0(1+g_1(t))g_1(t)t}{(1-L_0 t)^2} + \frac{M^3 L}{(1-L_0 t)^3}(1+|\theta| g_1(t))g_1(t)t, \\ h_2(t) &= g_2(t) - 1, \end{aligned}$$

and parameter r_A by

$$r_A = \frac{2}{2L_0 + L}.$$

By the preceding definitions, $h_1(0) = |\alpha|ML/2 - 1 < 0$ and $h_1(t) \to +\infty$ as $t \to (1/L_0)^-$. It then follows from the intermediate value theorem that h_1 has zeros in $(0, 1/L_0)$. Denote by r_1 the smallest such zero. Moreover, $h_2(0) = -1 < 0$ and $h_2(t) \to +\infty$ as $t \to (1/L_0)^-$. Denote by r_2 the smallest zero of h_2 in $(0, 1/L_0)$. Furthermore, define the radius of convergence r by

$$r = \min\{r_1, r_2\}. \tag{2.1}$$

Notice that

$$0 < r \leq r_A < 1/L_0, \tag{2.2}$$

and for each $t \in [0, r)$,

$$0 \leq g_i(t) < 1, \quad i = 1, 2. \tag{2.3}$$

Let $U(\gamma, \rho), \bar{U}(\gamma, \rho)$ stand, respectively for the open and closed balls in X with center $\gamma \in X$ and of radius $\rho > 0$.

Next, we present the local convergence analysis of method (1.5) using the above notation.

THEOREM 2.1. *Let $F : D \subset X \to Y$ be a twice Fréchet differentiable operator. Suppose there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M \geq 1$, $\delta, \theta, \alpha \in S$ such that $|\alpha| < 2/(ML)$,*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X), \tag{2.4}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\| \quad \text{for each } x \in D, \tag{2.5}$$

and for all $x, y \in D_0 := D \cap U(x^*, 1/L_0)$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \tag{2.6}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M, \tag{2.7}$$

$$\bar{U}(x^*, r) \subseteq D, \tag{2.8}$$

where the convergence radius r is defined by (2.1). Then the sequence $\{x_n\}$ generated by method (1.5) for $x_0 \in U(x^*, r) - \{x^*\}$ is well defined in $U(x^*, r)$, remains in $U(x^*, r)$ for each $n = 0, 1, \ldots$ and converges to the solution $x^* \in D$ of $F(x) = 0$. Moreover,

$$\|z_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.9}$$

$$\|x_{n+1} - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.10}$$

where the "g" functions are as defined previously. Furthermore, for $T \in [r, 2/L_0)$ the limit point x^* is the only solution of $F(x) = 0$ in $D^* := D \cap U(x^*, T)$.

Proof. We shall prove (2.9) and (2.10) by induction. Using the hypothesis $x_0 \in U(x^*, r) - \{x^*\}$, (2.1), (2.2) and (2.5), we have

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r < 1. \tag{2.11}$$

In view of (2.11) and the Banach Lemma on invertible operators [3, 16], z_0 and x_1 are well defined by method (1.5) for $n = 0, F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{2.12}$$

By (2.4) we can write

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + \tau(x_0 - x^*))(x_0 - x^*)\, d\tau. \tag{2.13}$$

By the convexity of D and $\|x^* + \tau(x_0 - x^*) - x^*\| = \tau\|x_0 - x^*\| < r$ we get $x^* + \tau(x_0 - x^*) \in U(x^*, r)$ for each $\tau \in [0, 1]$. Then, by (2.7) and (2.13),

$$\|F'(x^*)^{-1}F(x_0)\| \le M\|x_0 - x^*\|. \tag{2.14}$$

By (2.6) and the second Fréchet differentiablity of F we also have

$$\|F'(x^*)^{-1}F''(x_0)\| \le L. \tag{2.15}$$

By (2.12), (2.14) and (2.15) we have

$$\begin{aligned} \|L_F(x_0)\| &\le \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F''(x_0)\|\|F'(x_0)^{-1}F'(x^*)\| \\ &\le \frac{ML\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|}, \end{aligned} \tag{2.16}$$

leading to

$$\begin{aligned} \|A_0\| \le \frac{1}{2}\, \frac{ML\|x_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)^2} \bigg[|\alpha| + |\delta| \frac{ML\|x_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)^2} \\ + |\delta|^2 \frac{(ML\|x_0 - x^*\|)^2}{(1 - L_0\|x_0 - x^*\|)^4} \bigg]. \end{aligned} \tag{2.17}$$

