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FUNCTIONAL ANALYSIS

A characterization of simplicial spaces by an extension property

by

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Summary. Let $\mathcal{H}$ be a function space on a compact Hausdorff space K. We provide a characterization of the simpliciality of $\mathcal{H}$ via an extension property.

1. Introduction. Let X be a compact convex set in a real locally convex space. We write $\mathfrak{A}^c(X)$ for the space of all real continuous affine functions on X endowed with the supremum norm. The symbol $\mathfrak{A}^c(X)^+$ stands for the set of nonnegative elements of $\mathfrak{A}^c(X)$. In [4], a compact subset of X is called *hyper-extremal* if it is a union of compact faces (the class of hyper-extremal subsets of X in fact coincides with compact extremal subsets of X; see [4, p. 396] or [5, Proposition 2.69]). The set X is said to have the *property* (H) if

(1.1) there exists a constant $C \in \mathbb{R}$ such that for each hyper-extremal set $D \subseteq X$ and $a \in \mathfrak{A}^c(\overline{\text{co}}(D))^+$ there exists $b \in \mathfrak{A}^c(X)^+$ such that $b = a$ on D and $\|b\|_K \leq C\|a\|_D$.

It was shown by Batty [4] that X has the property (H) if and only if X is a Choquet simplex. A similar characterization was proven before by Andersen [2], but it was restricted to metrizable compact convex sets. In the present paper we provide a similar characterization for abstractly defined affine functions.

First we collect several definitions and basic facts of Choquet theory on function spaces. For details we refer the reader to [5]. In what follows, let

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K be a compact Hausdorff space and $\mathcal{H}$ be a *function space* on K, that is, a linear subspace of the space of continuous functions on K, endowed with the supremum norm, containing constant functions and separating the points of K.

For $x \in K$ we denote by $\mathcal{M}_x(\mathcal{H})$ the set of all *$\mathcal{H}$-representing measures for x*, that is,

$$\mathcal{M}_x(\mathcal{H}) = \Big\{\mu \in \mathcal{M}^1(K) : f(x) = \int_K f \,\mathrm{d}\mu \text{ for any } f \in \mathcal{H}\Big\}.$$

(Here $\mathcal{M}^1(K)$ stands for the set of all Radon probability measures on K.) The *Choquet boundary* $\mathrm{Ch}_{\mathcal{H}}(K)$ of $\mathcal{H}$ is the set

$$\mathrm{Ch}_{\mathcal{H}}(K) = \big\{x \in K : \mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}\big\},$$

where ε_x stands for the Dirac measure at a point $x \in K$.

A function f on K satisfying

$$f(x) \le \int_K f \,\mathrm{d}\mu, \quad x \in K,\ \mu \in \mathcal{M}_x(\mathcal{H}),$$

is termed *$\mathcal{H}$-convex.* A function f on K is *$\mathcal{H}$-concave* if $-f$ is $\mathcal{H}$-convex. If f is both $\mathcal{H}$-convex and $\mathcal{H}$-concave, then f is called *$\mathcal{H}$-affine.* The family of all continuous $\mathcal{H}$-affine functions is denoted by $\mathcal{A}^c(\mathcal{H})$. Then $\mathcal{A}^c(\mathcal{H})$ is again a function space. Moreover, it is closed and contains $\overline{\mathcal{H}}$.

We recall that given a pair of measures $\mu, \nu \in \mathcal{M}^1(K)$, we say that $\mu \prec_{\mathcal{H}} \nu$ if $\mu(f) \le \nu(f)$ for any $\mathcal{H}$-convex continuous function f on K (see [5, Definition 3.19]). A measure which is $\prec_{\mathcal{H}}$-maximal is called *$\mathcal{H}$-maximal.*

If $K = X$ is a compact convex set in a locally convex space and $\mathcal{H} = \mathfrak{A}^c(X)$, then $\mathcal{H}$ is a function space with $\mathcal{A}^c(\mathcal{H}) = \mathcal{H}$ and $\mathrm{Ch}_{\mathcal{H}}(X) = \operatorname{ext} X$, the set of all extreme points of X (see [5, Theorem 2.40]).

