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EXTENSIONS OF KANTOROVICH-TYPE THEOREMS FOR NEWTON'S METHOD

Abstract. We extend the applicability of Newton's method, so we can approximate a locally unique solution of a nonlinear equation in a Banach space setting in cases not covered before. To achieve this, we find a more precise set containing the Newton iterates than in earlier works.

1. Introduction. The most used iteration for generating a sequence approximating a locally unique solution x^* of a nonlinear equation

$$(1.1) \quad F(x) = 0,$$

is undoubtedly Newton's method defined for all $n = 0, 1, 2, \dots$ by

$$(1.2) \quad x_{n+1} = x_n - F'(x_n)^{-1}F(x_n),$$

where x_0 is an initial point and $F : \Omega \subseteq X \rightarrow Y$ is a continuously Fréchet differentiable operator between Banach spaces X and Y and Ω is a convex set.

There is an extensive literature on local as well as semilocal Kantorovich-type convergence results for Newton's method [1–14]. However, the convergence domain for Newton's method is small in general. In the present study, we show how to extend the convergence domain without adding hypotheses in the already existing works. To achieve this we provide a more precise location, where the Newton iterates lie, leading to smaller Lipschitz functions.

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The rest of the study is organized as follows: In Sections 2 and 3, we present the semilocal convergence analysis of Newton's method (1.2). Section 4 contains numerical examples.

2. Convergence analysis for Newton's method. In this section, we present the semilocal convergence analysis of Newton's method for Fréchet differentiable operators.

Let $L(X, Y)$ stand for the space of all bounded linear operators from a Banach space X to a Banach space Y . Define the balls $U(x, \rho) = \{y \in \Omega : \|x - y\| < \rho\}$ and $\bar{U}(x, \rho) = \{y \in \Omega : \|x - y\| \leq \rho\}$.

DEFINITION 2.1. Let $F : \Omega \subseteq X \rightarrow Y$ be a continuously Fréchet differentiable operator. Let $x_0 \in \Omega$ be such that $F'(x_0)^{-1} \in L(Y, X)$. We say that F' satisfies the L_0 -center Lipschitzian condition on Ω_0 if

$$(2.1) \quad \|F'(x_0)^{-1}(F'(x) - F'(x_0))\| \leq L_0(r)\|x - x_0\|$$

for all $r \in [0, R]$ and all $x \in \Omega_0 := U(x_0, r) \cap \Omega$, where $R > 0$ and $L_0 : [0, R] \rightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous and non-decreasing function with $L_0(0) = 0$.

Define

$$(2.2) \quad \bar{r}_0 = \sup\{t \in [0, R) : L_0(t) < 1\},$$

$$(2.3) \quad \Omega_1 = U(x_0, \bar{r}_0) \cap \Omega.$$

Notice that

$$(2.4) \quad \bar{r}_0 \leq R \quad \text{and} \quad \Omega_1 \subseteq \Omega_0.$$

DEFINITION 2.2. Let $F : \Omega \subseteq X \rightarrow Y$ be a continuously Fréchet differentiable operator. Let $x_0 \in \Omega$ be such that $F'(x_0)^{-1} \in L(Y, X)$. We say that F' satisfies the *restricted L-Lipschitzian condition* on Ω_1 if

$$(2.5) \quad \|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L(r)\|x - y\|$$

for all $r \in [0, \bar{r}_0]$ and all $x, y \in \Omega_1$, where $L : [0, \bar{r}_0) \rightarrow \mathbb{R}_+ \cup \{0\}$ is a continuous and non-decreasing function with $L(0) = 0$.

It is convenient for the semilocal convergence analysis that follows to introduce functions $\varphi_0 : [0, R] \rightarrow \mathbb{R}_+ \cup \{0\}$ and $\varphi : [0, \bar{r}_0] \rightarrow \mathbb{R}_+ \cup \{0\}$ defined by

$$(2.6) \quad \varphi_0(r) = b - r + \int_0^r L_0(t)(r - t) dt,$$

$$(2.7) \quad \varphi(r) = b - r + \int_0^r L(t)(r - t) dt$$

for some $b \geq 0$.

We now give the semilocal convergence analysis of Newton's method using the preceding notation.

