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CORRECTION TO THE PAPER "TWO CLASSES OF MEASURES"

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As pointed out to me by Professor A. Iwanik, the example of a non-perfect measure with countably additive products ([1], p. 337) contains an error: it may not be possible to choose the points $y_\eta^{(n)}$ appropriately. Professor A. Iwanik also observes that the construction can be repaired if every set in $[0, 1]$ of cardinality smaller than $2^{\aleph_0}$ is Lebesgue measurable.

Here I present another version of the construction for the product of two measures: there is a non-perfect measure μ such that every (indirect) product of μ with μ is countably additive. The construction needs no special set-theoretic assumptions.

Let $\mathscr{B}$ be the Borel σ-algebra in $I = [0, 1]$ and let ν be the Lebesgue measure on $\mathscr{B}$. Put

$$\mathscr{S} = \{H \subset I \times I \mid H \subset (V \times I) \cup (I \times V) \text{ for some } V \subset I \text{ of cardinality smaller than } 2^{\aleph_0}\}.$$

Let Ω be the first ordinal of cardinality $2^{\aleph_0}$, let $D_1, D_2, \ldots, D_\eta, \ldots$, $\eta < \Omega$, be all Borel subsets of $I \times I$ that are not in $\mathscr{S}$, and let $F_1, F_2, \ldots$, $F_\eta, \ldots, \eta < \Omega$, be all uncountable Borel subsets of I. As in [1], construct four transfinite sequences of points in I, namely $\{v_\eta\}$, $\{w_\eta\}$, $\{y_\eta^{(1)}\}$, and $\{y_\eta^{(2)}\}$, all indexed by $\eta < \Omega$, such that $v_\eta, w_\eta \in F_\eta$, $(y_\eta^{(1)}, y_\eta^{(2)}) \in D_\eta$, and no v_η belongs to the set

$$X = \{w_\eta \mid \eta < \Omega\} \cup \{y_\eta^{(1)} \mid \eta < \Omega\} \cup \{y_\eta^{(2)} \mid \eta < \Omega\}.$$

Put $\mathscr{A} = \mathscr{B} | X$ and $\mu = \nu | X$. Since the sets X and $I \setminus X$ are ν-thick, μ is not perfect.

To show that every product of μ with μ is countably additive, it is enough to prove (cf. [1], p. 338) that if λ is a measure on $\sigma(\mathscr{B} \otimes \mathscr{B})$ such that $\lambda(E \times I) = \nu E = \lambda(I \times E)$ for each $E \in \mathscr{B}$, and if $H \in \sigma(\mathscr{B} \otimes \mathscr{B})$ and $\lambda H > 0$, then $H \cap (X \times X) \neq \emptyset$. But from the lemma below it follows that if $H \notin \mathscr{S}$, then $H = D_\eta$ for some $\eta < \Omega$, and $(y_\eta^{(1)}, y_\eta^{(2)}) \in H \cap (X \times X)$.

LEMMA. *Let λ be a measure on $\sigma(\mathscr{B}\otimes\mathscr{B})$ such that $\lambda(E\times I) = \nu E = \lambda(I\times E)$ for $E\in\mathscr{B}$. If $H\in\sigma(\mathscr{B}\otimes\mathscr{B})$ and $\lambda H > 0$, then $H\notin\mathscr{S}$.*

Proof. Take any $V\subset I$ of cardinality smaller than $2^{\aleph_0}$. Denote the completion of ν by $\hat{\nu}$. There is a $\hat{\nu}$-disintegration $\{(\mathscr{B}, \nu_y)\}_{y\in I}$ of $\lambda|H$; that is, ν_y is a measure on $\mathscr{B}$ such that $\nu_y(I)\leqslant 1$ for each $y\in I$, the function $y\mapsto \nu_y E$ is $\hat{\nu}$-measurable for each $E\in\mathscr{B}$, and

$$\int_F \nu_y E\,d\hat{\nu}(y) = \lambda\big(H\cap(E\times F)\big) \quad \text{for } E, F\in\mathscr{B}.$$

There is a set $Z\in\mathscr{B}$ such that $\nu Z = 0$ and

$$\nu_y\{x\in I \mid (x, y)\notin H\} = 0 \quad \text{for } y\in I\setminus Z.$$

Put

$$A = \{y\in I \mid \nu_y(I\setminus C) = 0 \text{ for a countable set } C\subset I\}.$$

The set A is $\hat{\nu}$-measurable because

$$A = \bigcap_{k=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{r}\Big\{y \mid \nu_y\Big(I\setminus\bigcup_{i=1}^{j}[r_i, r_i+1/n]\Big)\leqslant 1/k\Big\},$$

where $\bigcup_r$ is the union over all j-tuples $r = (r_1, r_2, \ldots, r_j)$ of rational numbers. Approximate A by $A_0\subset A$, $A_0\in\mathscr{B}$, such that $\nu A_0 = \hat{\nu} A$. Distinguish two cases:

Case 1. $\lambda\big(H\cap(I\times A_0)\big) = 0$.

It follows that $\hat{\nu}\big(I\setminus(A\cup Z)\big) > 0$, hence $I\setminus(A\cup Z)$ has cardinality $2^{\aleph_0}$, and $I\setminus(A\cup Z\cup V)\neq\emptyset$. Take any $y\in I\setminus(A\cup Z\cup V)$: we have $\nu_y I > 0$, ν_y is nonatomic, and $\nu_y\{x \mid (x, y)\notin H\} = 0$. Hence the cardinality of $\{x \mid (x, y)\in H\}$ is $2^{\aleph_0}$, and there exists an $x\notin V$ such that $(x, y)\in H$. Therefore

$$(*)\qquad (x, y)\in H\setminus(V\times I)\cup(I\times V)\neq\emptyset.$$

Case 2. $\lambda\big(H\cap(I\times A_0)\big) > 0$.

Put

$$T = \{x\in I \mid \nu_y\{x\} > 0 \text{ for some } y\in A_0\setminus Z\},$$

$$\varkappa_y = \nu_y|T \text{ for } y\in I, \quad\text{and}\quad \varkappa E = \int_{A_0}\varkappa_y E\,d\hat{\nu}(y) \text{ for } E\in\mathscr{B}|T.$$

By Ramachandran's result [2], the measure $\varkappa$ on $\mathscr{B}|T$ is perfect (for it is a perfect mixture of discrete measures); hence there is a $T_0\subset T$, $T_0\in\mathscr{B}$, such that $\varkappa T_0 = \varkappa T$. Hence

$$\varkappa T_0 = \varkappa T = \int_{A_0}\varkappa_y T\,d\hat{\nu}(y) = \int_{A_0}\nu_y I\,d\hat{\nu}(y) = \lambda\big(H\cap(I\times A_0)\big) > 0,$$

and $\varkappa$ is nonatomic because $\varkappa\{x\} \leqslant \nu\{x\} = 0$ for $x \in T$. Hence T_0 has cardinality $2^{\aleph_0}$ and, consequently, there is an $x \in T_0 \setminus V$ such that $\nu_y\{x\} = 0$ for every $y \in V$. But $\nu_y\{x\} > 0$ for some $y \in A_0 \setminus Z$, and $y \notin V$ by the choice of x. Since $y \notin Z$, the point (x, y) is in H. Therefore again $(*)$ holds.

REFERENCES

[1] J. K. Pachl, *Two classes of measures*, Colloquium Mathematicum 42 (1979), p. 331-340.

[2] D. Ramachandran, *Mixtures of perfect probability measures*, The Annals of Probability 2 (1974), p. 495-500.

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