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ON THE RANKIN–SELBERG CONVOLUTION OF DEGREE 2 FUNCTIONS FROM THE EXTENDED SELBERG CLASS

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Abstract. Let $F(s)$ be a function of degree 2 from the extended Selberg class. Assuming certain bounds for the shifted convolution sums associated with $F(s)$, we prove that the Rankin–Selberg convolution $F\otimes\overline{F}(s)$ has holomorphic continuation to the half-plane $\sigma > \theta$ apart from a simple pole at $s = 1$, where $1/2 < \theta < 1$ depends on the above mentioned bounds.

1. Introduction. It is well known that given two normalized Hecke eigenforms f, g of weight k and level 1, with Fourier coefficients $a(n)$ and $b(n)$, respectively, the associated Rankin–Selberg convolution

$$L(s, f\otimes\overline{g}) = \sum_{n=1}^{\infty} \frac{a(n)\overline{b(n)}}{n^s}$$

has meromorphic continuation to the whole complex plane. Moreover, $L(s, f\otimes\overline{g})$ has a simple pole at $s = 1$ if and only if $f = g$; see Chapter 13 of Iwaniec [2]. It is also well

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known that the Rankin–Selberg convolution has been generalized and extended in several important directions.

Given a function $F(s)$ from the extended Selberg class $\mathcal{S}^\sharp$ with Dirichlet coefficients $a(n)$, see below for definitions, its Rankin–Selberg convolution is defined as

$$F\otimes\overline{F}(s) = \sum_{n=1}^{\infty} \frac{|a(n)|^2}{n^s}.$$

Very little is known about $F\otimes\overline{F}(s)$ in the general framework of $\mathcal{S}^\sharp$. Actually, as far as we know, the only result of general nature on $F\otimes\overline{F}(s)$ appears in Lemma C of [6], where we proved that if $F(s)$ has degree $1 < d_F < 2$, then $F\otimes\overline{F}(s)$ is holomorphic for $\sigma > 1-\delta$ apart from a simple pole at $s=1$, where δ is a certain positive constant. However, we have later shown in [8] that there exist no functions in $\mathcal{S}^\sharp$ with $1 < d_F < 2$, and apparently the techniques that we used in Lemma C are not applicable for similar investigations of functions with degree $d_F \geq 2$ without injecting new ideas. Hence the problem of the behavior of $F\otimes\overline{F}(s)$ for general functions $F(s)$ of degree $d_F \geq 2$ is widely open, although we may ask if any result can be obtained in the case $d_F = 2$, which is on the border of the range we treated previously, by elaborating on the techniques in [6].

In this paper we prove a conditional result about the meromorphic continuation of $F\otimes\overline{F}(s)$ for all functions of degree 2 in $\mathcal{S}^\sharp$. Our assumption concerns suitable bounds for the shifted convolution sums

$$S_k(x) = \sum_{n\leq x} a(n)\overline{a(n+k)}$$

with integers $k \geq 1$. These sums are classical objects, and non-trivial information about their behavior as $x\to\infty$ is known in several concrete cases. We shall assume bounds of type

$$S_k(x) \ll x^\theta,$$

see (1.3) below, while the Cauchy–Schwarz inequality immediately shows that, uniformly in x and k,

$$S_k(x) \ll x^{(\sigma_a(F\otimes\overline{F})+\varepsilon)/2}(x+k)^{(\sigma_a(F\otimes\overline{F})+\varepsilon)/2}, \tag{1.1}$$

where $\sigma_a(F\otimes\overline{F})$ is the abscissa of absolute convergence of $F\otimes\overline{F}(s)$ and $\varepsilon > 0$ is arbitrarily small. Actually, in general this bound is essentially optimal, since for example in the case of the divisor function $d(n)$ we have, for fixed k and $x\to\infty$,

$$\sum_{n\leq x} d(n)d(n+k) \sim xP_k(\log x)$$

with certain polynomials $P_k(X)$, and $\sigma_a(F\otimes\overline{F}) = 1$ in this case. However, in several concrete examples of degree 2 L-functions, $S_k(x)$ has no main term and non-trivial bounds are known with $\theta < 1$; see e.g. Corollary in Jutila [3]. Therefore, some links exist between the $S_k(x)$'s and $F\otimes\overline{F}(s)$, but obtaining information on the analytic continuation of $F\otimes\overline{F}(s)$ to the left of $\sigma_a(F\otimes\overline{F})$ from suitable bounds for $S_k(x)$, $k\geq 1$, is a non-trivial problem. Here we propose an approach to this question.

