APPLICATIONES MATHEMATICAE
46,2 (2019), pp. 175–189

Benali Aharrouch and Jaouad Bennouna (Fez)

EXISTENCE AND UNIQUENESS OF SOLUTION FOR A UNILATERAL PROBLEM IN SOBOLEV SPACES WITH VARIABLE EXPONENT

Abstract. We study the existence and uniqueness of the obstacle problem associated to the equation

$$-\operatorname{div}(a(x,u,\nabla u)+\phi(u))+g(x,u)=f-\operatorname{div} F$$

in the framework of Sobolev spaces with variable exponent, where $F \in (L^{r(\cdot)}(\Omega))^N$ and $f \in L^{q(\cdot)}(\Omega)$ with

$$\begin{cases} r(x) > \frac{N}{p(x)-1}, \quad r(x) \ge p'(x) & \forall x \in \Omega, \\ q(x) > \max\big(\frac{N}{p(x)}, 1\big), \quad q(x) \ge p'(x) & \forall x \in \Omega, \end{cases}$$

for a log-Lipschitz function $p : \overline{\Omega} \to [1,+\infty)$.

1. Introduction. Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \ge 2$), and $p(\cdot) : \overline{\Omega} \to [1,+\infty)$ be a function satisfying the log-Lipschitz continuous condition such that $1 < p_- \le p_+ < \infty$ (see Subsection 2.1).

The purpose of this paper is to study the obstacle and Dirichlet problem associated to the nonlinear elliptic equation

$$-\operatorname{div}(a(x,u,\nabla u)+\phi(u))+g(x,u)=f-\operatorname{div}(F), \tag{1.1}$$

2010 *Mathematics Subject Classification*: 35J87, 47J20, 35J66.
Key words and phrases: nonlinear obstacle problems, variable exponents, nonstandard growth, existence and uniqueness, L^∞ estimate.
Received 24 May 2018; revised 1 August 2018 and 24 February 2019.
Published online 23 July 2019.

DOI: 10.4064/am2372-2-2019

when

$$(1.2)\qquad f \in L^{q(\cdot)}(\Omega), \qquad q(x) > \max\left(\frac{N}{p(x)}, 1\right), \; q(x) \geq p'(x), \; \forall x \in \Omega,$$

$$(1.3)\qquad F \in (L^{r(\cdot)}(\Omega))^N, \quad r(x) > \frac{N}{p(x)-1}, \; r(x) \geq p'(x), \; \forall x \in \Omega,$$

and

- $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function,
- $\phi \in C^0(\mathbb{R}, \mathbb{R}^N)$,
- g is a Carathéodory function satisfying a sign condition.

The motivation for studying problem (1.1) comes from applications in elasticity [25] and non-Newtonian fluid mechanics [5, 21].

The solvability of (1.1) is very well understood in the case of p constant (see [11, 12, 15, 17, 20]). When $p(\cdot)$ is a variable exponent, the existence of solutions to problem (1.1) has been obtained [23, 27] under some restrictive conditions on F and f.

The novelty of this work is to refine and weaken the conditions on the data F and f and to show the existence and uniqueness of solution under conditions (1.2) and (1.3) on f and F.

The main tool used is the result of Stampacchia [22] which yields the boundedness of solutions; inspired by the idea of [7], we partition $\overline{\Omega}$ into a finite number of balls B_i such that for all continuous functions $f < g$ on Ω, we have $\sup(f) < \inf(g)$ on each B_i, and for which the conditions of [22, Lemma 4] are satisfied.

This paper is organized as follows: in Section 2, we collect the necessary preliminaries and specify some assumptions; in Section 3, the existence of a bounded solution to problem (1.1) is established; and in the last section, the uniqueness of solution is proved.

2. Preliminaries and assumptions

2.1. Preliminaries. Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$. We say that a real-valued continuous function $p(\cdot)$ is *log-Lipschitz continuous* in Ω if

$$-\log|x-y|\,|p(x)-p(y)| \leq C \quad \forall x, y \in \overline{\Omega} \text{ with } x \neq y \text{ and } |x-y| < 1/2,$$

with a positive constant C. We denote

$$C_+(\overline{\Omega}) = \{\text{log-Lipschitz continuous functions } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p_- \leq p_+ < N\},$$

where

$$p_- = \min\{p(x) : x \in \overline{\Omega}\}, \qquad p_+ = \max\{p(x) : x \in \overline{\Omega}\}.$$

For $p \in C_+(\overline{\Omega})$ we define the *variable exponent Lebesgue space*

$$L^{p(\cdot)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\};$$

under the norm

$$\|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}\,dx \le 1\Big\},$$

the space $L^{p(\cdot)}(\Omega)$ is a uniformly convex Banach space, and therefore reflexive. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$.

Proposition 2.1 (Generalized Hölder inequality [14, 24]).

(i) *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p_-} + \frac{1}{p'_-}\Big)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)}.$$

(ii) *For all $p_1, p_2 \in C_+(\overline{\Omega})$ such that $p_1(x) \le p_2(x)$ for all $x \in \Omega$, we have a continuous embedding*

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega).$$

Proposition 2.2 ([14, 24]). *Denote*

$$\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx \qquad \forall u \in L^{p(\cdot)}(\Omega).$$

Then the following assertions hold:

(i) $\|u\|_{p(\cdot)} < 1$ (*resp.* $= 1, > 1$) *if and only if* $\rho(u) < 1$ (*resp.* $= 1, > 1$).
(ii) $\|u\|_{p(\cdot)} > 1$ *implies* $\|u\|_{p(\cdot)}^{p_-} \le \rho(u) \le \|u\|_{p(\cdot)}^{p_+}$, *while* $\|u\|_{p(\cdot)} < 1$ *implies* $\|u\|_{p(\cdot)}^{p_+} \le \rho(u) \le \|u\|_{p(\cdot)}^{p_-}$.
(iii) *For a sequence* $(u_n)_n$ *in* $L^{p(\cdot)}(\Omega)$, $\|u_n\|_{p(\cdot)} \to 0$ *if and only if* $\rho(u_n) \to 0$, *and* $\|u_n\|_{p(\cdot)} \to \infty$ *if and only if* $\rho(u_n) \to \infty$.