Let A be a Borel subset of K. Then A is called *measure convex* if $x \in A$ whenever $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ with $\mu(K \setminus A) = 0$. If A is moreover closed, then it is called *$\mathcal{H}$-convex.* For a subset F of K, let

$$\overline{\mathrm{co}}^{\mathcal{H}}(F) = \bigcap \{C \subseteq K : C \supseteq F,\ C \text{ is } \mathcal{H}\text{-convex}\}.$$

The subset F is $\mathcal{H}$-convex if and only if $F = \overline{\mathrm{co}}^{\mathcal{H}}(F)$.

A Borel set $A \subseteq K$ is called *measure extremal* if for any $x \in A$ and any measure $\mu \in \mathcal{M}_x(\mathcal{H})$, μ is supported by A. Closed measure extremal sets are called *$\mathcal{H}$-extremal.* Finally, we say that A is a *Choquet set* if it is both measure convex and measure extremal.

The *upper envelope* of a bounded function f on K is defined as

$$f^*(x) = \inf\{s(x) : s \text{ is continuous and } \mathcal{H}\text{-concave},\ s \ge f\}, \qquad x \in K.$$

It is always an upper semicontinuous $\mathcal{H}$-concave function on K, and it coincides with f on $\mathrm{Ch}_{\mathcal{H}}(K)$ for f upper semicontinuous.

The *state space* $\mathbf{S}(\mathcal{H})$ of $\mathcal{H}$ is defined as

$$\mathbf{S}(\mathcal{H}) = \{\varphi \in \mathcal{H}^* : \varphi \geq 0,\ \varphi(1) = 1\}.$$

It is a w^*-compact convex subset of $\mathcal{H}^*$. The function space $\mathcal{H}$ is called *simplicial* if $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ is a Choquet simplex (see [5, Theorem 6.54]). As a canonical example of a simplicial function space serves the space $\mathfrak{A}^c(X)$ on a simplex X (see [5, Theorem 6.54]). A less obvious example is the following. Let $U \subset \mathbb{R}^d$ be an open bounded set. Then the space of all functions continuous on $\overline{U}$ that are harmonic on U is an example of a simplicial function space (see [5, Theorem 13.35]).

The *evaluation mapping* ϕ from K into $\mathbf{S}(\mathcal{H})$ is defined as

$$\phi : x \mapsto \phi_x, \qquad x \in K,$$

where ϕ_x maps a function $h \in \mathcal{H}$ to the real number $h(x)$. We further define a mapping $\Phi : \mathcal{H} \to \mathfrak{A}^c(\mathbf{S}(\mathcal{H}))$ for $h \in \mathcal{H}$ as

$$\Phi(h) : s \mapsto s(h), \qquad s \in \mathbf{S}(\mathcal{H}).$$

We point out several important properties of the mappings ϕ and Φ. For the proofs of these facts see e.g. [5, Proposition 4.26, Lemma 8.10 and Proposition 8.22].

The mapping ϕ is a homeomorphism of K into $\mathbf{S}(\mathcal{H})$, $\mathbf{S}(\mathcal{H}) = \overline{\mathrm{co}}(\phi(K))$ and $\phi(\mathrm{Ch}_{\mathcal{H}}(K)) = \operatorname{ext}\mathbf{S}(\mathcal{H})$. If $H \subseteq \mathbf{S}(\mathcal{H})$ is $\mathfrak{A}^c(\mathbf{S}(\mathcal{H}))$-measure extremal, then $\phi^{-1}(H \cap \phi(K))$ is measure extremal in K. Moreover, for each set $F \subseteq K$ we have

$$\overline{\mathrm{co}}^{\mathcal{H}}(F) = \phi^{-1}\big(\overline{\mathrm{co}}(\phi(F)) \cap \phi(K)\big).$$

The mapping Φ is positive, linear and norm-preserving. It is surjective if and only if $\mathcal{H}$ is closed, and in this case the inverse mapping is realized by

$$\Phi^{-1}(F) = F \circ \phi, \qquad F \in \mathfrak{A}^c(\mathbf{S}(\mathcal{H})).$$

If F is a closed subset of K then the space $\mathcal{H}|_F$ of all restrictions of functions from $\mathcal{H}$ to the set F is again a function space.