THEOREM 2.3. *Let $F : \Omega \subseteq X \rightarrow Y$ be a continuously Fréchet differentiable operator satisfying the L_0 -center Lipschitzian condition (2.1) on Ω_0 and the restricted L -Lipschitz condition (2.5) on Ω_1 for some $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}F(x_0)\| \leq b$ for some $b \geq 0$. Moreover, suppose:*

- (i) *The function φ defined by (2.7) has a unique zero r_- in $[0, \bar{r}_0)$ such that $\varphi(\bar{r}_0) \leq 0$.*
- (ii) *$L_0(r) \leq L(r)$ for all $[0, \bar{r}_0)$.*
- (iii) *$\bar{U}(x_0, r_-) \subseteq \Omega$.*

Then the following statements hold:

- (a) *The Newton sequence $\{s_n\}$ generated by*

$$s_0 = 0, \quad s_{n+1} = s_n - \frac{\varphi(s_n)}{\varphi'(s_n)}, \quad n = 0, 1, \dots,$$

is well defined in $[0, r_-]$ and converges monotonically to r_- .

- (b) *The Newton sequence $\{x_n\}$ generated by (1.2) is also well defined, remains in $\bar{U}(x_0, r_-)$ and converges to a unique zero of F in $U(x_0, \bar{r}_0)$. Moreover,*

$$\begin{aligned} \|x_n - x^*\| &\leq r_- - s_n, & \|x_{n+1} - x_n\| &\leq s_{n+1} - s_n, \\ \|x_n - x^*\| &\leq \mu^{2^n} (\bar{r}_0 - s_n), \end{aligned}$$

where $\mu = \|x_n - x^\|/\bar{r}_0$.*

Proof. Simply repeat the corresponding proofs in [14], but use the estimate (see (2.1))

$$\|F'(x)^{-1}F(x_0)\| \leq -\frac{1}{\varphi'(\|x - x_0\|)}$$

instead of the less precise estimate

$$\|F'(x)^{-1}F(x_0)\| \leq -\frac{1}{\psi'(\|x - x_0\|)}$$

(see [14, (2.8) and (ii)]). Moreover notice that the iterates $\{x_n\}$ lie in Ω_1 , which is a more precise location than Ω_0 (see (2.8)) used for the proof in [14, Proposition 4, p. 677]. ■

We have the following useful alternative for the uniqueness part.

PROPOSITION 2.4. *Under the hypothesis of Theorem 2.3, further suppose that there exists $\gamma \in [r_{-1}, r_0)$ such that*

$$\frac{1}{2}L(r_0)(\gamma + r_{-1}) < 1.$$

Then x^ is the only zero of F in $\Omega_2 = \Omega \cap \bar{U}(x_0, \gamma)$.*

Proof. The proof uses the standard arguments [3] and condition (2.1) instead of condition (2.8) of [14].

REMARK 2.5. (a) In order for us to compare the preceding results with the corresponding ones in [8–14], let us consider the L_1 -Lipschitzian condition on Ω_0 (not on Ω_1) with $L_1(0) = 0$ and the corresponding majorant function ψ (used in [14]). That is, we have

$$(2.8) \quad \|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq L_1(r)\|x - y\|$$

for all $r \in [0, R]$ and all $x, y \in \Omega_0$, where

$$(2.9) \quad \psi(r) = b - r + \int_0^r L_1(t)(r - t) dt.$$

In view of (2.1), (2.5)–(2.9), for each $r \in [0, r_0)$ we have

$$(2.10) \quad L_0(r) \leq L_1(r),$$

$$(2.11) \quad L(r) \leq L_1(r),$$

$$(2.12) \quad \varphi_0(r) \leq \psi(r),$$

$$(2.13) \quad \varphi(r) \leq \psi(r).$$

Define also the Newton iteration corresponding to ψ by

$$s_0 = 0, \quad s_{n+1} = s_n - \frac{\psi(s_n)}{\psi'(s_n)}, \quad n = 0, 1, \dots$$

Let s_- be the unique zero of the function ψ in $[0, R)$ such that $\psi(R) \leq 0$. These conditions imply the corresponding conditions of Theorem 2.3, but not necessarily vice versa. Hence, the new sufficient semilocal convergence conditions are at least just as weak.

