The Haar system in Besov-type spaces

by

WEN YUAN (Beijing), WINFRIED SICKEL (Jena) and DACHUN YANG (Beijing)

Abstract. Some Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ can be characterized in terms of the behavior of the Fourier–Haar coefficients. In this article, the authors discuss some necessary restrictions on the parameters $s$, $\tau$, $p$, $q$ and $n$ in order to have such a characterization. To do so, the authors measure the regularity of the characteristic function $\mathcal{X}$ of the unit cube in $\mathbb{R}^n$ via Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. Furthermore, the authors study necessary and sufficient conditions for the operation $\langle f, \mathcal{X} \rangle$ to generate a continuous linear functional on $B^{s,\tau}_{p,q}(\mathbb{R}^n)$.

1. Introduction. Besov-type spaces $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ are generalizations of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$. On the other hand, and more intuitively, they are relatives of bmo and $Q$-spaces, which have been introduced about 30 years ago in complex analysis with applications also in harmonic analysis and partial differential equations. As the most transparent special case, let us consider $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ with $p = q$, $\tau \in [0, 1/p)$ and $s \in (0, 1)$. Then a function $f$ belongs to $B^{s,\tau}_{p,p}(\mathbb{R}^n)$ if

$$
\sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{1}{|P|^\tau} \left\{ \int_{P} |f(x)|^p \, dx \right\}^{1/p} < \infty
$$

and

$$
\sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \int_{P} \int_{P} \frac{|f(x) - f(y)|^p}{|x - y|^{sp+n}} \, dx \, dy \right\}^{1/p} < \infty
$$

(see [77, 4.3.3]). Here and hereafter, $\mathcal{Q}$ denotes the collection of all dyadic cubes in $\mathbb{R}^n$. The main philosophy of these Besov-type spaces consists in characterizing regularity by means of controlling (weighted) differences of $f$ on cubes. This makes it clear that there must exist a connection with Morrey–Campanato spaces.

2010 Mathematics Subject Classification: Primary 42C15; Secondary 46E35.
Key words and phrases: Besov space, Besov-type space, characteristic function, orthonormal Haar system, smooth wavelets.

Received 28 August 2018; revised 5 July 2019.
Published online 27 January 2020.

DOI: 10.4064/sm180828-9-7

© Instytut Matematyczny PAN, 2020
To recall the definition of Besov-type spaces, we let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be such that
\begin{align}
\text{(1.1)} & \quad \text{supp } \mathcal{F}\varphi_0 \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2 \} \quad \text{and} \quad |\mathcal{F}\varphi_0(\xi)| \geq C > 0 \text{ if } |\xi| \leq 5/3 \quad \text{and} \\
\text{(1.2)} & \quad \text{supp } \mathcal{F}\varphi \subset \{ \xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2 \} \quad \text{and} \quad |\mathcal{F}\varphi(\xi)| \geq C > 0 \text{ if } 3/5 \leq |\xi| \leq 5/3,
\end{align}
where $C$ is a positive constant independent of $\varphi_0$ and $\varphi$. Observe that there exist positive constants $A$ and $B$ such that

$$A \leq \mathcal{F}\varphi_0(\xi) + \sum_{j=1}^{\infty} \mathcal{F}\varphi(2^{-j}\xi) \leq B \quad \text{for any } \xi \in \mathbb{R}^n.$$ 

In what follows, for any $j \in \mathbb{N}$, we let $\varphi_j(\cdot) := 2^{jn}\varphi(2^{j}\cdot)$.

For any given $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, denote by $Q_{j,k}$ the dyadic cube $2^{-j}([0,1)^n + k)$ and by $\ell(Q_{j,k})$ its side length. Let $Q := \{ Q_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n \}$, $Q^* := \{ Q \in Q : \ell(Q) \leq 1 \}$ and $j_Q := -\log_2 \ell(Q)$ for any $Q \in Q$.

**Definition 1.1.** Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $p, q \in (0, \infty]$ and $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be as in (1.1) and (1.2), respectively. The Besov-type space $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} := \sup_{P \in Q} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max\{j_P,0\}}^{\infty} 2^{jsq} \left[ \int_P |\varphi_j * f(x)|^p \, dx \right]^{q/p} \right\}^{1/q} < \infty$$

with the usual modifications made in case $p = \infty$ and/or $q = \infty$.

**Remark 1.2.** (i) It is known that $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ is a quasi-Banach space (see [77] Lemma 2.1]).

(ii) It is easy to see that $B_{p,q}^{s,0}(\mathbb{R}^n)$ coincides with the classical Besov space $B_{p,q}^{s}(\mathbb{R}^n)$.

(iii) We have monotonicity with respect to $s$ and $q$: $B_{p,q_0}^{s_0,\tau}(\mathbb{R}^n) \hookrightarrow B_{p,q_1}^{s_1,\tau}(\mathbb{R}^n)$ if $s_0 > s_1$ and $q_0, q_1 \in (0, \infty]$, as well as $B_{p,q_0}^{s,\tau}(\mathbb{R}^n) \hookrightarrow B_{p,q_1}^{s,\tau}(\mathbb{R}^n)$ if $q_0 \leq q_1$.

(iv) Let $s \in \mathbb{R}$ and $p \in (0, \infty]$. Then

$$B_{p,q}^{s,\tau}(\mathbb{R}^n) = B_{p,q}^{s+n(\tau-1/p)}(\mathbb{R}^n)$$

if either $q \in (0, \infty)$ and $\tau \in (1/p, \infty)$, or $q = \infty$ and $\tau \in [1/p, \infty)$ (see [75]). In case $s + n(\tau - 1/p) > 0$, $B_{p,q}^{s+n(\tau-1/p)}(\mathbb{R}^n)$ is a Hölder–Zygmund space with a transparent description in terms of differences (see, for instance, [56] Section 2.5.7)].
(v) Since $B_{s,\tau}^{s,\tau}(\mathbb{R}^n)$ with $\tau > 1/p$ is a classical Besov space, we will mainly consider the case of $\tau \in [0, 1/p]$ in this article.

As a generalization of the classical Besov spaces, the inhomogeneous Besov-type spaces $B_{s,\tau}^{s,\tau}(\mathbb{R}^n)$, restricted to the Banach space case, were first introduced by El Baraka [6, 7, 8]. Extension to quasi-Banach spaces was given in [72, 73]. Indeed, the homogeneous version of $B_{s,\tau}^{s,\tau}(\mathbb{R}^n)$ for full parameters was introduced in [72, 73] to cover both the Besov spaces and the (real-variable) $Q$ spaces as special cases. Recall that $Q$ spaces come originally from complex analysis (see [1, 10, 68, 69]) and their real-variable version has found a lot of applications in harmonic analysis (see, for instance, [9, 4, 5, 71, 26]) and partial differential equations (see, for instance, [70, 32, 31, 33, 30, 34]).

A systematic treatment of the inhomogeneous Besov-type spaces $B_{s,\tau}^{s,\tau}(\mathbb{R}^n)$ was later given in [77]. We refer the reader also to [45, 74, 75, 76, 51, 52] for further results on these spaces. In recent years, the Besov-type spaces and some of their special cases have also found interesting applications in some partial differential equations such as (fractional) Navier–Stokes equations (see, for instance, [70, 30, 34, 62, 64, 65, 27, 28, 29]).

Of particular importance for us is the following embedding into $C_{ub}(\mathbb{R}^n)$, the class of all complex-valued, uniformly continuous and bounded functions on $\mathbb{R}^n$. For its proof, we refer the reader to [77, Proposition 2.6(i)] and [51].

**Proposition 1.3.** Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, and $p, q \in (0, \infty]$.

1. If $s + n(\tau - 1/p) > 0$, then $B_{p,q}^{s,\tau}(\mathbb{R}^n) \hookrightarrow C_{ub}(\mathbb{R}^n)$.
2. Let $p \in (0, \infty)$, $q \in (0, \infty]$, $\tau \in (0, 1/p)$ and $s + n\tau - n/p = 0$. Then $B_{p,q}^{s,\tau}(\mathbb{R}^n) \not\subset C_{ub}(\mathbb{R}^n)$.
3. Let $p \in (0, \infty)$ and $q \in (0, \infty]$. Then $B_{p,q}^{0,1/p}(\mathbb{R}^n) \not\subset C_{ub}(\mathbb{R}^n)$.

Many classical function spaces can be described via the Haar system. Let us mention here at least the works of Haar [19], Schauder [46], Marcinkiewicz [37] and Ciesielski [2], related to $L^p$, of Ropela [43], Triebel [54, 55], Oswald [40, 41] and Kahane and Lemarie [22], related to (isotropic) Besov spaces, of Wojtaszczyk [66], related to Hardy spaces, of Kamont [23, 24, 25], treating anisotropic Besov spaces, and of Seeger and Ullrich [47, 48] and Garrigós et al. [12, 13], investigating Bessel potential and Triebel–Lizorkin spaces. Good sources are also the monographs by Wojtaszczyk [67, 8.3] and Triebel [61, Chapter 2].

Nowadays it is known that the Haar system is an unconditional Schauder basis in Besov spaces $B_{p,q}^s(\mathbb{R}^n)$ if $p, q \in (0, \infty)$ and

$$\max\{n(1/p-1), 1/p-1\} < s < \min\{1, 1/p\}$$

(see, for instance, [61, Theorem 2.21]). Already in [55] Triebel found that if $s < \max\{n(1/p-1), 1/p-1\}$ or $s > \min\{1, 1/p\}$,
then the Haar system is not an unconditional Schauder basis. For positive and negative results in borderline cases we refer the reader to Oswald [40, 41, 42] and Garrigós, Seeger and Ullrich [14].

The pairs \((p, q)\) with \(\max\{p, q\} = \infty\) have to be excluded because the associated Besov spaces are no longer separable. However, also for those pairs and the associated spaces there exists a characterization in terms of the Fourier–Haar coefficients (see [61, Theorem 2.21]). This is of particular interest because Besov-type spaces with \(\tau > 0\) are always nonseparable (for all \(s\), all \(p\) and all \(q\)). Finally, we mention that Triebel [63, 64] has established a characterization of some classes \(L^r B^s_{p,q}(\mathbb{R}^n)\) and \(L^r B^s_{p,q}(\mathbb{R}^n)\) (generalizations of Besov spaces also related to Morrey–Campanato spaces, see [62, 64]) in terms of the Haar system, and gave some sufficient conditions for this characterization. Recall that \(L^r B^s_{p,q}(\mathbb{R}^n) = B^{s,\tau}_{p,q}(\mathbb{R}^n)\) when \(\tau = \frac{1}{p} + \frac{r}{n}\) (see [64, Theorem 3.38]), which implies that \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\) can be characterized by the Haar system under certain conditions on the parameters \(p, q, s\) and \(\tau\). For the relations between \(L^r B^s_{p,q}(\mathbb{R}^n)\) and \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\) we refer the reader to [78] and [64, 2.7].

The main purpose of this article is to establish some necessary restrictions on the parameters \(s, \tau, p, q\) and \(n\) in order to have the characterization of Besov-type spaces via the Fourier–Haar coefficients. Two obvious restrictions come from the following properties:

- the regularity of the characteristic function \(\mathcal{X}\) of the unit cube in Besov-type spaces \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\);
- the operation \(\langle f, \mathcal{X} \rangle\) generates a continuous linear functional on \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\).

We will give answers in terms of restrictions on the parameters below for which the above two properties hold.

In a continuation [80] of this article, we will discuss sufficient conditions for characterizing \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\) in terms of Fourier–Haar coefficients including some applications to pointwise multipliers.

The structure of this article is as follows. Section 2 is devoted to a description of our main results (Theorems 2.1, 2.3, 2.5 and 2.8) and some comments. We give a necessary and sufficient condition on the parameters \(s, p, q\) and \(\tau\) in Theorem 2.1 so that the characteristic function \(\mathcal{X}\) belongs to \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\), while in Theorems 2.3 and 2.5 we also give an almost sharp condition on the parameters for the operation \(\langle f, \mathcal{X} \rangle\) to generate a continuous linear functional on \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\).

In Section 3 we recall some basic notation and properties of Besov-type spaces, as well as some tools used to prove the main results. Our main tool is the wavelet characterization of \(B^{s,\tau}_{p,q}(\mathbb{R}^n)\) in terms of sufficiently smooth Daubechies wavelets. However, also interpolation (±-method of Gustavsson and Peetre) and characterizations in terms of differences will be used.
Section 4, we give the proof of Theorem 2.1, while Section 5 is devoted to the proofs of Theorems 2.3, 2.5 and Corollary 2.7 as well as Theorem 2.8.

Finally, we make some convention on the notation used in this article. As usual, \( \mathbb{N} \) denotes the natural numbers, \( \mathbb{N}_0 \) the natural numbers including 0, \( \mathbb{Z} \) the integers and \( \mathbb{R} \) the real numbers. We also use \( \mathbb{C} \) to denote the complex numbers and \( \mathbb{R}^n \) the \( n \)-dimensional Euclidean space. All functions are assumed to be complex-valued, that is, we consider functions \( f: \mathbb{R}^n \to \mathbb{C} \).