Now, we define the *variable exponent Sobolev space*

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)} \qquad \forall u \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, and we define the *Sobolev exponent* by

$$p^*(x) = \frac{Np(x)}{N - p(x)} \qquad \text{for } p(x) < N.$$

PROPOSITION 2.3 ([14]).

(i) *If* $1 < p_- \le p_+ < \infty$, *then the spaces* $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *are separable and reflexive Banach spaces.*

(ii) *If* $q \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for any* $x \in \Omega$, *then the embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{q(\cdot)}(\Omega)$ *is continuous and compact.*

(iii) *Poincaré inequality: there exists a constant* $C > 0$ *such that*

$$\|u\|_{p(\cdot)} \le C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

(vi) *Sobolev–Poincaré inequality: there exists another constant* $C > 0$ *such that*

$$\|u\|_{p*(\cdot)} \le C\|\nabla u\|_{p(\cdot)} \quad \forall u \in W_0^{1,p(\cdot)}(\Omega).$$

REMARK 1. By Proposition 2.3(iii), the norms $\|\nabla u\|_{p(\cdot)}$ and $\|u\|_{1,p(\cdot)}$ are equivalent in $W_0^{1,p(\cdot)}(\Omega)$.

LEMMA 2.4 ([6]). *Let* $F : \mathbb{R} \to \mathbb{R}$ *be a uniformly Lipschitz function with* $F(0) = 0$ *and let* $p \in C_+(\overline{\Omega})$. *If* $u \in W_0^{1,p(\cdot)}(\Omega)$, *then* $F(u) \in W_0^{1,p(\cdot)}(\Omega)$. *Moreover, if the set* D *of discontinuity points of* F' *is finite, then*

$$\frac{\partial(F \circ u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}, \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\}. \end{cases}$$

LEMMA 2.5 ([6]). *Under assumptions* (H_1)–(H_6) *below, let* $(u_n)_n$ *be a sequence in* $W_0^{1,p(\cdot)}(\Omega)$ *such that* $u_n \rightharpoonup u$ *in* $W_0^{1,p(\cdot)}(\Omega)$ *and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u) \to 0.$$

Then $u_n \to u$ *in* $W_0^{1,p(\cdot)}(\Omega)$.

2.2. Assumptions. Let

(H_1) $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for some $\alpha > 0$,

$$a(x, s, \xi) \cdot \xi \ge \alpha|\xi|^{p(x)}, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^N.$$

(H_2) (1) $[a(x, s, \xi) - a(x, s, \xi')][\xi - \xi'] > 0$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, and all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \ne \xi'$,

(2) there is an increasing function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ and a non-negative function $\bar{\beta} \in L^{p'(\cdot)}(\Omega)$ with $|a(x, s, \xi)| \le \beta(|s|)[|\xi|^{p(x)-1} + \bar{\beta}(x)]$ for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$.

(H_3) $f \in L^{q(\cdot)}(\Omega)$, $F \in (L^{r(\cdot)}(\Omega))^N$, where $q(x) \ge 1$ and $r(x) \ge p'(x)$ for all $x \in \Omega$.

(H_4) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $\sup_{|s|\le n} |g(\cdot, s)| = h_n(\cdot) \in L^1(\Omega)$ and $g(x, s)s \ge 0$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

(H_5) $\phi : \mathbb{R} \to \mathbb{R}^N$ is continuous.
(H_6) $\psi \in L^\infty(\Omega)$ and $K(\psi) = \{v \in W_0^{1,p(\cdot)}(\Omega) : v \geq \psi$ a.e. in $\Omega\} \neq \emptyset$.

DEFINITION 2.6. For all $k > 0$ and $s \in \mathbb{R}$, the truncation function $T_k(\cdot)$ is defined by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k, \\ k \cdot \operatorname{sign}(s) & \text{if } |s| > k, \end{cases}$$

and we set

$$G_k(s) = s - T_k(s).$$

DEFINITION 2.7. A measurable function $u \in K(\psi)$ is called a *weak solution* of the unilateral problem (1.1) if $a(x, u, \nabla u) \in (L^{p'(\cdot)}(\Omega))^N$ and $g(x, u) \in L^1(\Omega)$, and for all $v \in K(\psi) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, u, \nabla u)\nabla(u - v)\, dx + \int_\Omega \phi(u)\nabla(u - v)\, dx + \int_\Omega g(x, u)(u - v)\, dx \leq \int_\Omega f(u - v)\, dx + \int_\Omega F\nabla(u - v)\, dx. \tag{2.1}$$

THEOREM 2.8. *Suppose that assumptions* (H_1)*–*(H_6) *hold, and let* $q(x) > \max\big(\frac{N}{p(x)}, p'(x)\big)$ *and* $r(x) > \frac{N}{p(x)-1}$. *Then any weak solution u to problem* (1.1) (*in the sense of definition* (2.7)) *is bounded.*

Proof. For fixed $k, h, \theta > 0$, define $\omega_n = \frac{1}{h} T_h(G_k(T_n(u))$ and $\overline{\omega}_n = \theta\omega_n$, where $k = \theta + \|\psi\|_{L^\infty(\Omega)}$.