Concerning the comparison between the majorizing sequences $\{r_n\}$ and $\{s_n\}$, further suppose that for all $u, v \in [0, R]$ with $u \leq v$,

$$(2.14) \quad -\frac{\varphi(u)}{\varphi'(u)} \leq -\frac{\psi(v)}{\psi'(v)}.$$

Then a simple inductive argument using (2.10)–(2.13) shows, for $n = 0, 1, \dots$,

$$(2.15) \quad r_n \leq s_n,$$

$$(2.16) \quad r_{n+1} - r_n \leq s_{n+1} - s_n,$$

$$(2.17) \quad r_- \leq s_-.$$

Moreover, strict inequality may hold in (2.15) (for $n = 2, 3, \dots$) and in (2.16) (for $n = 1, 2, \dots$) if it does in (2.11) or (2.10).

(b) It follows from the proof of Theorem 2.3 that the sequence $\{q_n\}$ defined by

$$q_0 = 0, \quad q_1 = q_0 - \frac{\varphi(q_0)}{\varphi'(q_0)}, \quad q_{n+1} = q_n - \frac{\varphi(q_n)}{\varphi'(q_n)}, \quad n = 1, 2, \dots,$$

is a more precise majorizing sequence than $\{r_n\}$ which converges under the same hypotheses such that

$$(2.18) \quad q_n \leq r_n,$$

$$(2.19) \quad q_{n+1} - q_n \leq r_{n+1} - r_n,$$

$$(2.20) \quad q_- = \lim_{n \rightarrow \infty} q_n = r_-.$$

(c) If condition (ii) of Theorem 2.3 is not satisfied, i.e.

$$(2.21) \quad L(r) < L_0(r),$$

then φ_0 can replace φ in Theorem 2.3.

(d) The uniqueness given in Proposition 2.4 also improves the corresponding one in [14], where L_1 was used instead of the more precise L_0 (see (2.10)).

It is worth noticing that in practice the computation of the original function L_1 requires the computation of the functions L_0 and L as special cases. That is, no hypotheses additional to [14] are needed to obtain these improvements.

3. Semilocal convergence II. In this section, we study the convergence of Newton's method for operators F that are $p \geq 2$ (p an integer) times Fréchet differentiable.

PROPOSITION 3.1. *Let $F : \Omega \subseteq X \rightarrow Y$ be a $p \geq 2$ times continuously Fréchet differentiable operator satisfying the L_0 -center Lipschitzian condition on Ω_0 for some $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in L(Y, X)$. Moreover, suppose:*

(i) $\|F'(x_0)^{-1}F(x_0)\| \leq b$ and $\|F'(x_0)^{-1}F^{(i)}(x_0)\| \leq c_i$, $i = 2, \dots, p$, for some $b, c_i \geq 0$.

(ii) *There exists a continuous non-decreasing function $L^{(p)} : [0, \bar{r}_0] \rightarrow \mathbb{R}_+ \cup \{0\}$ such that*

$$\|F'(x_0)^{-1}(F^{(p)}(y) - F^{(p)}(x))\| \leq L^{(p)}(r)\|y - x\|$$

for all $r \in [0, r_0]$ and all $x, y \in \Omega_1$.

(iii) *The function $\varphi : [0, \bar{r}_0] \rightarrow \mathbb{R}_+ \cup \{0\}$ defined by*

$$\varphi(r) = b - r + c_2 \frac{r^2}{2!} + \dots + c_p \frac{r^p}{p!} + \int_0^r L^{(p)}(t) \frac{(r-t)^p}{p!} dt$$

has a unique zero r_- in $[0, \bar{r}_0]$ and $\varphi(\bar{r}_0) \leq 0$, where

$$L(r) = \varphi''(r) = c_2 + \dots + c_p \frac{r^{p-2}}{(p-2)!} + \int_0^r L^{(p)}(t) \frac{(r-t)^{p-2}}{(p-2)!} dt.$$

- (iv) $L_0(r) \leq L(r)$ for all $r \in [0, \bar{r}_0)$,
- (v) $\bar{U}(x_0, r_-) \subseteq \Omega$.

Then the conclusions of Theorem 2.3 hold.