The extended Selberg class $\mathcal{S}^\sharp$ consists of the Dirichlet series $F(s)$ absolutely convergent for $\sigma > 1$ and such that $(s-1)^m F(s)$ extends to an entire function of finite order for some integer m, satisfying the functional equation $\Phi(s) = \omega\overline{\Phi(1-\overline{s})}$ with $|\omega| = 1$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s),$$

where $Q, \lambda_j > 0$ and $\Re\mu_j \geq 0$. Degree, conductor and ξ-invariant of $F \in \mathcal{S}^\sharp$ are respectively defined as

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \qquad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \qquad \xi_F = 2\sum_{j=1}^{r} (\mu_j - 1/2) = \eta_F + i d_F \theta_F,$$

say; here we deal only with functions of degree $d_F = 2$, and for simplicity we denote the conductor simply by q. We refer to our surveys [4], [5], [11], [12], [13] and [14] for definitions and the basic theory of the Selberg class. For $F \in \mathcal{S}^\sharp$ and an integer $k \geq 0$ we define

$$F_k(s) = F \otimes_k \overline{F}(s) = \sum_{n=1}^{\infty} \frac{a(n)\overline{a(n+k)}}{n^s}; \tag{1.2}$$

clearly $F \otimes \overline{F}(s) = F_0(s)$, and from the Cauchy–Schwarz inequality we see that $F_k(s)$ is absolutely convergent for $\sigma > \sigma_a(F \otimes \overline{F})$.

As outlined before, we assume that as $x \to \infty$

$$S_k(x) \ll_{\varepsilon,\theta} x^\theta \qquad \text{uniformly for } 1 \leq k \leq x^\varepsilon, \tag{1.3}$$

where $\theta \in \mathbb{R}$ and $\varepsilon > 0$ is arbitrarily small.

Theorem. *Let $F \in \mathcal{S}^\sharp$ with degree 2 satisfy (1.3) with some $1/2 < \theta < \sigma_a(F \otimes \overline{F})$. Then $\sigma_a(F \otimes \overline{F}) = 1$, and $F \otimes \overline{F}(s)$ has holomorphic continuation to $\sigma > \theta$ apart from a simple pole at $s = 1$.*

Recalling that the Ramanujan condition states that $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$, we have the following corollary.

Corollary. *Under the same hypotheses, there exists a constant $c_0 > 0$ such that*

$$\sum_{n \leq x} |a(n)|^2 \sim c_0 x. \tag{1.4}$$

Moreover, if $F(s)$ satisfies the Ramanujan condition, then $\sigma_a(F) = 1$.

Remarks.

1. It is interesting to note that the upper bound assumed in (1.3) implies the asymptotic formula (1.4).
2. We may assume a version of (1.3) with a main term of type $xP_d(\log x, k)$, where $P_d(X, k)$ is a polynomial of degree d, and get analogous consequences for $F \otimes \overline{F}(s)$.
3. In general, $\sigma_a(F \otimes \overline{F}) = 1$ does not imply that $\sigma_a(F) = 1$. Choose indeed $a(n) = n^{1/4}$ if n is a square and $a(n) = 0$ otherwise; in this case it is easily seen that $\sigma_a(F \otimes \overline{F}) = 1$ but $\sigma_a(F) = 5/8$; such coefficients do not satisfy the Ramanujan condition.

4. It is well known that the function $L(s, f \otimes \overline{f})$ introduced above may have, in addition to the pole at $s = 1$, also poles on the line $\sigma = 1/4$; see again Chapter 13 of [2]. Therefore we cannot expect $(s-1)F\otimes\overline{F}(s)$ to have a holomorphic continuation to a much wider right half-plane than $\sigma > \theta$.
5. A final remark on the above corollary, namely it is not true that $\sigma_a(F) = 1$ for every $F \in \mathcal{S}^\sharp$. Indeed, it follows from Theorem 1 of [7] that for every $F \in \mathcal{S}^\sharp$ we have $\sigma_a(F) \geq (d_F + 1)/2d_F$, and in [10] (see p. 1347) we have shown that there exist $F \in \mathcal{S}^\sharp$ with $\sigma_a(F)$ arbitrarily close to 1/2; our examples in [10] do not satisfy the Ramanujan condition.