By (2.1), (2.2), (2.3) (for $i = 1$), (2.16), (2.12), (2.14), (2.15) and (2.17), we obtain in turn

$$\begin{aligned} \|z_0 - x^*\| &\le \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\ &\quad + \|A_0\|\, \|F'(x_0)^{-1}F'(x^*)\|\, \|F'(x^*)^{-1}F(x_0)\| \\ &\le \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad \times \left\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \tau(x_0 - x^*)) - F'(x_0))\, d\tau\, (x_0 - x^*) \right\| \\ &\quad + \|A_0\|\, \|F'(x_0)^{-1}F'(x^*)\|\, \|F'(x^*)^{-1}F(x_0)\| \\ &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} + \frac{ML\|x_0 - x^*\|}{2(1 - L_0\|x_0 - x^*\|)^2} \\ &\quad \times \left[|\alpha| + \frac{|\delta| ML\|x_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)^2} + \frac{\delta^2 M^2 L^2\|x_0 - x^*\|^2}{(1 - L_0\|x_0 - x^*\|)^4} \right] \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r, \end{aligned} \tag{2.18}$$

which shows (2.9) for $n = 0$ and $z_0 \in U(x^*, r)$. Then, as in (2.14) we have

$$\|F'(x^*)^{-1}F(z_0)\| \le M\|z_0 - x^*\| \le M g_1(\|x_0 - x^*\|)\|x_0 - x^*\|. \tag{2.19}$$

Moreover,

$$\|B_0\| \le \frac{ML\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2} + \frac{|\theta|ML\|z_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2} \le \frac{ML\|x_0-x^*\| + |\theta|MLg_1(\|x_0-x^*\|)\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2}. \tag{2.20}$$

By the second substep of method (1.5) we can write

$$x_1 - x^* = (z_0 - x^* - F'(z_0)^{-1}F(z_0)) + (F'(z_n)^{-1} - F'(x_n)^{-1})F(z_n) - B_nF'(x_n)^{-1}F(z_n). \tag{2.21}$$

Then, by (2.1), (2.2), (2.3) (for $i = 1$), (2.12) (for $z_0 = x_0$) and (2.18)–(2.21), we obtain

$$\begin{aligned}
\|x_1 - x^*\| &\le \|F'(z_0)^{-1}F'(x^*)\| \\
&\quad \times \Big\| \int_0^1 F'(x^*)^{-1}(F'(x^* + \tau(z_0 - x^*)) - F'(z_0))(z_0 - x^*)d\tau \Big\| \\
&\quad + \|F'(z_0)^{-1}F'(x^*)\| \big[\|F'(x^*)^{-1}(F'(z_0) - F'(x^*))\| \\
&\quad + \|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\|\big] \\
&\quad \times \|F'(x_n)^{-1}F'(x^*)\| \, \|F'(x^*)^{-1}F(z_n)\| \\
&\quad + \|B_0\|\|F'(x_0)^{-1}F'(x^*)\| \, \|F'(x^*)^{-1}F(z_0)\| \\
&\le \frac{L\|z_0 - x^*\|^2}{2(1 - L_0\|z_0 - x^*\|)} \\
&\quad + \frac{ML_0(1 + g_1(\|x_0 - x^*\|))\|z_0 - x^*\|\|x_0 - x^*\|}{(1 - L_0\|z_0 - x^*\|)(1 - L_0\|x_0 - x^*\|)} \\
&\quad + \frac{[ML\|x_0 - x^*\| + |\theta|ML\|z_0 - x^*\|]M\|z_0 - x^*\|}{(1 - L_0\|x_0 - x^*\|)^3} \\
&\le \frac{Lg_1^2(\|x_0 - x^*\|)\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\
&\quad + \frac{ML_0(1 + g_1(\|x_0 - x^*\|))g_1(\|x_0 - x^*\|)\|x_0 - x^*\|^3}{(1 - L_0\|x_0 - x^*\|^2} \\
&\quad + \frac{M^2L(1 + |\theta|g_1(\|x_0 - x^*\|))g_1(\|x_0 - x^*\|)\|x_0 - x^*\|^2}{(1 - L_0\|x_0 - x^*\|)^3} \\
&= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \le \|x_0 - x^*\| < r,
\end{aligned}$$

which shows (2.10) for $n = 0$ and $x_1 \in U(x^*, r)$. Simply, replace x_0, z_0, x_1 by x_k, z_k, x_{k+1} in the preceding estimates to arrive at (2.9)–(2.10). In view of the estimate $\|x_{k+1} - x^*\| \le c\|x_k - x^*\| < r$, $c = g_2(\|x_0 - x^*\|) \in [0, 1)$, we conclude that $\lim_{k\to\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, r)$.