Let \( S(\mathbb{R}^n) \) be the collection of all Schwartz functions on \( \mathbb{R}^n \) equipped with the well-known topology determined by a countable family of seminorms and denote by \( S'(\mathbb{R}^n) \) its topological dual, the space of all bounded linear functionals on \( S(\mathbb{R}^n) \) equipped with the weak-* topology. The symbol \( \mathcal{F} \) refers to the Fourier transform, \( \mathcal{F}^{-1} \) to its inverse transform, both defined on \( S'(\mathbb{R}^n) \).

Recall that, for any \( \varphi \in S(\mathbb{R}^n) \) and \( \xi \in \mathbb{R}^n \),

\[
\mathcal{F}\varphi(\xi) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1}\varphi(\xi) := \mathcal{F}\varphi(-\xi),
\]

where, for any \( x := (x_1, \ldots, x_n) \) and \( \xi := (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \), \( x\xi := \sum_{i=1}^n x_i \xi_i \) and \( i := \sqrt{-1} \).

All function spaces which we consider in this article are subspaces of \( S'(\mathbb{R}^n) \), so they are spaces of equivalence classes with respect to almost everywhere equality. However, if such an equivalence class contains a continuous representative, then usually we work with this representative and call also the equivalence class a continuous function. We denote by \( C^\infty_c(\mathbb{R}^n) \) the set of all infinitely differentiable functions on \( \mathbb{R}^n \) with compact support.

The symbol \( C \) denotes a positive constant which depends only on the fixed parameters \( n, s, \tau, p, q \) and probably on auxiliary functions, unless otherwise stated; its value may vary from line to line. Sometimes we use “\( \lesssim \)” instead of “\( \leq \)”. The meaning of \( A \lesssim B \) is that there exists a constant \( C \in (0, \infty) \) such that \( A \leq CB \). The symbol \( A \asymp B \) will be used as an abbreviation of \( A \lesssim B \lesssim A \). Given two quasi-Banach spaces \( X \) and \( Y \), the operator norm of a linear operator \( T: X \to Y \) is denoted by \( \|T\|_{X \to Y} \). We shall often use the abbreviation

\[
(1.3) \quad \sigma_p := n \max\{0, 1/p - 1\}, \quad \forall p \in (0, \infty].
\]

For any \( a \in \mathbb{R} \), \( \lfloor a \rfloor \) denotes the largest integer not greater than \( a \).

2. Main results. First, we recall the definition of the Haar system. Let \( \tilde{X} \) denote the characteristic function of the interval \( [0, 1) \). The generator of the Haar system in dimension 1, denoted by \( \tilde{h} \), is given by

\[
\tilde{h}(t) := \begin{cases} 
1 & \text{when } t \in [0, 1/2), \\
-1 & \text{when } t \in [1/2, 1), \\
0 & \text{otherwise.}
\end{cases}
\]
The functions we are interested in are just tensor products of \( \tilde{X} \) and \( \tilde{h} \). For any given \( \varepsilon := (\varepsilon_1, \ldots, \varepsilon) \) with \( \varepsilon_i \in \{0, 1\} \), define

\[
(2.1) \quad h_\varepsilon(x) := \left[ \prod_{i: \varepsilon_i = 0} \tilde{X}(x_i) \right] \left[ \prod_{i: \varepsilon_i = 1} \tilde{h}(x_i) \right], \quad \forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n.
\]

This results in \( 2^n \) different functions. In case \( \varepsilon = (0, \ldots, 0) \) we always write \( X \) instead of \( h_{(0, \ldots, 0)} \). The other \( 2^n - 1 \) functions will be enumerated in an appropriate way and denoted by \( \{h_1, \ldots, h_{2^n - 1}\} \). These functions are the generators of the inhomogeneous Haar system in \( \mathbb{R}^n \). In what follows, for any \( i \in \{1, \ldots, 2^n - 1\} \), we let

\[
(2.2) \quad X_{j,m} := 2^{jn/2}X(2^j \cdot - m), \quad h_{i,j,m} := 2^{jn/2}h_i(2^j \cdot - m), \quad \forall j \in \mathbb{N}_0, \forall m \in \mathbb{Z}^n.
\]

Then the set

\[
H := \{X_{0,m}, h_{i,j,m} : i \in \{1, \ldots, 2^n - 1\}, j \in \mathbb{N}_0, m \in \mathbb{Z}^n\}
\]

forms the well-known orthonormal Haar wavelet system in \( \mathbb{R}^n \) (we shall call it just the Haar system).

For a function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) the Haar wavelet expansion is given by

\[
f = \sum_{m \in \mathbb{Z}^n} \langle f, X_{0,m} \rangle X_{0,m} + \sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \langle f, h_{i,j,m} \rangle h_{i,j,m}.
\]

When trying to characterize a space \( B_{p,q}^{s,\tau}(\mathbb{R}^n) \) by the associated Haar wavelet expansions as in [61, Theorem 2.21] there are two obvious necessary conditions:

- The Haar wavelet coefficients have to be well defined for any element \( f \in B_{p,q}^{s,\tau}(\mathbb{R}^n) \), i.e., the mappings \( f \mapsto \langle f, X_{0,m} \rangle \) and \( f \mapsto \langle f, h_{i,j,m} \rangle \) extend to continuous linear functionals on \( B_{p,q}^{s,\tau}(\mathbb{R}^n) \) for any \( m, i \) and \( j \).
- The partial sums

\[
S_N f := \sum_{m \in \mathbb{Z}^n} \langle f, X_{0,m} \rangle X_{0,m} + \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} \sum_{m \in \mathbb{Z}^n} \langle f, h_{i,j,m} \rangle h_{i,j,m}
\]

are uniformly bounded in \( B_{p,q}^{s,\tau}(\mathbb{R}^n) \), i.e.,

\[
\sup_{N \in \mathbb{N}_0} \sup_{\|f\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq 1} \|S_N f\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty.
\]

Of course, this forces \( H \subset B_{p,q}^{s,\tau}(\mathbb{R}^n) \).

In the framework of the classical Besov spaces, it is well known that \( \mathcal{X} \in B_{p,q}^{s}(\mathbb{R}^n) \) if and only if

\[
(2.3) \quad \text{either } s = 1/p \text{ and } q = \infty \text{ or } s < 1/p \text{ and } q \in (0, \infty]
\]
The Haar system in Besov-type spaces

(see [44] Lemma 2.3.1/3)). This means that, for fixed $p \in (0, \infty]$, the smallest Besov space which the function $X$ belongs to is $B^{1/p}_{p,\infty}(\mathbb{R}^n)$. Now we turn to the smoothness of $X$ and $h_{i,j,m}$ with respect to the scale $B^{s,\tau}_{p,q}(\mathbb{R}^n)$.

**Theorem 2.1.** Let $s \in \mathbb{R}$ and $p, q \in (0, \infty]$.

(i) Let $\tau \in (1/p, \infty)$. Then $X \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $s \leq n(1/p - \tau)$.

(ii) Let $\tau \in [0, 1/p]$. Then $X \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if either

\[ s = 1/p, \quad q = \infty \quad {\text{and}} \quad s \leq n(1/p - \tau), \]

or

\[ s < 1/p, \quad q \in (0, \infty] \quad {\text{and}} \quad s \leq n(1/p - \tau). \]

(iii) All elements of $H$ have the same smoothness properties with respect to Besov type spaces, i.e., $h_{i,j,m} \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $X_{0,m} \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $X \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$. Here $i, j, m$ are arbitrary (but as in $H$).

**Remark 2.2.**

(i) The most interesting thing of Theorem 2.1 is the influence of the Morrey parameter $\tau$. Locally, the smoothness of elements of $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ grows with $\tau$. If $\tau$ is large, discontinuous functions cannot belong to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. What concerns the smoothness of $X$, the Morrey parameter comes into play for $\tau \geq n^{-1}/np$.

(ii) Measuring the regularity of characteristic functions $X_{\Omega}$ of sets $\Omega \subset \mathbb{R}^n$ in Besov spaces has attracted some attention in recent decades (see, for instance, [11, 15, 16, 44, 49, 50, 57, 58, 59]). These investigations have found some applications in various areas such as pointwise multipliers for Besov and Triebel–Lizorkin spaces (see [15, 16, 44, 59]) and the Calderón inverse problem (see [11]). For a general set $\Omega$, the smoothness of $X_{\Omega}$ depends on the properties of the boundary. It turns out that the interrelations of smoothness and properties of the boundary is surprisingly complicated. We shall return to this problem in our forthcoming article [80].

Next we consider the mapping $f \mapsto \langle f, X \rangle$ and discuss under which restrictions it extends to a continuous linear functional on $B^{s,\tau}_{p,q}(\mathbb{R}^n)$. It seems appropriate to distinguish the cases $p \in [1, \infty]$ and $p \in (0, 1)$.

**Theorem 2.3.** Let $s \in \mathbb{R}$, $p \in [1, \infty]$, $q \in (0, \infty]$ and $\tau \in [0, \infty)$. The mapping $f \mapsto \langle f, X \rangle$ extends from $B^{s,\tau}_{p,q}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ to a continuous linear functional on $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if either

$$s = 1/p - 1, \quad \tau \in \left[0, \frac{n-1}{np}\right] \quad {\text{and}} \quad q \in (0, 1], \quad \text{or} \quad s > 1/p - 1, \quad \tau \in \left[0, \frac{n-1}{np}\right] \quad {\text{and}} \quad q \in (0, \infty].$$
or
\[ s > n/p - n\tau - 1, \quad \tau \in \left( \frac{n-1}{np}, \infty \right) \quad \text{and} \quad q \in (0, \infty). \]

Remark 2.4. (i) For classical Besov spaces \( B^s_{p,q}(\mathbb{R}^n) \) there is a convenient duality argument to deal with the existence of \( \langle f, \mathcal{X} \rangle \) (see [55]). It is based on the relation
\[ (B^s_{p,q}(\mathbb{R}^n))' = B^{-s}_{p',q'}(\mathbb{R}^n), \quad 1 \leq p, q < \infty, s \in \mathbb{R}. \]
Let us mention that this formula does not extend to the spaces \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \), \( \tau > 0 \).

(ii) Let again \( \tau = 0 \). As mentioned before, Triebel [55] had shown that \( f \mapsto \langle f, X \rangle \) will not extend to a continuous linear functional on \( B^{s,0}_{p,q}(\mathbb{R}^n) \) if \( p \in [1, \infty], q \in (0, \infty] \) and \( s < 1/p - n\tau - 1, \infty \). Kahane and Lemarié-Rieusset [22, Part II, Chapt. 6, Remark 2 on p. 349] have supplemented this by dealing with the special limiting case \( s = 1/2 \) and \( p = q = 2 \).

The behavior for \( p < 1 \) is surprisingly different. However, as in Theorems 2.1 and 2.3, the value \( \tau = \frac{n-1}{np} \) still plays a particular role.

Theorem 2.5. Let \( s \in \mathbb{R} \), \( p \in (0, 1) \) and \( q \in (0, \infty] \).

(i) Let \( \tau \in (\frac{n-1}{np}, \infty) \). Then \( f \mapsto \langle f, \mathcal{X} \rangle \) extends from \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) to a continuous linear functional on \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \) if and only if \( s \in (n/p - n\tau - 1, \infty) \).

(ii) Let \( \tau \in [0, \frac{n-1}{np}] \). If either
\[ s = (1-\tau p)n(1/p - 1) \quad \text{and} \quad q \in (0, p) \quad (q \in (0, 1] \text{ when } \tau = 0), \]
or
\[ s > (1-\tau p)n(1/p - 1) \quad \text{and} \quad q \in (0, \infty], \]
then the mapping \( f \mapsto \langle f, \mathcal{X} \rangle \) extends from \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) to a continuous linear functional on \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \). If either
\[ s = (1-\tau p)n(1/p - 1) \quad \text{and} \quad q \in (1, \infty], \]
or
\[ s < (1-\tau p)n(1/p - 1), \]
then the mapping \( f \mapsto \langle f, \mathcal{X} \rangle \) does not extend from \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \) to a continuous linear functional on \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \).

Remark 2.6. Summarizing, the only case which has been left open by Theorems 2.3 and 2.5 is given by
\[ p \in (0, 1), \quad s = (1-\tau p)n \left( \frac{1}{p} - 1 \right), \quad \tau \in \left( 0, \frac{n-1}{np} \right] \quad \text{and} \quad q \in (p, 1]. \]
Clearly, all regions, showing up in the restrictions, are convex, and the dependence on the parameters is continuous, there exist no jumps.
For a moment we turn to the classical situation of $\tau = 0$. Then $B_{p,q}^{s,0}(\mathbb{R}^n) = B_{p,q}^s(\mathbb{R}^n)$ and we obtain, essentially as a corollary of Theorems 2.3 and 2.5, the following final result.