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2. Proofs

2.1. Lemmas. We first obtain a suitable version of the basic transformation formula for the linear twists $F(s,\alpha)$ of functions $F \in \mathcal{S}^\sharp$ of degree 2, defined for $\alpha > 0$ and $\sigma > 1$ by

$$F(s,\alpha) = \sum_{n=1}^{\infty} \frac{a(n)e(-n\alpha)}{n^s},$$

obtained in Theorem 1.2 of [9]. We recall that θ_F is the internal shift defined in Section 1.

LEMMA 1. *Let $F \in \mathcal{S}^\sharp$ with degree 2 and conductor q, and let $\alpha > 0$. Then there exists a polynomial $Q(s)$ such that*

$$\begin{aligned} F(s,\alpha) = {} & \frac{Q(s)}{\alpha} F(s+1,\alpha) + (1+q\alpha)^{2s-1+2i\theta_F} \\ & \times \left\{ F\left(s, \frac{\alpha}{1+q\alpha}\right) - \frac{q\alpha+1}{\alpha} Q(s) F\left(s+1, \frac{\alpha}{1+q\alpha}\right)\right\} + H(s,\alpha), \end{aligned} \tag{2.1}$$

where $H(s,\alpha)$ is holomorphic for $\sigma > 0$ and differentiable with continuous derivative for $\alpha > 0$.

Proof. We start with the transformation formula in Theorem 1.2 of [9] with the choice $K = 1$, which we rewrite as

$$\begin{aligned} F(s,\alpha) = c_1 q^s \alpha^{2s-1+2i\theta_F} \left\{ \overline{F}\left(s + 2i\theta_F, -\frac{1}{q\alpha}\right) + c_2 \alpha Q_1(s) \overline{F}\left(s+1+2i\theta_F, -\frac{1}{q\alpha}\right)\right\} \\ + H_1(s,\alpha). \end{aligned} \tag{2.2}$$

Here $Q_1(s)$ is a certain polynomial, $c_j \neq 0$, $j = 1,2$, are constants depending on $F(s)$, and $H_1(s,\alpha)$ is holomorphic for $\sigma > 0$ and continuously differentiable for $\alpha > 0$. The fact that $H_1(s,\alpha)$ is continuously differentiable for $\alpha > 0$, which will be important later on in the paper, follows easily from the proof of the above mentioned Theorem 1.2, since $\frac{\partial}{\partial\alpha} H_1(s,\alpha)$ can be expressed, for $\sigma > 0$ and $\alpha > 0$, in terms of absolutely and uniformly convergent series of continuous functions in α. Note that formally Theorem 1.2 in [9] asserts that $H_1(s,\alpha)$ is holomorphic in a smaller region, but actually the condition on the region is needed only for the bound on $H_1(s,\alpha)$ stated immediately after. Since $F(s,\alpha)$ is 1-periodic in α, (2.2) yields

$$\begin{aligned}&\alpha^{2s-1+2i\theta_F}\Big\{\overline{F}\Big(s+2i\theta_F,-\frac{1}{q\alpha}\Big)+c_2\alpha Q_1(s)\overline{F}\Big(s+1+2i\theta_F,-\frac{1}{q\alpha}\Big)\Big\}\\&=(\alpha+1)^{2s-1+2i\theta_F}\\&\quad\times\Big\{\overline{F}\Big(s+2i\theta_F,-\frac{1}{q(\alpha+1)}\Big)+c_2(\alpha+1)Q_1(s)\overline{F}\Big(s+1+2i\theta_F,-\frac{1}{q(\alpha+1)}\Big)\Big\}\\&\quad+H_2(s,\alpha),\end{aligned}$$

where $H_2(s,\alpha)$ has the same properties as $H_1(s,\alpha)$. Lemma 1 follows now changing first $s+2i\theta_F\mapsto s$, then taking conjugates on both sides and finally changing $\alpha\mapsto 1/(q\alpha)$ and $s\mapsto\overline{s}$. ■

For later reference, we note that changing $s\mapsto\overline{s}$ and taking conjugates in (2.1) we obtain also

$$\begin{aligned}\overline{F}(s,-\alpha)=\frac{\overline{Q}(s)}{\alpha}\,\overline{F}(s+1,-\alpha)+(1+q\alpha)^{2s-1-2i\theta_F}\\\times\Big\{\overline{F}\Big(s,-\frac{\alpha}{1+q\alpha}\Big)-\frac{q\alpha+1}{\alpha}\,\overline{Q}(s)\overline{F}\Big(s+1,-\frac{\alpha}{1+q\alpha}\Big)\Big\}+\overline{H}(s,\alpha),\end{aligned}\tag{2.3}$$

where $\overline{H}(s,\alpha)$ has the same properties of $H(s,\alpha)$ and $\overline{Q}(s)$ is a polynomial.

