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UNCERTAINTY PRODUCT OF THE SPHERICAL ABEL–POISSON WAVELET

Abstract. The space and momentum variances of the spherical Abel–Poisson wavelet, as well as the limit of the uncertainty product as $\rho \to 0$, where ρ is a scale parameter, are computed. The values of these quantities coincide with a certain accuracy with those of the spherical Poisson wavelet with 1/2 substituted for the order parameter.

1. Introduction. Just as in physics (Heisenberg's uncertainty principle), there exist several uncertainty principles in mathematics. The uncertainty constant of a function is a measure for the trade-off between spatial and frequency localization. A practical example is a sound with a certain frequency that should last at least one period in order to be heard at the pitch [1].

The first formulations and proofs of an uncertainty principle for functions over the two-dimensional sphere appeared in the 1990s in the papers by Narcowich and Ward [21] and Freeden and Windheuser [7]. A generalization to higher dimensions was given in [22], though only for rotation-invariant functions.

In the 2002 paper [9] by Goh and Micchelli, the classical uncertainty principle for self-adjoint operators A, B,

$$|\langle [A,B]\,x,x\rangle| \le 2\,\Delta_x(A)\cdot\Delta_x(B), \tag{1}$$

where

$$[A,B]=AB-BA,\qquad \Delta_x(A)=\left(\|Ax\|^2-\frac{|\langle Ax,x\rangle|^2}{\|x\|^2}\,x\right)^{1/2},$$

was generalized to the case of linear operators with domain and range in

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the same complex Hilbert space. The authors proved that for any nonzero $x \in \mathcal{D}(AB) \cap \mathcal{D}(BA) \cap \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$,

$$|\langle [A,B]x,x\rangle| \le \Delta_x(A)\cdot\Delta_x(B^*) + \Delta_x(A^*)\cdot\Delta_x(B). \tag{2}$$

If each of A and B is normal or symmetric, inequality (2) reduces to (1). This is a generalization of the classical result since an operator is self-adjoint if it is simultaneously normal and symmetric.

Relying on the results from [9], Goh and Goodman [8] derived several uncertainty principles for distinct spaces, among them the following one that is now regarded as the canonical version of an uncertainty principle on the sphere [8, Theorem 5.1].

THEOREM 1.1. *Let F be a nonzero $\mathcal{C}^1$-function on the n-dimensional sphere $\mathcal{S}^n$. Then*

$$\left[\int_{\mathcal{S}^n} |f|^2\,d\sigma - \frac{|\int_{\mathcal{S}^n} x|f(x)|^2\,d\sigma(x)|^2}{\int_{\mathcal{S}^n}|f|^2\,d\sigma}\right]^{1/2}\cdot\left[\int_{\mathcal{S}^n} |\nabla_{\mathcal{S}^n} f|^2\,d\sigma - \frac{|\int_{\mathcal{S}^n}(\nabla_{\mathcal{S}^n} f)\bar f\,d\sigma|^2}{\int_{\mathcal{S}^n}|f|^2\,d\sigma}\right]^{1/2} \ge \frac{n}{2}\cdot\frac{|\int_{\mathcal{S}^n} x|f(x)|^2\,d\sigma(x)|^2}{\int_{\mathcal{S}^n}|f|^2\,d\sigma}. \tag{3}$$

The proof is based on the multivariate version of (1) (see [8, Theorem 4.1]):

$$\Big(\sum_{j=0}^n \Delta_x(A_j)^2\Big)\cdot\Big(\sum_{j=0}^n \Delta_x(B_j)^2\Big) \ge \frac{1}{4}\Big(\sum_{j=0}^n |\langle [A_j,B_j]x,x\rangle|\Big)^2,$$

where the (bounded and self-adjoint) operators A_j and (symmetric) operators B_j on $\mathcal{L}^2(\mathcal{S}^n)$ are given by

$$A-jf(x) = x_jf(x), \qquad B_jf(x) = iD_jf(x) - \frac{in}{2}\,x_jf(x)$$

for $x=(x_0,x_1,\dots,x_n)\in\mathcal{S}^n$.