**Corollary 2.7.** Let $s \in \mathbb{R}$ and $q \in (0, \infty]$.

(i) Let $p \in [1, \infty]$. Then the mapping $f \mapsto \langle f, \mathcal{X} \rangle$ extends from $B_{p,q}^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ to a continuous linear functional on $B_{p,q}^s(\mathbb{R}^n)$ if and only if either

$$s = 1/p - 1 \quad \text{and} \quad q \in (0, 1],$$

or

$$s > 1/p - 1 \quad \text{and} \quad q \in (0, \infty].$$

(ii) Let $p \in (0, 1)$. Then the mapping $f \mapsto \langle f, \mathcal{X} \rangle$ extends from $B_{p,q}^s(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ to a continuous linear functional on $B_{p,q}^s(\mathbb{R}^n)$ if and only if either

$$s = n(1/p - 1) \quad \text{and} \quad q \in (0, 1],$$

or

$$s > n(1/p - 1) \quad \text{and} \quad q \in (0, \infty].$$

Finally we turn to the mappings $f \mapsto \langle f, \mathcal{X}_{0,m} \rangle$ and $f \mapsto \langle f, h_{i,j,m} \rangle$.

**Theorem 2.8.** Theorems 2.3, 2.5 and Corollary 2.7 remain true on replacing $f \mapsto \langle f, \mathcal{X} \rangle$ by either $f \mapsto \langle f, \mathcal{X}_{0,m} \rangle$, $m \in \mathbb{Z}^n$, or $f \mapsto \langle f, h_{i,j,m} \rangle$, $i \in \{1, \ldots, 2^n - 1\}$, $j \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$.

**Remark 2.9.** (a) Oswald [40, 41, 42] discussed the properties of the Haar system in limiting cases with $p \in (0, 1)$. He was working on $[0, 1]^d$ instead of $\mathbb{R}^n$. In [42] he proved the following: If $p \in \left(\frac{d}{d+1}, 1\right)$, $s = d(1/p - 1)$ and $q \in (p, \infty)$, then:

(i) If $q \in (1, \infty)$, then the coefficient functionals of the Haar expansion cannot be extended to bounded linear functionals on $B_{p,q}^s([0, 1]^d)$.

(ii) If $q \in (p, 1]$, then the partial sum operators of the Haar expansion are not uniformly bounded on $B_{p,q}^s([0, 1]^d)$.

Clearly, (i) is the local counterpart of Corollary 2.7(ii). As mentioned in the Acknowledgement in [42], the counterexamples used to prove (ii) were communicated to Oswald by Ullrich and they have also been published in [14].

In addition, Oswald was able to show that $H$ restricted to $[0, 1]^d$ is a Schauder basis for $B_{p,q}^s([0, 1]^d)$ if $p \in \left(\frac{d}{d+1}, 1\right)$, $s = d(1/p - 1)$ and $q \in (0, p]$ (this result was also independently obtained by Garrigós, Seeger and Ullrich [14]).

(b) Recently, Garrigós, Seeger and Ullrich [14] settled all (!) the borderline cases of the Schauder basis properties for the Haar system in $B_{p,q}^s(\mathbb{R}^n)$. 


3. Besov-type spaces. In this section, we recall the characterizations of $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ in terms of sufficiently smooth Daubechies wavelets and differences, as well as their interpolation property, which will be used in our proofs of Theorems 2.1, 2.3 and 2.5 below.

3.1. Characterization by wavelets. Wavelet bases in Besov and Triebel–Lizorkin spaces are a well-developed concept (see, for instance, Meyer [38], Wojtaszczyk [67] and Triebel [59, 60]). Let $\tilde{\phi}$ be an orthonormal scaling function on $\mathbb{R}$ with compact support and of sufficiently high regularity. Let $\tilde{\psi}$ be a corresponding orthonormal wavelet. Then the tensor product ansatz yields a scaling function $\phi$ and associated wavelets $\{\psi_1, \ldots, \psi_{2^n-1}\}$, all defined now on $\mathbb{R}^n$ (see, for instance, [67, Proposition 5.2]). We suppose $\phi \in C^{N_1}(\mathbb{R}^n)$ and $\text{supp } \phi \subset [-N_2, N_2]^n$ (3.1) for certain natural numbers $N_1$ and $N_2$. This implies
\begin{equation}
\psi_i \in C^{N_1}(\mathbb{R}^n) \quad \text{and} \quad \text{supp } \psi_i \subset [-N_3, N_3]^n, \quad \forall i \in \{1, \ldots, 2^n-1\}
\end{equation}
for some $N_3 \in \mathbb{N}$. For any $k \in \mathbb{Z}^n$, $j \in \mathbb{N}_0$ and $i \in \{1, \ldots, 2^n-1\}$, we shall use the standard abbreviations
\begin{equation}
\phi_{j,k}(x) := 2^{jn/2}\phi(2^j x - k) \quad \text{and} \quad \psi_{i,j,k}(x) := 2^{jn/2}\psi_i(2^j x - k), \quad \forall x \in \mathbb{R}^n.
\end{equation}
Furthermore, it is well known that
\begin{equation}
\int_{\mathbb{R}^n} \psi_{i,j,k}(x)x^\gamma dx = 0 \quad \text{if } |\gamma| \leq N_1
\end{equation}
(see [67 Proposition 3.1]) and
\begin{equation}
\{\phi_{0,k} : k \in \mathbb{Z}^n\} \cup \{\psi_{i,j,k} : k \in \mathbb{Z}^n, j \in \mathbb{N}_0, i \in \{1, \ldots, 2^n-1\}\}
\end{equation}
yields an orthonormal basis of $L^2(\mathbb{R}^n)$ (see [38 Section 3.9] or [59 Section 3.1]). Thus, for any $f \in L^2(\mathbb{R}^n)$,
\begin{equation}
f = \sum_{k \in \mathbb{Z}^n} \lambda_k \phi_{0,k} + \sum_{i=1}^{2^n-1} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} \lambda_{i,j,k} \psi_{i,j,k}
\end{equation}
converges in $L^2(\mathbb{R}^n)$, where $\lambda_k := \langle f, \phi_{0,k} \rangle$ and $\lambda_{i,j,k} := \langle f, \psi_{i,j,k} \rangle$ with $\langle \cdot, \cdot \rangle$ denoting the inner product of $L^2(\mathbb{R}^n)$. For brevity we put
\begin{equation}
\lambda(f) := \{\lambda_k\}_k \cup \{\lambda_{i,j,k}\}_{i,j,k}.
\end{equation}
By means of such a wavelet system one can discretize the quasi-norm $\|\cdot\|_{B_{p,q}^{s,\tau}(\mathbb{R}^n)}$. Here we need some sequence spaces (see [77 Definition 2.2]).

**Definition 3.1.** Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$ and $p, q \in (0, \infty]$. Then the sequence space $b_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the space of all sequences $t := \ldots$
The Haar system in Besov-type spaces

\( \{t_{i,j,m}\}_{i\in\{1,\ldots,2^n-1\}, j\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C} \) (for short, \( \{t_{i,j,m}\}_{i,j,m} \)) such that \( \|t\|_{b^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty \), where

\[
\|t\|_{b^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max\{j_P,0\}}^{\infty} 2^{j(s+n/2-n/p)q} \sum_{i=1}^{2^n-1} \sum_{m: Q_{j,m} \subset P} |t_{i,j,m}|^p \right\}^{1/q}.
\]

As a special case of [36, Theorem 4.12] (see also [35]), we have the following wavelet characterization.

**Proposition 3.2.** Let \( s \in \mathbb{R}, \tau \in [0,\infty) \) and \( p,q \in (0,\infty] \). Let \( N_1 \in \mathbb{N}_0 \) satisfy

\[
(3.8) \quad N_1 + 1 > \max\{n + n/p - n\tau - s, 2\sigma_p + 2n + n\tau + 1, n(1+1/p + 1/2), n+s, -s + n/p\}.
\]

Let \( f \in \mathcal{S}'(\mathbb{R}^n) \). Then \( f \in B^{s,\tau}_{p,q}(\mathbb{R}^n) \) if and only if \( f \) can be represented in \( \mathcal{S}'(\mathbb{R}^n) \) as in (3.6) such that

\[
\|\lambda(f)\|_{b^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{m: Q_{0,m} \subset P} |\langle f, \phi_{0,m} \rangle|^p \right\}^{1/p} + \|\{\langle f, \psi_{i,j,m} \rangle\}_{i,j,m}\|_{b^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty.
\]

Moreover, \( \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \) is equivalent to \( \|\lambda(f)\|_{b^{s,\tau}_{p,q}(\mathbb{R}^n)} \) with the positive equivalence constants independent of \( f \).

**Remark 3.3.** (i) On the interpretation of \( \lambda(f) \), we observe that in general an element \( f \) of \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \) is not an element of \( L^2(\mathbb{R}^n) \), it might be a singular distribution. Thus, \( \langle f, \phi_{0,m} \rangle \) and \( \langle f, \psi_{i,j,m} \rangle \) require an interpretation, which has been given in the proof of Proposition 3.2 (see [36, Theorem 4.12] for all the details).

(ii) In [36], biorthogonal wavelet systems in the sense of Cohen et al. [3] have been considered. But here we do not need this generality, orthonormal wavelet systems are sufficient.

(iii) It is not claimed that the restriction in (3.8) is optimal.

(iv) For the case \( s \in (0,\infty) \), we refer the reader also to [77, Section 4.2].

**3.2. Characterization by differences.** Historically the characterization by differences (together with some characterizations by approximations) has been the first description of Besov spaces. In addition, such characterizations look also more transparent than the definition in terms of convolutions. For that reason the present authors have studied those characterizations with some care in [77]. To recall one of the results obtained in [77], we first need some notation.
For any $M \in \mathbb{N}$, a function $f : \mathbb{R}^n \to \mathbb{C}$ and $h, x \in \mathbb{R}^n$, let
\[
\Delta_h^M f(x) := \sum_{j=0}^{M} (-1)^j \binom{M}{j} f(x + (M - j)h),
\]
where $\binom{M}{j}$ for any $j \in \{0, \ldots, M\}$ denotes the binomial coefficient. For any $p \in (0, \infty]$, let $L^p(\mathbb{R}^n)$ denote the Lebesgue space of all measurable functions $f$ such that
\[
\|f\|_{L^p(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \, dx \right]^{1/p} < \infty,
\]
with the usual modification when $p = \infty$, and $L^p_{\text{loc}}(\mathbb{R}^n)$ the space of all measurable functions which belong locally to $L^p(\mathbb{R}^n)$. For any $\tau \in [0, \infty)$, $p \in (0, \infty]$ and $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, let
\[
(3.9) \quad \|f\|_{L^p_\tau(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{1}{|P|^\tau} \left[ \int_{P} |f(x)|^p \, dx \right]^{1/p},
\]
with the usual modification when $p = \infty$. Denote by $L^p_\tau(\mathbb{R}^n)$ the set of all functions $f$ satisfying $\|f\|_{L^p_\tau(\mathbb{R}^n)} < \infty$. Obviously, for any $p \in (0, \infty]$, $L^p_0(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Furthermore, we write
\[
\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \min \{\ell(P), 1\} \right\}^{2\min \{\ell(P), 1\}} \int_0^{t} t^{-sq} \sup_{t/2 \leq |h| < t} \left[ \int_P |\Delta_h^M f(x)|^p \, dx \right]^{q/p} \frac{dt}{t} \right\}^{1/q}.
\]

The following difference characterization was proved in [77] Theorems 4.7 and 4.9. Here we focus on the case $\tau \in [0, 1/p]$.

**Proposition 3.4.** Let $q \in (0, \infty]$ and $M \in \mathbb{N}$.

(i) Let $p \in [1, \infty]$, $s \in (0, M)$ and $\tau \in [0, 1/p]$. Then $f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p_\tau(\mathbb{R}^n)$ and $\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty$. Furthermore, $\|f\|_{L^p_\tau(\mathbb{R}^n)} + \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}$ are equivalent with the positive equivalence constants independent of $f$.

(ii) Let $p \in (0, 1)$, $s \in (\sigma_p, M)$ and $\tau \in [0, 1/p]$. Then $f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p_\tau(\mathbb{R}^n)$, $\sup_{P \in \mathcal{Q}, \ell(P) \geq 1} |P|^{-\tau} \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty$ with $s_0 \in (\sigma_p, s)$ and $\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} < \infty$. Furthermore,
\[
\|f\|_{L^p_\tau(\mathbb{R}^n)} + \sup_{P \in \mathcal{Q}, \ell(P) \geq 1} |P|^{-\tau} \|f\|_{B^{s_0,\tau}_{p,q}(\mathbb{R}^n)} + \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} + \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \quad \text{are equivalent with the positive equivalence constants independent of } f.
\]

### 3.3. Interpolation of Besov-type spaces.

The interpolation method we shall use is the $\pm$-method introduced by Gustavsson and Peetre [15][17].