Given a test function $\phi\in C_0^\infty((0,\infty))$ and $F\in\mathcal{S}^\sharp$ we define

$$h_\phi(s)=\int_0^\infty\phi(\alpha)\big(1-(1+q\alpha)^{2(s-1)}\big)\,\mathrm{d}\alpha,\tag{2.4}$$

where $q>0$ is the conductor of $F(s)$. Clearly, $h_\phi(s)$ is an entire function. The link between the shifted convolutions and the Rankin–Selberg convolution is provided by the following basic lemma.

Lemma 2. *Let $F\in\mathcal{S}^\sharp$ with $d=2$, $F_k(s)$ be as in* (1.2) *and $\sigma>\sigma_a(F\otimes\overline{F})$. Then for any test function $\phi\in C_0^\infty((0,\infty))$ we have*

$$h_\phi(s)F\otimes\overline{F}(s)=\sum_{k=1}^\infty A_k(s)F_k(s)+\sum_{k=1}^\infty B_k(s)\overline{F}_k(s)+H(s),\tag{2.5}$$

where $H(s)$ is holomorphic for $\sigma>\max(1/2,\sigma_a(F\otimes\overline{F})-1)$ and $A_k(s),B_k(s)$ are entire and bounded by $O_C(k^{-C})$ for every $C>0$, uniformly for s in any compact subset of $\mathbb{C}$.

Proof. For $\sigma>1$ we consider the integral

$$I(s)=\int_0^\infty\phi(\alpha)F(s,\alpha)\overline{F}(s,-\alpha)\,\mathrm{d}\alpha,\tag{2.6}$$

and we compute $I(s)$ in two different ways. First, a term-by-term integration shows that

$$\begin{aligned} I(s) &= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a(n)\overline{a(m)}}{n^s m^s} \int_0^\infty \phi(\alpha) e(-(n-m)\alpha)\,\mathrm{d}\alpha = \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a(n)\overline{a(m)}}{n^s m^s}\, \widehat{\phi}(n-m) \\ &= \widehat{\phi}(0) F\otimes\overline{F}(2s) + \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{a(n)\overline{a(n+k)}}{n^s(n+k)^s}\, \widehat{\phi}(-k) + \sum_{k=1}^\infty \sum_{n=1}^\infty \frac{\overline{a(n)}a(n+k)}{n^s(n+k)^s}\, \widehat{\phi}(k) \\ &= \widehat{\phi}(0) F\otimes\overline{F}(2s) + \Sigma(s) + \overline{\Sigma}(s), \end{aligned} \tag{2.7}$$

say. Moreover, since $\phi(\alpha)$ is a smooth test function, for $k \in \mathbb{Z}, k \neq 0$, and any $C > 0$ we have

$$\widehat{\phi}(k) \ll_C |k|^{-C}. \tag{2.8}$$

Thanks to the sharp decay of $\widehat{\phi}(k)$ in (2.8), for any $\varepsilon, C > 0$ we have

$$\begin{aligned} \Sigma(s) &= \sum_{n=1}^\infty \frac{a(n)}{n^s} \sum_{1\leq k\leq n^\varepsilon} \widehat{\phi}(-k) \frac{\overline{a(n+k)}}{(n+k)^s} + O\Big(\sum_{n=1}^\infty \frac{|a(n)|}{n^\sigma} \sum_{k>n^\varepsilon} \frac{|a(n+k)|}{|n+k|^\sigma k^{C/\varepsilon}} \Big) \\ &= \sum_{n=1}^\infty \frac{a(n)}{n^{2s}} \sum_{1\leq k\leq n^\varepsilon} \widehat{\phi}(-k) \overline{a(n+k)} \\ &\quad + \sum_{n=1}^\infty \frac{a(n)}{n^s} \sum_{1\leq k\leq n^\varepsilon} \widehat{\phi}(-k) \overline{a(n+k)} \Big(\frac{1}{(n+k)^s} - \frac{1}{n^s} \Big) + f_1(s), \end{aligned} \tag{2.9}$$