2. Characterization of simpliciality. We note that from the definition of $\mathcal{H}$-affine functions it follows that the sets of representing measures are the same with respect to both function spaces $\mathcal{H}$ and $\mathcal{A}^c(\mathcal{H})$, that is, for each $x \in K$ we have $\mathcal{M}_x(\mathcal{H}) = \mathcal{M}_x(\mathcal{A}^c(\mathcal{H}))$. From this it follows that $\mathcal{A}^c(\mathcal{A}^c(\mathcal{H})) = \mathcal{A}^c(\mathcal{H})$, and also that the classes of measure extremal and measure convex sets are the same with respect to both these function spaces.

We say that the function space $\mathcal{H}$ has the *property* (H) if

(2.1) there exists a constant $C \geq 0$ such that for each $\mathcal{H}$-extremal set $D \subseteq K$ and $a \in \mathcal{A}^c(\mathcal{H}|_{\overline{\mathrm{co}}^{\mathcal{H}}(D)})^+$ there exists $b \in \mathcal{A}^c(\mathcal{H})^+$ with $b|_D = a$ and $\|b\|_K \leq C\|a\|_D$.

In the rest of the paper we show that this property (H) characterizes the concept of simpliciality of a function space.

First we show the connection between our definition of property (H) and the property (H) of a compact convex set as defined by Batty.

Proposition 2.1. *If a function space $\mathcal{H}$ has the property (H) in the sense of (2.1) then $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ has the property (H) in the sense of (1.1).*

Proof. Let $X = \mathbf{S}(\mathcal{A}^c(\mathcal{H}))$. Let $D \subseteq X$ be a nonempty hyper-extremal set and $a \in \mathfrak{A}^c(\overline{\mathrm{co}}D)^+$. Then D is $\mathfrak{A}^c(X)$-extremal (see [5, Proposition 2.69]). We consider the closed function space $\mathcal{A}^c(\mathcal{H})$ and the above-defined mappings $\phi : K \to \mathbf{S}(X)$ and $\Phi : \mathcal{A}^c(\mathcal{H}) \to \mathfrak{A}^c(X)$. We denote $F = \phi^{-1}(D \cap \phi(K))$. Then F is $\mathcal{A}^c(\mathcal{H})$-extremal (see [5, Lemma 8.10]), and thus $\mathcal{H}$-extremal in K. Since the classes of $\mathcal{H}$-convex and $\mathcal{A}^c(\mathcal{H})$-convex sets coincide, $\overline{\mathrm{co}}^{\mathcal{H}}(F) = \overline{\mathrm{co}}^{\mathcal{A}^c(\mathcal{H})}(F)$. Let

$$\tilde{a}(x) = a(\phi(x)), \qquad x \in \overline{\mathrm{co}}^{\mathcal{H}}(F).$$

We claim that $\tilde{a} \in \mathcal{A}^c(\mathcal{H}|_{\overline{\mathrm{co}}^{\mathcal{H}}(F)})^+$. Obviously, $\tilde{a} \geq 0$. Let $x \in \overline{\mathrm{co}}^{\mathcal{H}}(F)$ and $\mu \in \mathcal{M}_x(\mathcal{H}|_{\overline{\mathrm{co}}^{\mathcal{H}}(F)})$. Then $\mu \in \mathcal{M}^1(\overline{\mathrm{co}}^{\mathcal{H}}(F))$, and since

$$\overline{\mathrm{co}}^{\mathcal{H}}(F) = \overline{\mathrm{co}}^{\mathcal{A}^c(\mathcal{H})}(F) = \phi^{-1}\big(\overline{\mathrm{co}}(\phi(F)) \cap \phi(K)\big),$$

the image $\phi\mu \in \mathcal{M}^1(X)$ under the mapping ϕ has support in $\overline{\mathrm{co}}(\phi(F)) \subset \overline{\mathrm{co}}(D)$. Further, the measure μ considered as a measure on K $\mathcal{H}$-represents x. Thus $\phi\mu$ $\mathfrak{A}^c(X)$-represents the point $\phi(x)$ (see [5, Proposition 4.26(c)]). Thus

$$\mu(\tilde{a}) = \mu(a \circ \phi) = \phi\mu(a) = a(\phi(x)) = \tilde{a}(x).$$

Hence $\tilde{a}$ is an $\mathcal{A}^c(\mathcal{H}|_{\overline{\mathrm{co}}^{\mathcal{H}}(F)})$-affine function on $\overline{\mathrm{co}}^{\mathcal{H}}(F)$.