To recall its definition, we consider a couple of quasi-Banach spaces (for
short, a quasi-Banach couple), $X_0$ and $X_1$, which are continuously embedded into a larger Hausdorff topological vector space $Y$. The space $X_0 + X_1$ is given by

$$X_0 + X_1 := \{ h \in Y : \exists h_i \in X_i, \; i \in \{0, 1\}, \text{ such that } h = h_0 + h_1 \},$$
equipped with the quasi-norm

$$\| h \|_{X_0 + X_1} := \inf \{ \| h \|_{X_0} + \| h \|_{X_1} : h = h_0 + h_1, \; h_0 \in X_1 \text{ and } h_1 \in X_1 \}.$$

**Definition 3.5.** Let $(X_0, X_1)$ be a quasi-Banach couple and $\Theta \in (0, 1)$. An $a \in X_0 + X_1$ is said to belong to $\langle X_0, X_1, \Theta \rangle$ if there exists a sequence $\{a_i\}_{i \in \mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i \in \mathbb{Z}} a_i$ with convergence in $X_0 + X_1$ and, for any finite subset $F \subset \mathbb{Z}$ and any bounded sequence $\{\varepsilon_i\}_{i \in \mathbb{Z}} \subset \mathbb{C}$,

$$\left\| \sum_{i \in F} \varepsilon_i 2^{i(1-\Theta)} a_i \right\|_{X_j} \leq C \sup_{i \in \mathbb{Z}} |\varepsilon_i|$$
for some non-negative constant $C$ independent of $F$, $\{\varepsilon_i\}_{i \in \mathbb{Z}}$ and $j \in \{0, 1\}$. The quasi-norm of $a \in \langle X_0, X_1, \Theta \rangle$ is defined as

$$\|a\|_{\langle X_0, X_1, \Theta \rangle} := \inf \{ C : C \text{ satisfies (3.10)} \}.$$

The following property is taken from [18, Proposition 6.1].

**Proposition 3.6.** Let $(A_0, A_1)$ and $(B_0, B_1)$ be any two quasi-Banach couples and $\Theta \in (0, 1)$.

(i) $\langle A_0, A_1, \Theta \rangle$ is a quasi-Banach space.

(ii) If $T$ is a linear continuous operator from $A_i$ into $B_i$, $i \in \{0, 1\}$, then $T$ maps $\langle A_0, A_1, \Theta \rangle$ continuously into $\langle B_0, B_1, \Theta \rangle$. Furthermore,

$$\|T\|_{\langle A_0, A_1, \Theta \rangle \rightarrow \langle B_0, B_1, \Theta \rangle} \leq \max\{\|T\|_{A_0 \rightarrow B_0}, \|T\|_{A_1 \rightarrow B_1}\}.$$

We also refer the reader to Nilsson [39] for more information on this interpolation method.

The following interpolation property of Besov-type spaces and the related sequence spaces via the $\pm$ method was obtained in [79, Theorem 2.12].

**Theorem 3.7.** Let $\theta \in (0, 1)$, $s_i \in \mathbb{R}$, $\tau_i \in [0, \infty)$ and $p_i, q_i \in (0, \infty]$, $i \in \{0, 1\}$, be such that $s = (1 - \theta)s_0 + \theta s_1$, $\tau = (1 - \theta)\tau_0 + \theta \tau_1$, 

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad \text{and} \quad \frac{1}{q} = \frac{1 - \theta}{q_0} + \frac{\theta}{q_1}.$$

If $\tau_0 p_0 = \tau_1 p_1 = \tau p$, then

$$\langle B^{s_0, \tau_0}_{p_0, q_0}(\mathbb{R}^n), B^{s_1, \tau_1}_{p_1, q_1}(\mathbb{R}^n), \theta \rangle = B^{s, \tau}_{p, q}(\mathbb{R}^n)$$

and

$$\langle b^{s_0, \tau_0}_{p_0, q_0}(\mathbb{R}^n), b^{s_1, \tau_1}_{p_1, q_1}(\mathbb{R}^n), \theta \rangle = b^{s, \tau}_{p, q}(\mathbb{R}^n).$$
4. Proof of Theorem 2.1. In this section, by employing the characterizations of Besov-type spaces via wavelets (see Proposition 3.2) and differences (see Proposition 3.4) as well as their interpolation property (see Theorem 3.7), we give the proof of Theorem 2.1.

First we have to introduce more notation. For any \( Q \in \mathcal{Q} \) and \( j \in \mathbb{Z} \), we let
\[
J_Q := \{ r \in \mathbb{Z}^n : |\text{supp} \phi_{0,r} \cap Q| > 0 \}
\]
and
\[
I_{Q,j} := \{ r \in \mathbb{Z}^n : \exists i \in \{1, \ldots, 2^n - 1\} \text{ such that } |\text{supp} \psi_{i,j,r} \cap Q| > 0 \}.
\]

Now we are ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** We show this theorem by four steps. Steps 1–3 will deal with \( X \), while in Step 4 the functions \( h_{i,j,m} \) are investigated.

**Step 1.** We first consider the case \( \tau \in (1/p, \infty) \). Here it is enough to use the monotonicity of \( B_{p,q}^{s,\tau}(\mathbb{R}^n) \) with respect to \( s \) and \( q \) (see Remark 1.2) and the fact \( X \in B_0^0(\mathbb{R}^n) \Leftrightarrow q = \infty \) (see [44, Lemma 4.6.3/2]). Because of \( B_{p,q}^{s,\tau}(\mathbb{R}^n) = B_{\infty,\infty}^{s+n(\tau-1/p)}(\mathbb{R}^n) \) (see Remark 1.2(iv)), we obtain \( X \in B_{p,q}^{s,\tau}(\mathbb{R}^n) \) if and only if \( s + n(\tau - 1/p) \leq 0 \).

**Step 2.** We prove sufficiency in case \( \tau \in [0, 1/p] \). We employ the wavelet characterization of \( B_{p,q}^{s,\tau}(\mathbb{R}^n) \) as given in Proposition 3.2. Thus, we have to check the finiteness of
\[
\sup_{P \in \mathcal{Q}, |P| \geq 1} \frac{1}{|P|^\tau} \left( \sum_{k \in \mathcal{J}_P} |\langle X, \phi_{0,k} \rangle|^p \right)^{1/p}
\]
\[+ \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j = \max\{j_P, 0\}}^{\infty} 2^{j(s+n/2)} \sum_{i=1}^{2^n-1} \left[ \sum_{k \in \mathcal{I}_{P,j}} 2^{-jn} |\langle X, \psi_{i,j,k} \rangle|^p \right]^{q/p} \right\}^{1/q},
\]
where \( \phi_{0,k} \) and \( \psi_{i,j,k} \) are as in (3.5). The first term of the above summation is always finite, hence we may concentrate on the second. Because of the moment conditions on \( \psi_{i,j,k} \) in (3.4), the scalar product \( \langle X, \psi_{i,j,k} \rangle \) is 0 if either \( \text{supp} \psi_{i,j,k} \subset Q_{0,0} \) or \( \text{supp} \psi_{i,j,k} \cap Q_{0,0} = \emptyset \). We define
\[\Omega_j := \{ r \in \mathbb{Z}^n : \exists i \in \{1, \ldots, 2^n - 1\} \text{ such that } \text{supp} \psi_{i,j,r} \cap \partial Q_{0,0} \neq \emptyset \}\]
and \( \omega_j := |\Omega_j| \) (the cardinality of \( \Omega_j \)). The properties of the wavelet system guarantee that
\[\omega_j \asymp 2^{j(n-1)}, \quad \forall j \in \mathbb{N}_0.
\]
Now we consider two different cases for the size of the dyadic cube \( P \).
Case 1. Assume that $P \in Q$ with $|P| \geq 1$. In this case, we need more information about the set $\mathcal{I}_{P,j}$ defined in (4.2). Let $P := Q_{m,\ell}$ with $m \in \mathbb{Z} \setminus \mathbb{N}$ and $\ell \in \mathbb{Z}^n$. Then we know that

$$\mathcal{I}_{P,j} \subset \bigcup_{|\ell - k| \leq M} \{ r \in \mathbb{Z}^n : \exists i \in \{1, \ldots, 2^n - 1\} \text{ such that } |\text{supp} \psi_{i,j,r} \cap Q_{m,k}| > 0 \},$$

where $M$ is a fixed natural number (depending on $N_2$ and $N_3$). It follows

$$|\Omega_j \cap \mathcal{I}_{P,j}| \lesssim |\Omega_j \cap \mathcal{I}_{Q_{0,0,j}}| \lesssim 2^{j(n-1)}, \quad \forall j \in \mathbb{N}_0.$$ 

In addition we shall use the obvious estimate

$$(4.3) \quad |\langle X, \psi_{i,j,k} \rangle| \leq 2^{j n/2} \int_{[0,1]^n} |\psi_i(2^j x - k)| \, dx$$

$$\leq 2^{-j n/2} (\max \{N_2, N_3\})^{n/2} \left[ \int_{\mathbb{R}^n} |\psi_i(y)|^2 \, dy \right]^{1/2} \lesssim 2^{-j n/2}.$$ 

Consequently, by (2.4), (2.5), (4.3) and the condition on $s$, for those cubes $P$ we have

$$(4.4) \quad \frac{1}{|P|^\tau} \left\{ \sum_{j = \max\{j_P,0\}}^{\infty} 2^{j(s+n/2)q} \sum_{i=1}^{2^n-1} \left[ \sum_{k \in \mathcal{I}_{P,j}} 2^{-j n} |\langle X, \psi_{i,j,k} \rangle|^p \right]^{q/p} \right\}^{1/q}$$

$$= \frac{1}{|P|^\tau} \left\{ \sum_{j=0}^{\infty} 2^{j(s+n/2)q} \sum_{i=1}^{2^n-1} \left[ \sum_{k \in \mathcal{I}_{P,j} \cap \Omega_j} 2^{-j n} |\langle X, \psi_{i,j,k} \rangle|^p \right]^{q/p} \right\}^{1/q}$$

$$\lesssim \left\{ \sum_{j=0}^{\infty} 2^{j(s+1/p)q} \right\}^{1/q} \lesssim \left\{ \sum_{j=0}^{\infty} 2^{j(s-1/p)q} \right\}^{1/q} < \infty.$$ 

Case 2. Assume now that $P \in Q$ with $|P| < 1$. We may write $P := Q_{m,\ell}$ with $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}^n$. For cubes of this size, we have

$$|\Omega_j \cap \mathcal{I}_{P,j}| \lesssim |\Omega_j \cap \mathcal{I}_{Q_{m,0,j}}| \lesssim 2^{j(m)n+(n-1)}, \quad \forall j \in \{m, m + 1, \ldots\}.$$ 

From this and (2.4) and (2.5), for those cubes $P$ we deduce that

$$\frac{1}{|P|^\tau} B\left\{ \sum_{j = \max\{j_P,0\}}^{\infty} 2^{j(s+n/2)q} \sum_{i=1}^{2^n-1} \left[ \sum_{k \in \mathcal{I}_{P,j}} 2^{-j n} |\langle X, \psi_{i,j,k} \rangle|^p \right]^{q/p} \right\}^{1/q}$$

$$= 2^{mn \tau} \left\{ \sum_{j=m}^{\infty} 2^{j(s+n/2)q} \sum_{i=1}^{2^n-1} \left[ \sum_{k \in \mathcal{I}_{P,j} \cap \Omega_j} 2^{-j n} |\langle X, \psi_{i,j,k} \rangle|^p \right]^{q/p} \right\}^{1/q}$$

$$\lesssim 2^{mn \tau} \left\{ \sum_{j=m}^{\infty} 2^{j(s+n/2)q} 2^{(j-m)(n-1)/p - j n q/p} 2^{-j n q/2} \right\}^{1/q}$$

$$\lesssim 2^{mn \tau} 2^{-m(n-1)/p} \left\{ \sum_{j=m}^{\infty} 2^{j(s-1/p)q} \right\}^{1/q}.$$
If either \( s = 1/p \) and \( q = \infty \), or \( s < 1/p \) and \( q \) is arbitrary, we conclude that

\[
\text{(4.5)} \quad \frac{1}{|P|} \left\{ \sum_{j = \max\{j_p, 0\}}^{\infty} 2^{j(s+n/2)q} \sum_{i=1}^{2^n-1} \left[ \sum_{k \in I_{j_p, j}} 2^{-jn} |\langle \mathcal{X}, \psi_{i,j,k} \rangle|^p \right]^{q/p} \right\}^{1/q} \\
\lesssim 2^{m(s+n\tau-n/p)},
\]

which is uniformly bounded in \( m \) for \( s + n\tau - n/p \leq 0 \). The estimates (4.4) and (4.5) together prove the sufficiency in cases (i) and (ii).

**Step 3.** We prove necessity in case \( \tau \in [0, 1/p] \). It seems difficult to apply the wavelet decomposition here because we do not know how many scalar products satisfy the inequality

\[
|\langle \mathcal{X}, \psi_{i,j,k} \rangle| \geq c 2^{-jn/2}
\]

with a positive constant \( c \) independent of \( j \) and \( k \). For that reason we switch to differences (see Proposition 3.4). Since in case \( p = \infty \) the claim is already known (see (2.3)), we may assume \( p < \infty \). In addition, the necessity of \( s + n(\tau - 1/p) \leq 0 \) follows from Proposition 1.3(i). It remains to deal with the relation between \( s \) and \( 1/p \). By the embedding in Remark 1.2(iii), we only need to show that \( \mathcal{X} \) is not in \( B_{p,q}^{1/p,\tau} (\mathbb{R}^n) \) with any given \( \tau \in [0, 1/p] \) and \( q \in (0, \infty) \).