say, where $f_1(s)$ is an entire function. We denote by $f_2(s)$ the second term in the last equation in (2.9); again thanks to (2.8), and to Cauchy–Schwarz inequality, we have

$$\begin{aligned} f_2(s) &\ll |s| \sum_{n=1}^\infty \frac{|a(n)|}{n^\sigma} \sum_{1\leq k\leq n^\varepsilon} k |\widehat{\phi}(-k)| \frac{|a(n+k)|}{n^{\sigma+1}} \\ &\ll |s| \sum_{k=1}^\infty k|\widehat{\phi}(-k)| \sum_{n\geq k^{1/\varepsilon}} \frac{|a(n)a(n+k)|}{n^{2\sigma+1}} \ll |s| \sum_{n=1}^\infty \frac{|a(n)|^2}{n^{2\sigma+1}}, \end{aligned}$$

hence $f_2(s)$ is holomorphic for $2\sigma > \sigma_a(F\otimes\overline{F}) - 1$. The first term in the last equation in (2.9) equals

$$\begin{aligned} \sum_{k=1}^\infty \widehat{\phi}(-k) \sum_{n\geq k^{1/\varepsilon}} \frac{a(n)a(n+k)}{n^{2s}} &= \sum_{k=1}^\infty \widehat{\phi}(-k) F_k(2s) \\ &\quad + O\Big(\sum_{k=1}^\infty |\widehat{\phi}(-k)| \sum_{n<k^{1/\varepsilon}} \frac{|a(n)a(n+k)|}{n^{2\sigma}} \Big) \\ &= \sum_{k=1}^\infty \widehat{\phi}(-k) F_k(2s) + O\Big(\sum_{k=1}^\infty |\widehat{\phi}(-k)| k^c \Big) \\ &= \sum_{k=1}^\infty \widehat{\phi}(-k) F_k(2s) + f_3(s), \end{aligned}$$

say, where $c = c(\sigma, \varepsilon)$ is a certain constant. Hence, once more thanks to (2.8), $f_3(s)$ is an entire function. Therefore, collecting the above results we have

$$\Sigma(s) = \sum_{k=1}^{\infty} \widehat{\phi}(-k) F_k(2s) + f_4(s), \tag{2.10}$$

where $f_4(s)$ is holomorphic for $2\sigma > \sigma_a(F\otimes\overline{F}) - 1$. In a completely analogous way we also obtain

$$\overline{\Sigma}(s) = \sum_{k=1}^{\infty} \widehat{\phi}(k) \overline{F}_k(2s) + f_5(s), \tag{2.11}$$

again $f_5(s)$ being holomorphic for $2\sigma > \sigma_a(F\otimes\overline{F}) - 1$. From (2.7), (2.10) and (2.11) we finally obtain

$$I(s) = \widehat{\phi}(0) F\otimes\overline{F}(2s) + \sum_{k=1}^{\infty} \widehat{\phi}(-k) F_k(2s) + \sum_{k=1}^{\infty} \widehat{\phi}(k) \overline{F}_k(2s) + f(s), \tag{2.12}$$

where $f(s)$ is holomorphic for $2\sigma > \sigma_a(F\otimes\overline{F}) - 1$.

Next we compute the integral in (2.6) using the transformation formulae (2.1) and (2.3). With obvious notation, for $\sigma > 1$ we rewrite (2.1) and (2.3) as

$$F(s,\alpha) = \sum_{j=1}^{4} G_j(s,\alpha) \qquad \text{and} \qquad \overline{F}(s,-\alpha) = \sum_{\ell=1}^{4} K_\ell(s,\alpha), \tag{2.13}$$

hence plugging (2.13) into (2.6) we obtain

$$I(s) = \sum_{j=1}^{4}\sum_{\ell=1}^{4} \int_0^\infty \phi(\alpha) G_j(s,\alpha) K_\ell(s,\alpha)\, \mathrm{d}\alpha = \sum_{j=1}^{4}\sum_{\ell=1}^{4} I_{j,\ell}(s), \tag{2.14}$$

say. Clearly,

$$I_{j,\ell}(s) \text{ is holomorphic for } \sigma > 0 \text{ for every } j,\ell \in \{1,3,4\}. \tag{2.15}$$

