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DIFFERENTIAL GEOMETRY

# Revisiting Liebmann's theorem in higher codimension

by

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**Summary.** We deal with compact surfaces immersed with flat normal bundle and parallel normalized mean curvature vector field in a space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$. Such a surface is called an *LW-surface* when it satisfies a linear Weingarten condition of the type $K = aH + b$ for some real constants $a$ and $b$, where $H$ and $K$ denote the mean and Gaussian curvatures, respectively. In this setting, we extend the classical rigidity theorem of Liebmann (1899) showing that a non-flat LW-surface with non-negative Gaussian curvature must be isometric to a totally umbilical round sphere.

**1. Introduction and statement of the main result.** The study of surfaces immersed in a 3-dimensional Riemannian space form $\mathbb{Q}_c^3$ of constant sectional curvature $c$ plays an important role in the theory of submanifolds. In relation to this topic, in 1897 Hadamard [3] proved that an ovaloid, that is, a compact connected surface with positive Gaussian curvature, in the 3-dimensional Euclidean space $\mathbb{R}^3$ is a topological sphere. In view of this result, it was natural to look for conditions which allowed one to conclude that such a surface was necessarily a totally umbilical round sphere. In 1899 Liebmann [5] obtained his celebrated rigidity result, which states that every compact connected surface in $\mathbb{R}^3$ with constant Gaussian curvature is necessarily a totally umbilical round sphere.

Later on, there have been different generalizations of Liebmann's theorem from several points of view for surfaces, and more generally hypersurfaces, in the Euclidean space [4, 7, 8, 9, 10], or in the hyperbolic space or an open hemisphere [6]. In [1], as an application of the Gauss–Bonnet theorem

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along with a formula involving the Gaussian curvatures of the first and second fundamental forms of the surface, Aledo, Alías and Romero established a new direct proof of these results.

Here, we consider a wide class of surfaces $M^2$ immersed in a $(2+p)$-dimensional space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$, which extend those of constant Gaussian curvature, the so-called *linear Weingarten surfaces* or simply *LW-surfaces*. We recall that a surface is said to be an LW-surface when its mean curvature $H$ and its Gaussian curvature $K$ satisfy a linear relation of the type $K = aH + b$ for some constants $a, b \in \mathbb{R}$. These surfaces were originally introduced by Weingarten [11, 12] in the context of the problem of finding all surfaces of the Euclidean space isometric to a prescribed surface of revolution. In this setting, we obtain the following rigidity result which can be regarded as an extension of the previously mentioned ones:

THEOREM 1.1. *Let $M^2$ be a compact non-flat LW-surface immersed in a Riemannian space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$, with flat normal bundle, parallel normalized mean vector field and such that its Gaussian curvature $K$ and mean curvature $H$ satisfy $K = aH + b$ with $a^2 + 2(2b - c) \geq 0$. If $K$ is non-negative on $M^2$, then $M^2$ is a totally umbilical round sphere.*

The proof of Theorem 1.1 is given in Section 3. Before, in Section 2 we recall some basic facts concerning the geometry of surfaces immersed in a space form.

**2. Preliminaries.** Let $M^2$ be a connected surface immersed in a space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature $c$. We will use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq 2+p, \quad 1 \leq i, j, k, \ldots \leq 2, \quad 3 \leq \alpha, \beta, \gamma, \ldots \leq 2+p.$$

We choose a local orthonormal frame field $\{e_1, e_2, e_3, \ldots, e_{2+p}\}$ along $M^2$, where $\{e_i\}_{i=1,2}$ are tangent to $M^2$ and $\{e_\alpha\}_{\alpha=3,\ldots,2+p}$ are normal to $M^2$. Let $\{\omega_B\}$ be the corresponding dual coframe, and $\{\omega_{BC}\}$ the connection 1-forms on $\mathbb{Q}_c^{2+p}$. The second fundamental form $h$, the curvature tensor $R$ and the normal curvature tensor $R^\perp$ of $M^2$ can be given by

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2} \sum_{k,l} R_{\alpha\beta kl}^\perp \omega_k \wedge \omega_l.$$

Moreover, the components $h^\alpha_{ijk}$ of the covariant derivative $\nabla h$ satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ki}\omega_{kj} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.1}$$

The *Gauss equation* is

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \tag{2.2}$$

In particular, the components of the Ricci tensor $R_{ik}$ are given by

$$R_{ik} = c\delta_{ik} + 2\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk}, \tag{2.3}$$

where $H^\alpha = \frac{1}{2}\sum_i h^\alpha_{ii}$ are the components of the mean curvature vector field $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha$.