**Substep 3.1.** First we assume that \( p > (n - 1)/n \), that is, \( \sigma_p < 1/p \). In this situation we can employ Proposition 3.4. By using the abbreviations introduced there and by choosing \( P = Q_{0,0} \), we find that

\[
\sup_{t/2 \leq |h| < t} \left\{ \int_{Q_{0,0}} \left| \Delta_h^M \mathcal{X}(x) \right|^p dx \right\}^{1/p} \\
\geq \sup_{t/2 \leq -h_1 < t} \left\{ \int_{x_i \in [0, t/2]} \left| \Delta_h^M \mathcal{X}(x) \right|^p dx \right\}^{1/p} \geq (t/2)^{1/p}
\]

for any \( t \in (0, 1) \). This immediately implies that \( \| \mathcal{X} \|_{B_{p,q}^{1/p,\tau} (\mathbb{R}^n)} = \infty \) for any \( q \in (0, \infty) \).

**Substep 3.2.** Now we consider the case \( p \leq (n - 1)/n \), that is, \( \sigma_p \geq 1/p \). In addition, we may assume \( \tau \in (0, 1/p] \), due to the known result when \( \tau = 0 \) (see (2.3)). We prove \( \mathcal{X} \notin B_{p,q}^{1/p,\tau} (\mathbb{R}^n) \) in this case by contradiction.

Assume that \( \mathcal{X} \in B_{p,q}^{1/p,\tau} (\mathbb{R}^n) \) with some \( p \leq (n - 1)/n \) and \( q \in (0, \infty) \). We argue by employing the \( \pm \) interpolation method of Gustavsson–Peetre, in particular we shall use

\[
\langle B_{p,q}^{1/p,\tau} (\mathbb{R}^n), B_{p_1,\infty}^{1/p_1,\tau_1} (\mathbb{R}^n), \theta \rangle = B_{p_0,q_0}^{1/p_0,\tau_0} (\mathbb{R}^n)
\]
with $\theta \in (0, 1)$, $\tau_0 = (1 - \theta) \tau + \theta \tau_1$,
\[
\frac{1}{p_0} = \frac{1 - \theta}{p} + \frac{\theta}{p_1}, \quad \frac{1}{q_0} = \frac{1 - \theta}{q} \quad \text{and} \quad \frac{\tau}{p_1} = \frac{\tau_1}{p}
\]
(see Theorem 3.7). We choose $p_1 > (n - 1)/n$ and define
\[
\tau_1 := \tau p/p_1.
\]
Then, by Step 2, we know that $\mathcal{X}$ belongs to $B^{1/p_1, \tau_1}_p(\mathbb{R}^n)$, which, together with the assumption $\mathcal{X} \in B^{1/p_0, \tau}_p(\mathbb{R}^n)$ and the above interpolation formula, implies that $\mathcal{X} \in B^{1/p_0, \tau_0}_p(\mathbb{R}^n)$ for some $q_0 \in (0, \infty)$. Taking $p_1$ as large and $\theta$ as close to 0 as we want, we arrive at a situation where also $p_0 > (n - 1)/n$. But because of $q_0 < \infty$ this is in contradiction to Substep 3.1.

Combining Substeps 3.1 and 3.2, we find that $\mathcal{X} \notin B^{1/p_0, \tau}_p(\mathbb{R}^n)$ whenever $q \in (0, \infty)$. This finishes the proof of Theorem 2.1 restricted to $\mathcal{X}$.

**Step 4.** By the translation invariance of Besov-type spaces we have $\mathcal{X} \in B^{s, \tau}_{p,q}(\mathbb{R}^n)$ if and only if $\mathcal{X}_{0,m} \in B^{s, \tau}_{p,q}(\mathbb{R}^n)$. Now we turn to the functions $h_{i,j,m}$. The “if” part follows completely analogously to Steps 1–3 by concentrating on $h_{i,0,0}$. Next we deal with the mapping $f \mapsto f(2 \cdot)$. Essentially as a consequence of the flexibility in choosing the system $\{\varphi_j\}_{j \in \mathbb{N}_0}$ in the definition of the spaces $B^{s, \tau}_{p,q}(\mathbb{R}^n)$ we conclude that this mapping is bounded on $B^{s, \tau}_{p,q}(\mathbb{R}^n)$ for all admissible parameters. By taking into account the translation invariance of Besov-type spaces this shows that the functions $h_{i,j,m}$ belong to $B^{s, \tau}_{p,q}(\mathbb{R}^n)$ whenever $\mathcal{X}$ is in $B^{s, \tau}_{p,q}(\mathbb{R}^n)$.

For the “only if” part we argue as follows. By the translation and rotation invariance of $B^{s, \tau}_{p,q}(\mathbb{R}^n)$ it will be sufficient to deal with $h_{1,j,0}$. The support of this function is $[0, 2^{-j}].$ There exists at least one dyadic subcube $Q_{j+1, k}$ such that $h_{1,j,0} = 1$ on this cube. Without loss of generality we may assume $Q_{j+1,k} = Q_{j+1,0}$. Let $\varrho$ be a compactly supported smooth function such that
\[
\text{supp} \varrho \subset Q_{j+1,0} \cup \{x : x_1 \leq 0, \ldots, x_n \leq 0\}
\]
and $\varrho = 1$ on $[2^{-j-3}, 2^{-j-2}] \times [-2^{-j-3}, 2^{-j-3}]$. Those smooth functions $\varrho$ are pointwise multipliers for $B^{s, \tau}_{p,q}(\mathbb{R}^n)$ (see [77, 6.1.1]). Hence, if we assume $h_{1,j,0} \in B^{s, \tau}_{p,q}(\mathbb{R}^n)$ then $\varrho \cdot h_{1,j,0} \in B^{s, \tau}_{p,q}(\mathbb{R}^n)$. But locally, more exactly on $[2^{-j-3}, 2^{-j-2}] \times [-2^{-j-3}, 2^{-j-3}]$, the product $\varrho \cdot h_{1,j,0}$ behaves like a characteristic function. Following the arguments in Steps 1–3 one can show that this can only be true if $\mathcal{X}$ itself belongs to $B^{s, \tau}_{p,q}(\mathbb{R}^n)$. This finishes the proof of Theorem 2.1.

**5. Proofs of Theorems 2.3, 2.5 and 2.8 as well as Corollary 2.7.** This section is devoted to the proofs of Theorems 2.3, 2.5 and 2.8. More precisely, in Section 5.1 we give the proof of the sufficient conditions for
the existence of $\langle f, \mathcal{X} \rangle$, while the proof of the necessary conditions is presented in Section 5.2. Section 5.3 is devoted to the proofs of Theorems 2.3 and 2.5. Finally, the proofs of Corollary 2.7 and Theorem 2.8 are presented, respectively, in Sections 5.4 and 5.5.

To prove these theorems, we shall first discuss a reasonable way to define $\langle f, \mathcal{X} \rangle$ for any $f \in B_{p,q}^{s,\tau}(\mathbb{R}^n)$. Fix $s$, $p$, $q$ and $\tau$, and let the wavelet system
\[
\{ \phi_{0,k}, \psi_{i,j,k} : k \in \mathbb{Z}^n, j \in \mathbb{N}_0, i \in \{1, \ldots, 2^n - 1\} \}
\]
be admissible for $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ in the sense of Proposition 3.2. Then the wavelet decomposition of $f$ is given by
\[
f = \sum_{k \in \mathbb{Z}^n} \langle f, \phi_{0,k} \rangle \phi_{0,k} + \sum_{i=1}^{2^n-1} \sum_{j \in \mathbb{N}_0} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{i,j,k} \rangle \psi_{i,j,k} = \lim_{N \to \infty} S_N f
\]
with convergence in $S'(\mathbb{R}^n)$, where
\[
S_N f := \sum_{k \in \mathbb{Z}^n} \langle f, \phi_{0,k} \rangle \phi_{0,k} + \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} \sum_{k \in \mathbb{Z}^n} \langle f, \psi_{i,j,k} \rangle \psi_{i,j,k}.
\]
Observe that both the above summations $\sum_{k \in \mathbb{Z}^n}$ are locally finite and hence $S_N f \in C^{N_1}(\mathbb{R}^n)$ (due to (3.1) and (3.2)). Then we define
\[
\langle f, \mathcal{X} \rangle := \lim_{N \to \infty} \langle S_N f, \mathcal{X} \rangle = \lim_{N \to \infty} \int_{[0,1]^n} S_N f(x) \, dx
\]
whenever this limit exists.

### 5.1. Sufficient conditions for the existence of $\langle f, \mathcal{X} \rangle$.

Now we turn to the sufficient condition for this existence which at the same time guarantees the independence of $\langle f, \mathcal{X} \rangle$ from the chosen wavelet system.

**Theorem 5.1.** Let $q \in (0, \infty]$ and $\tau \in [0, \infty)$.

(i) Let $p \in [1, \infty]$. If $s = 1/p - 1$, then $f \mapsto \langle f, \mathcal{X} \rangle$ extends to a continuous linear functional on $B_{p,1}^{s,\tau}(\mathbb{R}^n)$ which coincides on $B_{p,1}^{s,\tau}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, $r \in [1, \infty]$, with $\int_{\mathbb{R}^n} f(x) \mathcal{X}(x) \, dx$.

(ii) Let $p \in (0, 1)$ and $\tau \in \left[0, \frac{n-1}{np}\right]$. If $s = (1 - p \tau)n(1/p - 1)$ and $q \in (0, p)$, then $f \mapsto \langle f, \mathcal{X} \rangle$ extends to a continuous linear functional on $B_{p,q}^{s,\tau}(\mathbb{R}^n)$ which coincides on $B_{p,q}^{s,\tau}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, $r \in [1, \infty]$, with $\int_{\mathbb{R}^n} f(x) \mathcal{X}(x) \, dx$.

(iii) Let $p \in (0, 1)$ and $\tau = 0$. If $s = n(1/p - 1)$ and $q \in (0, 1)$, then $f \mapsto \langle f, \mathcal{X} \rangle$ extends to a continuous linear functional on $B_{p,1}^{s,0}(\mathbb{R}^n)$ which coincides on $B_{p,1}^{s,0}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, $r \in [1, \infty]$, with $\int_{\mathbb{R}^n} f(x) \mathcal{X}(x) \, dx$. 
Proof. Step 1 (Proof of (i)). Let $p \in [1, \infty]$. With $P := [0, 1]^n$, we find
\[
\langle S_N f, \mathcal{X} \rangle = \sum_{k \in J_P} \langle f, \phi_{0,k} \rangle \langle \phi_{0,k}, \mathcal{X} \rangle + \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} \sum_{k \in I_{P,j}} \langle f, \psi_{i,j,k} \rangle \langle \psi_{i,j,k}, \mathcal{X} \rangle,
\]
where $J_P$ and $I_{P,j}$ are, respectively, as in (4.1) and (4.2) with $Q$ replaced by $P$. Now we look for sufficient conditions guaranteeing the existence of the limit as $N \to \infty$. Clearly, the sum $\sum_{k \in J_P} \langle f, \phi_{0,k} \rangle \langle \phi_{0,k}, \mathcal{X} \rangle$ is a well-defined complex number. Furthermore, because of the moment condition (3.4), we conclude that
\[
\langle \psi_{i,j,k}, \mathcal{X} \rangle = 2^{-jn/2} \int_{[0,2j]^n} \psi_i(x - k) \, dx = 0
\]
possibly except when
(5.2) $|\text{supp} \psi_i(\cdot - k) \cap [0, 2j]^n| \cdot |\text{supp} \psi_i(\cdot - k) \cap (\mathbb{R}^n \setminus [0, 2j]^n)| > 0$.

Let us denote by $K_{i,j}$ the set of all $k$ satisfying (5.2). Because of the compact support of our generators of the wavelet system, there exists a finite positive constant $c_1$, independent of $j \in \mathbb{N}_0$, such that the cardinality of $K_{i,j}$ is bounded by $c_12^{j(n-1)}$. We fix a positive constant $c_2$ such that
\[
c_2 \geq \max_{k \in \mathbb{Z}^n} \left| \int_{[0,2j]^n} \psi_i(x - k) \, dx \right|, \quad \forall j \in \mathbb{N}_0.
\]
Using these observations, we obtain
(5.3) $\left| \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} \sum_{k \in I_{P,j}} \langle f, \psi_{i,j,k} \rangle \langle \psi_{i,j,k}, \mathcal{X} \rangle \right| \leq c_2 \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} 2^{-jn/2} \sum_{k \in K_{i,j}} |\langle f, \psi_{i,j,k} \rangle|.$

From the Hölder inequality for $p > 1$, it follows that
(5.4) $\left| \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} \sum_{k \in I_{P,j}} \langle f, \psi_{i,j,k} \rangle \langle \psi_{i,j,k}, \mathcal{X} \rangle \right| \leq c_1^{1/p'} c_2 \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} 2^{-jn/2} 2^{j(n-1)/p'} \left( \sum_{k \in K_{i,j}} |\langle f, \psi_{i,j,k} \rangle|^p \right)^{1/p}.$

Observe that if $k \in K_{i,j}$, then, because of the compact supports of the wavelets, there exists a natural number $D$, independent of $i$ and $j$, such that
$Q_{j,k} \subset \bigcup_{|m| \leq D} Q_{0,m}.$
This implies that
\[
\sum_{i=1}^{2^n-1} \sum_{j=0}^{N} 2^{-jn/2}2^{j(n-1)/p'} \left( \sum_{k \in K_{i,j}} |\langle f, \psi_{i,j,k} \rangle|^p \right)^{1/p} 
\leq c_3 \max_{|m| \leq D} \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} 2^{-jn/2}2^{j(n-1)/p'} \left( \sum_{k: Q_{j,k} \subset Q_{0,m}} |\langle f, \psi_{i,j,k} \rangle|^p \right)^{1/p} 
\leq c_4 \|f\|_{B^{1/p-1,\tau}_{p,1}(\mathbb{R}^n)}
\]
for some positive constants $c_3$ and $c_4$ independent of $f$ (see Proposition [3.2]). Thus, if $p \in [1, \infty]$, then $\lim_{N \to \infty} \langle S_N f, \mathcal{X} \rangle$ exists for any $f \in B^{1/p-1,\tau}_{p,1}(\mathbb{R}^n)$.