Switching summation and integration we have

$$I_{1,2}(s) = Q(s) \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a(n)\overline{a(m)}}{n^{s+1} m^s} \int_0^\infty \frac{\phi(\alpha)}{\alpha} (1+q\alpha)^{2s-1-2i\theta_F} e\Big(-n\alpha + m\frac{\alpha}{1+q\alpha}\Big)\, \mathrm{d}\alpha,$$

and, writing $\alpha^{-1}(1+q\alpha)^{2s-1-2i\theta_F} = \rho_1(\alpha,s) + i\rho_2(\alpha,s)$ with real functions $\rho_j(\alpha,s)$, $j=1,2$, we express the above integral as $I_1(s,n,m) + iI_2(s,n,m)$ with

$$\int_0^\infty \phi(\alpha)\rho_j(\alpha,s) e\Big(-n\alpha + m\frac{\alpha}{1+q\alpha}\Big)\, \mathrm{d}\alpha. \tag{2.16}$$

Since $\rho_j(\alpha,s)$ is clearly continuously differentiable in α, the total variation $V_j(s)$ of $\phi(\alpha)\rho_j(\alpha,s)$ for $\alpha \in (0,\infty)$ is uniformly bounded for s in any compact subset of $\mathbb{C}$. Let

$$f_{n,m}(\alpha) = -n\alpha + m\frac{\alpha}{1+q\alpha} \qquad \text{and} \qquad V(s) = \max(V_1(s), V_2(s)).$$

It is easily seen, by checking its second derivative, that $f_{n,m}(\alpha)$ has a monotonic first derivative for $\alpha > 0$ for every $n,m \geq 1$. Therefore, for $n,m \geq 1$ we may apply the first and second derivative tests to the integrals $I_j(s,n,m)$ in (2.16), see Lemmas 5.1.2

and 5.1.3 of Huxley [1]. Since

$$f'_{n,m}(\alpha) = -n + \frac{m}{(1+q\alpha)^2} \qquad \text{and} \qquad f''_{n,m}(\alpha) = -\frac{2mq}{(1+q\alpha)^3},$$

for $j = 1, 2$ we obtain for suitable constants $0 < a < b$:

$$I_j(s,n,m) \ll \begin{cases} V(s)/n & \text{if } 1 \le m \le an \\ V(s)/\sqrt{m} & \text{if } an < m < bn \\ V(s)/m & \text{if } m \ge bn, \end{cases}$$

where a, b and the implied constant in the $\ll$-symbol may depend on $\phi(\alpha)$ and $F(s)$. As a consequence, for $j = 1, 2$ we have

$$\begin{aligned} &\sum_{n=1}^\infty \sum_{m=1}^\infty \left| \frac{a(n)\overline{a(m)}}{n^{s+1}m^s} I_j(s,n,m) \right| \\ &\ll \sum_{n=1}^\infty \frac{|a(n)|}{n^{\sigma+2}} \sum_{m \le an} \frac{|a(m)|}{m^\sigma} + \sum_{n=1}^\infty \frac{|a(n)|}{n^{\sigma+1}} \sum_{an<m<bn} \frac{|a(m)|}{m^{\sigma+1/2}} + \sum_{n=1}^\infty \frac{|a(n)|}{n^{\sigma+1}} \sum_{m \ge bn} \frac{|a(m)|}{m^{\sigma+1}} \\ &\ll \sum_{n=1}^\infty \frac{|a(n)|}{n^{2\sigma+1-\varepsilon}} + \sum_{n=1}^\infty \frac{|a(n)|}{n^{2\sigma+1/2-\varepsilon}} + \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{|a(n)a(m)|}{n^{\sigma+1}m^{\sigma+1}}, \end{aligned}$$

and all the sums in the last row are uniformly convergent in any compact subset of the half-plane $2\sigma > 1/2$. Hence

$$I_{1,2}(s) \text{ is holomorphic for } 2\sigma > 1/2, \tag{2.17}$$

and similar arguments show that

$$I_{2,1}(s),\ I_{2,3}(s) \text{ and } I_{3,2}(s) \text{ are holomorphic for } 2\sigma > 1/2. \tag{2.18}$$