Note that $v = T_n(u) - \overline{\omega}_n \in K(\psi) \cap L^\infty(\Omega)$. Taking v as a test function in (2.1), we obtain

$$\int_\Omega a(x, u, \nabla u) \cdot \nabla(\overline{\omega}_n + u - T_n(u))\, dx + \int_\Omega \phi(u) \cdot \nabla(\overline{\omega}_n + u - T_n(u)) + \int_\Omega g(x, u)(\overline{\omega}_n + u - T_n(u)) \leq \int_\Omega f(\overline{\omega}_n + u - T_n(u))\, dx + \int_\Omega F\nabla(\overline{\omega}_n + u - T_n(u))\, dx. \tag{2.2}$$

Setting $\overline{\omega} = \frac{\theta}{h} T_h(u - G_k(u))$, we have

$$\overline{\omega}_n \to \overline{\omega} \quad \text{strongly in } W_0^{1.p(\cdot)}(\Omega)$$

and

$$\nabla\overline{\omega} = \frac{\theta}{h} \nabla\chi_{\{k \leq |u| \leq k+h\}} \quad \text{a.e. in } \Omega.$$

Passing to the limit in (2.2), we get

$$\begin{aligned}(2.3)\quad &\frac{1}{h}\int_{\{k<|u|<k+h\}} a(x,u,\nabla u)\cdot\nabla u\,dx+\frac{1}{h}\int_{\{k<|u|<k+h\}}\phi(u)\cdot\nabla u\,dx\\ &\quad+\frac{1}{h}\int_{\Omega} g(x,u)T_h(u-G_k(u))\,dx\\ &\le\frac{1}{h}\int_{\Omega} f(x)T_h(u-G_k(u))\,dx+\frac{1}{h}\int_{\Omega} g(x,u)T_h(u-G_k(u))\,dx\\ &\quad+\int_{\{k<|u|<k+h\}} F\cdot\nabla u\,dx.\end{aligned}$$

By (H_5), we may assume that $\phi=(\phi_1,\dots,\phi_N)$, where $\phi_i\in C(\mathbb{R})$ for $1\le i\le N$.

Let $\tilde\phi_i(t)=\int_0^t\chi_{\{k<|\eta|<k+h\}}\phi_i(\eta)d\eta$ and set $\tilde\phi=(\tilde\phi_1,\dots,\tilde\phi_N)$. Then it is easy to see that $\tilde\phi\in(W_0^{1,p(\cdot)}(\Omega))^N$. Thus, for the second term of the left-hand side in (2.3), using Lemma 2.4, we have

$$\begin{aligned}(2.4)\quad \int_{\{k<|u|<k+h\}}\phi(u)\cdot\nabla u\,dx&=\int_{\Omega}\chi_{\{k<|u|<k+h\}}\phi(u)\cdot\nabla u\,dx\\ &=\int_{\Omega}\operatorname{div}\tilde\phi(u)\,dx=0.\end{aligned}$$

Combining (2.3) with (2.4), it follows from (H_1) and (H_4) that

$$(2.5)\quad \alpha\int_{\{k<|u|<k+h\}}|\nabla u|^{p(x)}\,dx\le\int_{\{|u|>k\}}|f(x)|\,|G_k(u)|\,dx+\int_{\{k<|u|<k+h\}}|F|\,|\nabla u|\,dx.$$

Letting h tend to infinity in (2.5), we obtain

$$(2.6)\quad \alpha\int_{A(k)}|\nabla u|^{p(x)}\,dx\le\int_{A(k)}|f(x)|\,|G_k(u)|\,dx+\int_{A(k)}|F|\,|\nabla u|\,dx,$$

where $A(k)=\{x\in\Omega:|u(x)|>k\}$.

By using the Young inequality in the second term on the right hand side of (2.6) we have

$$\int_{\Omega}F\cdot\nabla G_k(u)\,dx\le c_1\int_{A(k)}|F|^{p'(x)}\,dx+c_2\int_{\Omega}|\nabla G_k(u)|^{p(x)}\,dx.$$

Now combining the last two formulas, using Proposition 2.2, and taking

$c' = \alpha - c_2 > 0$, we get

$$\begin{aligned} c' \int_\Omega |\nabla G_k(u)|^{p(x)}\,dx &\le c_1 \int_{A(k)} |F|^{p'(x)}\,dx + \int_\Omega |f|\cdot|G_k(u)|\,dx \\ &\le c_1 \int_{A(k)} |F|^{p'(x)}\,dx + c_3\|f\chi_{A_k}\|_{p'_*(\cdot)}\cdot\|G_k(u)\|_{p_*(\cdot)} \\ &\le c_1 \int_{A(k)} |F|^{p'(x)}\,dx + c_3\|f\chi_{A_k}\|_{p'_*(\cdot)}\cdot\|\nabla G_k(u)\|_{p(\cdot)} \\ &\le c_1 \int_{A(k)} |F|^{p'(x)}\,dx + c_3\|f\chi_{A_k}\|_{p'_*(\cdot)}\Big(\int_\Omega |\nabla G_k(u)|^{p(x)}\,dx\Big)^{1/\gamma_1} \end{aligned}$$

with

$$\gamma_1 = \begin{cases} p^- & \text{if } \|\nabla G_k(u)\|_{p(\cdot)} \ge 1, \\ p^+ & \text{if } \|\nabla G_k(u)\|_{p(\cdot)} < 1. \end{cases}$$

Using Young's inequality, we obtain

$$\begin{aligned} c' \int_\Omega |\nabla G_k(u)|^{p(x)}\,dx \le c_1 \int_{A(k)} |F|^{p'(x)}\,dx &+ c'_1\|f\chi_{A_k}\|_{p'_*(\cdot)}^{\gamma'_1} \\ &+ \frac{c'}{2}\int_\Omega |\nabla G_k(u)|^{p(x)}\,dx. \end{aligned}$$