**Step 2 (Proof of (ii)).** Let $p \in (0, 1)$. Obviously $f \mapsto \langle f, \mathcal{X} \rangle$ makes sense for any $f \in L^1(\mathbb{R}^n)$. Thus, to show that $\langle f, \mathcal{X} \rangle$ can be extended to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, it suffices to prove the embedding $B^{s,\tau}_{p,q}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$. It is also clear that we may assume $\text{supp} \ f \subset [-1,2]^n$, because smooth functions with compact support are pointwise multipliers on $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ (see [77 Theorem 6.1]).

Now we will obtain a sufficient condition by studying the embedding $B^{s,\tau}_{p,q}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$. In [21] Theorem 3.8(i), Haroske et al. showed that $B^{s,\tau}_{p,q}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$ when $p \in (0, 1)$, $s = (1 - p \tau)n(1/p - 1)$ and $q \in (0, p]$.

Looking into the details of their proof, we find that one can sharpen their result as follows: there exists a positive constant $c$ such that
\[
\int_{[-1,2]^n} |f(x)| \, dx \leq c \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}
\]
for any $f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ with support contained in $[-1,2]^n$. Thus, $\langle f, \mathcal{X} \rangle$ makes sense for any $f \in B^{s,\tau}_{p,q}(\mathbb{R}^n)$ when $p \in (0, 1)$, $s = (1 - p \tau)n(1/p - 1)$ and $q \in (0, p]$.

**Step 3 (Proof of (iii)).** We argue as in Step 2 but using the continuous embeddings
\[
B^{n/p-n,0}_{p,1}(\mathbb{R}^n) = B^{n/p-n}_{p,1}(\mathbb{R}^n) \hookrightarrow B^{0,0}_{1,1}(\mathbb{R}^n) = B^0_{1,1}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n), \quad \forall p \in (0, 1)
\]
(see, for instance, [53]). The proof of Theorem 5.1 is thus complete. ■

We now consider another variant of extending $f \mapsto \langle f, \mathcal{X} \rangle$ to $B^{s,\tau}_{p,q}(\mathbb{R}^n)$.

**Theorem 5.2.** Let $s \in \mathbb{R}$, $p, q \in (0, \infty)$ and $\tau \in [0, \infty)$. If $s > n/p - n\tau - 1$, then $f \mapsto \langle f, \mathcal{X} \rangle$ extends to a continuous linear functional on $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ which coincides on $B^{s,\tau}_{p,q}(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$, $r \in [1, \infty]$, with $\int_{\mathbb{R}^n} f(x)\mathcal{X}(x) \, dx$. 
Proof. We only need to modify the proof of Theorem 5.1 after (5.3).
Using Proposition 3.2 we know that
\[ |\langle f, \psi_{i,j,k} \rangle| \lesssim 2^{-j(s+n\tau+n/2-n/p)} \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \]
for any \( i \in \{1, \ldots, 2^n - 1\}, j \in \mathbb{N}_0, k \in \mathbb{Z}^n \) and \( f \in B^{s,\tau}_{p,q}(\mathbb{R}^n) \). Using (5.3), we conclude, with \( s > n/p - n\tau - 1 \), that
\[
\left| \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} \sum_{k \in I_{p,j}} \langle f, \psi_{i,j,k} \rangle \langle \psi_{i,j,k}, \mathcal{X} \rangle \right| 
\lesssim \sum_{i=1}^{2^n-1} \sum_{j=0}^{N} 2^{-jn/2} 2^{j(n-1)/2} 2^{-j(s+n\tau+n/2-n/p)} \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)} \lesssim \|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)},
\]
which completes the proof of Theorem 5.2. 

5.2. Necessary conditions for the existence of \( \langle f, \mathcal{X} \rangle \). Next we turn to negative results concerning the existence of the limit in (5.1). Therefore we will construct several families of test functions.

To prove these negative statements we need several specific properties of the generators of our wavelet system. First, we need a wavelet system which is admissible in the sense of Proposition 3.2 for \( B^{s,\tau}_{p,q}(\mathbb{R}^n) \) with \( p, q, \tau \) fixed and
\[
\min\{n/p-n\tau-1, \max(n[1/p-1], 1/p-1)\} - 1 \leq s \leq \min\{1/p, n(1/p-\tau)\} + 1.
\]
In addition it should be of Daubechies type (see [67, 4.1, 4.2]). Let \( \tilde{\phi} \) be a scaling function and \( \tilde{\psi} \) an associated wavelet (both on \( \mathbb{R} \)). Here we can work also with a shift of these two functions without changing the wavelet system. We claim that we may choose \( \tilde{\psi} \in C^{N_1}(\mathbb{R}) \) such that:

(a) There exist integers \( K < 0 \) and \( L > 0 \) such that
\[ \text{supp} \tilde{\psi} \subset [K, L] \].
(b) There exists a natural number \( j_0 \) such that \( 2^{-j_0} L \leq 1 \) and
\[
\int_{0}^{1} \tilde{\psi}(2^{j_0} t) \, dt \neq 0. \tag{5.5}
\]
Part (a) can be found in [67, 4.1]. To show (b), we assume for contradiction that, for any generator \( \tilde{\psi}(\cdot - m), m \in \mathbb{Z}, \) of this wavelet system and for any \( j \in \mathbb{N} \) satisfying \( L \leq 2^j \), we have
\[
\int_{0}^{1} \tilde{\psi}(2^j t - m) \, dt = 0.
\]
Then, in the one-dimensional case,

\[ \mathcal{X} = \sum_{k \in \mathbb{Z}} \langle \mathcal{X}, \tilde{\phi}_{0,k} \rangle \tilde{\phi}_{0,k} + \sum_{j=0}^{j_0-1} \sum_{k \in \mathbb{Z}} \langle \mathcal{X}, \tilde{\psi}_{j,k} \rangle \tilde{\psi}_{j,k}, \]

where \( j_0 \) is as in (5.5), \( \tilde{\phi}_{0,k}(x) := \tilde{\phi}(x-k) \) and \( \tilde{\psi}_{j,k}(x) := 2^{j/2} \tilde{\psi}(2^j x - k) \) for any \( x \in \mathbb{R}, j \in \mathbb{N}_0 \) and \( k \in \mathbb{Z} \). Since the summations on the right-hand side are all finite, Proposition 3.2 shows that \( \mathcal{X} \) belongs to \( \mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}) \) for any \( s \leq 1/p - \tau + 1 \). But this contradicts Theorem 2.1. Thus, (5.5) holds for some \( j_0 \). Without loss of generality, we may assume that the integral in (5.5) is positive.

In addition we choose the scaling function \( \tilde{\phi} \in C_{11}^{N_1}(\mathbb{R}) \) (again the degree of freedom we have is the shift) such that \( \text{supp} \tilde{\phi} \subset [0, \tilde{L}] \) with \( \tilde{L} := L - K \) and

\[ \int_{0}^{\tilde{L}} \tilde{\phi}(t) \, dt > 0 \]

(see [67, 4.1, 4.2]). Let \( \psi_1(x) := \tilde{\psi}(x_1)\tilde{\phi}(x_2)\cdots\tilde{\phi}(x_n), \quad \forall x := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n. \)

Let \( j_0 \) be defined as in (b) and let \( \{\lambda_{\ell}\}_{\ell=1}^{\infty} \) be a given sequence of real numbers. We define our first family of test functions as

\[ f_N(x) := \sum_{\ell=1}^{N} \lambda_{\ell} \sum_{k=(0,k_2,\ldots,k_n)}^{0 \leq k < 2^\ell-\tilde{L}} \psi_{1,\ell,k}(x), \quad \forall x \in \mathbb{R}^n, \forall N \in \mathbb{N} \cap [j_0, \infty), \]

where \( \psi_{1,\ell,k} \) is as in (3.3) with \( i = 1 \) and \( j = \ell \). Clearly, \( f_N \) is as smooth as the elements of the wavelet system and has compact support. More exactly, the support is concentrated near a part of the boundary of \( [0, 1]^n \). For us, the following estimates of \( \|f_N\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} \) are important.

**Lemma 5.3.** Let \( s \in \mathbb{R}, p, q \in (0, \infty] \) and \( \tau \in [0, \infty) \). Then, for any \( N \in \mathbb{N} \cap [j_0, \infty), \)

\[ \|f_N\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \left\{ \begin{array}{ll} \left( \sum_{j=j_0}^{N} 2^{j(s+n/2-1/p)q} |\lambda_j|^q \right)^{1/q} & \quad \text{if } \tau \in \left[ 0, N-\frac{1}{n} \frac{1}{p} \right], \\
\sup_{J \in \{j_0, \ldots, N\}} 2^{J(n\tau-n-1/p)} \left\{ \sum_{j=1}^{N} 2^{j(s+n/2-1/p)q} |\lambda_j|^q \right\}^{1/q} & \quad \text{if } \tau \in \left( N-\frac{1}{n} \frac{1}{p}, \infty \right). \end{array} \right. \]

with the positive equivalence constants independent of \( N \) and \( \{\lambda_j\}_{j=j_0}^{\infty} \).

**Proof.** To estimate \( \|f\|_{\mathcal{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} \), we proceed by using Proposition 3.2. Here the following observations will be applied:
• All $Q_{j,k}$, associated to a non-zero coefficient $\lambda_j$, are subsets of $[0,1]^n$. Thus, it will be enough to consider $P \subset Q_{0,0} := [0,1]^n$.
• If $P := Q_{j,k} \subset [0,1]^n$, then it is enough to consider those $k$ such that $P \cap \partial [0,1]^n \neq \emptyset$, where $\partial [0,1]^n$ denotes the boundary of $[0,1]^n$.
• For $J \in \mathbb{N}_0$ fixed, the cube $Q_{J,0}$ leads to the largest contribution, more exactly,
\[
\max_{k \in \mathbb{Z}^n} \sum_{m : Q_{j,m} \subset Q_{J,k}} | \langle f_N, \psi_{i,j,m} \rangle |^p \leq \sum_{m : Q_{j,m} \subset Q_{J,0}} | \langle f_N, \psi_{i,j,m} \rangle |^p.
\]
• If $J > N$, then there exists no cube $Q_{j,m}$, associated to a non-zero coefficient and contained in $P := Q_{J,k}$.

Then, by the orthogonality of the wavelet system, we have
\[
\|f_N\|_{B^p_{q,\tau}(\mathbb{R}^n)} \lesssim \sup_{P \subset Q} \frac{1}{|P|^\tau} \left\{ \sum_{j = \max\{j_p,j_0\}}^N 2^{j(s + n/2 - n/p)q} \left( \sum_{m : Q_{j,m} \subset P} | \langle f_N, \psi_{1,j,m} \rangle |^p \right)^{q/p} \right\}^{1/q}
\lesssim \sup_{J \in \{0,\ldots,j_0-1\}} 2^{Jn\tau} \left\{ \sum_{j = j_0}^N 2^{j(s + n/2 - n/p)q} \left( \sum_{m : Q_{j,m} \subset Q_{J,0}} | \langle f_N, \psi_{1,j,m} \rangle |^p \right)^{q/p} \right\}^{1/q}
+ \sup_{J \in \{0,\ldots,j_0-1\}} 2^{Jn\tau} \left\{ \sum_{j = j_0}^N 2^{j(s + n/2 - n/p)q} \left( \sum_{m : Q_{j,m} \subset Q_{J,0}} | \langle f_N, \psi_{1,j,m} \rangle |^p \right)^{q/p} \right\}^{1/q}
=: S^- + S^+.
\]

Checking the number of cubes $Q_{j,m} \subset Q_{J,0}$ with $m := (0,m_2,\ldots,m_n)$, we find that
\[
S^+ \lesssim \sup_{J \in \{0,\ldots,j_0-1\}} 2^{Jn\tau} \left\{ \sum_{j = j_0}^N 2^{j(s + n/2 - n/p)q} |\lambda_j|^q 2^{(j-J)(n-1)q/p} \right\}^{1/q}
\lesssim \sup_{J \in \{0,\ldots,j_0-1\}} 2^{J(n\tau - \frac{n-1}{p})} \left\{ \sum_{j = j_0}^N 2^{j(s + n/2 - 1/p)q} |\lambda_j|^q \right\}^{1/q}.
\]
Similarly,
\[
S^- \lesssim \sup_{J \in \{j_0,\ldots,N\}} 2^{Jn\tau} \left\{ \sum_{j = J}^N 2^{j(s + n/2 - n/p)q} |\lambda_j|^q 2^{(j-J)(n-1)q/p} \right\}^{1/q}
\lesssim \sup_{J \in \{j_0,\ldots,N\}} 2^{J(n\tau - \frac{n-1}{p})} \left\{ \sum_{j = J}^N 2^{j(s + n/2 - 1/p)q} |\lambda_j|^q \right\}^{1/q}.
\]
Now we have to distinguish two cases: $0 \leq \tau \leq \frac{n-1}{n} \frac{1}{p}$ and $\frac{n-1}{n} \frac{1}{p} < \tau$. In the
first case we obtain
\[ \|f_N\|_{B^{s,T}_{p,q}(\mathbb{R}^n)} \lesssim \left\{ \sum_{j=j_0}^N 2^{j(s+n/2-1/p)q} |\lambda_j|^q \right\}^{1/q}, \]
whereas in the second case we conclude that
\[ \|f_N\|_{B^{s,T}_{p,q}(\mathbb{R}^n)} \lesssim \sup_{J \in \{j_0, \ldots, N\}} 2^{J(n\tau-n/2-1/p)} \left\{ \sum_{j=J}^N 2^{j(s+n/2-1/p)q} |\lambda_j|^q \right\}^{1/q}. \]
This finishes the proof of Lemma 5.3.

