BULLETIN OF THE POLISH
ACADEMY OF SCIENCES
MATHEMATICS
Online First version

DIFFERENTIAL GEOMETRY

Revisiting Liebmann's theorem in higher codimension

by

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Summary. We deal with compact surfaces immersed with flat normal bundle and parallel normalized mean curvature vector field in a space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$. Such a surface is called an *LW-surface* when it satisfies a linear Weingarten condition of the type $K = aH + b$ for some real constants a and b, where H and K denote the mean and Gaussian curvatures, respectively. In this setting, we extend the classical rigidity theorem of Liebmann (1899) showing that a non-flat LW-surface with non-negative Gaussian curvature must be isometric to a totally umbilical round sphere.

1. Introduction and statement of the main result. The study of surfaces immersed in a 3-dimensional Riemannian space form $\mathbb{Q}_c^3$ of constant sectional curvature c plays an important role in the theory of submanifolds. In relation to this topic, in 1897 Hadamard [3] proved that an ovaloid, that is, a compact connected surface with positive Gaussian curvature, in the 3-dimensional Euclidean space $\mathbb{R}^3$ is a topological sphere. In view of this result, it was natural to look for conditions which allowed one to conclude that such a surface was necessarily a totally umbilical round sphere. In 1899 Liebmann [5] obtained his celebrated rigidity result, which states that every compact connected surface in $\mathbb{R}^3$ with constant Gaussian curvature is necessarily a totally umbilical round sphere.

Later on, there have been different generalizations of Liebmann's theorem from several points of view for surfaces, and more generally hypersurfaces, in the Euclidean space [4, 7, 8, 9, 10], or in the hyperbolic space or an open hemisphere [6]. In [1], as an application of the Gauss–Bonnet theorem

2010 *Mathematics Subject Classification*: Primary 53C42; Secondary 53A10, 53C20.
Key words and phrases: space forms, compact LW-surfaces, parallel normalized mean curvature vector field, totally umbilical round spheres.
Received 29 May 2019.
Published online 17 June 2019.

DOI: 10.4064/ba190514-30-5

along with a formula involving the Gaussian curvatures of the first and second fundamental forms of the surface, Aledo, Alías and Romero established a new direct proof of these results.

Here, we consider a wide class of surfaces M^2 immersed in a $(2+p)$-dimensional space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$, which extend those of constant Gaussian curvature, the so-called *linear Weingarten surfaces* or simply *LW-surfaces*. We recall that a surface is said to be an LW-surface when its mean curvature H and its Gaussian curvature K satisfy a linear relation of the type $K = aH + b$ for some constants $a, b \in \mathbb{R}$. These surfaces were originally introduced by Weingarten [11, 12] in the context of the problem of finding all surfaces of the Euclidean space isometric to a prescribed surface of revolution. In this setting, we obtain the following rigidity result which can be regarded as an extension of the previously mentioned ones:

THEOREM 1.1. *Let M^2 be a compact non-flat LW-surface immersed in a Riemannian space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature $c \in \{-1, 0, 1\}$, with flat normal bundle, parallel normalized mean vector field and such that its Gaussian curvature K and mean curvature H satisfy $K = aH + b$ with $a^2 + 2(2b - c) \geq 0$. If K is non-negative on M^2, then M^2 is a totally umbilical round sphere.*

The proof of Theorem 1.1 is given in Section 3. Before, in Section 2 we recall some basic facts concerning the geometry of surfaces immersed in a space form.

2. Preliminaries. Let M^2 be a connected surface immersed in a space form $\mathbb{Q}_c^{2+p}$ of constant sectional curvature c. We will use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq 2+p, \qquad 1 \leq i, j, k, \ldots \leq 2, \qquad 3 \leq \alpha, \beta, \gamma, \ldots \leq 2+p.$$

We choose a local orthonormal frame field $\{e_1, e_2, e_3, \ldots, e_{2+p}\}$ along M^2, where $\{e_i\}_{i=1,2}$ are tangent to M^2 and $\{e_\alpha\}_{\alpha=3,\ldots,2+p}$ are normal to M^2. Let $\{\omega_B\}$ be the corresponding dual coframe, and $\{\omega_{BC}\}$ the connection 1-forms on $\mathbb{Q}_c^{2+p}$. The second fundamental form h, the curvature tensor R and the normal curvature tensor $R^\perp$ of M^2 can be given by

$$\omega_{i\alpha} = \sum_j h_{ij}^\alpha \omega_j, \quad h = \sum_{i,j,\alpha} h_{ij}^\alpha \omega_i \otimes \omega_j \otimes e_\alpha,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l,$$

$$d\omega_{\alpha\beta} = \sum_\gamma \omega_{\alpha\gamma} \wedge \omega_{\gamma\alpha} - \frac{1}{2} \sum_{k,l} R_{\alpha\beta kl}^\perp \omega_k \wedge \omega_l.$$