The results from [21], [7], and [22] are all special cases of Theorem 1.1.

Further papers yield different proofs for (3) (see, e.g., [10]) or its weaker version

$$\left[\int_{\mathcal{S}^n} |f|^2\,d\sigma - \frac{|\int_{\mathcal{S}^n} x|f(x)|^2\,d\sigma(x)|^2}{\int_{\mathcal{S}^n}|f|^2\,d\sigma}\right]^{1/2}\cdot\left[\int_{\mathcal{S}^n} |\nabla_{\mathcal{S}^n} f|^2\,d\sigma\right]^{1/2} \ge \frac{n}{2}\cdot\Big|\int_{\mathcal{S}^n} x|f(x)|^2\,d\sigma(x)\Big| \tag{4}$$

(see [2] together with [3], or [5]). The term $\|\nabla_{\mathcal{S}^n}F\|_2$ in (4) is equal to and often replaced by $\sqrt{\langle -\Delta_{\mathcal{S}^n}F,F\rangle}$. A slightly modified version of (4) is a special case of a general result for compact manifolds [23].

In the recent years, several papers have been published about spherical uncertainty principles in some special settings, such as for weighted function spaces [6, 24] or in the Clifford algebra setting [4]. The results from [6, 24], reduced to the nonweighted case, do not essentially differ from (4).

The present research is devoted to spherical wavelets (for the theory of wavelet transforms see [12]). If it does not affect other important features of the wavelet, its uncertainty product should be kept as small as possible so that the localization of the wavelet transform reflects mostly the localization of the analyzed signal and not that of the wavelet. Thus, it is important to compute the uncertainty product of the most popular wavelets in order to be able to compare them with respect to this feature.

In [14] a general class of wavelets

$$\Psi_\rho(x) = \sum_{l=0}^{\infty} [\rho^a q_\nu(l)]^c e^{-\rho^a q_\nu(l)} \cdot \frac{l+\lambda}{\lambda} \mathcal{C}_l^\lambda(\cos\vartheta) \tag{5}$$

is considered, where ϑ is the first spherical coordinate of $x \in \mathcal{S}^n$, a and c are positive constants, and q_ν is a polynomial of degree ν, positive and increasing on $(1,\infty]$. $\mathcal{C}_l^\lambda$ denotes the Gegenbauer polynomial of order $\lambda = (n-1)/2$ and degree l. It is shown that the uncertainty product of Ψ_ρ behaves as $\rho \to 0$ like

$$U(\Psi_\rho) \le \mathcal{O}(\rho^{a/(2\nu)}).$$

In the general case, its increase for $\rho \to 0$ is undesirable. However, there exist wavelet families constructed according to (5) with bounded uncertainty product, namely the Poisson wavelets

$$g_\rho^m = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} (\rho l)^m e^{-\rho l} \cdot \frac{l+\lambda}{\lambda} \mathcal{C}_l^\lambda(\cos\vartheta), \qquad m \in \mathbb{N},\ \Sigma_n = \int_{\mathcal{S}^n} d\sigma$$

(see [16] for the proof), and the Gauss–Weierstrass wavelet on the two-dimensional sphere

$$\Psi_\rho^G(x) = \sum_{l=0}^{\infty} \frac{2l+1}{4\pi} \sqrt{2\rho l(l+1)}\, e^{-\rho l(l+1)} P_l(\cos\vartheta),$$

P_l being the Legendre polynomials (see [17]).