From (2.3) we get the relation

$$2K = -2 + 4H^2 - S, \tag{2.4}$$

where $K$ stands for the Gaussian curvature of $M^2$, $H = |\mathbf{H}|$ is the mean curvature function and $S = |h|^2 = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$ is the squared norm of the second fundamental form $h$ of $M^2$.

Assuming that $M^2$ has flat normal bundle (that is, $R^\perp = 0$), by exterior differentiation of (2.1) we obtain the *Ricci identity*

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl}. \tag{2.5}$$

Moreover, the *Codazzi equation* is given by

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik}. \tag{2.6}$$

**3. Proof of Theorem 1.1.** In what follows, we will deal with surfaces $M^2$ of $\mathbb{Q}^{2+p}_c$ having *parallel normalized mean curvature vector field*, which means that the mean curvature function $H$ is positive and the corresponding normalized mean curvature vector field $\mathbf{H}/H$ is parallel as a section of the normal bundle.

In this context, we can choose a local orthonormal frame $\{e_1, \dots, e_{2+p}\}$ such that $e_3 = \mathbf{H}/H$. Thus,

$$H^3 = \frac{1}{2}\,\mathrm{tr}(h^3) = H \quad \text{and} \quad H^\alpha = \frac{1}{2}\mathrm{tr}(h^\alpha) = 0,\ \alpha \geq 4, \tag{3.1}$$

where $h^\alpha$ stands for the $2\times 2$ matrix $(h^\alpha_{ij})$.

We will consider the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,$$

where $\Phi^\alpha_{ij} = h^\alpha_{ij} - H^\alpha\delta_{ij}$. Consequently,

$$\Phi^3_{ij} = h^3_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^\alpha_{ij} = h^\alpha_{ij},\ 4 \leq \alpha \leq 2+p. \tag{3.2}$$

Let $|\Phi|^2 = \sum_{i,j,\alpha}(\Phi_{ij}^\alpha)^2$ be the square of the length of $\Phi$. From (2.4), it is not difficult to verify that $\Phi$ is traceless with

$$|\Phi|^2 = S - 2H^2 = 2(c + H^2 - K). \tag{3.3}$$

In order to prove Theorem 1.1, we will also need the following key lemma which is obtained by just adapting the proof of [13, Proposition 2.2]:

LEMMA 3.1. *Let $M^2$ be an LW-surface immersed in $\mathbb{Q}_c^{2+p}$, with $K = aH + b$ for some $a, b \in \mathbb{R}$ such that $a^2 + 8(b - c) \geq 0$. Then*

$$|\nabla h|^2 \geq 4|\nabla H|^2. \tag{3.4}$$

*Moreover, if equality holds in* (3.4) *on $M^2$, then $H$ is constant on $M^2$.*

Now, we are in a position to present the proof of Theorem 1.1.

*Proof of Theorem 1.1.* We have

$$\tfrac{1}{2}\Delta S = \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2, \tag{3.5}$$

where the Laplacian $\Delta h_{ij}^\alpha$ of $h_{ij}^\alpha$ is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$. Using the Codazzi equation (2.6) in (3.5) we obtain

$$\tfrac{1}{2}\Delta S = |\nabla h|^2 + \sum_{i,j,k,\alpha} h_{ij}^\alpha h_{kijk}^\alpha. \tag{3.6}$$

Thus, from (2.5), (3.1) and (3.6), we conclude that

$$\begin{aligned}\tfrac{1}{2}\Delta S = |\nabla h|^2 + \sum_{i,j} nH_{ij}^{n+1} h_{ij}^{n+1} + \sum_{i,j,m,k,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \\ + \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk}.\end{aligned} \tag{3.7}$$