Moreover, the components h^α_{ijk} of the covariant derivative ∇h satisfy

$$\sum_k h^\alpha_{ijk}\omega_k = dh^\alpha_{ij} + \sum_k h^\alpha_{ki}\omega_{kj} + \sum_k h^\alpha_{kj}\omega_{ki} + \sum_\beta h^\beta_{ij}\omega_{\beta\alpha}. \tag{2.1}$$

The *Gauss equation* is

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + \sum_\alpha (h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk}). \tag{2.2}$$

In particular, the components of the Ricci tensor R_{ik} are given by

$$R_{ik} = c\delta_{ik} + 2\sum_\alpha H^\alpha h^\alpha_{ik} - \sum_{\alpha,j} h^\alpha_{ij}h^\alpha_{jk}, \tag{2.3}$$

where $H^\alpha = \frac{1}{2}\sum_i h^\alpha_{ii}$ are the components of the mean curvature vector field $\mathbf{H} = \sum_\alpha H^\alpha e_\alpha$.

From (2.3) we get the relation

$$2K = -2 + 4H^2 - S, \tag{2.4}$$

where K stands for the Gaussian curvature of M^2, $H = |\mathbf{H}|$ is the mean curvature function and $S = |h|^2 = \sum_{i,j,\alpha}(h^\alpha_{ij})^2$ is the squared norm of the second fundamental form h of M^2.

Assuming that M^2 has flat normal bundle (that is, $R^\perp = 0$), by exterior differentiation of (2.1) we obtain the *Ricci identity*

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = \sum_m h^\alpha_{mj}R_{mikl} + \sum_m h^\alpha_{im}R_{mjkl}. \tag{2.5}$$

Moreover, the *Codazzi equation* is given by

$$h^\alpha_{ijk} = h^\alpha_{ikj} = h^\alpha_{jik}. \tag{2.6}$$

3. Proof of Theorem 1.1. In what follows, we will deal with surfaces M^2 of $\mathbb{Q}^{2+p}_c$ having *parallel normalized mean curvature vector field*, which means that the mean curvature function H is positive and the corresponding normalized mean curvature vector field $\mathbf{H}/H$ is parallel as a section of the normal bundle.

In this context, we can choose a local orthonormal frame $\{e_1, \ldots, e_{2+p}\}$ such that $e_3 = \mathbf{H}/H$. Thus,

$$H^3 = \tfrac{1}{2}\,\mathrm{tr}(h^3) = H \quad \text{and} \quad H^\alpha = \tfrac{1}{2}\mathrm{tr}(h^\alpha) = 0,\ \alpha \geq 4, \tag{3.1}$$

where h^α stands for the 2×2 matrix (h^α_{ij}).

We will consider the symmetric tensor

$$\Phi = \sum_{\alpha,i,j} \Phi^\alpha_{ij}\omega_i \otimes \omega_j \otimes e_\alpha,$$

where $\Phi^\alpha_{ij} = h^\alpha_{ij} - H^\alpha\delta_{ij}$. Consequently,

$$\Phi^3_{ij} = h^3_{ij} - H\delta_{ij} \quad \text{and} \quad \Phi^\alpha_{ij} = h^\alpha_{ij},\ 4 \leq \alpha \leq 2+p. \tag{3.2}$$

Let $|\Phi|^2 = \sum_{i,j,\alpha} (\Phi_{ij}^\alpha)^2$ be the square of the length of Φ. From (2.4), it is not difficult to verify that Φ is traceless with

$$|\Phi|^2 = S - 2H^2 = 2(c + H^2 - K). \tag{3.3}$$

In order to prove Theorem 1.1, we will also need the following key lemma which is obtained by just adapting the proof of [13, Proposition 2.2]:

LEMMA 3.1. *Let M^2 be an LW-surface immersed in $\mathbb{Q}_c^{2+p}$, with $K = aH + b$ for some $a, b \in \mathbb{R}$ such that $a^2 + 8(b - c) \geq 0$. Then*

$$|\nabla h|^2 \geq 4|\nabla H|^2. \tag{3.4}$$

Moreover, if equality holds in (3.4) on M^2, then H is constant on M^2.

Now, we are in a position to present the proof of Theorem 1.1.

Proof of Theorem 1.1. We have

$$\tfrac{1}{2}\Delta S = \sum_{i,j,\alpha} h_{ij}^\alpha \Delta h_{ij}^\alpha + \sum_{i,j,k,\alpha} (h_{ijk}^\alpha)^2, \tag{3.5}$$

where the Laplacian Δh_{ij}^α of h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum_k h_{ijkk}^\alpha$. Using the Codazzi equation (2.6) in (3.5) we obtain

$$\tfrac{1}{2}\Delta S = |\nabla h|^2 + \sum_{i,j,k,\alpha} h_{ij}^\alpha h_{kijk}^\alpha. \tag{3.6}$$