By Hölder's inequality and Proposition 2.2, we get

$$\begin{aligned} \frac{c'}{2}\int_\Omega |\nabla G_k(u)|^{p(x)}\,dx &\le c_1 \int_{A(k)} |F|^{p'(x)}\,dx + c'_1\Big(\int_{A(k)} |f|^{p'_*(x)}\Big)^{\gamma'_1/\gamma_2}\,dx \\ &\le c_1 \int_{A(k)} |F|^{p'(x)}\,dx + c'_1\|\,|f|^{p'_*}\|_{L^{s_2(\cdot)/p'_*(\cdot)}(\Omega)}^{\gamma'_1/\gamma_2}\cdot\|\chi_{A_k}\|_{L^{\frac{s_2(\cdot)}{s_2(\cdot)-p'_*(\cdot)}}(\Omega)}^{\gamma'_1/\gamma_2} \\ &\le c_1 \int_{A(k)} |F|^{p'(x)}\,dx + c'_3(\Phi(k))^{\frac{\gamma'_1}{\gamma_2\cdot\gamma_5}} \\ &\le c_1\big\|\,|F|^{p'(x)}\big\|_{s_1(\cdot)/p'(\cdot)}\cdot(\Phi(k))^{\frac{1}{\gamma_6}} + c'_3(\Phi(k))^{\frac{\gamma'_1}{\gamma_2\cdot\gamma_5}} \end{aligned} \tag{2.7}$$

with $s_1(x) > r(x)$ and $s_2(x) > q(x)$ for $x \in \Omega$, $\Phi(k) = \mathrm{meas}(A(k))$, and

$$\gamma_2 = \begin{cases} p'^{-}_* & \text{if } \|f\chi_{A_k}\|_{p'_*(\cdot)} \ge 1, \\ p'^{+}_* & \text{if } \|f\chi_{A_k}\|_{p'_*(\cdot)} < 1, \end{cases}$$

$$\gamma_5 = \begin{cases} \big(\frac{s_2(x)}{s_2(x)-p'_*(x)}\big)^- & \text{if } \|\chi_{A_k}\|_{\frac{s_2(\cdot)}{s_2(\cdot)-p'_*(\cdot)}} \ge 1, \\ \big(\frac{s_2(x)}{s_2(x)-p'_*(x)}\big)^+ & \text{if } \|\chi_{A_k}\|_{\frac{s_2(\cdot)}{s_2(\cdot)-p'_*(\cdot)}} < 1, \end{cases}$$

$$\gamma_6 = \begin{cases} \left(\frac{s_1(x)}{s_1(x)-p'(x)}\right)^- & \text{if } \|\chi_{A_k}\|_{\frac{s_1(\cdot)}{s_1(\cdot)-p'(\cdot)}} \geq 1, \\ \left(\frac{s_1(x)}{s_1(x)-p'(x)}\right)^+ & \text{if } \|\chi_{A_k}\|_{\frac{s_1(\cdot)}{s_1(\cdot)-p'(\cdot)}} < 1. \end{cases}$$

In view of the Sobolev inequality and Proposition 2.2, we have

$$\int_\Omega |\nabla G_k(u)|^{p(x)}\,dx \geq c_4\Big(\int_\Omega |G_k(u)|^{p_*(x)}\Big)^{\gamma_4/\gamma_3}\,dx, \tag{2.8}$$

where

$$\gamma_3 = \begin{cases} (p_*)^- & \text{if } \|G_k(u)\|_{p_*(\cdot)} \geq 1, \\ (p_*)^+ & \text{if } \|G_k(u)\|_{p_*(\cdot)} < 1. \end{cases} \qquad \gamma_4 = \begin{cases} p^- & \text{if } \|\nabla G_k(u)\|_{p(\cdot)} \geq 1, \\ p^+ & \text{if } \|\nabla G_k(u)\|_{p(\cdot)} < 1. \end{cases}$$

So by (2.7) and (2.8), we obtain

$$\int_\Omega |G_k(u)|^{p_*(x)}\,dx \leq c\max\Big((\Phi(k))^{\frac{\gamma_1'\cdot\gamma_3}{\gamma_2\cdot\gamma_5\cdot\gamma_4}};(\Phi(k))^{\frac{\gamma_3}{\gamma_6\cdot\gamma_4}}\Big). \tag{2.9}$$

Choose h such that $h-k>1$ and in $A_h=\{x\in\Omega : |u|>h\}$ we have $h-k<G_k(u)$. Then in view of (2.9) we get

$$\Phi(h) \leq \frac{C}{(h-k)^{p_*^-}}\max\Big((\Phi(k))^{\frac{\gamma_1'\cdot\gamma_3}{\gamma_2\cdot\gamma_5\cdot\gamma_4}};(\Phi(k))^{\frac{\gamma_3}{\gamma_6\cdot\gamma_4}}\Big).$$

First, let p^+ be a constant satisfying $p^+ < \min_{x\in\overline{\Omega}}(1+1/N)p(x)$, which implies that

$$p^+ < \min_{x\in\overline{\Omega}}\frac{Np(x)}{N-p(x)}.$$

Then $\gamma_3/\gamma_4>1$ and $\gamma_1'/\gamma_2>1$. By a suitable choice of $s_1(\cdot)$ and $s_2(\cdot)$, we have $\beta=\frac{\gamma_1'\cdot\gamma_3}{\gamma_2\cdot\gamma_5\cdot\gamma_4}>1$. By Lemma 4 of Stampacchia [22], there exists a constant C such that $\|u\|_\infty\leq C$.