Now we turn to the calculation of \( \langle f_N, \mathcal{X} \rangle \).

**Lemma 5.4.** For any \( N \in \mathbb{N} \cap [j_0, \infty) \),
\[
(5.9) \quad \langle f_N, \mathcal{X} \rangle \lesssim \sum_{j=j_0}^N \lambda_j 2^{j(n/2-1)}
\]
with the positive equivalence constants independent of \( N \) and \( \{\lambda_j\}_{j=j_0}^\infty \), where \( j_0 \) is as in (5.5).

**Proof.** By the definition of \( f_N \) and the choice of \( j_0 \), we conclude that, for any \( N \in \mathbb{N} \cap [j_0, \infty) \),
\[
\langle f_N, \mathcal{X} \rangle = \int_{\mathbb{R}^n} \left[ \sum_{j=j_0}^N \lambda_j \sum_{k=(0,k_2,\ldots,k_n)} \psi_{1,j,k}(x) \mathcal{X}(x) \right] dx
\]
\[
= \sum_{j=j_0}^N \lambda_j 2^{j/2} \int_{0}^{2^{-j_0}} \psi_1(2^j t) dt \sum_{k=(0,k_2,\ldots,k_n)} \prod_{i=2}^n \phi(2^j x_i - k_i) dx_i
\]
\[
= \sum_{j=j_0}^N \lambda_j 2^{j(n/2)} \left[ \int_{0}^{2^{-j_0}} \psi_1(2^j t) dt \right] \sum_{k=(0,k_2,\ldots,k_n)} 2^{-(n-1) \sum_{i=2}^n 2^{j-k_i}} \prod_{i=2}^n \phi(x_i) dx_i.
\]
Employing our assumption concerning the support of \( \phi \) and the definition of \( j_0 \), we obtain, for any \( N \in \mathbb{N} \cap [j_0, \infty) \),
\[
\langle f_N, \mathcal{X} \rangle = \sum_{j=j_0}^N \lambda_j 2^{j(n/2)} \left[ \int_{0}^{2^{-j_0} L} \psi_1(2^j t) dt \right] \sum_{k=(0,k_2,\ldots,k_n)} 2^{-(n-1) \sum_{i=2}^n 2^{j-k_i}} \prod_{i=2}^n \phi(x_i) dx_i
\]
\[
\lesssim 2^{j_0} \sum_{j=j_0}^N \lambda_j 2^{j(n/2-1)} \left[ \int_{0}^{2^{-j_0} L} \psi_1(2^j t) dt \right] \lesssim \sum_{j=j_0}^N \lambda_j 2^{j(n/2-1)}.
\]
This proves Lemma 5.4.
To deal with the case $p \in (0, 1)$, we need a second sequence of test functions. First we need a preparation. Let $\alpha \in (0, n)$ and $j \in \mathbb{N}$. We would like to distribute $\lfloor 2^j \alpha \rfloor$ dyadic cubes $Q_{j,m}$ in $Q_{0,0} := [0,1)^n$ in a rather specific way (not uniformly). We claim that there exists a set $A_j \subset \mathbb{N}_0^n$ of cardinality $\lfloor 2^j \alpha \rfloor$ such that

$$Q_{j,m} \subset [0,1)^n \quad \text{for any } m \in A_j,$$

and there exists a positive constant $\tilde{C}$, independent of $j \in \mathbb{N}$, such that

$$\sum_{\{m \in A_j : Q_{j,m} \subset Q_{J,0}\}} 1 \leq \tilde{C} 2^{(j-J)\alpha}, \quad \forall J \in \{0, \ldots, j\}. \quad (5.11)$$

The construction of such an $A_j$ is rather easy. Clearly

$$[0,1)^n = Q_{j,0} \cup \bigcup_{J=1}^j (Q_{J-1,0} \setminus Q_{J,0}).$$

We will define $A_j$ via first defining the subset of $A_j$ in each $Q_{J-1,0} \setminus Q_{J,0}$ with $J \in \{1, \ldots, j\}$. First, we put 0 into the set $A_j$, which means that $Q_{j,0}$ is chosen. Inside $Q_{J-1,0} \setminus Q_{J,0}$, there exist $2^n - 1$ dyadic cubes in $Q_j$ and we need to select $\lfloor 2^{\alpha} - 1 \rfloor$ of them; which we take is unimportant. We proceed by induction with induction hypothesis

$$\lfloor 2^{(j-J+1)\alpha} - 2^{(j-J)\alpha} \rfloor + 1 \quad (5.12)$$

where $\tilde{C} \geq 1$ is a constant independent of $j$ and $J$ and will be determined later. It will be sufficient to look at the step from $J$ to $J-1$. Of course there exist $2^n - 1$ cubes $Q_{J,K}$ such that

$$Q_{J-1,0} = Q_{J,0} \cup \bigcup_K Q_{J,K}.$$ 

Altogether we have $(2^n - 1)2^{(j-J)n}$ cubes $Q_{j,m}$ in $Q_{J-1,0} \setminus Q_{J,0}$. We decide for an almost uniform distribution: in each $Q_{J,K} \subset Q_{J-1,0} \setminus Q_{J,0}$, we select

$$\left\lfloor \frac{2^{(j-J+1)\alpha} - 2^{(j-J)\alpha}}{2^n - 1} \right\rfloor + 1$$

cubes $Q_{j,m}$. This guarantees the lower bound for $|\{m \in A_j : Q_{j,m} \subset Q_{J-1,0}\}|$ as in (5.12) with $J$ replaced by $J - 1$. Now we deal with the corresponding
upper bound. Notice that
\[ \tilde{C}2^{(j-J)\alpha} + (2^n - 1) \left\{ \left[ \frac{2^{(j-J+1)\alpha} - 2^{(j-J)\alpha}}{2^n - 1} \right] + 1 \right\} \]
\[ \leq \tilde{C}2^{(j-J)\alpha} + (2^\alpha - 1)2^{(j-J)\alpha} + 2^n - 1. \]

With
\[ \tilde{C} \geq \frac{2^n + 2^\alpha - 2}{2^\alpha - 1} \]
we conclude
\[ \tilde{C}2^{(j-J)\alpha} + (2^n - 1) \left\{ \left[ \frac{2^{(j-J+1)\alpha} - 2^{(j-J)\alpha}}{2^n - 1} \right] + 1 \right\} \leq \tilde{C}2^{(j-J+1)\alpha}, \]
\forall J \in \{0, \ldots, j - 1\},
which implies that \(|\{m \in A_j : Q_{j,m} \subset Q_{J-1,0}\}|\) satisfies the upper bound in (5.12) with \(J\) replaced by \(J - 1\).

Since (5.12) is just (5.11), we know that (5.11) is fulfilled by \(A_j\) determined as above. On the other hand, since
\[ \left[ \frac{2^{(j-J+1)\alpha} - 2^{(j-J)\alpha}}{2^n - 1} \right] + 1 \leq \left[ 2^{(j-J)\alpha} \right] + 1 \leq 2\left[ 2^{(j-J)\alpha} \right], \quad \forall J \in \{0, \ldots, j\}, \]
one also finds that (5.10) is fulfilled. This proves the previous claim. Let us mention that our construction was inspired by a similar one in [20, proof of Theorem 3.1, Substep 2.4].

Below we need to indicate the dimension \(n\) in which our construction took place. So we will use the notation \(A_j^n\) instead of \(A_j\).

Now we can introduce our second family of test functions. For fixed \(\alpha \in (0, n - 1)\), \(j_1 \geq j_0\) and a given sequence \(\{\lambda_j\}_{j=j_1}^\infty\) of real numbers, we define
\[ T_j := \{m := (0, m_2, \ldots, m_n) : (m_2, \ldots, m_n) \in A_j^{n-1}, \max\{m_2, \ldots, m_n\} < 2^j - \tilde{L} \} \]
for any \(j \in \{j_1, j_1 + 1, \ldots\}\), and
\[ g_N(x) := \sum_{j=j_1}^N \lambda_j \sum_{m \in T_j} \psi_{1,j,m}(x), \quad \forall x \in \mathbb{R}^n, \forall N \in \mathbb{N} \cap [j_1, \infty), \]
where \(A_j^{n-1} \subset \mathbb{N}_0^{n-1}\) is a set constructed as \(A_j = A_j^n\) above but with \(n\) replaced by \(n - 1\).

As \(f_N\), the function \(g_N\) is as smooth as the generators of the wavelet system and it has compact support. We have the following estimate.
ŁEMMA 5.5. Let $s \in \mathbb{R}$, $p, q \in (0, \infty]$ and $\tau \in (0, \frac{n-1}{np})$. If $\alpha = np\tau$, then, for any $N \in \mathbb{N} \cap [j_1, \infty)$,

\begin{equation}
\left\| g_N \right\|_{B^s_{p,q}(\mathbb{R}^n)} \lesssim \left\{ \sum_{j=j_1}^{N} 2^j (s+n/2+n\tau-n/p) q |\lambda_j|^q \right\}^{1/q}
\end{equation}

with the positive equivalence constants independent of $N$ and $\{\lambda_j\}_{j=j_1}^{\infty}$, where $j_1 \geq j_0$ and $j_0$ is as in (5.5).

Proof. First, observe that $\alpha = np\tau < n - 1$ as required in our previous construction. As in the case of $f_N$, we conclude that

\begin{align*}
\left\| g_N \right\|_{B^s_{p,q}(\mathbb{R}^n)} &\lesssim \sup_{J \in \{j_1, \ldots, N\}} 2^{Jn\tau} \left\{ \sum_{j=J}^{N} 2^j (s+n/2-n/p) q \left( \sum_{\{m: Q_{j,m} \subset Q_{j,0}\}} |\langle g_N, \psi_{1,j,m} \rangle|^p \right)^{q/p} \right\}^{1/q} \\
&\quad + \sup_{J \in \{0, \ldots, j_1 - 1\}} 2^{Jn\tau} \left\{ \sum_{j=j_1}^{N} 2^j (s+n/2-n/p) q \left( \sum_{\{m: Q_{j,m} \subset Q_{j,0}\}} |\langle g_N, \psi_{1,j,m} \rangle|^p \right)^{q/p} \right\}^{1/q} \\
&=: S^- + S^+.
\end{align*}

Using the properties of $A^n_j$, namely, (5.10) and (5.11) with $n$ replaced by $n - 1$, we find that

\begin{align*}
S^+ &\lesssim \sup_{J \in \{0, \ldots, j_1 - 1\}} 2^{Jn\tau} \left\{ \sum_{j=J}^{N} 2^j (s+n/2-n/p) q |\lambda_j|^q 2^{(j-J)\alpha q/p} \right\}^{1/q} \\
&\lesssim \sup_{J \in \{0, \ldots, j_1 - 1\}} 2^{J(n\tau - \alpha/p)} \left\{ \sum_{j=j_1}^{N} 2^j (s+n/2+\alpha/p-n/p) q |\lambda_j|^q \right\}^{1/q} \\
&\lesssim \left\{ \sum_{j=j_1}^{N} 2^j (s+n/2+n\tau-n/p) q |\lambda_j|^q \right\}^{1/q}.
\end{align*}

Similarly,

\begin{align*}
S^- &\lesssim \sup_{J \in \{j_1, \ldots, N\}} 2^{Jn\tau} \left\{ \sum_{j=J}^{N} 2^j (s+n/2-n/p) q |\lambda_j|^q 2^{(j-J)\alpha q/p} \right\}^{1/q} \\
&\lesssim \left\{ \sum_{j=j_1}^{N} 2^j (s+n/2+n\tau-n/p) q |\lambda_j|^q \right\}^{1/q}.
\end{align*}

This proves the estimate of Lemma 5.5.

The parameter $j_1$ does not play any role in Lemma 5.5, but it will be used for the estimate of $\langle g_N, x \rangle$. 