The integral $I_{2,4}(s)$ can be treated using only the first derivative test. Switching summation and integration we have

$$I_{2,4}(s) = \sum_{n=1}^\infty \frac{a(n)}{n^s} \int_0^\infty \phi(\alpha)(1+q\alpha)^{2s-1+2i\theta_F}\overline{H}(s,\alpha)e\Big(-n\frac{\alpha}{1+q\alpha}\Big)\,\mathrm{d}\alpha$$

and, again splitting $(1+q\alpha)^{2s-1+2i\theta_F}\overline{H}(s,\alpha)$ into real and imaginary parts, we rewrite the above integral as $I_1(s,n) + iI_2(s,n)$ with

$$I_j(s,n) = \int_0^\infty \phi(\alpha)\rho_j(\alpha,s)e\Big(-n\frac{\alpha}{1+q\alpha}\Big)\,\mathrm{d}\alpha$$

and certain real functions $\rho_j(\alpha,s)$. Here the $\rho_j(\alpha,s)$'s have the same properties as in (2.16), thanks to the corresponding properties of $\overline{H}(s,\alpha)$ in (2.3). Moreover, by the first derivative test we have

$$I_j(s,n) \ll \frac{V(s)}{n},$$

where $V(s)$ is the maximum of the total variations of $\phi(\alpha)\rho_j(\alpha,s)$, $j = 1, 2$, for $\alpha \in (0,\infty)$. As a consequence we have

$$I_{2,4}(s) \ll \sum_{n=1}^\infty \frac{|a(n)|}{n^{\sigma+1}}.$$

Since a completely similar argument can be applied to $I_{4,2}(s)$, we deduce that

$$I_{2,4}(s) \text{ and } I_{4,2}(s) \text{ are holomorphic for } \sigma > 0. \tag{2.19}$$

Finally we deal with $I_{2,2}(s)$. Here we have

$$\begin{aligned} I_{2,2}(s) &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a(n)\overline{a(m)}}{n^s m^s} \int_0^\infty \phi(\alpha)(1+q\alpha)^{2(2s-1)} e\Big(-(n-m)\frac{\alpha}{1+q\alpha}\Big)\,\mathrm{d}\alpha \\ &= F\otimes\overline{F}(2s)\int_0^\infty \phi(\alpha)(1+q\alpha)^{2(2s-1)}\,\mathrm{d}\alpha + J_{2,2}(s) + \overline{J}_{2,2}(s), \end{aligned} \tag{2.20}$$

where

$$J_{2,2}(s) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{a(n)}{n^s}\frac{\overline{a(n+k)}}{(n+k)^s} \int_0^\infty \phi(\alpha)(1+q\alpha)^{2(2s-1)} e\Big(k\,\frac{\alpha}{1+q\alpha}\Big)\,\mathrm{d}\alpha \tag{2.21}$$

and $\overline{J}_{2,2}(s) = \overline{J_{2,2}(\overline{s})}$. After the substitution $\frac{\alpha}{1+q\alpha} \mapsto \alpha$ the integral in (2.21) becomes the Fourier transform at $-k$ of a certain smooth function with compact support in $(0,\infty)$ which we denote by $\phi(\alpha,s)$. By the same argument leading to (2.10) and (2.11) we therefore obtain

$$J_{2,2}(s) = \sum_{k=1}^{\infty} \widehat{\phi}(-k,s)F_k(2s) + g_1(s) \quad \text{and} \quad \overline{J}_{2,2}(s) = \sum_{k=1}^{\infty} \widehat{\phi}(k,s)\overline{F}_k(2s) + g_2(s), \tag{2.22}$$

where $\widehat{\phi}(k,s)$ is entire and bounded by $O_C(k^{-C})$ for every $C>0$, uniformly for s in any compact subset of $\mathbb{C}$, and $g_1(s), g_2(s)$ are holomorphic for $2\sigma > \sigma_a(F\otimes\overline{F}) - 1$. Hence (2.20) and (2.22) show that

$$\begin{aligned} I_{2,2}(s) &= F\otimes\overline{F}(2s)\int_0^\infty \phi(\alpha)(1+q\alpha)^{2(2s-1)}\,\mathrm{d}\alpha \\ &\quad + \sum_{k=1}^{\infty}\widehat{\phi_1}(-k,s)F_k(2s) + \sum_{k=1}^{\infty}\widehat{\phi_1}(k,s)\overline{F}_k(2s) + g_3(s) \end{aligned} \tag{2.23}$$

with $g_3(s)$ holomorphic for $2\sigma > \sigma_a(F\otimes\overline{F}) - 1$.