By (2.1), there exists a function $\tilde{b} \in \mathcal{A}^c(\mathcal{H})^+$ such that $\|\tilde{b}\|_K \leq C\|\tilde{a}\|_F$ and $\tilde{b} = \tilde{a}$ on F. Let $b = \Phi(\tilde{b}) \in \mathfrak{A}^c(X)^+$. Then $b = a$ on D.

Indeed, let $s \in D$ be given. We find an $\mathfrak{A}^c(X)$-maximal measure $\mu \in \mathcal{M}^1(X)$ which $\mathfrak{A}^c(X)$-represents s (see [5, Theorem 3.65]). Since D is $\mathfrak{A}^c(X)$-extremal, μ is supported by D and, by maximality, by $\phi(K)$ (see [5, Propositions 3.64 and 4.26(d)]). Let $\tilde{\mu} \in \mathcal{M}^1(K)$ satisfy $\phi\tilde{\mu} = \mu$. Then $\tilde{\mu}$ is supported by $\phi^{-1}(D \cap \phi(K)) = F$. Thus

$$\begin{aligned} a(s) = \mu(a) = (\phi\tilde{\mu})(a) &= \int_F a \circ \phi \,\mathrm{d}\tilde{\mu} = \int_F \tilde{a} \,\mathrm{d}\tilde{\mu} = \int_F \tilde{b} \,\mathrm{d}\tilde{\mu} \\ &= \int_{\phi(F)} \Phi(\tilde{b}) \,\mathrm{d}\phi\tilde{\mu} = \mu(b) = b(s). \end{aligned}$$

Obviously we have

$$\|b\|_X = \|\tilde{b}\|_K \leq C\|\tilde{a}\|_F \leq C\|a\|_D.$$

Thus $X = \mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ satisfies (H) in the sense of (1.1). ∎

In [4], the proof that simplices have the property (H) is deduced from the facts that the closed convex hull of a dilated subset of a simplex is a face (see [3, p. 114]), and that affine continuous functions on a face of a simplex may be extended with preservation of norm (the proof is similar to that of [1, Theorem II.5.19]). (We recall that a closed subset D of a compact convex set X is said to be *dilated* if whenever μ is a maximal probability measure on X that $\mathfrak{A}^c(X)$-represents a point $x \in D$ then μ is supported by D. Thus a closed set $D \subset X$ is dilated provided it is measure extremal. On the other hand, the set $\{0, 1/2, 1\}$ is a dilated subset of $[0, 1]$ which is not measure extremal.) The following two lemmas are analogous results in the context of function spaces.

Lemma 2.2. *Let $\mathcal{H}$ be a simplicial function space and F be an $\mathcal{H}$-extremal subset of K. Then $\overline{\text{co}}^{\mathcal{H}}(F)$ is a Choquet set.*

Proof. If $F \subset K$ is a nonempty $\mathcal{H}$-extremal set, the characteristic function χ_F is an upper semicontinuous $\mathcal{H}$-convex function. We show that $\chi_F^* = 1$ on $\overline{\text{co}}^{\mathcal{H}}(F)$.

Indeed, let $x \in \overline{\text{co}}^{\mathcal{H}}(F)$ be given. By [5, Proposition 8.18], there exists a measure $\mu \in \mathcal{M}_x(\mathcal{H})$ supported by F. Then by [5, Lemma 3.21],

$$1 = \mu(F) = \mu(\chi_F) \leq \sup\{\nu(\chi_F) : \nu \in \mathcal{M}_x(\mathcal{H})\} = \chi_F^*(x) \leq 1,$$

which implies $\chi_F^*(x) = 1$.