Proof. Let $r_x = \|x - x_0\|$. It is a straightforward application of Taylor's theorem to show that in both cases $p = 2$ and $p \geq 3$ we have

$$\|F'(x_0)^{-1}F''(x)\| \leq L(r_x).$$

Hence, L satisfies the hypotheses of Theorem 2.3. ■

PROPOSITION 3.2. *Let $F : \Omega \subseteq X \rightarrow Y$ be an infinitely many times continuously Fréchet differentiable operator satisfying the L_0 -center Lipschitzian condition on Ω_0 for some $x_0 \in \Omega$ such that $F'(x_0)^{-1} \in L(Y, X)$. Moreover, suppose:*

- (i) $\|F'(x_0)^{-1}F(x_0)\| \leq b$ and $\|F'(x_0)^{-1}F^i(x_0)\| \leq c_i$, $i \geq 2$, for some $b, c_i \geq 0$.
- (ii) The function $\varphi : [0, \bar{r}_0] \rightarrow \mathbb{R}_+ \cup \{0\}$ is defined by

$$\varphi(r) = b - r + \sum_{p \geq 2} c_p \frac{r^p}{p!}$$

assuming that the series converges and has a unique zero r_- in $[0, \bar{r}_0)$.

- (iii) $L_0(r) \leq L(r)$ for all $r \in [0, \bar{r}_0)$, where

$$L(r) = \varphi''(r) = \sum_{p \geq 2} c_p \frac{r^{p-2}}{(p-2)!}.$$

- (iv) $\bar{U}(x_0, r_-) \subset \Omega$.

Then the conclusions of Theorem 2.3 hold.

Proof. By the expansion of F'' at x_0 , we again get

$$\|F'(x_0)^{-1}F''(x)\| \leq L(\|x - x_0\|)$$

for all $x \in \Omega_1$. Hence L satisfies the hypotheses of Theorem 2.3. ■

4. Numerical examples. We present two numerical examples, where the function ψ in (2.9) has no real zero. Hence the older results do not apply [5–14], but the function φ has solutions, so the new results apply to solve equations.

In both examples, L_0, L and L_1 are constant functions. Notice that in this case the functions φ_0, φ and ψ are reduced to

$$\varphi_0(r) = \frac{L_0}{2}r^2 - r + b, \quad \varphi(r) = \frac{L}{2}r^2 - r + b, \quad \psi(r) = \frac{L_1}{2}r^2 - r + n.$$

Therefore, the equations

$$\varphi_0(r) = 0, \quad \varphi(r) = 0, \quad \psi(r) = 0$$

each have real solutions provided that the respective Newton–Kantorovich-type conditions [8]

$$(4.1) \quad 2L_0b \leq 1,$$

$$(4.2) \quad 2Lb \leq 1,$$

$$(4.3) \quad 2L_1b \leq 1$$

hold.

EXAMPLE 4.1. Let $X = Y = \mathbb{R}$, $x_0 = 1$, $\Omega = \{x : |x - x_0| \leq 1 - \beta\}$, $\beta \in [0, 1/2)$, $R = 1 - \beta$, $\bar{r}_0 = 1/L_0$. Define a function F on Ω by

$$(4.4) \quad F(x) = x^3 - \beta.$$

Using the hypotheses of Theorem 2.3, we get

$$b = \frac{1 - \beta}{3},$$

and

$$\begin{aligned} |F'(x_0)^{-1}(F'(x) - F'(x_0))| &= |x^2 - x_0^2| = |x + x_0||x - x_0| \\ &= |(x - x_0) + 2|x_0||x - x_0| \\ &\leq (|x - x_0| + 2|x_0|)|x - x_0| \\ &\leq (1 - \beta + 2)|x - x_0| = (3 - \beta)|x - x_0| \end{aligned}$$

for each $x \in \Omega_0$. So, we can choose $L_0 = 3 - \beta$. Moreover, we have

$$\begin{aligned} |F'(x_0)^{-1}(F'(x) - F'(y))| &= |x^2 - y^2| = |x + y||x - y| \\ &= |(x - x_0) + (y - x_0) + 2x_0||x - y| \\ &\leq (|x - x_0| + |y - x_0| + 2|x_0|)|x - y| \\ &\leq (2(1 - \beta) + 2)|x - y| = 2(1 - \beta)|x - y| \end{aligned}$$