Finally, to show the uniqueness part, let $y^* \in D^*$ be such that $F(y^*) = 0$. Set $Q = \int_0^1 F'(y^* + \tau(x^* - y^*))\, d\tau$. Then in view of (1.5) we get

$$\|F'(x^*)^{-1}(Q - F'(x^*))\| \le \frac{L_0}{2}\|x^* - y^*\| = \frac{L_0}{2}T < 1.$$

Hence $Q^{-1} \in L(Y, X)$. Then from the identity

$$0 = F(x^*) - F(y^*) = Q(x^* - y^*)$$

we deduce that $x^* = y^*$. ∎

REMARK 2.2. (a) When $w_0(t) = L_0 t, w(t) = Lt$, the radius $r_A = \frac{2}{2L_0+L}$ was obtained by Argyros [3] as the convergence radius for Newton's method under conditions (2.4)–(2.6). Notice that the convergence radius for Newton's method given independently by Rheinboldt [19] and Traub [20] is

$$\rho = \frac{2}{3L} < r_1. \tag{2.22}$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $D = U(0, 1)$. Then $L_0 = e - 1 < l = e$, so $\rho = 0.24252961 < r_1 = 0.324947231$.

Moreover, the new error bounds [3] are

$$\|x_{n+1} - x^*\| \le \frac{L}{1 - L_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [19, 20] were

$$\|x_{n+1} - x^*\| \le \frac{L}{1 - L\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Obviously, the new error bounds are more precise if $L_0 < L$. Clearly, we do not expect the radius of convergence of method (1.2) given by r to be larger than r_A.

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [3, 4, 13].

(c) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [3, 4, 13]

$$F'(x) = P(F(x)),$$

where $P : Y \to Y$ is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. As an example define $F(x) = e^x - 1$. Then we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.5) is not affected if we use the new conditions instead of the old ones [21]. Moreover, for the error bounds in practice we can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{\ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}} \qquad \text{for each } n = 1, 2, \dots,$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{\ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}} \qquad \text{for each } \ n = 0, 1, \dots$$

instead of the error bounds obtained in Theorem 2.1.

(e) In view of (2.5) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\le 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 1 + w_0(\|x - x^*\|)\end{aligned}$$

condition (2.7) can be dropped and can be replaced by

$$v(t) = 1 + w_0(t) \quad \text{or} \quad v(t) = 1 + w_0(r_0),$$

since $t \in [0, r_0)$.

3. Semilocal convergence. We now show how to improve the semilocal convergence analysis given in [21]. First of all we provide the conditions $(\mathcal{C})$ given in [21] in affine invariant form:

$(\mathcal{C}_1)$ $\|F'(x_0)^{-1}F(x_0)\| \le \eta$.
$(\mathcal{C}_2)$ $\|F'(x_0)^{-1}F''(x)\| \le K$.
$(\mathcal{C}_3)$ $\|F'(x_0)^{-1}(F''(x) - F''(y))\| \le w(\|x - y\|)$ for all $x, y \in D_0 \subseteq D$ where $w(t)$ is a non-decreasing continuous function defined on the interval $(0, +\infty)$ with $w(0) \ge 0$.
$(\mathcal{C}_4)$ There exists a non-negative function $\Phi \in C[0, 1]$ satisfying $\Phi(q) \le 1$ and $w(qt) \le \Phi(q)w(t)$ for each $q \in [0, 1]$ and $t \in (0, +\infty)$.