The Abel–Poisson wavelet

$$\Psi_\rho^A = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} \sqrt{2\rho l}\, e^{-\rho l} \cdot \frac{l+\lambda}{\lambda} \mathcal{C}_l^\lambda(\cos\vartheta), \tag{6}$$

which is rotation-invariant, completes the list of classical examples. Its uncertainty product is investigated in the present paper. Since in the Gegenbauer series of the Abel–Poisson wavelet the square root of ρl appears, more sophisticated methods are required than those used in [16] for Poisson wavelets. On the other hand, the exponent of the exponential function in (6) is a multiple of l. This makes the computation easier than in the case of the Gauss–Weierstrass wavelet [17], and enables investigating the general case of wavelets over the n-dimensional sphere.

In this paper, the so-called space and momentum variances of the Abel–Poisson wavelet are computed explicitly and exactly. The uncertainty product of a function is the product of the square roots of the variances. In the case of the Abel–Poisson wavelet, its limit for $\rho \to 0$ is finite. Up to a certain order in the series expansion with respect to ρ, the values of the variances and the uncertainty product coincide with those for Poisson wavelets [16] with $1/2$ substituted for the order parameter m (Poisson wavelets are defined for $m \geq 1$).

The paper is organized as follows. After an introduction of the necessary notions and statements in Section 2, the main result of the paper, Theorem 3.1, is proven in Section 3.

2. Preliminaries. Let $\mathcal{S}^n$ denote the n-dimensional unit sphere in $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with hyperspherical variables $(\vartheta, \vartheta_2, \ldots, \vartheta_{n-1}, \varphi)$. Integrable zonal (i.e. rotation-invariant with respect to the x_1-axis) functions on the sphere have the Gegenbauer expansion

$$f(x) = \sum_{l=0}^{\infty} \widehat{f}(l) \mathcal{C}_l^\lambda(\cos\vartheta)$$

with the Gegenbauer coefficients

$$\widehat{f}(l) = c(l,\lambda) \int_{-1}^{1} f(t)\mathcal{C}_l^\lambda(t)(1-t^2)^{\lambda-1/2}\, dt,$$

where λ is an index related to the space dimension by

$$\lambda = \frac{n-1}{2}$$

and c is a constant that depends on l and λ. $\mathcal{C}_l^\lambda$, $l \in \mathbb{N}_0$, are the Gegenbauer polynomials of order $\lambda \in \mathbb{R}$ and degree $l \in \mathbb{N}_0$.

The *space and momentum variances* of a $\mathcal{C}^2(\mathcal{S}^n)$-function f satisfying $\int_{\mathcal{S}^n} x|f(x)|^2\, d\sigma(x) \neq 0$ are given by [20]

$$\mathrm{var}_S(f) = \left(\frac{\int_{\mathcal{S}^n} |f(x)|^2\, d\sigma(x)}{\left|\int_{\mathcal{S}^n} x|f(x)|^2\, d\sigma(x)\right|} \right)^2 - 1,$$

$$\mathrm{var}_M(f) = -\frac{\int_{\mathcal{S}^n} \Delta^* f(x) \cdot \bar{f}(x)\, d\sigma(x)}{\int_{\mathcal{S}^n} |f(x)|^2\, d\sigma(x)},$$

where Δ^* is the Laplace–Beltrami operator on $\mathcal{S}^n$. The quantity

$$U(f) = \sqrt{\mathrm{var}_S(f)} \cdot \sqrt{\mathrm{var}_M(f)} \tag{7}$$

is called the *uncertainty product* of f.

REMARK. The uncertainty product defined by (7) corresponds to the uncertainty principle given by (4) (see Theorem 2.2 below).

The uncertainty product of zonal functions may be computed from their Gegenbauer coefficients [13, Lemma 4.2].