Now let $p\in C_+(\overline{\Omega})$ be such that $p(x)<\frac{Np(x)}{N-p(x)}$ and $p(x)<(1+1/N)p(x)$. By the continuity of $p(\cdot)$ on $\overline{\Omega}$ there exist constants $\delta_1,\delta_2>0$ such that

$$\max_{y\in\overline{B(x,\delta_1)\cap\Omega}} p(y) < \min_{y\in\overline{B(x,\delta_1)\cap\Omega}}\frac{Np(y)}{N-p(y)} \qquad \text{for all } x\in\overline{\Omega}, \tag{2.10}$$

$$\max_{y\in\overline{B(x,\delta_2)\cap\Omega}} p(y) < \inf_{y\in\overline{B(x,\delta_2)\cap\Omega}}\Big(1+\frac{1}{N}\Big)p(y) \qquad \text{for all } x\in\overline{\Omega}. \tag{2.11}$$

Since $\overline{\Omega}$ is compact, we can cover it with a finite number of balls $(B_j)_{j=1}^k$ and there exists a constant $\lambda>0$ such that

$$\min(\delta_1,\delta_2)>|\Omega_i|>\lambda, \qquad \Omega_i=B_i\cap\Omega, \qquad \text{for } i=1,\dots,k. \tag{2.12}$$

We denote by p_j^+ and p_{*j}^+ the local maxima of p and $p_*=\frac{Np}{N-p}$ on $\overline{\Omega_j}$ respectively (and by p_j^- and p_{*j}^- the respective local minima). By (2.9) and the fact that $p_{*i}^{-} < p_* = \frac{Np(\cdot)}{N-p(\cdot)}$ on Ω_i, we have

(2.13)
$$\int_{\Omega_i} |G_k(u)|^{p_{*i}^{-}}\,dx \le c_4' \max\Big((\Phi_i(k))^{\frac{(\gamma_1^i)'\cdot\gamma_3^i}{\gamma_2^i\cdot\gamma_5^i\cdot\gamma_4^i}}; (\Phi_i(k))^{\frac{\gamma_3^i}{\gamma_6^i\gamma_4^i}}\Big) \quad \text{for } i=1,\dots,k,$$

with $\Phi_i(k) = \mathrm{meas}(\{x\in\Omega_i : |u|>k\})$, and γ_j^i the restriction of γ_j to Ω_i.

Choose h such that $h-k>1$ and in $A_h^i = \{x\in\Omega_i : |u|>h\}$ we have $h-k<G_k(u)$. Then in view of (2.13) we obtain

$$\Phi(h) \le \frac{C}{(h-k)^{p_{*i}^{-}}} \max\Big((\Phi_i(k))^{\frac{(\gamma_1^i)'\cdot\gamma_3^i}{\gamma_2^i\cdot\gamma_5^i\cdot\gamma_4^i}}; (\Phi_i(k))^{\frac{\gamma_3^i}{\gamma_6^i\gamma_4^i}}\Big) \quad \text{for } i=1,\dots,k.$$

It follows from (2.11) that

$$\frac{\gamma_3^j}{\gamma_4^j} > 1 \quad\text{and}\quad \frac{(\gamma_1^j)'}{\gamma_2^j} > 1 \quad \text{for all } x\in\overline{\Omega} \text{ and } j=1,\dots,k,$$

which gives $\frac{\gamma_3^j}{\gamma_4^j}\frac{(\gamma_1^j)'}{\gamma_2^j} > 1$ and by a suitable choice of $s_1(\cdot)$ and $s_2(\cdot)$ we have

$$\frac{(\gamma_1^i)'\cdot\gamma_3^i}{\gamma_2^i\cdot\gamma_5^i\cdot\gamma_4^i} > 1 \quad \text{for all } x\in\overline{\Omega} \text{ and } i=1,\dots,k.$$

By [22, Lemma 4] we get $\|u\|_\infty \le C$. ∎

3. Existence of solution for the unilateral problem (1.1)

THEOREM 3.1. *Suppose that assumptions* (H_1)–(H_6) *hold, and let* $q(x) > \max\big(\frac{N}{p(x)}, p'(x)\big)$ *and* $r(x) > \frac{N}{p(x)-1}$ *for all* $x\in\Omega$. *Then there exists a weak solution* u *to problem* (1.1) (*in the sense of definition* (2.7)).

Proof. We divide the proof into three steps.

STEP 1: *A priori estimate.* Let us define

(3.1) $$a_n(x,s,\xi) = a(x,T_n(s),\xi), \quad \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R},\ \forall\xi\in\mathbb{R}^N,$$

and

(3.2) $$\phi_n(s) = \phi(T_n(s)), \quad g_n(x,s) = T_{1/n}(g(x,s)), \quad \text{a.e. } x\in\Omega,\ \forall s\in\mathbb{R},\ \forall\xi\in\mathbb{R}^N.$$

We consider the following approximate problem: find $u_n\in K(\psi)$ such that

(3.3) $$\langle -\operatorname{div}(a_n(x,u_n,\nabla u_n)), u_n-v\rangle + \langle -\operatorname{div}(\phi_n(u)), u_n-v\rangle + (g_n(x,u_n), u_n-v) \le (f, u_n-v) + \langle -\operatorname{div}(F), u_n-v\rangle \quad \forall v\in K(\psi).$$

By the classical result by Leray and Lions [16], for each $n\in\mathbb{N}$, there exists a weak solution $u_n\in K(\psi)\cap L^\infty(\Omega)$ of (3.3). By the same argument as before, we derive that

$$\|u_n\|_{L^\infty(\Omega)} \le M, \tag{3.4}$$

and thus

$$a_n(x, u_n, \nabla u_n) = a(x, u_n, \nabla u_n) \quad \text{and} \quad \phi_n(u_n) = \phi(u_n). \tag{3.5}$$