Based on the conditions $(\mathcal{C})$ the semilocal convergence of method (1.5) was analyzed in [21, Theorem 1]. Next, we show how to improve Theorem 1 using the following conditions $(\mathcal{H})$:

$(\mathcal{H}_1)$ $= (\mathcal{C}_1)$.
$(\mathcal{H}_2)$ $= (\mathcal{C}_2)$.
$(\mathcal{H}_3)$ $\|F'(x_0)^{-1}(F''(x) - F''(y))\| \le w_0(\|x - y\|)$ for all $x, y \in D_1 := D_0 \cap U(x_0, 1/K)$, where $w_0(t)$ is a non-decreasing continuous function defined on $(0, +\infty)$ with $w_0(0) \ge 0$.
$(\mathcal{H}_4)$ There exists a non-negative function $\Phi_0 \in C[0, 1]$ satisfying $\Phi_0(q) \le 1$ and $w_0(qt) \le \Phi_0(q)w_0(t)$ for all $q \in [0, 1]$ and $t \in (0, +\infty)$.

It follows from $(\mathcal{C}_3)$, $(\mathcal{C}_4)$,$(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ that since $D_1 \subseteq D_0$, the following inequalities hold:

$$w_0(t) \le w(t) \quad \text{for each } t \in (0, +\infty), \tag{3.1}$$

$$\Phi_0(q) \le \Phi(q) \quad \text{for each } q \in [0, 1]. \tag{3.2}$$

Let us present an academic example to show that inequalities (3.1) and (3.2) can be strict.

Example 3.1. Let $X = Y = \mathbb{R}$, $x_0 = 1$, $\xi \in (0, 0.13)$, $D_0 = D = U(x_0, 1 - \xi)$ and define a function F on D_0 by

$$F(x) = \frac{x^4}{12} - \xi. \tag{3.3}$$

Using the conditions $(\mathcal{C})$, $(\mathcal{H})$ and (3.3), we obtain $K = \frac{1}{3}(2 - \xi)^2, w_0(t) = \frac{1}{3}t(t + 2(1/K + 1))$ and $w(t) = \frac{1}{3}t(t + 2(2 - \xi))$. Then, for $\xi \in (0, 0.13)$, we find $w_0(t) < w(t)$ for each $t \in (0, +\infty)$, since

$$\frac{3}{(2 - \xi)^2} < 1 - \xi. \tag{3.4}$$

Moreover, choose $\Phi_0(q) = \beta_0 q$, $\Phi(q) = \beta q$ with $1 < \beta_0 < \beta$. It follows that D_1 is a proper subset of D_0. The proof of Theorem 2.1 can be repeated using the conditions $(\mathcal{H})$ instead of the stronger conditions $(\mathcal{C})$ by simply noticing that the iterates $\{x_n\}$ of method (1.5) lie in D_1, which is a more precise location than D_0. This modification leads to tighter functions and majorizing sequences. Let us redefine the old functions and sequences as well as define the new functions and sequences, so we can compare them for $\alpha = 1$:

$$\begin{aligned}
g_1(t) &= 1 + \frac{1}{2}t + \frac{\delta}{2}t^2 + \frac{\delta^2}{2}t^3,\\
g_2(t) &= \left[\frac{1}{2} + \frac{\delta}{2}t + \frac{\delta^2}{2}t^2 + \frac{1}{2}g_1(t)^2\right]t,\\
\varphi(t,u) &= |\theta| t\psi(t,u)^2 + t^2[1 + |\theta|\psi(t,u)]\psi(t,u)\\
&\quad + J_2 u[1 + t + |\theta| t\psi(t,u)]\psi(t,u)\\
&\quad + t^2\left[\frac{1}{2} + \frac{\delta}{2}t + \frac{\delta^2}{2}t^2\right][1 + t + |\theta| t\psi(t,u)]\psi(t,u)\\
&\quad + \frac{t}{2}[1 + t + |\theta| t\psi(t,u)]^2\psi(t,u)^2,\\
\psi(t,u) &= \frac{\delta}{2}t^2 + \frac{\delta^2}{2}t^3 + t\left(\frac{1}{2}t + \frac{\delta}{2}t^2 + \frac{\delta^2}{2}t^3\right)^2\\
&\quad + J_1 u + \frac{1}{2}t\left(\frac{1}{2}t + \frac{\delta}{2}t^2 + \frac{\delta^2}{2}t^3\right),
\end{aligned}$$