LEMMA 2.1. *Let f be a zonal square integrable and continuously differentiable function on $\mathcal{S}^n$ given by its Gegenbauer expansion*

$$f(x) = \sum_{l=0}^{\infty} \widehat{f}(l)\, \mathcal{C}_l^\lambda(\cos\vartheta).$$

Its space and momentum variances are equal to

$$\mathrm{var}_S(f) = \left(\frac{\sum_{l=0}^{\infty} \frac{\lambda}{l+\lambda}\binom{l+2\lambda-1}{l} |\widehat{f}(l)|^2}{\sum_{l=0}^{\infty} \binom{l+2\lambda}{l} \frac{\lambda^2 |\overline{\widehat{f}(l)}\,\widehat{f}(l+1) + \widehat{f}(l)\,\overline{\widehat{f}(l+1)}|}{(l+\lambda)(l+\lambda+1)}} \right)^2 - 1, \tag{8}$$

$$\mathrm{var}_M(f) = \frac{\sum_{l=1}^{\infty} \frac{l\lambda(l+2\lambda)}{l+\lambda}\binom{l+2\lambda-1}{l} |\widehat{f}(l)|^2}{\sum_{l=0}^{\infty} \frac{\lambda}{l+\lambda}\binom{l+2\lambda-1}{l} |\widehat{f}(l)|^2}, \tag{9}$$

whenever the series are convergent.

This lemma corrects [13, formula (26)]. In order to compute the space variance of a function, one should take the absolute value of the expression

$$\overline{\widehat{f}(l)}\,\widehat{f}(l+1) + \widehat{f}(l)\,\overline{\widehat{f}(l+1)} \tag{10}$$

instead of (10). In [13], there is a mistake in the formula defining the space variance at the beginning of Section 4, where the numerator is a vector and not the norm of the vector.

According to the spherical uncertainty principle, the uncertainty product of a spherical function is bounded from below by $n/2$.

THEOREM 2.2. *For $f \in \mathcal{L}^2(\mathcal{S}^n) \cap \mathcal{C}^1(\mathcal{S}^n)$, $U(f) \geq n/2$.*

3. The uncertainty product of the Abel–Poisson wavelet. Recall that the Abel–Poisson wavelet is given by (6):

$$\varPsi_\rho^A(x) = \frac{1}{\Sigma_n} \sum_{l=0}^{\infty} \sqrt{2\rho l}\, e^{-\rho l} \cdot \frac{l+\lambda}{\lambda} \mathcal{C}_l^\lambda(\cos\vartheta),$$

where $x = (\vartheta, \vartheta_2, \ldots, \vartheta_{n-1}, \varphi)$ in hyperspherical coordinates. Formally, it is a Poisson wavelet of order $1/2$. Similarly to the case of Poisson wavelets [16], its uncertainty product is bounded in the limit $\rho \to 0$, as the following theorem states.

THEOREM 3.1. *The uncertainty product of the Abel–Poisson wavelet satisfies*

$$\lim_{\rho\to 0} U(\Psi_\rho^A) = \frac{1}{2}\sqrt{\frac{(n+1)(n+2)(n^2-3n+3)}{n(n-1)}}. \tag{11}$$

Its space and momentum variances domain are given by

$$\begin{aligned}\operatorname{var}_S(\Psi_\rho^A) &\\ = \{\rho^n &+ 2[2(n-1)(1-e^{-2\rho})^n\kappa(\rho)\rho^2 - (4n^2-6n+3)\rho^n]\,e^{2\rho}\\ &- [4(n-1)(1-e^{-2\rho})^n\kappa(\rho)\rho^2 + (4n-5)\rho^n]e^{4\rho}\}^{-2}\\ &\cdot \{16\,(n-1)^2[n-1+(n+1)\,e^{2\rho}]^2\rho^{2n}e^{2\rho}\} - 1,\end{aligned} \tag{12}$$

where κ is a bounded function, and

$$\operatorname{var}_M(\Psi_\rho^A) = \frac{n(n+1)[n+(n+3)e^{2\rho}+e^{4\rho}]e^{2\rho}}{[n-1+(n+1)e^{2\rho}](e^{2\rho}-1)^2}. \tag{13}$$