Now we prove that $\overline{\text{co}}^{\mathcal{H}}(F)$ is $\mathcal{H}$-extremal. To this end, let $x \in \overline{\text{co}}^{\mathcal{H}}(F)$ and $\mu \in \mathcal{M}_x(\mathcal{H})$ be given. Let $\nu \in \mathcal{M}^1(K)$ be an $\mathcal{H}$-maximal measure satisfying $\mu \prec_{\mathcal{H}} \nu$ (see [5, Theorem 3.65]). Then $\nu \in \mathcal{M}_x(\mathcal{H})$. Since $\mathcal{H}$ is simplicial, χ_F^* is $\mathcal{H}$-affine (see [5, Theorem 6.5]). Thus by [5, Theorem 3.68],

$$1 = \chi_F^*(x) = \nu(\chi_F^*) = \nu(\chi_F),$$

hence $\nu(F) = 1$. By [5, Proposition 8.24],

$$\operatorname{spt} \mu \subset \overline{\text{co}}^{\mathcal{H}}(\operatorname{spt} \nu) \subset \overline{\text{co}}^{\mathcal{H}}(F).$$

Thus $\overline{\text{co}}^{\mathcal{H}}(F)$ is an $\mathcal{H}$-extremal set and the proof is complete. ∎

Lemma 2.3. *Let $\mathcal{H}$ be simplicial, and D be a closed Choquet subset of K. Then any $a \in \mathcal{A}^c(\mathcal{H}|_D)^+$ may be extended to a function in $\mathcal{A}^c(\mathcal{H})^+$ with the same norm.*

Proof. We define functions

$$s(x) = \begin{cases} a(x), & x \in D, \\ 0, & x \in K \setminus D, \end{cases} \qquad t(x) = \begin{cases} a(x), & x \in D, \\ \|a\|, & x \in K \setminus D. \end{cases}$$

Then it is easy to verify that s is $\mathcal{H}$-convex and upper semicontinuous, while t is $\mathcal{H}$-concave and lower semicontinuous. We prove the desired properties for s. Concerning the $\mathcal{H}$-convexity, let $x \in K$ and $\mu \in \mathcal{M}_x(\mathcal{H})$. We want to show that $s(x) \le \int_K f \, d\mu$. If $x \in D$, then $\operatorname{spt} \mu \subseteq D$, since D is a Choquet set. But s coincides with a on D, and so the desired inequality is satisfied due to the fact that $a \in \mathcal{A}^c(\mathcal{H})$. If $x \in K \setminus D$, then $s(x) = 0$, and since s is nonnegative on K, we are done.

Now we show that s is upper semicontinuous. Let $x \in K$. If $x \in D$, then for given $\varepsilon > 0$ there exists a neighborhood U of x such that $s = a \le a(x) + \varepsilon = s(x) + \varepsilon$ on $U \cap D$. Since on $U \setminus D$ we have $s = 0 < s(x) + \varepsilon$, we see that $s \le s(x) + \varepsilon$ on U. On the other hand, if $x \in K \setminus D$, then since D is closed, s is constant on some neighborhood of x, and the upper semicontinuity of s is proven.

Now, since $\mathcal{H}$ is simplicial, by the Edwards in-between theorem (see [5, Theorem 6.6]) there exists a function $f \in \mathcal{A}^c(\mathcal{H})$ such that $s \le f \le t$. Then f is clearly a nonnegative extension of a with the same norm. ∎

We obtained the following characterization of simpliciality of a function space.

THEOREM 2.4. *Let $\mathcal{H}$ be a function space on a compact Hausdorff space K. Then $\mathcal{H}$ is simplicial if and only if $\mathcal{H}$ has the property* (H) *in the sense of* (2.1).

Proof. It follows by Lemmas 2.2 and 2.3 that every simplicial space has the property (H).

On the other hand, if $\mathcal{H}$ has the property (H) in the sense of (2.1) then $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ has the property (H) in the sense of (1.1) by Proposition 2.1. Thus by [4, Theorem 4] the state space $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ is a Choquet simplex, so $\mathcal{H}$ is simplicial. ∎

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