for all $x, y \in \Omega_0$, so we can choose $L_1 = 2(2 - \beta)$. Furthermore, for each $x, y \in \Omega_1 = U(x, \bar{r}_0) \cap U(x_0, 1 - \beta) = U(x_0, \bar{r}_0)$ (since $\bar{r}_0 < 1 - \beta$) we obtain

$$\begin{aligned} |F'(x_0)^{-1}(F'(x) - F'(y))| &\leq (|x - x_0| + |y - x_0| + 2|x_0|)|x - y| \\ &\leq (2 + 2\bar{r}_0)|x - y| = 2\left(1 + \frac{1}{3 - \beta}\right)|x - y|, \end{aligned}$$

so we can choose $L(r) = 2\left(1 + \frac{1}{3 - \beta}\right)$. Notice that

$$L_0 < L < L_1 \quad \text{and} \quad \bar{r}_0 < R \quad \text{for all } \beta \in [0, 1/2).$$

The Newton–Kantorovich condition (4.3) is not satisfied, since

$$(4.5) \quad \frac{4}{3}(1 - \beta)(2 - \beta) > 1 \quad \text{for all } \beta \in [0, 1/2).$$

Hence, there is no guarantee that Newton's method (1.2) converges to

$x^* = \sqrt[3]{\beta}$, starting at $x_0 = 1$. However, our corresponding condition (4.2) is true for all $\beta \in I = [0.4619832, 1/2)$. Hence, the conclusions of our Theorem 2.3 can be applied to solve the equation $F(x) = 0$ for all $\beta \in I$.

EXAMPLE 4.2. Let $X = Y = \mathcal{C}[0, 1]$, the space of continuous real-valued functions defined on $[0, 1]$. We shall use the max-norm. Let $\Omega = \{x \in X : \|x\| \leq R\}$ such that $R > 0$. Define F on Ω by [7, 10]

$$(4.6) \quad F(x)(s) = x(s) - f(s) - \delta \int_0^1 K(s, t)x(t)^3 dt, \quad x \in X, s \in [0, 1],$$

where $f \in X$ is a given function, δ is a real constant and the kernel K is the Green's function defined by

$$K(s, t) = \begin{cases} (1-s)t & \text{if } t \leq s, \\ s(1-t) & \text{if } s \leq t. \end{cases}$$

It follows from (4.6) that, for each $x \in \Omega$, $F'(x)$ is a linear operator defined by

$$[F'(x)(v)](s) = v(s) - 3\delta \int_0^1 K(s, t)x(t)^2v(t) dt, \quad v \in X, s \in [0, 1].$$

Let us choose $x_0(s) = f(s) = 1$. It follows that $\|I - F'(x_0)\| \leq 3|\delta|/8$. If $|\delta| < 8/3$, then $F'(x_0)^{-1}$ exists and

$$\|F'(x_0)^{-1}\| \leq \frac{8}{8 - 3|\delta|}.$$

We also get $\|F(x_0)\| \leq |\delta|/8$, so

$$b = \|F'(x_0)^{-1}F(x_0)\| \leq \frac{|\delta|}{8 - 3|\delta|}.$$

Moreover, for $x, y \in \Omega$, we obtain

$$\|F'(x) - F'(y)\| \leq \frac{1 + 3|\delta|\|x + y\|}{8}\|x - y\| \leq \frac{1 + 6R|\delta|}{8}\|x - y\|$$

and

$$\|F'(x) - F'(1)\| \leq \frac{1 + 3|\delta|(\|x\| + 1)}{8}\|x - 1\| \leq \frac{1 + 3|\delta|(1 + R)}{8}\|x - 1\|.$$

Choosing $\delta = 1.175$ and $R = 2$, we have $b = 0.26257\dots$, $L_1 = 2.76875\dots$, $L_0 = 1.8875\dots$, $1/L_0 = 0.529801\dots$ and $L = 1.47314\dots$

Using these values, we find that condition (4.3) is not satisfied, since

$$1.4539813 > 1.$$

However, our condition (4.2) is satisfied, since

$$0.7736047 < 1.$$

Hence, the convergence of Newton's method is guaranteed by Theorem 2.3.

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