Proof. Substituting

$$\widehat{\Psi_\rho^A}(l) = \frac{\lambda+l}{\lambda}\sqrt{2\rho l}\,e^{-\rho l}$$

into the expressions (8) and (9) we obtain

$$\operatorname{var}_S(\Psi_\rho^A) = \left(\frac{\sum_{l=1}^\infty \frac{l+\lambda}{\lambda}\binom{l+2\lambda-1}{l} l e^{-2\rho l}}{\sum_{l=1}^\infty \binom{l+2\lambda}{l}\cdot 2\sqrt{l(l+1)}\,e^{-\rho(2l+1)}}\right)^2 - 1,$$

$$\operatorname{var}_M(\Psi_\rho^A) = \frac{\sum_{l=1}^\infty \frac{l(l+\lambda)(l+2\lambda)}{\lambda}\binom{l+2\lambda-1}{l} l e^{-2\rho l}}{\sum_{l=1}^\infty \frac{l+\lambda}{\lambda}\binom{l+2\lambda-1}{l} l e^{-2\rho l}}.$$

Set

$$S_{n,m}(\rho) = \sum_{l=1}^\infty \binom{l+2\lambda-1}{l} l^m e^{-2\rho l}.$$

Then

$$\operatorname{var}_S(\Psi_\rho^A) = \left[\frac{e^\rho\cdot A(\rho)}{B(\rho)}\right]^2 - 1, \tag{14}$$

$$\operatorname{var}_M(\Psi_\rho^A) = \frac{C(\rho)}{A(\rho)} \tag{15}$$

with

$$A(\rho) = \frac{1}{\lambda}S_{n,2}(\rho) + S_{n,1}(\rho), \tag{16}$$

$$C(\rho) = \frac{1}{\lambda}S_{n,4}(\rho) + 3S_{n,3}(\rho) + 2\lambda S_{n,2}(\rho), \tag{17}$$

and

$$B(\rho) = \sum_{l=0}^{\infty} \frac{l+2\lambda}{\lambda} \binom{l+2\lambda-1}{l} \sqrt{l(l+1)}\, e^{-2\rho l}.$$

In order to estimate $B(\rho)$ by $S_{n,m}(\rho)$ note that for $l \in \mathbb{N}$,

$$l + \frac{1}{2} - \frac{1}{8l} \le \sqrt{l(l+1)} \le l + \frac{1}{2} - \frac{1}{8l} + \frac{1}{16l^2}.$$

Thus,

$$\frac{l+2\lambda}{\lambda}\sqrt{l(l+1)} \ge \frac{l^2}{\lambda} + \left(\frac{1}{2\lambda}+2\right)l + \left(1 - \frac{1}{8\lambda}\right) - \frac{1}{4l}$$

and

$$\begin{aligned} \frac{l+2\lambda}{\lambda}\sqrt{l(l+1)} &\le \frac{l}{\lambda}\left(l + \frac{1}{2} - \frac{1}{8l} + \frac{1}{16l^2}\right) + 2\left(l + \frac{1}{2\lambda}\right) \\ &= \frac{l^2}{\lambda} + \left(\frac{1}{2\lambda}+2\right)l + \left(1 - \frac{1}{8\lambda}\right) + \frac{1}{16\lambda l}. \end{aligned}$$

Consequently,

$$B(\rho) = \frac{1}{\lambda} S_{n,2}(\rho) + \left(\frac{1}{2\lambda}+2\right) S_{n,1}(\rho) + \left(1 - \frac{1}{8\lambda}\right) S_{n,0}(\rho) + R(\rho) \tag{18}$$

with

$$|R(\rho)| \le \frac{1}{4} \cdot S_{n,-1}(\rho).$$

It can be proven by induction on $n = 2\lambda + 1$ that

$$S_{n,0}(\rho) = \frac{1}{(1-e^{-2\rho})^{n-1}} - 1. \tag{19}$$

The values of $S_{n,1}$, $S_{n,2}$, $S_{n,3}$, and $S_{n,4}$ are obtained from (19) by the recurrence relation