$$
\begin{aligned}
J_1 &= \int_0^1 \Phi(t)(1-t)\,dt,\\
J_2 &= \int_0^1 \Phi(t)\,dt,\\
\eta_0 &= \eta, & \eta_{n+1} &= d_n\eta_n,\\
a_0 &= K\eta, & a_{n+1} &= K\beta_{n+1}\eta_{n+1},\\
b_0 &= \eta w(\eta), & b_{n+1} &= \beta_{n+1}\eta_{n+1}w(\eta_{n+1}),\\
\beta_0 &= 1, & \beta_{n+1} &= h(a_n)\beta_n,\\
d_0 &= h(a_0)\varphi(a_0,b_0), & d_{n+1} &= h(a_{n+1})\varphi(a_{n+1},b_{n+1}),
\end{aligned}
$$

$$
p(t) = g_1(t) + [1+t+|\theta|tg_2(t)]g_2(t), \qquad h(t) = \frac{1}{1-tp(t)}.
$$

The new functions and sequences denoted with a bar, like $\bar{p}(t)$ instead of $p(t)$, are defined similarly but with Φ_0, w_0 replacing Φ and w, respectively. Then, Theorem 1 in [21] can be rewritten in the following improved setting.

THEOREM 3.2. *Let $F : D \subset X \to Y$ be a twice Fréchet differentiable operator on an open convex set $D_0 \subseteq D$. Suppose:*

(i) *The conditions* $(\mathcal{H})$ *hold for some $x_0 \in D_0$.*
(ii) *For $\bar{a}_0 = K\eta$, $\bar{b}_0 = \eta w_0(\eta)$, $\bar{d}_0 = \bar{h}(\bar{a}_0)\bar{\Phi}(\bar{a}_0, \bar{b}_0)$, $\bar{a}_0 < \bar{s}$ and $\bar{h}(\bar{a}_0)\bar{d}_0 < 1$, where $\bar{s}$ is the positive zero of the equation $t\bar{p}(t) - 1 = 0$.*
(iii) *$\bar{U}(x_0, \bar{R}\eta) \subseteq D_0$, where $\bar{R} = \bar{p}(\bar{a}_0)/(1-\bar{d}_0)$.*

Then the sequence $\{x_n\}$ generated for $x_0 \in D_0$ by method (1.5) *is well defined in $\bar{U}(x_0, \bar{R}\eta)$, remains in $\bar{U}(x_0, \bar{R}\eta)$ for each $n = 0, 1, \ldots$ and converges to a unique solution of $F(x) = 0$ in $U(x_0, 2/K - \bar{R}\eta) \cap D_0$. Moreover,*

$$
\|x_n - x^*\| \le \bar{p}(\bar{a}_0)\eta\bar{\lambda}^n\bar{\gamma}^{(3^n-1)/2}\frac{1}{1-\bar{\lambda}\bar{\gamma}^{3n}},
$$

where $\bar{\gamma} = \bar{h}(\bar{a}_0)\bar{d}_0$ and $\bar{\lambda} = 1/\bar{h}(\bar{a}_0)$.

REMARK 3.3. If $w_0(t) = w(t)$ for each $t \in [0, +\infty)$ and $\Phi_0(q) = \Phi(q)$ for each $q \in [0, 1]$, then Theorem 3.2 reduces to Theorem 1 of [21]. Otherwise, strict inequality holds in (3.1) or (3.2) (see also Example 3.1). Thus, the new Theorem 3.2 improves the old one with advantages as already stated in the introduction.

4. Numerical examples

EXAMPLE 4.1. Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0,1)$, $x^* = (0,0,0)^T$. Define a function F on D for $w = (x,y,z)^T$ by

$$F(w) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

Then the Fréchet derivative is given by

$$F'(w) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that $F'(x^*) = F'(x^*)^{-1} = \operatorname{diag}\{1,1,1\}, L_0 = e - 1 < L = e^{1/L_0} = M$. Then, for method (1.5) with $\alpha = \frac{1}{ML}, \delta = \theta = 0.5$, the parameters are

$$\xi_1 = 2.9515, \quad r_1 = 0.0927, \quad r_2 = 0.0526 = r < r_A = 0.3827.$$

Example 4.2. Returning to the motivational example of the introduction, we have $L_0 = L = 96.6629073$ and $M = 1.6631$ for $x \in D$. The parameters for method (1.5) with $\alpha = 1/(ML)$, $\delta = \theta = 0.5$, are

$$\xi_1 = 2.9686, \; r_2 = 0.0007, \; r_1 = 0.0006 = r < r_A = 0.0069.$$

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