Thus, from (2.5), (3.1) and (3.6), we conclude that

$$\begin{aligned} \tfrac{1}{2}\Delta S = |\nabla h|^2 &+ \sum_{i,j} nH_{ij}^{n+1} h_{ij}^{n+1} + \sum_{i,j,m,k,\alpha} h_{ij}^\alpha h_{mi}^\alpha R_{mkjk} \\ &+ \sum_{i,j,k,m,\alpha} h_{ij}^\alpha h_{km}^\alpha R_{mijk}. \end{aligned} \tag{3.7}$$

Consequently, taking a (local) orthonormal frame $\{e_1, e_2\}$ on M^2 such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$ for every α, from (3.7) we obtain the Simons-type formula

$$\tfrac{1}{2}\Delta S = |\nabla h|^2 + \sum_i \lambda_i^3 (2H)_{ii} + \frac{1}{2} \sum_{i,j,\alpha} R_{ijij} (\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{3.8}$$

We define an appropriate modified Cheng–Yau operator by

$$L = \Box - \tfrac{1}{2} a\Delta, \tag{3.9}$$

where the square operator is defined by

$$\Box f = \sum_{i,j} (2H\delta_{ij} - h_{ij}^3) f_{ij} \tag{3.10}$$

for each $f \in C^\infty(M)$.

Setting $f = 2H$ in (3.10), we obtain

$$\begin{aligned}\Box(2H) &= 2H\Delta(2H) - \sum_i \lambda_i^3 (2H)_{ii} \\ &= \tfrac{1}{2}\Delta(2H)^2 - \sum_i (2H)_i^2 - \sum_i \lambda_i^3 (2H)_{ii} \\ &= \Delta R + \tfrac{1}{2}\Delta S - 4|\nabla H|^2 - \sum_i \lambda_i^3 (2H)_{ii}.\end{aligned} \tag{3.11}$$

Consequently, inserting (3.8) into (3.11) we get

$$\Box(2H) = \Delta R + |\nabla h|^2 - 4|\nabla H|^2 + \frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{3.12}$$

Since $R = aH + b$, from (3.9) and (3.12) we have

$$L(2H) = |\nabla h|^2 - 4|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2. \tag{3.13}$$

From the Gauss equation we have

$$R_{ijij} = c + \sum_\beta \lambda_i^\beta \lambda_j^\beta. \tag{3.14}$$

Hence, using (3.14), (2.4) and (3.3) we have

$$\begin{aligned}\frac{1}{2}\sum_{i,j,\alpha} R_{ijij}(\lambda_i^\alpha - \lambda_j^\alpha)^2 &= \sum_\alpha R_{1212}(\lambda_1^\alpha - \lambda_2^\alpha)^2 \\ &= \Big(c + \sum_\beta \lambda_1^\beta \lambda_2^\beta\Big)\sum_\alpha (\lambda_1^\alpha - \lambda_2^\alpha)^2 \\ &= 2\bigg(c + \sum_\beta \bigg(\frac{|h^\beta|^2}{2} - |\Phi^\beta|^2\bigg)\bigg)|\Phi|^2 \\ &= 2\bigg(c + \frac{S}{2} - |\Phi|^2\bigg)|\Phi|^2 \\ &= |\Phi|^2(-|\Phi|^2 + 2H^2 + 2c) = 2K|\Phi|^2.\end{aligned} \tag{3.15}$$

Thus, using Lemma 3.1 and since we are supposing that K is non-negative on M^2, from (3.13) and (3.15) we get

$$L(H) \geq K|\Phi|^2 \geq 0. \tag{3.16}$$

On the other hand, from (3.9) and (3.10) it is not difficult to verify that

$$L(H) = \operatorname{div}_M(P(\nabla H)), \tag{3.17}$$

where $P = (2H + a/2)I - h^3$ and I denotes the identity in the algebra of smooth vector fields on M^2.

From (3.16) and (3.17), integrating $L(H)$ on M^2, which is supposed be compact, we obtain $L(H) = 0$ on M^2. So, returning to (3.13) we get $|\nabla h|^2 = 4|\nabla H|^2$ on M^2. Thus, using once more Lemma 3.1 we conclude that H is constant on M^2. Consequently, since $K = aH + b$ and M^2 is also assumed be non-flat, from (3.16) we infer that $|\Phi|$ vanishes identically, and therefore M^2 is totally umbilical.

Consequently, since M^2 is totally umbilical and taking into account (3.1), we get

$$h^\alpha = \langle \mathbf{H}, e_\alpha \rangle I = H^\alpha I = 0$$

for every $\alpha > 3$. This implies that the first normal subspace

$$N_1 = \{e_\alpha \in \mathfrak{X}^\perp(M);\ h^\alpha = 0\}^\perp$$

is parallel and has dimension 1. Therefore, we can apply [2, Proposition 4.1] to reduce the codimension of M^2 to 1 and we conclude that it must be isometric to a totally umbilical round sphere. ■

Acknowledgements. The first author was partially supported by INCTMat/CAPES, Brazil. The second author was partially supported by CNPq, Brazil, grant 303977/2015-9.

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