$$S_{n,m+1}(\rho) = -\frac{1}{2} S'_{n,m}(\rho). \tag{20}$$

In order to estimate $S_{n,-1}(\rho)$ note that

$$S_{n,-1}(\rho) = \sum_{l=1}^{\infty} \left[P_{n-3}(l) + \frac{(n-2)!}{l} \right] e^{-2\rho l},$$

where P_{n-3} is a polynomial of degree $n-3$. Thus, $S_{n,-1}$ can be expressed as a linear combination of $S_{2,m}$, $m = -1, 0, \ldots, n-3$:

$$S_{n,-1}(\rho) = \sum_{m=-1}^{n-3} \kappa_{n,m} \cdot S_{2,m}(\rho). \tag{21}$$

It follows from (19) and (20) that for $m \in \mathbb{N}_0$,

$$S_{2,m}(\rho) = \frac{Q(n, e^{-2\rho})}{(1-e^{-2\rho})^{m+1}}, \tag{22}$$

where Q is a polynomial. Further,

$$S_{2,-1}(\rho) = \sum_{l=1}^{\infty} \frac{e^{-2\rho l}}{l}.$$

Since the function $t \mapsto e^{-2\rho t}/t$ is decreasing on $(0,\infty)$,

$$\sum_{l=2}^{\infty} \frac{e^{-2\rho l}}{l} \le \int_1^{\infty} \frac{e^{-2\rho t}}{t}\, dt.$$

Thus,

$$0 \le S_{2,-1}(\rho) \le e^{-2\rho} + \int_1^{\infty} \frac{e^{-2\rho t}}{t}\, dt = e^{-2\rho} + \Gamma(0, 2\rho). \tag{23}$$

It follows from (21)–(23) and

$$\frac{1}{(1-e^{-2\rho})^m} = \mathcal{O}\left(\frac{1}{\rho^m}\right) \quad \text{as } \rho \to 0$$

that

$$S_{n,-1}(\rho) = \mathcal{O}\left(\frac{1}{\rho^{n-2}}\right) \quad \text{as } \rho \to 0.$$

Therefore, the rest term in (18) can be written as

$$R(\rho) = \frac{\kappa(\rho)}{\rho^{n-2}}$$

with a bounded function κ. Formulae (12) and (13) are obtained from (14) and (15) by substituting (16)–(18) with the series $S_{n,m}$, $m = 1, 2, 3, 4$, computed from (19) via (20). In the computation, $S_{n,0}$ is replaced by $(1-e^{-2\rho})^{1-n}$. The difference between the two expressions can be absorbed by the rest term $R(\rho)$, and the replacement simplifies the calculations significantly.

(11) follows directly from (12) and (13). ∎

Remark. The variances of the Abel–Poisson wavelet can be written as

$$\operatorname{var}_S(\Psi_\rho^A) = \frac{n^2-3n+3}{n(n-1)} \cdot \rho^2 - \frac{2n^2-4n+3+2^{n+1}n(n-1)\kappa(\rho)}{n^2(n-1)} \cdot \rho^3 + \mathcal{O}(\rho^4),$$

$$\operatorname{var}_M(\Psi_\rho^A) = \frac{n^2+3n+2}{4\rho^2} + \frac{n^2-1}{2n\rho} + \mathcal{O}(1)$$

as $\rho \to 0$. Note that the first term of the space variance as well as the first two terms of the momentum variance are equal to those computed for Poisson wavelets [16, Theorem 3.1] with 1/2 substituted for m. Consequently, the limit of $U(\Psi_\rho^A)$ as $\rho \to 0$ coincides with that for a Poisson wavelet with $m = 1/2$. Note that formally the Abel–Poisson wavelet is a Poisson wavelet of order 1/2. Therefore, the results concerning the variances and the uncertainty product are not surprising.

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