From (2.14), (2.15), (2.17), (2.18), (2.19) and (2.23) we finally get

$$\begin{aligned} I(s) &= F\otimes\overline{F}(2s)\int_0^\infty \phi(\alpha)(1+q\alpha)^{2(2s-1)}\,\mathrm{d}\alpha \\ &\quad + \sum_{k=1}^{\infty}\widehat{\phi_1}(-k,s)F_k(2s) + \sum_{k=1}^{\infty}\widehat{\phi_1}(k,s)\overline{F}_k(2s) + g(s), \end{aligned} \tag{2.24}$$

where $g(s)$ is holomorphic for $2\sigma > \max(1/2, \sigma_a(F\otimes\overline{F}) - 1)$. In view of (2.8) and of the properties of $\widehat{\phi}(k,s)$ reported after (2.22), Lemma 2 now follows by comparing (2.12) with (2.24) and changing $2s \mapsto s$. ∎

The next lemma contains the properties of the function $h_\phi(s)$, defined in (2.4), required in the proof of our theorem. Note that such properties do not depend on the value of the conductor q.

Lemma 3. *Let $\phi \in C_0^\infty((0,\infty))$, $\phi(x) \geq 0$ but not identically vanishing. Then the entire function $h_\phi(s)$ has a simple zero at $s=1$ and $h_\phi(s) \neq 0$ for $s \in \mathbb{R}\setminus\{1\}$.*

Proof. Clearly we have $h_\phi(1) = 0$ and

$$-h'_\phi(1) = 2\int_0^\infty \phi(\alpha)\log(1+q\alpha)\,\mathrm{d}\alpha > 0,$$

hence $h_\phi(s)$ has a simple zero at $s=1$. Moreover, it is clear from (2.4) that $h_\phi(s) > 0$ for $s < 1$, and $h_\phi(s) < 0$ for $s > 1$. ∎

2.2. Proof of the theorem and its corollary. Let $\varepsilon > 0$ be as in (1.3). Since clearly $\sigma_a(F\otimes\overline{F}) \le 2$, by partial summation, (1.1) and (1.3) we have

$$\sum_{n=1}^\infty \frac{a(n)\overline{a(n+k)}}{n^s} = s\int_1^\infty S_k(x)x^{-s-1}\,\mathrm{d}x$$
$$\ll |s|\int_1^{k^{1/\varepsilon}} x^{1+\varepsilon}(x+k)^{1+\varepsilon}x^{-\sigma-1}\,\mathrm{d}x + |s|\int_{k^{1/\varepsilon}}^\infty x^{\theta-\sigma-1}\,\mathrm{d}x \ll |s|k^{2/\varepsilon}$$

provided $\sigma \ge \theta+\delta$, for any $\delta > 0$. Hence $F_k(s)$ is holomorphic for $\sigma > \theta$ and satisfies $F_k(s) \ll k^{2/\varepsilon}$ uniformly for s in any compact subset of the half-plane $\sigma > \theta$; the same holds for $\overline{F}_k(s)$. Applying Lemma 2 with $C = 2/\varepsilon + 2$ and recalling that $\theta > 1/2$, we deduce that the right hand side of (2.5) is holomorphic for $\sigma > \max(\theta, \sigma_a(F\otimes\overline{F}) - 1)$. Hence from Lemma 3 we see that $F\otimes\overline{F}(s)$ is meromorphic in the same half-plane, and its only real singularity in such half-plane is at most a simple pole at $s=1$. Since by the hypothesis of the theorem we infer that $\max(\theta, \sigma_a(F\otimes\overline{F})-1) < \sigma_a(F\otimes\overline{F})$, from Landau's theorem on Dirichlet series with non-negative coefficients we deduce that $F\otimes\overline{F}(s)$ has a simple pole at $s=1$ and $\sigma_a(F\otimes\overline{F}) = 1$; the theorem is therefore proved. ∎

Moving on to the corollary, the asymptotic formula (1.4) follows from the theorem by a standard Tauberian theorem, since the coefficients of $F\otimes\overline{F}(s)$ are non-negative. Moreover, under the Ramanujan condition, by the Cauchy–Schwarz inequality and (1.4) we have

$$\sum_{n\le x}|a(n)| \ll x \qquad \text{and} \qquad x \ll \sum_{n\le x}|a(n)|^2 \ll x^\varepsilon \sum_{n\le x}|a(n)|$$

for every $\varepsilon > 0$, hence $\sigma_a(F) = 1$. ∎

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