As $\psi \in K(\psi) \cap L^\infty(\Omega)$, taking $v = \psi$ as a test function in (3.3), by (3.5) we obtain

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla(u_n - \psi)\,dx + \int_\Omega \phi(u_n)\nabla(u_n - \psi)\,dx + \int_\Omega g(x, u_n)(u_n - \psi)\,dx \le \int_\Omega f(u_n - \psi)\,dx + \int_\Omega F\nabla(u_n - \psi)\,dx.$$

Noting that $\int_\Omega \phi(u_n)\nabla u_n\,dx = 0$, by Young's inequality, (3.5) and (H_1)–(H_4) we obtain

$$\int_\Omega |\nabla u_n|^{p(x)}\,dx \le C_1.$$

By Proposition 2.2 and the last inequality,

$$\|\nabla u_n\|_{L^{p(\cdot)}(\Omega)} \le C_2. \tag{3.6}$$

Then it follows from the results of [10] that there exists a subsequence of (u_n) (still denoted by (u_n)) such that

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in } (L^{p(\cdot)}(\Omega))^N, \tag{3.7}$$

$$u_n \to u \quad \text{strongly in } L^{p(\cdot)}(\Omega) \text{ and a.e. in } \Omega, \tag{3.8}$$

$$u_n \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty(\Omega). \tag{3.9}$$

By (3.8), we obtain

$$g_n(x, u_n) \to g(x, u) \quad \text{a.e. in } \Omega.$$

By assumption (H_4) and (3.4), for any measurable set $E \subset \Omega$,

$$\int_E |g_n(x, u_n)|\,dx \le \int_E h_M(x)\,dx.$$

Using Vitali's theorem, we conclude that

$$g_n(x, u_n) \to g(x, u) \quad \text{strongly in } L^1(\Omega). \tag{3.10}$$

STEP 2: *Almost everywhere convergence of the gradient.* By (H_2), to obtain the convergence of the gradient, it suffices to prove

$$\limsup_{n\to\infty} \int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)](\nabla u_n - \nabla u)\,dx \le 0. \tag{3.11}$$

The left-hand side of (3.11) can be written as

$$\begin{aligned}(3.12)\qquad &\int_\Omega [a(x,u_n,\nabla u_n)-a(x,u,\nabla u)](\nabla u_n-\nabla u)\,dx\\ &= \int_\Omega a(x,u_n,\nabla u_n)(\nabla u_n-\nabla u)\,dx-\int_\Omega a(x,u_n,\nabla u)(\nabla u_n-\nabla u)\,dx\\ &= A_n-B_n.\end{aligned}$$

The term B_n goes to zero as $n\to\infty$. Indeed, by (H_2), (3.4), (3.8), and Lebesgue's dominated convergence theorem, we have

$$(3.13)\qquad a(x,u_n,\nabla u)\to a(x,u,\nabla u)\quad \text{strongly in } (L^{p'(\cdot)}(\Omega))^N;$$

this convergence together with (3.7) implies $\lim_{n\to\infty} B_n=0$.

Next, we claim that $\limsup_{n\to\infty} A_n\le 0$. Indeed, as $u_n\in K(\psi)$, and $u_n\to u$ almost everywhere, we deduce that $u\ge\psi$ a.e. in Ω and $u\in L^\infty(\Omega)$. Thus we can take u as a test function in (3.3). By (3.5), we obtain

$$\begin{aligned}\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n-u)\,dx+\int_\Omega \phi(u_n)\nabla(u_n-u)\,dx+\int_\Omega g(x,u_n)(u_n-u)\,dx\\ \le\int_\Omega f(u_n-u)\,dx+\int_\Omega F\nabla(u_n-u)\,dx.\end{aligned}$$

Letting $n\to\infty$ we get

$$(3.14)\qquad \limsup_{n\to\infty} A_n\le 0.$$

By (3.13), and (3.14) and using Lemma 2.5 we conclude that

$$(3.15)\qquad \nabla u_n\to\nabla u\quad \text{a.e. in } \Omega.$$

Step 3: *Passage to the limit.* Let us take $v\in K(\psi)\cap L^\infty(\Omega)$ as a test function in (3.3):

$$\begin{aligned}(3.16)\qquad &\int_\Omega a(x,u_n,\nabla u_n)\nabla(u_n-v)\,dx+\int_\Omega\phi(u_n)\nabla(u_n-v)\,dx\\ &\qquad+\int_\Omega g(x,u_n)(u_n-v)\,dx\\ &\qquad\le\int_\Omega f(u_n-v)\,dx+\int_\Omega F\nabla(u_n-v)\,dx.\end{aligned}$$

By (H_3) and the assumptions of the theorem it is easy to get

$$(3.17)\qquad \int_\Omega f(u_n-v)\,dx\to\int_\Omega f(u-v)\,dx,$$

$$(3.18)\qquad \int_\Omega F\nabla(u_n-v)\,dx\to\int_\Omega F\nabla(u-v)\,dx.$$

Also by (H_4), (H_5) and (3.4),

$$\int_{\Omega} \phi(u_n)\nabla(u_n - v)\,dx \to \int_{\Omega} \phi(u)\nabla(u - v)\,dx \tag{3.19}$$

and

$$\int_{\Omega} g(x,u_n)(u_n - v)\,dx \to \int_{\Omega} g(x,u)(u - v)\,dx. \tag{3.20}$$

For the first term in (3.16), by (3.15) and (H_2) we obtain

$$a(x,u_n,\nabla u_n) \to a(x,u,\nabla u) \quad \text{strongly in } (L^{p'(\cdot)}(\Omega))^N. \tag{3.21}$$