Lemma 5.6. Let $j_1 \geq j_0$ be sufficiently large with $j_0$ as in (5.5). Then, for any $N \in \mathbb{N} \cap [j_1, \infty)$,

$$
\langle g_N, \mathcal{X} \rangle \asymp \sum_{j=j_1}^{N} \lambda_j 2^{j(\alpha-n/2)}
$$

(5.15)

with the positive equivalence constants independent of sufficiently large $N$ and $\{\lambda_j\}_{j=j_1}^{\infty}$.

Proof. Since the cardinality of $A^{n-1}_j$ equals $\lfloor 2^{j\alpha} \rfloor$, we know that $|T_j| < 2^{j\alpha}$ for any $j \in \{j_1, j_1 + 1, \ldots\}$.

On the other hand, let $j_1$ be large enough such that $2^{j_1} > \tilde{L}$. For each $j \geq j_1$, let $J_j \in \{1, \ldots, j\}$ be the unique number satisfying $2^{j-J_j} \leq 2^j - \tilde{L} < 2^{j-J_j+1}$. Then, by the property of $A^{n-1}_j$ (5.11 with $n$ replaced by $n - 1$), we conclude that

$$
|T_j| \geq \left| \left\{ m := (0, m_2, \ldots, m_n) : (m_2, \ldots, m_n) \in A^{n-1}_j, \max\{m_2, \ldots, m_n\} < 2^{j-J_j} \right\} \right|
$$

$$
\geq \lfloor 2^{(j-J_j)\alpha} \rfloor \geq 2^{(j-J_j)\alpha} - 1 > 2^{-\alpha}(2^j - \tilde{L})^{\alpha} - 1.
$$

Altogether we obtain

$$
2^{-\alpha}(2^j - \tilde{L})^{\alpha} - 1 < |T_j| < 2^{j\alpha}.
$$

From this and an argument similar to that used in the proof of Lemma 5.4 we find that (5.15) holds if

$$
2^{-\alpha}(2^j - \tilde{L})^{\alpha} - 1 \asymp 2^{j\alpha},
$$

which is true when $j \in \{j_1, j_1 + 1, \ldots\}$ and $j_1$ is sufficiently large. This proves Lemma 5.6. □

5.3. Proofs of Theorems 2.3 and 2.5. We are now ready to prove Theorems 2.3 and 2.5.

Step 1 (Sufficiency). The “if” cases in Theorems 2.3 and 2.5 are covered by Theorems 5.1 and 5.2.

Step 2 (Necessity). To show that $\langle f, \mathcal{X} \rangle$ cannot be extended to a Besov-type space $B^{s,\tau}_{p,q}(\mathbb{R}^n)$, it suffices to find a sequence $\{f_N\}_N$ of functions in $B^{s,\tau}_{p,q}(\mathbb{R}^n)$ such that $\{\|\langle f_N, \mathcal{X} \rangle \| / \|f_N\|_{B^{s,\tau}_{p,q}(\mathbb{R}^n)}\}_N$ is unbounded. Below we only concentrate on limiting cases (if there is one), because for the remaining parameter constellations, one can use the elementary embedding mentioned in Remark 1.2.
**Substep 2.1.** Let \( p \in [1, \infty] \), \( s = 1/p - 1 \), \( q \in (1, \infty) \) and \( \tau \in [0, \frac{n-1}{np}] \). We choose a sequence \( \{\mu_j\}_{j=j_0}^{\infty} \) of positive real numbers such that

\[
\sum_{j=j_0}^{\infty} \mu_j = \infty \quad \text{and} \quad \left\{ \sum_{j=j_0}^{\infty} \mu_j^q \right\}^{1/q} < \infty,
\]

where \( j_0 \) is as in (5.5). Now let \( \lambda_j := 2^{-j(n/2-1)} \mu_j \) for any \( j \in \{j_0, j_0 + 1, \ldots\} \). Then the associated sequence \( \{f_N\}_{N=j_0}^{\infty} \) defined in (5.7), is bounded in \( B_{p,q}^{1/p-1,\tau}(\mathbb{R}^n) \) (see Lemma 5.3), while \( \left\{ \int_{\mathbb{R}^n} f_N(x) X(x) \, dx \right\}_{N=j_0}^{\infty} \) is unbounded (see Lemma 5.4). Thus, in this case, \( \{f_N\}_{N=j_0}^{\infty} \) is as desired.

**Substep 2.2.** Let \( p \in (0, \infty] \), \( s = n/p - n \tau - 1 \), \( q \in (0, 1] \) and \( \tau \in (\frac{n-1}{np}, \infty) \). Define \( \lambda_j := j2^{-j(n/2-1)} \) for any \( j \in \mathbb{N}_0 \), and let \( \{f_N\}_{N=j_0}^{\infty} \) be as in (5.7). Then Lemma 5.4 tells us that, for any \( N \in \{j_0, j_0 + 1, \ldots\} \),

\[
|\langle f_N, X \rangle| \gtrsim \sum_{j=j_0}^{N} j \gtrsim \frac{N^2 - j_0^2}{2}.
\]

Meanwhile, (5.8) gives that, for any \( N \in \{j_0, j_0 + 1, \ldots\} \),

\[
\|f_N\|_{B_{p,q}^{n/p-n\tau-1,\tau}(\mathbb{R}^n)} \lesssim \sup_{J \in \{j_0, \ldots, N\}} 2^{J(n\tau - \frac{n-1}{p})} \left\{ \sum_{j=J}^{N} 2^{j(n-1)/q} j^q \right\}^{1/q}.
\]

Since \( \frac{n-1}{p} - n \tau < 0 \), it follows that, for any \( J \in \{j_0, j_0 + 1, \ldots\} \) and \( N \in \{J, J + 1, \ldots\} \),

\[
\sum_{j=J}^{N} 2^{j(n-1)/q} j^q \lesssim J^q 2^{J(n-1)/q}.
\]

This implies that, for any \( N \in \{j_0, j_0 + 1, \ldots\} \),

\[
\|f_N\|_{B_{p,q}^{n/p-n\tau-1,\tau}(\mathbb{R}^n)} \lesssim \sup_{J \in \{j_0, \ldots, N\}} 2^{J(n\tau - \frac{n-1}{p})} J^q 2^{J(n-1)/q} \lesssim N.
\]

Therefore,

\[
\frac{|\langle f_N, X \rangle|}{\|f_N\|_{B_{p,q}^{n/p-n\tau-1,\tau}(\mathbb{R}^n)}} \to \infty
\]
as \( N \to \infty \). Thus, in this case, the sequence \( \{f_N\}_{N=j_0}^{\infty} \) is as desired.

**Substep 2.3.** Let \( p \in (0, 1) \), \( q \in (1, \infty] \), \( \tau \in (0, \frac{n-1}{np}) \) and \( s = (1 - \tau p)n (1/p - 1) \). We shall work with the family \( \{g_N\}_{N=j_1}^{\infty} \) defined in (5.13) with \( \alpha = pn \tau \). Using Lemma 5.5 we obtain, for any \( N \in \{j_1, j_1 + 1, \ldots\} \),

\[
\|g_N\|_{B_{p,q}^{n,\tau}(\mathbb{R}^n)} \gtrsim \left\{ \sum_{j=j_1}^{N} 2^{j(-n/2 + pn \tau)q} |\lambda_j|^q \right\}^{1/q}.
\]
On the other hand, Lemma 5.6 implies that, for any \( N \in \{j_1, j_1 + 1, \ldots\} \),

\[
\langle g_N, \mathcal{X} \rangle \simeq \sum_{j = j_1}^{N} \lambda_j 2^{j(pm\tau - n/2)}.
\]

(5.17)

If \( \{\mu_j\}_{j = j_1}^{\infty} \) is as in Substep 2.1 with \( j_0 \) replaced by \( j_1 \), and if we choose \( \lambda_j \) such that \( \lambda_j := 2^{-j(pm\tau - n/2)} \mu_j \) for any \( j \in \{j_1, j_1 + 1, \ldots\} \), one finds that \( \{ \psi \langle g_N, \mathcal{X} \rangle / \|g_N\|_{B_{p,q}^s(\mathbb{R}^n)} \}_{N=j_1}^{\infty} \) is unbounded. Thus, in this case, \( \{g_N\}_{N=j_1}^{\infty} \) is as desired.

**Substep 2.4.** Let \( p \in (0, 1), q \in (1, \infty], \tau = n - 1/np \) and \( s = 1/p - 1 \). Here we can employ the same example as in Substep 2.1 to obtain the desired sequence.

**Substep 2.5.** Let \( p \in (0, 1), q \in (1, \infty], \tau = 0 \) and \( s = n(1/p - 1) \). We choose

\[
h_N(x) := \sum_{j = j_0}^{N} \lambda_j \psi_1,j,0(x), \quad \forall x \in \mathbb{R}^n, \forall N \in \{j_0, j_0 + 1, \ldots\}.
\]

(5.18)

As in Lemma 5.3 for any \( N \in \{j_0, j_0 + 1, \ldots\} \),

\[
\|h_N\|_{B_{p,q}^s(\mathbb{R}^n)} \simeq \left\{ \sum_{j = j_0}^{N} 2^{j(s+n/2-n/p)q}|\lambda_j|^q \right\}^{1/q} \simeq \left\{ \sum_{j = j_0}^{N} 2^{j(-n/2)q}|\lambda_j|^q \right\}^{1/q}.
\]

As in Lemma 5.4 for any \( N \in \{j_0, j_0 + 1, \ldots\} \) we obtain

\[
\langle h_N, \mathcal{X} \rangle \simeq \sum_{j = j_0}^{N} \lambda_j 2^{j(-n/2)}.
\]

Then, letting \( \{\mu_j\}_{j = j_0}^{\infty} \) be as in Substep 2.1 and \( \lambda_j := 2^{n/2j} \mu_j \) for any \( j \in \{j_0, j_0 + 1, \ldots\} \), we can argue as before to obtain

\[
\frac{\langle h_N, \mathcal{X} \rangle}{\|h_N\|_{B_{p,q}^{s(1/p-1),0}(\mathbb{R}^n)}} \to \infty, \quad N \to \infty.
\]

Thus \( \{h_N\}_{N=j_0}^{\infty} \) is as desired.

Altogether the proofs of Theorems 2.3 and 2.5 are complete.

**5.4. Proof of Corollary 2.7.** If \( p \in [1, \infty] \), then Corollary 2.7 follows directly from Theorem 2.3. In case \( p \in (0, 1) \), the “if” assertion is contained in Theorem 2.5. For the “only if” assertion, we refer the reader to Substep 2.5 in the above proofs of Theorems 2.3 and 2.5.

**5.5. Proof of Theorem 2.8.** Sufficiency can be proved as in case of \( \mathcal{X} \). Necessity follows from a modification of our test functions. As in Step 4 of the proof of Theorem 2.1 in Section 4, it will be enough to deal with the existence of \( \langle f, h_{1,j,0} \rangle \). Recall \( h_{1,j,0} = 1 \) on \( Q_{j+1,0} \). Now, to define the
modified test functions we only select those wavelets $\psi_{1,\ell,k}$ whose supports lie between the hyperplanes $x_1 = 0$ and $x_1 = 2^{-j-1}$ and have a nontrivial intersection with the set $\{x \in \mathbb{R}^n : x_2 < 0, \ldots, x_n < 0\}$. This leads to the family

$$\tilde{f}_N(x) := \sum_{\ell = \max\{j_0, j\}}^N \lambda_{\ell} \sum_{\substack{k = (0,k_2,\ldots,k_n) \in \mathbb{Z}^n \\text{ with } 0 \leq k_i < 2^{\ell-1}-\tilde{L}, \ i \in \{2,\ldots,n\}}} \psi_{1,\ell,k}(x), \quad \forall x \in \mathbb{R}^n, \ \forall N \in \mathbb{N} \cap [j_0, \infty).$$

In a similar way our second family $\{g_N\}_{N=j_1}^\infty$ has to be modified. Now we can proceed as in the proofs of Theorems 2.3 and 2.5. We omit further details.

Acknowledgements. The authors would like to thank Professor Peter Oswald who made the manuscript [42] available to them. They also thank the referee for careful reading and several valuable comments which improved the presentation of this article.

This project is supported by the National Natural Science Foundation of China and the Deutsche Forschungsgemeinschaft (Grant No. 11761131002). Wen Yuan is also partially supported by the National Natural Science Foundation of China (Grant No. 11871100) and the Alexander von Humboldt Foundation. Dachun Yang is also partially supported by the National Natural Science Foundation of China (Grant Nos. 11971058 and 11671185).

References


[33] P. Li and Z. Zhai, Riesz transforms on $Q$-type spaces with application to quasi-geostrophic equation, Taiwanese J. Math. 16 (2012), 2107–2132.


34 W. Yuan et al.


Wen Yuan, Dachun Yang (corresponding author)
Laboratory of Mathematics and Complex Systems
(Ministry of Education of China)
School of Mathematical Sciences
Beijing Normal University
Beijing 100875, People’s Republic of China
E-mail: wenyuan@bnu.edu.cn
dcyang@bnu.edu.cn

Winfried Sickel
Mathematisches Institut
Friedrich-Schiller-Universität Jena
Jena 07743, Germany
E-mail: winfried.sickel@uni-jena.de