Consequently, taking a (local) orthonormal frame $\{e_1, e_2\}$ on $M^2$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$ for every $\alpha$, from (3.7) we obtain the Simons-type formula

$$\tfrac{1}{2}\Delta S = |\nabla h|^2 + \sum_i \lambda_i^3 (2H)_{ii} + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{3.8}$$

We define an appropriate modified Cheng–Yau operator by

$$L = \square - \tfrac{1}{2}a\Delta, \tag{3.9}$$

where the square operator is defined by

$$\square f = \sum_{i,j}(2H\delta_{ij} - h_{ij}^3) f_{ij} \tag{3.10}$$

for each $f \in C^\infty(M)$.

Setting $f = 2H$ in (3.10), we obtain

$$\square(2H) = 2H\Delta(2H) - \sum_i \lambda_i^3 (2H)_{ii} \tag{3.11}$$
$$= \tfrac{1}{2}\Delta(2H)^2 - \sum_i (2H)_i^2 - \sum_i \lambda_i^3 (2H)_{ii}$$
$$= \Delta R + \tfrac{1}{2}\Delta S - 4|\nabla H|^2 - \sum_i \lambda_i^3 (2H)_{ii}.$$

Consequently, inserting (3.8) into (3.11) we get

$$\square(2H) = \Delta R + |\nabla h|^2 - 4|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{3.12}$$

Since $R = aH + b$, from (3.9) and (3.12) we have

$$L(2H) = |\nabla h|^2 - 4|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{3.13}$$

From the Gauss equation we have

$$R_{ijij} = c + \sum_\beta \lambda_i^\beta \lambda_j^\beta. \tag{3.14}$$

Hence, using (3.14), (2.4) and (3.3) we have

$$\frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 = \sum_\alpha R_{1212}(\lambda_1^\alpha - \lambda_2^\alpha)^2 \tag{3.15}$$
$$= \Big(c + \sum_\beta \lambda_1^\beta \lambda_2^\beta\Big) \sum_\alpha (\lambda_1^\alpha - \lambda_2^\alpha)^2$$
$$= 2\bigg(c + \sum_\beta \Big(\frac{|h^\beta|^2}{2} - |\Phi^\beta|^2\Big)\bigg)|\Phi|^2$$
$$= 2\Big(c + \frac{S}{2} - |\Phi|^2\Big)|\Phi|^2$$
$$= |\Phi|^2(-|\Phi|^2 + 2H^2 + 2c) = 2K|\Phi|^2.$$

Thus, using Lemma 3.1 and since we are supposing that $K$ is non-negative on $M^2$, from (3.13) and (3.15) we get

$$L(H) \geq K|\Phi|^2 \geq 0. \tag{3.16}$$

On the other hand, from (3.9) and (3.10) it is not difficult to verify that

$$L(H) = \mathrm{div}_M(P(\nabla H)), \tag{3.17}$$

where $P = (2H + a/2)I - h^3$ and $I$ denotes the identity in the algebra of smooth vector fields on $M^2$.

From (3.16) and (3.17), integrating $L(H)$ on $M^2$, which is supposed be compact, we obtain $L(H) = 0$ on $M^2$. So, returning to (3.13) we get $|\nabla h|^2 = 4|\nabla H|^2$ on $M^2$. Thus, using once more Lemma 3.1 we conclude that $H$ is constant on $M^2$. Consequently, since $K = aH + b$ and $M^2$ is also assumed be non-flat, from (3.16) we infer that $|\Phi|$ vanishes identically, and therefore $M^2$ is totally umbilical.

Consequently, since $M^2$ is totally umbilical and taking into account (3.1), we get

$$h^\alpha = \langle \mathbf{H}, e_\alpha \rangle I = H^\alpha I = 0$$

for every $\alpha > 3$. This implies that the first normal subspace

$$N_1 = \{e_\alpha \in \mathfrak{X}^\perp(M);\, h^\alpha = 0\}^\perp$$

is parallel and has dimension 1. Therefore, we can apply [2, Proposition 4.1] to reduce the codimension of $M^2$ to 1 and we conclude that it must be isometric to a totally umbilical round sphere. ■

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