According to (3.7), we have

$$\int_{\Omega} a(x,u_n,\nabla u_n)\nabla(u_n - v)\,dx \to \int_{\Omega} a(x,u,\nabla u)\nabla(u - v)\,dx. \tag{3.22}$$

Finally, by (3.17), (3.19), (3.22) we conclude that u is a weak solution to problem (1.1). ∎

4. Uniqueness of solution for (1.1). In this section, we discuss the uniqueness of weak solutions to problem (1.1). We make the following assumptions:

(H_6) ϕ is a locally Lipschitz continuous function.
(H_7) For every $k > 0$, there exists $\bar{c}_k \in L^{p'(\cdot)}(\Omega)$ and a constant $\beta_k > 0$ such that

$$|a(x,s_1,\xi) - a(x,s_2,\xi)| \le |s_1 - s_2|[\beta_k|\xi|^{p(x)-1} + \bar{c}_k(x)] \quad \text{for a.e. } x \in \Omega \tag{4.1}$$

for all $\xi \in \mathbb{R}^N$ and s_1, s_2 with $|s_1|, |s_2| \le k$.
(H_8) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is strictly increasing with respect to the second variable.

THEOREM 4.1. *Suppose that* $1 < p(\cdot) < N$. *Assume that* (H_1)–(H_8) *hold. Then problem* (1.1) *admits a unique weak solution* $u \in K(\psi) \cap L^\infty(\Omega)$.

Proof. The existence is proved in Theorem 3.1. Now, to prove uniqueness, assume that $u_1, u_2 \in K(\psi) \cap L^\infty(\Omega)$ are two weak solutions to (1.1), so

$$\begin{aligned} &\int_{\Omega} a(x,u_1,\nabla u_1)\nabla(u_1 - v)\,dx + \int_{\Omega} \phi(u_1)\nabla(u_1 - v)\,dx \\ &\quad + \int_{\Omega} g(x,u_1)(u_1 - v)\,dx \\ &\le \int_{\Omega} f(u_1 - v)\,dx + \int_{\Omega} F\nabla(u_1 - v)\,dx, \quad \forall v \in K(\psi) \cap L^\infty(\Omega), \end{aligned} \tag{4.2}$$

and

$$\int_\Omega a(x,u_2,\nabla u_2)\nabla(u_2-v)\,dx+\int_\Omega \phi(u_2)\nabla(u_2-v)\,dx + \int_\Omega g(x,u_2)(u_2-v)\,dx \le \int_\Omega f(u_2-v)\,dx+\int_\Omega F\nabla(u_2-v)\,dx,\quad \forall v\in K(\psi)\cap L^\infty(\Omega). \tag{4.3}$$

Denote

$$v_{1\varepsilon}=u_1-T_\varepsilon((u_1-u_2)^+),\qquad v_{2\varepsilon}=u_2+T_\varepsilon((u_1-u_2)^+). \tag{4.4}$$

It is is easy to check that $v_{1\varepsilon}, v_{2\varepsilon}\in K(\psi)\cap L^\infty(\Omega)$. Thus, we can choose $v=v_{1\varepsilon}$ and $v=v_{2\varepsilon}$ as test functions in (4.2) and (4.3) to obtain

$$\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[a(x,u_1,\nabla u_1)-a(x,u_2,\nabla u_2)]\nabla(u_1-u_2)\,dx + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[\phi(u_1)-\phi(u_2)]\nabla(u_1-u_2)\,dx + \frac{1}{\varepsilon}\int_{\Omega_\varepsilon}T_\varepsilon((u_1-u_2)^+)(g(x,u_1)-g(x,u_2))\,dx\le 0, \tag{4.5}$$

where $\Omega_\varepsilon=\{x\in\Omega: 0<u_1-u_2<\varepsilon\}$. Denote the three terms on the left-hand side by $J_1(\varepsilon), J_2(\varepsilon), J_3(\varepsilon)$. Then

$$J_1(\varepsilon)=\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[a(x,u_1,\nabla u_1)-a(x,u_1,\nabla u_2)]\nabla(u_1-u_2)\,dx +\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[a(x,u_1,\nabla u_2)-a(x,u_2,\nabla u_2)]\nabla(u_1-u_2)\,dx. \tag{4.6}$$

By Theorem 3.1, $\|u_1\|_{L^\infty(\Omega)}, \|u_2\|_{L^\infty(\Omega)}\le M$. Therefore, using (H_7), we have

$$\left|\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[a(x,u_1,\nabla u_2)-a(x,u_2,\nabla u_2)]\nabla(u_1-u_2)\,dx\right| \le \int_{\Omega_\varepsilon}[\beta_M|\nabla u_2|^{p(x)-1}+c_k(x)]\nabla(u_1-u_2)\,dx.$$

It follows that

$$\lim_{\varepsilon\to\infty}\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[a(x,u_1,\nabla u_2)-a(x,u_2,\nabla u_2)]\nabla(u_1-u_2)\,dx=0. \tag{4.7}$$

Combining (4.6)–(4.7) with (H_2) yields

$$\limsup_{\varepsilon\to\infty} J_1(\varepsilon)\ge 0. \tag{4.8}$$

For the term J_2 we have, in view of (H_7),

$$\bigg|\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}[\phi(u_1)-\phi(u_2)]\nabla(u_1-u_2)\,dx\bigg| \le k_M \int_{\Omega_\varepsilon}|\nabla(u_1-u_2)|\,dx,$$

where k_M is the Lipschitz constant of ϕ on $[-M,M]$, and thus

$$\lim_{\varepsilon\to\infty} J_2(\varepsilon)=0. \tag{4.9}$$

By (H_9), it is easy to see that

$$\begin{aligned}\lim_{\varepsilon\to\infty} J_3(\varepsilon) &= \int_{\{u_1\ge u_2\}}(g(x,u_1)-g(x,u_2))\,dx \\ &= \int_{\{u_1>u_2\}}(g(x,u_1)-g(x,u_2))\,dx.\end{aligned} \tag{4.10}$$

Letting $\varepsilon\to\infty$, it follows from (4.8)–(4.10) that

$$\int_{\{u_1>u_2\}}(g(x,u_1)-g(x,u_2))\,dx \le 0. \tag{4.11}$$

Hence, $|\{u_1>u_2\}|=0$, that is, $u_1\le u_2$ a.e. in Ω and changing the roles of u_1 and u_2, we obtain $u_2\le u_1$ a.e. in Ω, which gives $u_1=u_2$ a.e. in Ω. ∎

References

[1] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. 164 (2002), 213–259.

[2] E. Acerbi, G. Mingione and G. A. Seregin, *Regularity results for parabolic systems related to a class of non-Newtonian fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21 (2004), 5–60.

[3] L. Aharouch and J. Bennouna, *Existence and uniqueness of solutions of unilateral problems in Orlicz spaces*, Nonlinear Anal. 72 (2010), 3553–3565.

[4] B. Aharrouch, M. Boukhrij and J. Bennouna, *Existence of solutions for a class of degenerate elliptic equations in $P(x)$-Sobolev spaces*, Topol. Methods Nonlinear Anal. 51 (2018), 389–411.

[5] S. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (2006), 19–36.

[6] M. B. Benboubker, E. Azroul and A. Barbara, *Quasilinear elliptic problems with nonstandard growth*, Electron. J. Differential Equations 2011, art. 62, 16 pp.

[7] M. Bendahmane and P. Wittbold, *Renormalized solutions for nonlinear elliptic equations with variable exponents and L^1 data*, Nonlinear Anal. 70 (2009), 567–583.

[8] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vázquez, *An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 22 (1995), 241–273.

[9] L. Boccardo, D. Giachetti, J. I. Diaz and F. Murat, *Existence and regularity of renormalized solutions for some elliptic problems involving derivatives of nonlinear terms*, J. Differential Equations 106 (1993), 215–237.

[10] L. Boccardo, F. Murat and J. P. Puel, *Existence of bounded solutions for nonlinear elliptic unilateral problems*, Ann. Mat. Pura Appl. 152 (1988), 183–196.

[11] S. Challal, A. Lyaghfouri and J. F. Rodrigues, *On the A-obstacle problem and Hausdorff measure of its free boundary*, Ann. Mat. Pura Appl. 191 (2012), 113–165.

[12] S. Challal, A. Lyaghfouri, J. F. Rodrigues and R. Teymurazyan, *On the regularity of the free boundary for quasilinear obstacle problems*, Interfaces Free Bound. 16 (2014), 359–394.

[13] Y. Chen, S. Levine and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. 66 (2006), 1383–1406.

[14] X. L. Fan and D. Zhao, *On the generalised Orlicz–Sobolev space $W^{k,p(x)}(\Omega)$*, J. Gansu Educ. College 12 (1998), 1–6.

[15] V. Ferone, M. R. Posteraro and J. M. Rakotoson, *L^∞-estimates for nonlinear elliptic problems with p-growth in the gradient*, J. Inequal. Appl. 3 (1999), 109–125.

[16] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod and Gauthier-Villars, Paris, 1969.

[17] D. Marcon, J. F. Rodrigues and R. Teymurazyan, *Homogenization of obstacle problems in Orlicz–Sobolev spaces*, Portugal. Math. 75 (2018), 267–283.

[18] M. Rakotoson, *Existence of bounded solutions of some degenerate quasilinear elliptic equations*, Comm. Partial Differential Equations 12 (1987), 633–676.

[19] J. F. Rodrigues, M. Sanchón and J. M. Urbano, *The obstacle problem for nonlinear elliptic equations with variable growth and L^1-data*, Monatsh. Math. 154 (2008), 303–322.

[20] J. F. Rodrigues and R. Teymurazyan, *On the two obstacles problem in Orlicz–Sobolev spaces and applications*, Complex Var. Elliptic Equations 56 (2011), 769–787.

[21] M. Růžička, *Electrorheological Fluids: Modelling and Mathematical Theory*, Lecture Notes in Math. 1748, Springer, Berlin, 2000.

[22] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier (Grenoble) 15 (1965), 189–258.

[23] C. Yazough, E. Azroul and H. Redwane, *Existence of solutions for some nonlinear elliptic unilateral problems with measure data*, Electron. J. Qual. Theory Differential Equations 2013, art. 43, 21 pp.

[24] D. Zhao, W. J. Qiang, X. L. Fan, *On generalized Orlicz spaces $L^{p(x)}(\Omega)$*, J. Gansu Sci. 9 (1997), 1–7.

[25] V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), 675–710 (in Russian).

[26] W. Zou and F. Li, *Existence of solutions for degenerate quasilinear elliptic equations*, Nonlinear Anal. 73 (2010), 3069–3082.

[27] W. Zou, W. Wang and Y. Bi, *On a class of nonlinear obstacle problems with non-standard growth*, Math. Methods Appl. Sci. 38 (2015), 2911–2921.

Benali Aharrouch, Jaouad Bennouna
Sidi Mohamed Ben Abdellah University
Faculty of Sciences Dhar El Mahraz
Laboratory LAMA, Department of Mathematics
P.O. Box 1796 Atlas
Fez, Morocco
E-mail: bnaliaharrouch@gmail.com
jbennouna@hotmail.com